NOTES ON THE THEORY AND APPLICATION OF FOURIER TRANSFORMS. I–II*

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Introduction

We propose to publish under the above title a series of notes. The results are of a varied nature, but the methods we employ are very similar and consist, roughly speaking, in conformally mapping the unit circle into a half plane, and considering the Fourier transforms of functions defined on the boundary of the half plane. The notes may be read independently.

I. On a theorem of Carleman

1. The chief object of this note is to give a simple proof of the following theorem which is substantially the same as one due to Carleman.† Let $A_0=1, A_1, \cdots, A_\nu, \cdots$ be a set of positive numbers, and let C_A denote the set of functions defined in the interval $(-\infty, \infty)$, infinitely many times differentiable in that range, and satisfying the inequalities

$$(*)\qquad \int_{-\infty}^{\infty} |f^{(\nu)}(x)|^2 dx \leqq B^\nu A_\nu^2 \qquad (\nu = 0, 1, 2, \cdots),$$

where B is a constant which may depend on $f(x)$. We say that the class C_A is quasi-analytic if a function of C_A is defined completely over $(-\infty, \infty)$ by the values of its derivatives $f^{(\nu)}(x)$ $(\nu=0, 1, 2, \cdots)$ at a single point x_0, or, what is the same thing, if the equations

$$f^{(\nu)}(x_0) = 0 \qquad (\nu = 0, 1, 2, \cdots),$$

together with the condition $f(x) \subset C_A$, imply that $f(x)$ vanishes identically. The theorem is the following:

THEOREM I. *A necessary and sufficient condition that C_A should be quasi-analytic is that the integral*

$$(1)\qquad \int_0^\infty \log\left(\sum_{\nu=0}^\infty \frac{x^{2\nu}}{A_\nu^2}\right)\frac{dx}{1+x^2}$$

* Presented to the Society, February 25, 1933; received by the editors January 19, 1933. [Mathematical science has suffered an irreparable loss in the untimely death of R. E. A. C. Paley. He was killed on April 7, 1933, at the age of twenty-five, in an accident that occurred during a skiing excursion near Banff, Alberta. J. D. Tamarkin.]

† T. Carleman, *Les Fonctions Quasi-Analytiques*, 1926. We have slightly modified Carleman's definition of C_A. We consider $\int_{-\infty}^{\infty} |f^{(\nu)}(x)|^2 dx$ instead of $\max_{-1\leqq x\leqq 1} |f^{(\nu)}(x)|$, the problem in this form being more adaptable to our attack, but the difference is not at all essential.

should diverge, or, what is the same thing, that the least non-increasing majorant of the series

$$\sum_{\nu=0}^{\infty} 1/(A_\nu)^{1/\nu}$$

should diverge.

The equivalence of the two conditions has been established by Carleman in his book.* In this paper we shall concern ourselves only with the first one.

2. We begin by proving the following theorem.

THEOREM II. *Let $\phi(x)$ be a real non-negative function not equivalent to zero, defined for $-\infty < x < \infty$, and of integrable square in this range. A necessary and sufficient condition that there should exist a real- or complex-valued function $F(x)$ defined in the same range, vanishing for $x \geqq x_0$ for some number x_0, and such that the Fourier transform $G(x)$ of $F(x)$ should satisfy $|G(x)| = \phi(x)$, is that*

$$\int_{-\infty}^{\infty} \frac{|\log \phi(x)|}{1+x^2}\,dx < \infty . \tag{2}$$

We observe that the theorem is similar to one due to de la Vallée Poussin.† He concerns himself with the Fourier coefficients of a periodic function all of whose derivatives vanish at some fixed point. Here we demand rather more of the function, and we undertake actually to fix the modulus of the transform, subject of course to the convergence of (2).

3. Suppose first that the integral (2) converges. We write for $z = x + iy$, $y > 0$,

$$\lambda(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\log \phi(x')y}{(x-x')^2+y^2}\,dx',$$

which is harmonic in the half plane $y > 0$. Let $\mu(z)$ be its conjugate, and write

$$h(z) = \exp\,[\lambda(z) + i\mu(z)].$$

It is well known, by an argument of the Fatou type, that, for almost all x,

$$\lim_{y\to 0} \lambda(x+iy) = \log \phi(x),$$

or, what is the same thing,

$$\lim_{y\to 0} |h(x+iy)| = \phi(x).$$

We observe first that, by the convexity property,

* Carleman, loc. cit., pp. 50ff.

† Carleman, loc. cit., pp. 76 and 91.

$$| h(x+iy) | = e^{\lambda(z)} \leqq \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\phi(x')y}{(x-x')^2+y^2}dx'$$

$$\leqq \frac{1}{\pi}\left\{\int_{-\infty}^{\infty}\phi(x')^2dx'\right\}^{1/2}\left\{\int_{-\infty}^{\infty}\left(\frac{y}{(x-x')^2+y^2}\right)^2dx'\right\}^{1/2}$$

and tends to zero as $y\to\infty$, uniformly in x. This shows that $h(x+iy)$ is uniformly bounded in any half-plane $y \geqq y_0 > 0$.

Next

$$\int_{-\infty}^{\infty} | h(x+iy) |^2 dx \leqq \frac{1}{\pi}\int_{-\infty}^{\infty}dx \int_{-\infty}^{\infty}\frac{\phi(x')^2 y}{(x-x')^2+y^2}dx'$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\phi(x')^2dx'\int_{-\infty}^{\infty}\frac{ydx}{(x-x')^2+y^2}$$

$$= \int_{-\infty}^{\infty}\phi(x')^2dx',$$

and is therefore uniformly bounded in y. Now let $0<y_0<y<y_1$. Cauchy's theorem gives

$$\begin{aligned} -\ 2\pi i h(x+iy) &= \int_{-N}^{N}\frac{h(x'+iy_0)}{(x-x')+i(y-y_0)}dx' - \int_{-N}^{N}\frac{h(x'+iy_1)}{(x-x')+i(y-y_1)}dx' \\ &+ \int_{y_0}^{y_1}\frac{h(N+iy')}{(x-N)+i(y-y')}dy' - \int_{y_0}^{y_1}\frac{h(-N+iy')}{(x+N)+i(y-y')}dy'. \end{aligned}$$

Making first N and then y_1 tend to infinity in the last formula we obtain

$$h(x+iy) = -(2\pi i)^{-1}\int_{-\infty}^{\infty}\frac{h(x'+iy_0)}{(x-x')+i(y-y_0)}dx'. \tag{3}$$

Now let $H_y(\xi)$ denote the Fourier transform

$$\underset{A\to\infty}{\text{l.i.m.}}\ (2\pi)^{-1/2}\int_{-A}^{A}h(x+iy)e^{ix\xi}\,dx$$

of $h(x+iy)$. Since the Fourier transform of

$$-\ (2\pi i)^{-1}[x+i(y-y_0)]^{-1}$$

is

$$(2\pi)^{-1/2}e^{(y-y_0)\xi}$$

for negative ξ, and vanishes for positive ξ, it follows that

$$H_y(\xi) = \begin{cases} (2\pi)^{1/2}H_{y_0}(\xi)\cdot(2\pi)^{-1/2}e^{(y-y_0)\xi}, & \xi < 0, \\ 0, & \xi > 0. \end{cases}$$

Thus we have $H_y(\xi)=0$ for $\xi>0$ and all positive y, and this gives

$$H_{y_0}(\xi) = \begin{cases} e^{(y_0-y)\xi}H_y(\xi), & \xi < 0, \\ 0, & \xi > 0. \end{cases}$$

Let us keep y fixed and make y_0 tend to zero. Since

$$\int_{-\infty}^{\infty} | h(x+iy_0) |^2 dx$$

is bounded, it follows that

$$\int_{-\infty}^{0} | H_y(\xi)e^{(y_0-y)\xi} |^2 d\xi \tag{4}$$

is bounded and increasing as y_0 decreases to zero, and thus the integral (4) tends to a limit as $y_0\to 0$, and $H_y(\xi)e^{(y_0-y)\xi}$ tends to $H_y(\xi)e^{-y\xi}$ in the mean of order 2. The Fourier transform of the function which coincides with $H_y(\xi)e^{(y_0-y)\xi}$ for $-\infty<\xi<0$, and which vanishes for $\xi>0$, is $h(x+iy_0)$, and hence $h(x+iy_0)$ tends in mean of order 2 to a function $G(x)$ as $y_0\to 0$. We have shown that the Fourier transform of $h(x+iy_0)$ (with y_0 fixed and positive) vanishes for $\xi>0$, and it follows that the same is true of the Fourier transform $F(\xi)$ of $G(x)$. We have already seen that $|G(x)|=\phi(x)$.

Now suppose that $F(x')$ vanishes for $x'>x_0$, where we may suppose without loss of generality that $x_0=0$. We are to show that the integral (2) converges. We write

$$G(x) = \underset{N\to\infty}{\text{l.i.m.}}\ (2\pi)^{-1/2}\int_{-N}^{N} F(x')e^{-ixx'}dx',$$

$$\psi(z) = \underset{N\to\infty}{\text{l.i.m.}}\ (2\pi)^{-1/2}\int_{-N}^{N} F(x')e^{-izx'}dx', \quad \Im z > 0.$$

The function $\psi(z)$ is readily seen to be analytic in the half plane $\Im z>0$. Suppose that we invert the half plane $\Im z>0$ into the circle $|\zeta|<1$, $\zeta=re^{i\theta}$, and that $G(x)$ becomes $\Gamma(e^{i\theta})$ and $\psi(z)$ becomes $\gamma(\zeta)$. Then it is easily seen that

$$\int_{-\pi}^{\pi} | \Gamma(e^{i\theta}) |^2 d\theta = 2\int_{-\infty}^{\infty} \frac{| G(x) |^2}{1+x^2}dx,$$

so that Γ certainly is of class L^2. Also a simple computation shows that if $re^{i\phi}$ is the inverse of $x'+iy'$, then

$$(2\pi)^{-1}\int_{-\pi}^{\pi}\Gamma(e^{i\theta})\frac{1-r^2}{1-2r\cos(\theta-\phi)+r^2}d\theta = \pi^{-1}\int_{-\infty}^{\infty}G(x)\frac{y'dx}{(x-x')^2+y'^2}$$

$$= \pi^{-1}\int_{-\infty}^{\infty}\frac{y'dx}{(x-x')^2+y'^2}\ \underset{N\to\infty}{\text{l.i.m.}}\ (2\pi)^{-1/2}\int_{-N}^{0}F(\xi)e^{-ix\xi}d\xi$$

$$= \lim_{N\to\infty}\pi^{-1}\int_{-\infty}^{\infty}\frac{y'dx}{(x-x')^2+y'^2}(2\pi)^{-1/2}\int_{-N}^{0}F(\xi)e^{-ix\xi}d\xi$$

$$= \lim_{N\to\infty}\pi^{-1}\int_{-N}^{0}\frac{F(\xi)d\xi}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}\frac{e^{-ix\xi}y'dx}{(x-x')^2+y'^2}$$

$$= \lim_{N\to\infty}(2\pi)^{-1/2}\int_{-N}^{0}F(\xi)e^{-ix'\xi+y'\xi}d\xi = \psi(x'+iy') = \gamma(re^{i\phi}),$$

so that γ is in fact the Poisson integral of $\Gamma(e^{i\theta})$. Then

$$(5)\quad \begin{aligned}&(2\pi)^{-1}\int_{-\pi}^{\pi}\log^+|\gamma(re^{i\theta})|\,d\theta\\ &\qquad\leqq (2\pi)^{-1}\int_{-\pi}^{\pi}|\gamma(re^{i\theta})|^2d\theta \leqq (2\pi)^{-1}\int_{-\pi}^{\pi}|\Gamma(e^{i\theta})|^2d\theta.\end{aligned}$$

It is known by a theorem of Ostrowski* and Nevanlinna that the boundedness of the integral (5) implies that of the integral

$$(2\pi)^{-1}\int_{-\pi}^{\pi}|\log|\gamma(re^{i\theta})||\,d\theta.$$

Since finally $\log|\gamma(re^{i\theta})|$ tends almost everywhere to $\log|\Gamma(e^{i\theta})|$ we have

$$(2\pi)^{-1}\int_{-\pi}^{\pi}|\log|\Gamma(e^{i\theta})||\,d\theta < \infty,$$

and inverting again to the half plane this implies that

$$\int_{-\infty}^{\infty}\frac{|\log|G(x)||}{1+x^2}dx < \infty,$$

which is the same as (2).

4. Returning now to the proof of Carleman's theorem we observe first that if the integral (1) converges, and

$$\phi(x)^2 = (10)^{-1}(1+x^2)^{-1}\left[\sum_{\nu=0}^{\infty}\frac{x^{2\nu}}{A_\nu^2}\right]^{-1},$$

* See, e. g., A. Ostrowski, *Über die Bedeutung der Jensenschen Formel für einige Fragen der komplexen Funktionentheorie*, Acta Szeged, vol. 1 (1923), pp. 80–87.

then

$$\int_{-\infty}^{\infty} \frac{|\log \phi(x)|}{1+x^2}\,dx < \infty, \quad \int_{-\infty}^{\infty} \phi(x)^2 dx < \infty,$$

so that we can find a function $F(x) \subset L^2$ which vanishes for $x>0$ but not identically and with its Fourier transform $G(x)$ satisfying $|G(x)| = \phi(x)$. Finally we have

$$\int_{-\infty}^{\infty} |F^{(\nu)}(x)|^2 dx = \int_{-\infty}^{\infty} |G(x)|^2 x^{2\nu} dx$$
$$= \int_{-\infty}^{\infty} \phi(x)^2 x^{2\nu} dx \leqq \int_{-\infty}^{\infty} [10(1+x^2)]^{-1}\left(\frac{x^{2\nu}}{A_\nu^2}\right)^{-1} x^{2\nu} dx \leqq A_\nu^2 .$$

Thus the divergence of the integral (1) is certainly necessary for the quasi-analyticity of C_A.

Suppose now that $f(x)$ vanishes with all its derivatives at $x=0$, but does not vanish identically. We are to show that the integral (1) converges. Let $F(x)$ be identical with $f(x)$ for negative x and vanish identically for positive x, and let $G(x)$ be its Fourier transform. Then, assuming, as we may, that, in formula (*), $B=1$, we have

$$A_\nu^2 \geqq \int_{-\infty}^{\infty} |f^{(\nu)}(x)|^2 dx \geqq \int_{-\infty}^{\infty} |F^{(\nu)}(x)|^2 dx \geqq \int_{-\infty}^{\infty} |G(x)|^2 x^{2\nu} dx.$$

It follows that

$$\log\left(\sum_{\nu=0}^{\infty} \frac{r^{2\nu}}{A_\nu^2}\right) \leqq \log\left(\sum_{\nu=0}^{\infty}\left[\int_{-\infty}^{\infty} |G(x)|^2 \left(\frac{x}{r}\right)^{2\nu} dx\right]^{-1}\right)$$
$$\leqq \log\left(\sum_{\nu=0}^{\infty}\left[\int_{2r}^{2r+1} |G(x)|^2 \left(\frac{x}{r}\right)^{2\nu} dx\right]^{-1}\right)$$
$$\leqq \log\left(2\left[\int_{2r}^{2r+1} |G(x)|^2 dx\right]^{-1}\right) \leqq 2\int_{2r}^{2r+1} |\log(2^{-1/2}|G(x)|)|\,dx.$$

Hence

$$\int_1^{\infty} \log\left(\sum_{\nu=0}^{\infty}\frac{r^{2\nu}}{A_\nu^2}\right)\frac{dr}{r^2} \leqq 2\int_1^{\infty}\frac{dr}{r^2}\int_{2r}^{2r+1} |\log(2^{-1/2}|G(x)|)|\,dx$$
$$\leqq 2\int_2^{\infty} |\log(2^{-1/2}|G(x)|)|\,dx \int_{x/2-1/2}^{x/2}\frac{dr}{r^2}$$
$$\leqq 20\int_2^{\infty} |\log(2^{-1/2}|G(x)|)|\,x^{-2}dx < \infty.$$

Thus the divergence of (1) is also sufficient for quasi-analyticity.

II. On conjugate functions

1. We prove the following theorem.

Theorem. *Let $f(\theta)$ be an odd function defined in $(-\pi, \pi)$ and suppose that it is absolutely integrable and non-decreasing in this range. Let $\tilde{f}(\theta)$ be the conjugate function. Then $\tilde{f}(\theta) \subset L$.*

We may assume, without loss of generality, that, in addition to the above properties, $f(\theta)$ satisfies the condition of assuming only integral values, for it differs by at most 1 from a function having all these properties. Then the function conjugate to this step function will differ from $\tilde{f}(\theta)$ by a function which is certainly of class L^2.

2. Let us invert the circle $|z|<1$ into the half plane $\Im Z>0$, so that the point $e^{i\theta}$ inverts into $X(\theta)=\tan\,\theta/2$. Suppose that $f(\theta)$ inverts into $n(X)$. Then $n(X)$ is also non-decreasing in $(-\infty, \infty)$, and

$$\int_{-\pi}^{\pi} |f(\theta)|\, d\theta = 2\int_{-\infty}^{\infty} \frac{|n(X)|}{1+X^2}\, dX. \tag{1}$$

Let the points $\pm\lambda_\nu$, $\lambda_1 \leqq \lambda_2 \leqq \cdots$, be those at which $n(X)$ increases by 1 (if $n(X)$ increases by more than 1 then λ_n is taken an appropriate number of times). We may assume without loss of generality that λ_n is never zero, or, what is the same thing, that $n(X)$ is zero in the neighborhood of the origin. Then the convergence of the second integral (1) implies that of the series

$$\sum_{n=1}^{\infty} \lambda_n^{-1}. \tag{2}$$

3. Now consider the branch of the function

$$\frac{i}{\pi} \sum_{\nu=1}^{\infty} \log\left(1 - \frac{Z^2}{\lambda_\nu^2}\right) \tag{3}$$

which takes the value 0 at the origin, and is regular in the half plane $\Im Z \geqq 0$, $Z \neq \pm\lambda_\nu$ (the function (3) certainly exists in virtue of (2)). We observe that the real part of the function (3) coincides with $n(X)$ on the real axis, and is indeterminate at the points λ_ν. Further it may easily be seen that the imaginary part of the function (3) on the real axis differs only by a constant

$$Ci = -\pi i \int_{-\infty}^{\infty} \frac{\sum_{\nu=1}^{\infty} \log\left(1 - \dfrac{X^2}{\lambda_\nu^2}\right)}{\pi(1+X^2)}\, dX$$

since

$$\int_{-\infty}^{\infty}\left\{\Re\left[\frac{1}{\pi}\sum_{\nu=1}^{\infty}\log\left(1-\frac{X^2}{\lambda_\nu^2}\right)\right]+C\right\}\frac{2dX}{1+X^2}=\int_{-\pi}^{\pi}\tilde{f}(\theta)d\theta=0,$$

from the transform to the conjugate $\tilde{f}(\theta)$ of $f(\theta)$. Thus it is sufficient to establish the existence of the integral

$$\frac{\pi}{2}\int_{-\pi}^{\pi}|\tilde{f}(\theta)|\,d\theta-\pi^2C=\int_{-\infty}^{\infty}\left|\Re\left\{\sum_{\nu=1}^{\infty}\log\left(1-\frac{X^2}{\lambda_\nu^2}\right)\right\}\right|\frac{dX}{1+X^2}$$

$$\leqq\int_{-\infty}^{\infty}\left|\Re\left\{\sum_{\nu=1}^{\infty}\log\left(1-\frac{X^2}{\lambda_\nu^2}\right)\right\}\right|\frac{dX}{X^2}. \tag{4}$$

The integral (4) does not exceed

$$\sum_{\nu=1}^{\infty}\int_{-\infty}^{\infty}\left|\log\left|1-\frac{X^2}{\lambda_\nu^2}\right|\right|\frac{dX}{X^2}\leqq\sum_{\nu=1}^{\infty}\lambda_\nu^{-1}\int_{-\infty}^{\infty}\left|\log\left|1-X^2\right|\right|\frac{dX}{X^2}<\infty,$$

in virtue of (2), and our theorem is proved.

4. We observe finally that the restriction that $f(\theta)$ should be an odd function is an essential one. For suppose that the theorem were true without this restriction. Let $f(\theta)$ be a function which is positive and which increases in $(-\pi, \pi)$, satisfying the two conditions

$$\int_{-\pi}^{\pi}f(\theta)d\theta<\infty,\qquad\int_{-\pi}^{\pi}f(\theta)\log^+f(\theta)d\theta=\infty.$$

Then, on the assumption that the theorem is true in the extended form, we should have

$$\int_{-\pi}^{\pi}|\tilde{f}(\theta)|\,d\theta<\infty.$$

But, by a theorem of M. Riesz,* the last inequality, together with the condition $f(\theta)>0$, implies that

$$\int_{-\pi}^{\pi}f(\theta)\log^+f(\theta)d\theta<\infty,$$

giving a contradiction.

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