PFAFFIAN SYSTEMS OF SPECIES ONE*

BY
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This paper begins a study of the minimum number of differentials in terms of which a Pfaffian system can be expressed, together with the related subjects of reduced and canonical forms for such systems. The particular case treated here is that in which the minimum number of differentials exceeds the number of equations by unity. Such a system is said to be of *species one*. After passive (completely integrable) systems it is the simplest type. It is characterized by the following property: the adjunction of a single (suitably chosen) equation to it gives a passive system.

It is shown how to reduce any system of species one to a form involving the minimum number of differentials.

A system is called *nested* if the above reduction can be effected simultaneously for it and all of its derived systems. It is shown how to reduce any nested system of species one to a canonical form. It results that any such system can be written as the sum of a set of *special* systems, and that it is characterized by a finite set of arithmetical invariants. There is also given a necessary and sufficient condition for the existence of a nested system of species one having a given set of integers as invariants.

A further study of systems of species one in relation to their derived systems requires a theory of systems of higher species, which it is hoped to develop in a subsequent article.

The methods employed are largely those developed by Cartan and expounded by Goursat, with whose book† we suppose the reader is familiar.

1. **Generalities on the species of a Pfaffian system.** A *Pfaffian* is a form linear in the differentials of a finite set of variables, the coefficients being analytic functions of the variables. A *Pfaffian system* is obtained by equating to zero a linearly independent (and therefore finite) set of Pfaffians.

The variables are denoted by $x^1, x^2, \cdots, x^n$.

The *class* of a system is the minimum number of variables in terms of which a system equivalent to it can be expressed. A system comprising a single equation is always of odd class $2\sigma+1$. The minimum number of differentials which can appear in an equivalent equation is known to be $\sigma+1$,

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* Presented to the Society, December 30, 1931; received by the editors January 20, 1932.

† Goursat, E., *Leçons sur le Problème de Pfaff*, Paris, 1922.

and is consequently determined by the class. This is not true in general for systems comprising more than one equation.

Consider the family of varieties

$$f^1(x) = \text{const.}, \cdots, f^k(x) = \text{const.}, \tag{1.1}$$

one of which passes through every point of a region. The rank of the matrix

$$\left\| \frac{\partial f}{\partial x} \right\|$$

is assumed to be k. If the above family is integral for the Pfaffian system

$$\omega^1 = 0, \cdots, \omega^r = 0, \tag{1.2}$$

the ω's must be linear homogeneous combinations of the df's; and conversely. Hence, *the minimum number of f's defining a family of integral varieties one of which passes through every point of a region is the same as the minimum number of differentials in terms of which an equivalent system can be expressed.*

Let the minimum number of differentials for system (1.2) be denoted by $r+\sigma$. The non-negative integer σ so defined will be called the species of the system.* The justification for this name is that the species enables us to differentiate between systems not differing in class or genus. Thus for every system of two equations of class five, expressed in characteristic variables, the genus, as defined by Cartan, is unity, whereas the species may be either one or two.

A function which is not identically equal to a constant and whose constancy is implied by (1.2) is called an *integral* of that system. Sometimes by ellipsis this term is also used to designate an *integral variety*, which we shall always designate by its full name, reserving the name integral for the concept to which it was attached by Poincaré.

The maximum number of independent integrals which the system (1.2) can possess is r and is attained only when the system is *passive*,† i.e., when it is equivalent to a system of the form

* The notion of species is inherent in the paper of E. von Weber, *Theorie der Systeme Pfaff'scher Gleichungen*, Mathematische Annalen, vol. 55 (1902), pp. 386–440. Although he does not mention the invariance of the minimum number of differentials, von Weber considers the possibility of reducing a given system to a form containing a specified number of differentials. In the terminology of the present paper, he gives necessary and sufficient conditions that the species do not exceed a specified value for systems whose *Stufe* (=class minus number of equations) does not exceed six.

† The usual term is *completely integrable* or *in involution*. We prefer to extend Riquier's terminology to Pfaffian systems. The justification of this lies in the fact that a Pfaffian system passive in the sense just defined is equivalent to a system of partial differential equations which is passive in Riquier's sense.

In the general theory of partial differential equations, the term passive is applicable to systems for which an existence theorem has not been proved and for which the name completely integrable would be at least temporarily a misnomer.

$$dx^1 = 0, \cdots, dx^r = 0. \tag{1.3}$$

Passive systems are therefore systems of r equations whose class is r. Their species is zero.

It is known that the differentials appearing in a non-passive system can always be made fewer than the variables by at least unity. A consequence of this is that no system of r equations can be of class $r+1$, and that *the maximum value of the species for a system of class p is $p-r-1$.*

2. **The maximum system of species zero implied by a given system is determined by a maximum set of independent integrals.** The latter can be found from the last derived system of (1.2). A more direct method of finding it is furnished by

THEOREM 1. *The integrals of the Pfaffian system*

$$\omega^1 = 0, \cdots, \omega^r = 0$$

are the non-trivial solutions of the linear homogeneous partial differential equation of the first order

$$\omega^1 \cdots \omega^r df = 0. \tag{2.1}$$

The proof is almost immediate. Since the equations of the Pfaffian system are independent, we have

$$\omega^1 \cdots \omega^r \neq 0. \tag{2.2}$$

Equation (2.1) is then a necessary and sufficient condition* for the existence of multipliers λ such that

$$df = \lambda_1 \omega^1 + \cdots + \lambda_r \omega^r. \tag{2.3}$$

This proves the theorem.

Suppose f^1 is a non-trivial solution of (2.1). If we have also

$$\omega^2 \cdots \omega^r df^1 = 0, \tag{2.4}$$

from (2.3) and (2.2) we deduce that the λ_1 in (2.3) is zero. If all r expressions of which the left member of (2.4) is the prototype were zero, equation (2.3) would show that f^1 is a constant, contrary to hypothesis. It is only a matter of notation, therefore, to assume that f^1 does not satisfy (2.4).

When the notation has been properly adjusted, system (1.2) is accordingly equivalent to

$$df^1 = 0, \omega^2 = 0, \cdots, \omega^r = 0, \tag{2.5}$$

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* Cartan, É., Bulletin de la Société Mathématique de France, vol. 29 (1901), p. 250.

and the integrals of (1.2) and (2.5) are the same. But the latter are by Theorem 1 the solutions of

$$\omega^2 \cdots \omega^r df^1 df = 0, \tag{2.6}$$

where f^1 is given and f is to be determined.

If (1.2) has more than one integral in its complete set, system (2.6) will have a solution f^2 for which $df^1 df^2 \neq 0$. From the discussion given above in the case of f^1, we know that adjusting the notation will make

$$\omega^3 \cdots \omega^r df^1 df^2 \neq 0,$$

and (1.2) is equivalent to

$$df^1 = 0,\ df^2 = 0, \omega^3 = 0, \cdots, \omega^r = 0.$$

If another integral f^3 exists, it is a solution of

$$\omega^3 \cdots \omega^r df^1 df^2 df = 0.$$

We continue until we reach an equation which has only a trivial solution. In this way, the complete set of q integrals is found. At the same time, we have a method of writing (1.2) in the form

$$dx^1 = 0, \cdots, dx^q = 0, \omega^{q+1} = 0, \cdots, \omega^r = 0, \tag{2.7}$$

which puts in evidence its maximum system of species zero written in the form (1.3). If $q=r$, the system is passive and the method reduces it to the form (1.3).

3. **Determination of the species.** The system

$$\omega^1 = 0, \cdots, \omega^r = 0, df^{r+1} = 0, \cdots, df^{r+k} = 0, \tag{3.1}$$

where

$$\omega^1 \cdots \omega^r df^{r+1} \cdots df^{r+k} \neq 0, \tag{3.2}$$

is passive if and only if (1.2) can be expressed in terms of $r+k$ differentials. The problem of finding a minimum set of differentials is therefore equivalent to that of finding a minimum passive Pfaffian system which implies the given one.

If we put

$$\omega^1 \cdots \omega^r \omega'^\alpha = \Omega^\alpha, \tag{3.3}$$

the conditions of passivity* of (3.1) are

$$\Omega^\alpha df^{r+1} \cdots df^{r+k} = 0. \tag{3.4}$$

* Cartan, E., *Leçons sur les Invariants Intégraux*, Paris, 1922, p. 101.

To determine the species, therefore, we form (3.1) and (3.2) for $k=0, 1, \cdots$ until the first consistent system is reached. The k at this stage is the species.

4. **Reduced form for systems of species one.** A system (1.2) is in *reduced form* if it involves the minimum number of differentials and is solved for r of them. The set of variables whose differentials appear in a reduced form will likewise be described by the adjective reduced.

Consider a system of species one expressed in the minimum number of differentials. Since the system is not passive, there are at least two of the $r+1$ differentials whose vanishing is not implied by the system. Call them dx^r and dx^{r+1}. Algebraically considered, (1.2) is a linear and homogeneous system of rank r in $r+1$ unknowns dx, and the unknown dx^{r+1} in particular can be chosen arbitrarily. Hence (1.2) can be written in the form

$$dx^1 - A^1 dx^{r+1} = 0, \cdots, dx^r - A^r dx^{r+1} = 0. \tag{4.1}$$

If (1.2) is written in the form (2.7), the system

$$\omega^{q+1} = 0, \cdots, \omega^r = 0 \tag{4.2}$$

can be put in the form (4.1) because q of the $r+1$ differentials can be eliminated by means of $dx^1 = \cdots = dx^q = 0$. Hence (1.2) can be written as

$$dx^1 = 0, \cdots, dx^q = 0, dx^{q+1} - A^{q+1}dx^{r+1} = 0, \cdots, dx^r - A^r dx^{r+1} = 0, \tag{4.3}$$

where the A's are all different from zero, and *the maximum system of species zero is*

$$dx^1 = 0, \cdots, dx^q = 0. \tag{4.4}$$

5. **Reduced variables for systems of species one.** If $k=1$, system (3.4) is

$$\omega^1 \cdots \omega^r \omega'^\alpha df = 0 \qquad (\alpha = 1, 2, \cdots, r), \tag{5.1}$$

and is equivalent to a system of linear homogeneous partial differential equations of the first order in a single unknown function f. From §3, if (5.1) has a solution not satisfying (2.1), the system (1.2) is of species one or zero. Hence we have

THEOREM 2. *A non-passive Pfaffian system* (1.2) *is of species one if and only if the auxiliary system* (5.1) *has a solution which is not an integral of* (1.2).

Let f^{r+1} be a solution of (5.1) which does not satisfy (2.1). The system

$$\omega^1 = 0, \cdots, \omega^r = 0, df^{r+1} = 0 \tag{5.2}$$

is passive. Its integrals are a set of reduced variables for (1.2).

It was seen in §4 that if $f^1, \cdots, f^{r+1}$ constitute a set of reduced variables any one of them which is not an integral of the system can be made to play the role of x^{r+1} in the discussion of reduced form. Adjoining $dx^{r+1}=0$ to a reduced form of the equations obviously gives a passive system of $r+1$ equations. Hence any reduced variable which is not an integral can play the role of f^{r+1} in (5.2) and will therefore satisfy (5.1). Since the integrals obviously satisfy (5.1), *all the reduced variables satisfy* (5.1).

If (5.1) has only $r+1$ independent solutions, any $r+1$ independent solutions will be a set of reduced variables because any set of reduced variables will be functions of those $r+1$ solutions, and the differentials of the reduced variables will be linear homogeneous combinations of the differentials of the $r+1$ solutions, so that the system can be expressed in terms of the latter set of differentials. Hence we have

THEOREM 3. *For a system of species one the auxiliary system* (5.1) *has at least* $r+1$ *independent solutions. If it has exactly* $r+1$, *any set of* $r+1$ *independent solutions is a set of reduced variables.*

An example is

$$dx^1 - x^2dx^5 = 0,\ dx^2 - x^1dx^3 - x^4dx^5 = 0. \tag{5.3}$$

The auxiliary system is

$$x^1\frac{\partial f}{\partial x^2} + \frac{\partial f}{\partial x^3} = 0,\ \frac{\partial f}{\partial x^4} = 0.$$

It has just three solutions: $x^1, x^5, x^2-x^1x^3$. By the use of them, system (5.3) can be written in the reduced form

$$dx^1 - x^2dx^5 = 0,\ d(x^2 - x^1x^3) - (x^4 - x^2x^3)dx^5 = 0.$$

The auxiliary system can, however, have more than $r+1$ solutions. An example is furnished by the system

$$dx^1 - x^2dx^3 = 0, \tag{5.4}$$

whose auxiliary system has the $r+2$ solutions x^1, x^2, x^3.

Since the forms (3.1) are of degree $r+2$, except when they are zero, they cannot have more than $r+2$ linear factors. Consequently, *the number of independent solutions of the auxiliary system never exceeds* $r+2$ *unless* (1.2) *is passive.* When this maximum number of solutions is attained is stated by

THEOREM 4. *The auxiliary system for a system of species one has* $r+2$ *independent solutions if and only if the class is* $r+2$.

If the class is $r+2$, let $x^1, \cdots, x^{r+2}$ be characteristic variables. When the forms (3.1) are expressed in terms of these x's, they become

(5.5) $$\Omega^\alpha = B^\alpha dx^1 \cdots dx^{r+2} \qquad (\alpha = 1, 2, \cdots, r),$$

because they are of degree $r+2$ in $r+2$ differentials. Hence the auxiliary system (5.1) has the $r+2$ x's for solutions.

Conversely, let $x^1, \cdots, x^{r+2}$ be independent solutions of (5.1). Equations (5.5) then hold. From them we find the characteristic system* of (1.2) to be

$$dx^1 = 0, \cdots, dx^{r+2} = 0.$$

Hence the class is $r+2$, and the theorem is proved.

When the auxiliary system has $r+2$ solutions, $r+1$ of them chosen at random do not necessarily form a set of reduced variables. Thus x^1 and x^3 do not form such a set for (5.4). When one solution f^{r+1}, other than an integral, has been found for the auxiliary system, the set of reduced variables is completed by finding the integrals of (5.2). This can be accomplished by the method developed in §2; for example, by solving

(5.6) $$\omega^1 \cdots \omega^r df^{r+1} df = 0.$$

For the example (5.4) with $f^{r+1} = x^3$, system (5.6) is

$$x^3 \frac{\partial f}{\partial x^1} + \frac{\partial f}{\partial x^2} = 0,$$

of which a solution is $x^1 - x^2x^3$. Hence a set of reduced variables containing the variable x^3 is x^3, $x^1 - x^2x^3$, and the equation can be written

$$d(x^1 - x^2x^3) + x^2dx^3 = 0.$$

To put a given system in reduced form, write it first in the form (2.7), and then consider the auxiliary system for (4.2). When one solution of the latter has been found, the set of reduced variables can be completed as indicated above for (5.6).

6. **Properties of the reduced form for systems of species one.** For a system in the form (4.3) the derived forms ω', with $dx^1, \cdots, dx^r$ eliminated by means of (4.3), are

(6.1) $$\omega'^1 = 0, \cdots, \omega'^q = 0, \quad \omega'^{q+1} = -\partial A^{q+1} dx^{r+1}, \cdots, \omega'^r = -\partial A^r dx^{r+1},$$

where ∂ denotes the differential formed on the assumption

(6.2) $$x^1 = \text{const.}, \cdots, x^{r+1} = \text{const.}$$

Now one of the ∂A's must be different from zero. Otherwise all the derived

* Cartan, loc. cit. in §3 of this paper.

forms would be zero and the system would be passive. Hence the characteristic system contains the equations

$$dx^1 = 0, \cdots, dx^{r+1} = 0, \ \partial A^{q+1} = 0, \cdots, \partial A^r = 0.$$

Its rank, the class p of the system, is accordingly the number of independent variables in the set

$$x^1, \cdots, x^{r+1}; \ A^{q+1}, \cdots, A^r. \tag{6.3}$$

The rank of the matrix

$$\left\| \begin{matrix} \dfrac{\partial A^{q+1}}{\partial x^{r+2}}, & \cdots, & \dfrac{\partial A^{q+1}}{\partial x^n} \\ \cdots, & \cdots, & \cdots \\ \dfrac{\partial A^r}{\partial x^{r+2}}, & \cdots, & \dfrac{\partial A^r}{\partial x^n} \end{matrix} \right\| \tag{6.4}$$

is therefore $p-r-1$.

The number of linearly independent solutions of

$$\lambda_{q+1}\partial A^{q+1} + \cdots + \lambda_r \partial A^r = 0$$

is the number of unknowns minus the rank of (6.4). This number, increased by q corresponding to equations (4.4), gives the number of equations r^1 in the first derived system:

$$r^1 = 2r + 1 - p. \tag{6.5}$$

7. **The genus of a Pfaffian system.** The genus of a Pfaffian system, as defined by Cartan,* satisfies the equation

$$n - \gamma = r + s_1 + s_2 + \cdots, \tag{7.1}$$

where the sum on the right extends over all the characters s. The number γ is the maximum dimensionality of a non-singular integral variety.

In studying Pfaffian systems, however, it is customary to consider changes of variables which do not preserve dimensionality, i.e., the number of dimensions of the representative space may change. Hence γ is not an invariant with respect to the transformations considered. This leads to some confusion, which is perhaps only increased by defining a "true" genus.†

On the other hand, the non-zero characters are invariant under the trans-

* Cartan, E., *Sur l'intégration des systèmes d'équations aux différentielles totales,* Annales Scientifiques de l'Ecole Normale Supérieure, (3), vol. 18 (1901), p. 262.

† Cf. Goursat, p. 362; Cartan, p. 291.

formations in question. Consequently, the left member of (7.1) is invariant. Its significance is *the minimum number of independent relations defining a non-singular integral variety*. Its least value is r. We shall write

$$g = n - \gamma - r = s_1 + s_2 + \cdots . \tag{7.2}$$

Because of its invariance, we shall employ g rather than γ, and in order not to multiply names needlessly, we shall call g the genus rather than γ. The least value of g is zero and occurs for a passive system. For a non-passive system, the value of γ computed in characteristic variables is at least one. Hence, *the genus of a non-passive system satisfies the inequalities*

$$0 < g \leqq p - r - 1. \tag{7.3}$$

An integral variety defined by fewer than $g+r$ relations is necessarily singular.

8. **The genus of systems of species one.** The system* whose rank is the character $s = s_1$ of system (4.3) is readily found by use of (6.1) to be

$$\begin{aligned} &\left(\frac{\partial A^i}{\partial x^{r+2}} dx^{r+2} + \cdots + \frac{\partial A^i}{\partial x^p} dx^p\right)\delta x^{r+1} \\ &\quad - dx^{r+1}\left(\frac{\partial A^i}{\partial x^{r+2}} \delta x^{r+2} + \cdots + \frac{\partial A^i}{\partial x^p} \delta x^p\right) = 0 \qquad (i = q+1, \cdots, r), \end{aligned} \tag{8.1}$$

where the variables are characteristic (i.e., $n = p$), the dx's are given, and the ∂x's are the unknowns.

If $dx^{r+1} \neq 0$, multiplying the second, third, $\cdots$, $(p-r)$th columns of the matrix of (8.1) by dx^{r+2}/dx^{r+1}, dx^{r+3}/dx^{r+1}, $\cdots$, dx^p/dx^{r+1}, respectively, and adding to the first reduces it to a column of zeros. Hence the rank of (8.1) is that of (6.4), namely, $p-r-1$. Thus the value of the character is

$$s = p - r - 1, \tag{8.2}$$

a formula which of course only applies to non-passive systems. Since the genus is at least equal to s, inequality (7.3) gives

$$s = g, \tag{8.3}$$

a result which we state as

THEOREM 5. *For any system of species one the genus and character are equal.*

Because of (8.3) equation (6.5) can be written

$$s = r - r^1. \tag{8.4}$$

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* Goursat, p. 290.

9. **Systems of species one whose first derived system has species not exceeding one.** If the first r^1 equations of a system S written in reduced form (4.1) constitute its derived system S^1, the auxiliary systems (5.1) for S and S^1 have in common the solution x^{r+1}, which is not an integral of S or S^1.

Conversely, if the auxiliary systems of S and S^1 have in common a solution x^{r+1} which is not an integral, let a set of reduced variables containing x^{r+1} be found for S^1. By means of it S^1 is put in the form

$$S^1:\ dx^1 - A^1dx^{r+1} = 0, \cdots, dx^{r^1} - A^{r^1}dx^{r+1} = 0. \tag{9.1}$$

Clearly $x^1, \cdots, x^{r^1}$ are integrals of the passive system obtained by adjoining $dx^{r+1}=0$ to S. Since any set of $r+1$ independent integrals of that passive system is a set of reduced variables for S, a set of reduced variables for S containing $x^1, \cdots, x^{r^1}, x^{r+1}$ can be found, whereby the equations Σ which it is necessary to adjoin to S^1 in order to get S can be given the form

$$\Sigma:\ dx^{r^1+1} - A^{r^1+1}dx^{r+1} = 0, \cdots, dx^r - A^rdx^{r+1} = 0. \tag{9.2}$$

The conditions that S^1 be the derived system of S can be written

$$\omega^1 \cdots \omega^r\omega'^\alpha = 0 \qquad (\alpha = 1, 2, \cdots, r^1). \tag{9.3}$$

Substitution from (9.1) and (9.2) gives

$$dx^1 \cdots dx^{r+1}dA^\alpha = 0 \qquad (\alpha = 1, 2, \cdots, r^1). \tag{9.4}$$

Consequently $A^1, \cdots, A^{r^1}$ are functions of $x^1, \cdots, x^{r+1}$ alone:

$$A^1 = A^1(x^1, \cdots, x^{r+1}), \cdots, A^{r^1} = A^{r^1}(x^1, \cdots, x^{r+1}). \tag{9.5}$$

Because of the statement just preceding (6.3), relations (9.5) show that the number of independent variables in the set

$$x^1, \cdots, x^{r+1}, A^{r^1+1}, \cdots, A^r \tag{9.6}$$

is the class p. From (6.5) the class is $2r-r^1+1$. Since this is the number of variables (9.6), those variables form an independent set.

If S^1 is passive, any one of its reduced variables is an integral of S and therefore satisfies the auxiliary system for S.

If S^1 is not passive and its class is not r^1+2, by Theorems 3 and 4 any reduced variable for S^1 is a function of the $x^1, \cdots, x^{r^1}, x^{r+1}$ in (9.1) and therefore satisfies the auxiliary system of S.

If S^1 is of class r^1+2, one of the coefficients in (9.1) must involve one of the variables $x^{r^1+1}, \cdots, x^r$. Suppose one involves x^r. A change of variable will make the coefficient in question x^r. At the same time the form of the last equation of Σ can be preserved by using the equations of S to eliminate the excessive differentials. Direct calculation then shows that x^r is a characteristic variable and consequently a reduced variable for S^1; and from the form of (9.1), (9.2) x^r is reduced for S.

In every case, therefore, *if a variable which is not an integral is reduced for both S^1 and S, every variable reduced for S^1 is reduced for S also.*

As a result of the developments in this section we have

THEOREM 6. *Given a system S of species one whose first derived system S^1 has species not exceeding one. Any reduced form of S^1 can be made the first r^1 equations of a reduced form of S if and only if the auxiliary systems of S and S^1 have in common a solution which is not an integral.*

Simultaneous reduction is not always possible. The system

$$\begin{gathered} dx^1 - x^2dx^4 = 0, \\ dx^2 - x^5dx^4 = 0,\ dx^2 - x^6dx^3 - x^4dx^5 = 0 \end{gathered} \tag{9.7}$$

is of species one. Its derived system is the first equation and is also of species one. But S and S^1 cannot be thrown simultaneously into reduced form because their auxiliary systems have no solution in common.

10. **Nested systems of species one. Canonical form.** If the auxiliary systems (5.1) formed for a system and all its derived systems, except the last, have in common a solution which is not an integral, the system will be called a *nested system of species one.* All the derived systems, except the last, are systems of the same sort.

By a *canonical form* of a Pfaffian system we mean an equivalent system in which the variables are all independent and each equation is in the canonical form for a single equation.*

We next prove

THEOREM 7. *Every nested system of species one can be reduced to a canonical form having the following properties:*

(i) *The system S is in reduced form.*

(ii) *The first derived system S^1 is obtained by omitting the last $r-r^1$ equations of S, the second derived system S^2 by omitting the last r^1-r^2 equations of S^1, etc.*

(iii) *Every variable occurs at most once as a coefficient and at most once as a differential with the exception of the variable whose differential occurs in all the second terms.*

(iv) *Any variable which occurs as a coefficient but not as a differential in S^i occurs as a differential in S^{i-1}.*

(v) *The only variables which do not occur as coefficients are those whose dif-*

* See Goursat, p. 58, for a definition of the latter.

ferentials occur in the last derived system and the variable whose differential occurs in all the second terms.

(vi) *The only variables whose differentials do not occur at all are the coefficients in the equations which are in* S *but not in* S^1.

It is of course clear that every derived system will be in a canonical form having the same properties.

The theorem is immediate for $r=1$.

We now assume the theorem for $r-1$ equations and proceed by induction.

The given system S is not passive, but since we are to apply induction and one of the derived systems may be passive, it is necessary to consider the passive case and to remark that a passive system can obviously be thrown into a canonical form having all the enumerated properties.

Since S is not passive, the number of equations in its derived system does not exceed $r-1$. The first derived system can therefore be supposed written in canonical form as the first r^1 equations of S:

$$dx^1 = 0, \cdots, dx^q = 0,\ dx^{q+1} - x^\alpha dx^{r+1} = 0, \cdots, \quad dx^{r^1} - x^\beta dx^{r+1} = 0. \tag{10.1}$$

Theorem 6 can be applied to show that the remaining equations of S can be simultaneously given the form

$$dx^{r^1+1} - A^{r^1+1}dx^{r+1} = 0, \cdots, dx^r - A^r dx^{r+1} = 0. \tag{10.2}$$

Let the variables which occur as coefficients but not as differentials in S^1 be denoted by

$$\bar{x}^{r^1+1}, \cdots, \bar{x}^{r^1+s^1}. \tag{10.3}$$

Relations (9.5) give

$$\bar{x}^{r^1+1} = F^{r^1+1}(x^1, \cdots, x^{r+1}), \cdots, \bar{x}^{r^1+s^1} = F^{r^1+s^1}(x^1, \cdots, x^{r+1}). \tag{10.4}$$

It must be possible to solve (10.4) for s^1 of the variables

$$x^{r^1+1}, \cdots, x^r, \tag{10.5}$$

for otherwise their elimination would lead to a relation among

$$x^1, \cdots, x^{r^1}, \bar{x}^{r^1+1}, \cdots, \bar{x}^{r^1+s^1}, x^{r+1},$$

that is, among the variables in terms of which S^1 is expressed. This would contradict S^1's being in canonical form. In particular, we must have

$$s^1 \leqq r - r^1. \tag{10.6}$$

Hence by means of (10.4) we can express the differentials of s^1 of the variables (10.5), which it is merely a matter of notation to suppose are

$$(10.7)\qquad x^{r^1+1}, \cdots, x^{r^1+s^1},$$

as linear, homogeneous combinations of

$$dx^1, \cdots, dx^{r^1}, d\bar{x}^{r^1+1}, \cdots, d\bar{x}^{r^1+s^1}, dx^{r^1+s^1+1}, \cdots, dx^{r+1}.$$

By use of the equations of S^1 these become combinations of

$$d\bar{x}^{r^1+1}, \cdots, d\bar{x}^{r^1+s^1}, dx^{r^1+s^1+1}, \cdots, dx^{r+1}.$$

When these values have been substituted in (10.2), equations (10.2) can be put by solving into a similar form in which the differentials of the variables (10.7) are replaced by those of (10.3). So all x's occurring as coefficients in S^1 occur as differentials in the new form of S. Moreover, the A's in the new (10.2) when taken with the x's form a set of independent variables because of the result about the variables (9.6). The induction is therefore complete.

If the derived system is vacuous, the demonstration holds as given, but also follows immediately from the reduced form and the result about the variables (9.6).

It is clear that *any system written in the above canonical form is nested and of species one.*

The number s^1 appearing in above discussion and in (10.6) is the number of equations in S^1 but not in S^2, for it gives the number of ω''s which vanish by virtue of S but not by virtue of S^1. Hence we have

$$(10.8)\qquad s^1 = r^1 - r^2.$$

In the same way we denote by s^i the number of variables which occur as coefficients but not as differentials in S^i. From (10.8) we have therefore

$$(10.9)\qquad s^i = r^i - r^{i+1}.$$

From (8.4) it is seen that the s's are the first characters of the successive derived systems, s^0 being s. From (8.4) and (10.6) we get

$$(10.10)\qquad s \geqq s^1 \geqq s^2 \geqq \cdots \geqq 0.$$

We find it convenient to define a set of integers t by means of the formulas

$$(10.11)\qquad t^i = s^i - s^{i+1}.$$

These t's are the second differences of the numbers r. Even for a nested system of species one they do not necessarily satisfy inequalities like (10.10), but they are non-negative because the s's satisfy (10.10).

Let l be the least number such that the derived system S^l is passive or vacuous. Since

$$r^l = r^{l+1},$$

we have

$$s^i = t^i = 0 \qquad (i \geqq l). \tag{10.12}$$

As a result of the foregoing, the numbers s, t are all determined for a nested system of species one when the r's are given. Conversely, if $t, t^1, \cdots, t^{l-1}$ and r^l are given, the s's are determined by (10.11) and (10.12). The other r's are then determined by (10.9).

11. **A nested system of species one as a sum of special systems.** Consider a nested system of species one written in the canonical form of the preceding section. From it construct a set of systems in the following manner. Make $\omega^r = 0$, which is the last equation in S, the last equation of one system. If x^r, whose differential is the first term of ω^r, is not a coefficient in S^1, the new system will consist of the single equation $\omega^r = 0$ and will be called a system T_0. If x^r is the coefficient in an equation of S^1, we take that equation, say $\omega^{r^1} = 0$, as the next to the last in the new system. If x^{r^1} is not a coefficient in S^2, we close the new system and call it a system T_1. If x^{r^1} is a coefficient in S^2, we take the corresponding equation, say $\omega^{r^2} = 0$, as the third from the last in the new system. And so on. If the system is closed with an equation from S^i, it will be called a system T_i. It contains $i+1$ equations.

We next take the equation $\omega^{r-1} = 0$, if it is not in S^1, and form in the above manner the system T which it determines. And so on until all the equations in $\Sigma = S - S^1$ are exhausted. When this stage is reached, all the equations of S, except those in its passive system, have been used. This is for the following reason. Because of the nature of the canonical form, the coefficient in an equation of $S^i - S^{i+1}$ occurs as a differential in an equation of $S^{i-1} - S^i$; the coefficient in the latter occurs as a differential in $S^{i-2} - S^{i-1}$; and so on until $S - S^1$ is reached.

The passive system is to be taken as a separate system. For it there is already the notation S^l.

The systems T are nested, of species one and in canonical form. In addition, *each derived system is obtained from the preceding by omitting its last equation,* and the class exceeds the number of equations by two. Consequently, each system T is special and of the type first reduced to canonical form by von Weber.* A passive system is also special. Therefore we have

* See Goursat, pp. 321–8.

THEOREM 8. *Every nested system of species one is equivalent to a set of special systems no two of which have an equation in common.*

Let t^i be the number of special systems T_i. There is one and only one equation in $S-S^1$ corresponding to each system T. Hence the total number of equations in $S-S^1$ is the same as the total number of systems T, a fact which is stated by the equation

$$r - r^1 = t^0 + t^1 + \cdots + t^{l-1}.$$

In the same way, there are just as many equations in S^1-S^2 as there are systems T with index at least unity. Hence we get the sequence of relations

$$\begin{aligned} r^1 - r^2 &= t^1 + \cdots + t^{l-1}, \\ r^2 - r^3 &= t^2 + \cdots + t^{l-1}, \\ \cdots \quad & \quad \cdots \quad \cdots \\ r^{l-1} - r^l &= t^{l-1}. \end{aligned}$$

From these and (10.9) it readily follows that *the t's are the numbers defined by* (10.11).

There is only one symbol common to two of the partial systems, namely, dx^{r+1}, and it plays the same role throughout. It is clear, therefore, that a nested system of species one can be broken up by the above process into a sum of special systems in only one way and that two nested systems of species one are equivalent if and only if their sets of special systems can be transformed one into the other.

From the canonical form of a special system whose class is two more than the number of equations and which has no integrals it follows immediately that *two such systems are equivalent if and only if they consist of the same number of equations.** Therefore two nested systems of species one are equivalent if and only if they have the same number of integrals and the same sequence of numbers t. From the results at the end of §10 this amounts to having the same number of integrals and the same sequence of first characters; or to having the same sequence of numbers r.

THEOREM 9. *A nested system of species one is completely characterized by a finite set of arithmetic invariants, which can be taken as the number of equations in the successive derived systems.*

If $(r, r^1, \cdots, r^l)$ is defined as the *symbol* of a Pfaffian system, *a nested system of species one is completely characterized by its symbol.*

The following question arises: what conditions must the non-negative

* Goursat, p. 328.

integers in $(r, r^1, \cdots, r^l)$ satisfy in order that it may be the symbol of a nested system of species one? In the first place, to have a meaning at all, it must not contain two equal r's because of the definition of l. In the second place, the characters formed as the first differences of the numbers in the symbol must be non-increasing and non-negative in accordance with (10.10); and from the fact that no two r's are equal they must therefore be positive. These two conditions are also sufficient. For the satisfaction of (10.10) insures that the t's will be non-negative. Corresponding to each t^i we write t^i special systems of $i+1$ equations and class $i+3$, each in canonical form with the dx^{r+1} in common and all the other variables distinct. To these we adjoin the differentials of r^l other variables equated to zero. The result is a nested system of species one having the specified symbol.

THEOREM 10. *There is a nested system of species one having* $(r, r^1, \cdots, r^l)$ *as symbol if and only if the first differences of the* r*'s form a non-increasing sequence of positive integers.*

If the given system is special and of class $r+2$, the invariants have the following values:

$$l = r - q, \; s^0 = s^1 = \cdots = s^{l-1} = 1, \; t^0 = t^1 = \cdots = t^{l-2} = 0, \; t^{l-1} = 1.$$

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