THE BOUNDARY VALUES OF ANALYTIC FUNCTIONS. II*

BY

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Let

$$(0.1) \qquad f_1(z), f_2(z), \cdots$$

be a uniformly bounded sequence of functions analytic for $|z|<1$. By a theorem of Fatou,† $\lim_{r\to 1} f_n(re^{it})$ exists almost everywhere on the interval $0 \leqq t < 2\pi$, defining a boundary function $F_n(e^{it}) = \lim_{r\to 1} f_n(re^{it})$ almost everywhere on $|z|=1$, $z=e^{it}$. A new sequence

$$(0.2) \qquad F_1(z), F_2(z), \cdots$$

is thus determined. What are the relations between these two sequences? More generally, let the sequence (0.1) consist of functions meromorphic for $|z|<1$. In §2 below, a boundary function $\mathcal{J}_n(z)$ will be defined at every point of $|z|=1$ for any function $f_n(z)$ meromorphic for $|z|<1$. A new sequence

$$(0.3) \qquad \mathcal{J}_1(z), \mathcal{J}_2(z), \cdots$$

is thus determined. What are the relations between the sequences (0.1) and (0.3)?

The following questions are closely related to these two. Let $f(z)$ be a bounded function, analytic for $|z|<1$, with Fatou boundary function $F(z)$, as defined above. Let P be a point on $|z|=1$. Then what are the relations between $f(z)$ and $F(z)$ in a neighborhood of P? More generally let $f(z)$ be meromorphic for $|z|<1$. In §2 below, a boundary function $\mathcal{J}(z)$ of $f(z)$ will be defined at every point of $|z|<1$. What are the relations between $f(z)$ and $\mathcal{J}(z)$ in a neighborhood of P?

The purpose of this paper is to treat these four questions. Before treating them, however, a number of definitions, some new and some old, will be made in the following two sections.

1. Metric density and approximate continuity

In a previous paper‡ applications of the concepts of mean metric density and approximate continuity to complex function theory were made by the

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† P. Fatou, Acta Mathematica, vol. 30 (1906), pp. 366–367.

‡ These Transactions, vol. 34 (1932), pp. 153–170.

author. The following lemma will be used in discussing further applications.

LEMMA 1.1. *Let E be a point set on the interval $-1<x<1$ having lower and upper mean metric density δ_l, δ_u respectively at $x=0$. Let E become E' under the transformation $x'=\Psi(x)$ and let E' have lower and upper mean metric density δ_l', δ_u' respectively at $\Psi(0)=0$.*

(a) *If $\Psi'(x)=d\Psi(x)/dx$ is continuous for $-1<x<1$, $\Psi'(0)>0$, then*

$$\delta_l' = \delta_l,\ \delta_u' = \delta_u. \tag{1.11}$$

(b) *If $\Psi(x)=x^\nu$ for $x\geqq 0$, $\nu\geqq 1$, and $\Psi(x)=-|x|^\nu$ for $x\leqq 0$, then $\delta_u=1$ implies that $\delta_u'=1$.*

The simple proof of this lemma will be omitted here.*

COROLLARY. *If E has lower and upper metric densities on the right δ_{lr}, δ_{ur}, respectively at $x=0$ and if E' has lower and upper metric densities on the right δ_{lr}', δ_{ur}' respectively at $x=0$,*

$$\delta_{lr}' = \delta_{lr},\ \delta_{ur}' = \delta_{ur}, \tag{1.12}$$

in Case (a) *and $\delta_{ur}=1$ implies $\delta_{ur}'=1$ in Case* (b).

We can suppose that E has no points to the left of the origin, when the corollary follows immediately from the lemma.

Let $F(z)$ be a measurable function defined almost everywhere on $|z|=1$. The idea of approximate continuity will be slightly extended as follows. If the set of those points at which $|F(z)-\alpha|\leqq\epsilon$ has upper mean metric density 1 at z_0 for some complex number α and for all positive numbers ϵ, $F(z)$ will be said to be quasi-approximately continuous at z_0 with limit value α there. $F(z)$ may be quasi-approximately continuous at a point with several limit values there.

Let $F_1(z)$, $F_2(z)$, $\cdots$ be a sequence of measurable functions defined on a set of positive measure E on $|z|=1$. The sequence is said to converge in measure to a (measurable) function $F(z)$ when the measure of the set of those points for which $|F(z)-F_n(z)|\geqq\epsilon$ approaches 0 with $1/n$ for every positive number ϵ. If the sequence is uniformly bounded, one necessary and sufficient condition for this is that $F(z)$ be bounded and measurable on E and that

$$\lim_{n\to\infty}\int_E |F(z) - F_n(z)|\,|dz| = 0,$$

and another that every subsequence of the sequence $\{F_n(z)\}$ contain a further subsequence converging almost everywhere on E to $F(z)$.†

* Cf. the proof of Lemma 2.1 in the previous paper.

† F. Riesz, Paris Comptes Rendus, vol. 148 (1909), pp. 1303–1305.

LEMMA 1.2. *Let $\{F_n(z)\}$ be a sequence of measurable functions defined on a measurable set E on $|z|=1$, $mE>0$. A necessary and sufficient condition that the sequence converge in measure on E to the measurable function $F(z)$ is that*

$$\lim_{n\to\infty} \underline{B}\{\,|F(z)-F_n(z)|\,,E_n\} = 0^*$$

for every sequence $\{E_n\}$ of measurable point sets on $|z|=1$ such that $E_n\subset E$, $n=1, 2, \cdots$, and such that

$$\liminf_{n\to\infty} mE_n > 0.$$

This result is an immediate consequence of the definition of convergence in measure.

It will be seen in §5 that the concepts of approximate continuity and convergence in measure are related to each other.

2. CLUSTER VALUES OF FUNCTIONS AND OF SEQUENCES

In the following, points of the extended plane, or of the sphere corresponding to it by stereographic projection, will be considered. "Closed," "open," etc., used of point sets of the plane, will refer to the corresponding point sets on the sphere. The point ∞ is then in no way exceptional, and is allowable as a value assumed by a function.

Let $f(z)$ be a single-valued function defined in a domain† γ bounded by a simple closed Jordan curve Γ (i.e., a one-to-one and continuous image of the perimeter of a circle). Let P be a point on Γ. Then if there is a complex number α and a sequence of points $\{z_n\}$, in γ, such that

$$\lim_{n\to\infty} z_n = P, \quad \lim_{n\to\infty} f(z_n) = \alpha, \tag{2.01}$$

α is called a cluster value of $f(z)$ in γ at P. The set of all cluster values of $f(z)$ in γ at P is called the cluster set of $f(z)$ in γ at P. This set is closed and connected if $f(z)$ is continuous in γ. The function $\mathcal{J}(z)$, defined for every point P on Γ, as the cluster set of $f(z)$ in γ at P will be called the cluster boundary function of $f(z)$. It is evidently multiple-valued, in general. The function $f(z)$ is said to have the cluster value α on a given path to P if there exists a sequence of points $\{z_n\}$ on that path, so that (2.01) is satisfied. If γ is the interior of the unit circle, $|z|<1$, the path will be called non-tangential if it is contained in some angle with vertex at P whose sides are chords of $|z|=1$.

* Throughout this paper if $F(z)$ is a function defined on a set E, $\underline{B}\{\,|F(z)|, E\}$ will denote the greatest lower bound of $|F(z)|$ on E, and $O\{F(z), E\}$ will denote the oscillation of $F(z)$ on E, i.e. the least upper bound of $|F(P)-F(Q)|$ for P, Q any two points of E.

† In this paper, any open connected point set will be called a domain.

If the path is a continuous curve C* and if there is only a single cluster value of $f(z)$ on C:

$$\lim_{z\to P} f(z) = \alpha \tag{2.02}$$

when z approaches P on C, $f(z)$ is said to have the convergence value α at P.

If there is a complex number α, a sequence of points $\{z_n\}$ on Γ in a neighborhood of P, all on one side of P and different from P, such that

$$\lim_{n\to\infty} z_n = P, \quad \lim_{n\to\infty} \mathcal{J}(z_n) = \alpha \tag{2.03}$$

(choosing one definite value of $\mathcal{J}(z_n)$ for each value of n), α is called a cluster value of $f(z)$ on Γ at P on the side in question. The cluster sets of $f(z)$ on Γ at P on each side are then defined as the set of all the cluster values of $f(z)$ on that side, and the cluster set of $f(z)$ on Γ at P is the sum of these two sets. If $f(z)$ is continuous in γ, the cluster sets of $f(z)$ on Γ at P on each side are closed and connected. If E is a point set on Γ which has P as a limit point, and if in (2.03) the points $\{z_n\}$ all belong to E, the set of all values α thus determined will be called the cluster set of $f(z)$ on Γ on E at P. These ideas were introduced by Painlevé.†

It does not seem to have been realized that the above definitions are analogues of certain definitions for sequences of functions, defined in the interior of the unit circle. Let

$$f_1(z), f_2(z), \cdots \tag{2.04}$$

be a sequence of single-valued functions defined for $|z|<1$, with cluster boundary functions $\mathcal{J}_1(z)$, $\mathcal{J}_2(z)$, $\cdots$ respectively on $|z|=1$. Then if there is a complex number α, a subsequence $\{f_{a_n}(z)\}$, and a sequence of points $\{z_{a_n}\}$ in $|z|<1$, such that

$$\lim_{n\to\infty} f_{a_n}(z_{a_n}) = \alpha, \tag{2.05}$$

α will be called a cluster value of the sequence (2.04) in $|z|<1$. If $g_n(z)$ is defined by

$$g_n(z) = f_{a_n}\left(\frac{z - z_{a_n}}{\bar{z}_{a_n} z - 1}\right)\ddagger \tag{2.06}$$

and if

* That is, C is determined by $z=\psi(t)$ where $\psi(t)$ is continuous for $0\leqq t\leqq 1$, $z=\psi(t)$ is in γ for $0\leqq t<1$, $\psi(1)=P$.

† P. Painlevé, Paris Comptes Rendus, vol. 131 (1900), p. 489.

‡ The conjugate complex number of ξ is denoted by $\bar{\xi}$.

$$\lim_{n\to\infty} g_n(z) = \alpha \tag{2.07}$$

uniformly in every closed subregion of $|z| <1$, α will be called a convergence value of the sequence. The sets of all cluster and convergence values of the sequence in $|z| <1$ will be called the cluster and convergence sets in $|z| <1$, respectively. The former set is closed.

If there is a complex number α, a subsequence $\{f_{a_n}(z)\}$, and a sequence of points $\{z_{a_n}\}$ on $|z| =1$, such that

$$\lim_{n\to\infty} \mathcal{J}_{a_n}(z_{a_n}) = \alpha \tag{2.08}$$

(choosing one definite value for $\mathcal{J}_{a_n}(z_{a_n})$ for each value of n), α will be called a cluster value of the sequence (2.04) on $|z| =1$. The set of all cluster values of the sequence on $|z| =1$, which we designate as the cluster set of the sequence on $|z| =1$, is closed.

Let $\{A_n\}$ be a set of open arcs on $|z| =1$. Then if α is a cluster value of the sequence (2.04) in $|z| <1$ in accordance with the definition given above and if under the transformation

$$z' = \frac{z - z_{a_n}}{\bar{z}_{a_n} z - 1}, \tag{2.09}$$

the arc A_{a_n} becomes the arc A_{a_n}' such that

$$\liminf_{n\to\infty} mA_{a_n}' > 0, \tag{2.10}$$

α will be called a cluster value of the sequence in $|z| <1$ with respect to the arcs $\{A_n\}$. If in particular

$$\lim_{n\to\infty} mA_{a_n}' = 2\pi, \tag{2.11}$$

α will be called a strong cluster value of the sequence in $|z| <1$ with respect to the arcs $\{A_n\}$. If

$$\limsup_{n\to\infty} |z_{a_n}| < 1,$$

the conditions (2.10) and (2.11) are equivalent to

$$\liminf_{n\to\infty} mA_{a_n} > 0, \quad \lim_{n\to\infty} mA_{a_n} = 2\pi,$$

respectively. If

$$A_n = A_1, n > 1, mA_1 < 2\pi, \tag{2.12}$$

then in the case (2.10), a convergent subsequence of the sequence $\{z_{a_n}\}$ must approach (i) a point of $|z|<1$, or (ii) a point of A_1, or (iii) an end point of A_1, remaining in the angle between A_1 and some chord through that end point. In the case (2.11), still assuming (2.12), a convergent subsequence of the sequence $\{z_{a_n}\}$ must approach (i) a point of A_1, or (ii) an end point of A_1, approaching the end point tangentially—on the same side of the end point as A_1. Conversely the conditions given are sufficient that α be a cluster value or a strong cluster value of the sequence $\{f_n(z)\}$ with respect to the arcs $\{A_n\}$ respectively. There is only slight modification of these criteria if $\liminf_{n\to\infty} mA_n>0$. The sets of all cluster values and strong cluster values with respect to a set of arcs will be called the cluster set and the strong cluster set of the sequence with respect to those arcs, respectively. It is not hard to show that the latter is a closed subset of the former. If α is a cluster value with respect to a set of arcs on $|z|=1$, i.e. if (2.05) and (2.10) are satisfied, and if (2.07) is also satisfied, α will be called a convergence value of the sequence with respect to those arcs. The convergence set with respect to the arcs will then be the set of all these convergence values. The convergence set with respect to a set of arcs is a subset of the strong cluster set with respect to the arcs. For if α is a convergence value with respect to the arcs $\{A_n\}$ we can suppose that $\liminf_{n\to\infty} mA_{a_n}>0$ (or we could use the sequence determined by (2.06)). By (2.07) there is then a subsequence $\{f_{b_n}(z)\}$ of $\{f_{a_n}(z)\}$ and a sequence of points $\{\xi_{b_n}\}$ such that $\lim_{n\to\infty}|\xi_{b_n}|=1$ and such that $\lim_{n\to\infty} f_{b_n}(\xi_{b_n})=\alpha$. We can suppose that ξ_{b_n} is so chosen that the distance from ξ_{b_n} to the midpoint of the arc A_{b_n} approaches 0 with $1/n$. Then α is a strong cluster value of the sequence with respect to the arcs $\{A_n\}$, by the criterion suggested above.

If α is a cluster value of the sequence (2.04) on $|z|=1$ in accordance with (2.08) and if the point z_{a_n} lies on the arc A_{a_n} for all values of n, α will be called a cluster value of the sequence on $|z|=1$ on the arcs $\{A_n\}$. The set of all these cluster values will be called the cluster set of the sequence on the arcs considered. This set is closed.

A point α will be said to be assumed by the sequence (2.04) if every function of the sequence except for at most a finite number assumes the value α. A point α will be said to be exceptional to or omitted by the sequence if at most a finite number of the functions assume the value α.

3. The properties of the boundary functions of a uniformly bounded convergent sequence of analytic functions

Let

$$f_1(z),\ f_2(z),\ \cdots \tag{3.01}$$

be a uniformly bounded sequence of functions analytic for $|z| <1$, with Fatou boundary functions

$$F_1(z), F_2(z), \cdots \tag{3.02}$$

respectively. What are the relations between these two sequences? The sequence (3.01) forms a normal family.* If it is uniformly convergent in every closed subregion of $|z| <1$ to the limit function $f(z)$, we can reduce the problem to that in which the limit function vanishes identically by substituting the sequence $\{f_n(z)-f(z)\}$ for $\{f_n(z)\}$. This will be convenient in much of what follows. The problem solved in this section is the following. Necessary and sufficient conditions are found on the sequence (3.02) that the sequence (3.01) converge uniformly in every closed subregion of $|z| <1$ to a function $f(z)$, where, if convenient, the limit function $f(z)$ is supposed to vanish identically. It will be seen that any domain bounded by a simple closed rectifiable Jordan curve could be used instead of $|z| <1$ as the domain of definition of the functions of the sequence (3.01).

LEMMA 3.1. *Let $f_1(z), f_2(z), \cdots$ be a uniformly bounded sequence of functions analytic for $|z| <1$, with Fatou boundary functions $F_1(z)$, $F_2(z), \cdots$ respectively, $|F_n(z)| \leqq 1$, $n=1, 2, \cdots$. Then if there is a sequence of points $\{z_n\}$ such that*

$$\limsup_{n\to\infty} |z_n| < 1, \quad \lim_{n\to\infty} |f_n(z_n)| = 1, \tag{3.101}$$

it follows that

$$\lim_{n\to\infty} |f_n(z)| = 1 \tag{3.102}$$

uniformly in every closed subregion of $|z| <1$ and that the sequence $\{|F_n(z)|\}$ converges in measure to 1 *on $|z| =1$.*

The sequence $\{f_n(z)\}$ forms a normal family, as remarked above, and any limit function $f(z)$ must satisfy the two inequalities

$$|f(z)| \leqq 1, \quad |f(z_0)| = 1, \quad |z_0| < 1, \tag{3.103}$$

where z_0 is a limit point of the sequence $\{z_n\}$. It follows from the maximum principle that $|f(z)| \equiv 1$. Then every limit function of the sequence is a constant of modulus 1. If the sequence $\{|f_n(z)|\}$ did not converge uniformly to 1 in every closed subregion of $|z| <1$, there would be a closed subregion R, and a positive number $\rho<1$, such that

$$|f_{a_n}(\xi_{a_n})| \leqq \rho \qquad (n = 1, 2, \cdots), \tag{3.104}$$

* P. Montel, *Leçons sur les Familles Normales*, Paris, 1927, p. 21.

for some subsequence $\{f_{a_n}(z)\}$ and a sequence $\{\xi_{a_n}\}$ of points in R. From $\{f_{a_n}(z)\}$ can be extracted a further subsequence $\{f_{b_n}(z)\}$ such that $\lim_{n\to\infty} |f_{b_n}(z)| = 1$ uniformly in R, contradicting (3.104). The sequence $\{|f_n(z)|\}$ therefore converges uniformly to 1 in every closed subregion of $|z|<1$. In particular $\lim_{n\to\infty} |f_n(0)| = 1$. Now by the Cauchy integral formula,

$$(3.105)\qquad |f_n(0)| \leqq \frac{1}{2\pi}\int_{|z|=1} |F_n(z)||dz| \leqq 1,$$

so that

$$(3.106)\quad 0 = \lim_{n\to\infty} [1 - |f_n(0)|] \geqq \lim_{n\to\infty} \frac{1}{2\pi}\int_{|z|=1} [1 - |F_n(z)|]|dz| = 0,$$

which proves that the sequence $\{|F_n(z)|\}$ converges in measure to 1 on $|z| = 1$.*

THEOREM 3.1. *Let $f_1(z), f_2(z), \cdots$ be a uniformly bounded sequence of functions analytic for $|z|<1$, with Fatou boundary functions $F_1(z), F_2(z), \cdots$ respectively, $|F_n(z)| \leqq 1$, $n=1, 2, \cdots$.*

(a) *If there is a sequence of points $\{z_n\}$ such that*

$$(3.111)\qquad \limsup_{n\to\infty} |z_n| < 1, \quad \lim_{n\to\infty} f_n(z_n) = \alpha, \quad |\alpha| = 1,$$

it follows that

$$(3.112)\qquad \lim_{n\to\infty} f_n(z) = \alpha$$

uniformly in every closed subregion of $|z|<1$ and that the sequence $\{F_n(z)\}$ converges in measure to α on $|z|=1$.†

(b) *If the sequence $\{F_n(z)\}$ converges in measure to α, $|\alpha| \leqq 1$, on a measurable set E on $|z|=1$, $mE>0$, the conclusions of* (a) *hold.*

If we take $|\alpha| = 1$ in (b), the condition of (b) is necessary and sufficient that (3.112) hold, so Theorem 3.1 solves the given problem in a very particular case.

(a) We can assume that $\alpha = 1$. The result (a) is then simply Lemma 3.1 applied to the sequence $\{\phi_n(z)\}$, where

$$(3.113)\qquad \phi_n(z) = e^{f_n(z)-1}.$$

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* Cf. §1.

† A similar theorem was proved by J. L. Walsh (who uses the term quasi-convergence instead of convergence in measure), and applied in another connection, these Transactions, vol. 32 (1930), pp. 378–379.

(b) To prove (b) it is only necessary to prove that the hypothesis of (b) implies (3.112). By a theorem of Khintchine and Ostrowski* the actual convergence of the sequence $\{F_n(z)\}$ almost everywhere on E implies (3.112). The proof as given in Bieberbach's book also proves the more general result desired. A still more general result will be useful, however. *Let $E_n(\epsilon)$ be the set of those points on $|z|=1$ for which $|F_n(z)-\alpha|\leqq\epsilon$. Then it is sufficient in* (b) *if*

$$\lim_{\epsilon\to 0}\left\{\limsup_{n\to\infty}\epsilon^{mE_n(\epsilon)}\right\}=0. \tag{3.114}$$

This follows from the Ostrowski-Nevanlinna inequality, or the proof referred to above can be modified to prove this also (by choosing the constant A used in it properly). It is sufficient for (3.114) that

$$\liminf_{\epsilon\to 0}\left[\liminf_{n\to\infty} mE_n(\epsilon)\right]>0,$$

and it is this special case which will be used most in the applications in this paper.

COROLLARY. *In the above theorem if $w_n=f_n(z_n)$ for large values of n: $n\geqq n(\rho)$, is outside every circle C_ρ tangent to $|w|=1$ at $w=\alpha$, of radius $\rho<1$, the same will be true of the values of $w=f_n(z)$ for z in any fixed closed subregion of $|z|<1$ and the measure of the set of those points on $|z|=1$ at which $w=F_n(z)$ is inside C_ρ approaches* 0 *with* $1/n$ *for every value of* $\rho<1$.

We can suppose that $\alpha=1$. The corollary is simply the theorem applied to the sequence $\{\phi[f_n(z)]\}$ where $\phi(w)$ is defined by

$$\phi(w)=e^{(w+1)/(w-1)}.$$

THEOREM 3.2. *Let $f_1(z), f_2(z), \cdots$ be a uniformly bounded sequence of functions analytic for $|z|<1$, with Fatou boundary functions $F_1(z)$, $F_2(z), \cdots$ respectively. Suppose that $f_n(z)\neq 0$, $n=1, 2, \cdots$.*

(a) *If there is a sequence of points $\{z_n\}$ such that*

$$\limsup_{n\to\infty}|z_n|<1,\quad \lim_{n\to\infty} f_n(z_n) = 0, \tag{3.21}$$

it follows that

$$\lim_{n\to\infty} f_n(z) = 0 \tag{3.22}$$

* See, for example, L. Bieberbach, *Lehrbuch der Funktionentheorie*, 2d edition, vol. II, 1931, pp. 157–158.

uniformly in every closed subregion of $|z|<1$ *and that the sequence* $\{1/\log F_n(z)\}$* *converges in measure to* 0 *on* $|z|=1$ *whatever branch of* $\log F_n(z)$ *is chosen.*

(b) *If the sequence* $\{1/\log F_n(z)\}$ *converges in measure to* 0 *on a measurable set E of positive measure on* $|z|=1$, *it follows that*

$$\lim_{n\to\infty} 1/\log f_n(z) = 0 \tag{3.23}$$

uniformly in every closed subregion of $|z|<1$ *and that the sequence* $\{1/\log F_n(z)\}$ *is convergent in measure to* 0 *on* $|z|=1$, *where the branch of* $\log F_n(z)$ *is determined by that of* $\log f_n(z)$ *in* (3.23).

The uniformly bounded sequence $\{f_n(z)\}$ forms a normal family. Let $f(z)$ be a limit function of the family: $f(z)=\lim_{n\to\infty} f_{a_n}(z)$. Then $f(z_0)=0$ in Case (a), where z_0 is a limit point of the sequence $\{z_n\}$. Then by a well known theorem of Hurwitz, if $f(z)\not\equiv 0$, $f_{a_n}(z)$ must vanish in a neighborhood of z_0 for all large values of n. Since this is not true, $f(z)\equiv 0$. Thus every limit function of the family vanishes identically, and (3.22) is proved by an argument similar to that used in proving (3.102) in the proof of Lemma 3.1.

We can suppose that $|f_n(z)|<1$, $n=1, 2, \cdots$. Define the function $\phi_n(z)$, analytic for $|z|<1$, by

$$\phi_n(z) = \frac{\log f_n(z) + 1}{\log f_n(z) - 1}\cdot \tag{3.24}$$

Then $|\phi_n(z)|<1$ and $\phi_n(z)$ has the boundary function $\Phi_n(z)$:

$$\Phi_n(z) = \frac{\log F_n(z) + 1}{\log F_n(z) - 1}\cdot \tag{3.25}$$

Now $\lim_{n\to\infty} \phi_n(z)=1$ and $\lim_{n\to\infty} 1/\log f_n(z)=0$ are equivalent statements, so the theorem is an immediate consequence of Theorem 3.1. We note a generalization of (b) corresponding to one of Theorem 3.1 (b) which will be used in proving the next theorem. *Let* $E_n(\epsilon)$ *be the set of those points on* $|z|=1$ *for which* $|\log F_n(z)|>1/\epsilon$. *It is sufficient in* (b) *if*

$$\liminf_{\epsilon\to 0}\left[\liminf_{n\to\infty} mE_n(\epsilon)\right] > 0.$$

* Choose a branch of $\log f_n(z)$ at some point of $|z|<1$ and continue it analytically throughout $|z|<1$, determining a single-valued analytic function which has finite radial boundary values wherever $F_n(z)\neq 0$. Since $f_n(z)\not\equiv 0$, $F_n(z)=0$ at most on a set of measure 0 by a theorem of F. and M. Riesz. Then this branch of $\log f_n(z)$ has a finite-valued and single-valued boundary function defined almost everywhere on $|z|=1$ which will be denoted by $\log F_n(z)$. The function $\log F_n(z)$ has infinitely many branches differing by integral multiples of 2π.

The condition that the sequence $\{1/\log F_n(z)\}$ converge in measure to 0 on $|z|=1$ is equivalent to

(3.26) $$\lim_{n\to\infty} \underline{B}\{\,|\,1/\log F_n(z)\,|\,, E_n\} = 0,$$

for every sequence $\{E_n\}$ of measurable sets on $|z|=1$ such that

$$\liminf_{n\to\infty} mE_n > 0,$$

by Lemma 1.2. In this form the result can be more readily compared with that in the next theorem.

An example of the theorem in which $|F_n(z)|=1$ except at $z=1$, $F_n(1)=0$, for all values of n is given by

(3.27) $$f_n(z) = e^{n(z+1)/(z-1)} \qquad (n = 1, 2, \cdots).$$

Theorem 3.2 enables us to solve a particular case of the problem proposed at the beginning of this section which includes the particular case of Theorem 3.1 obtained by setting $|\alpha|=1$ in (b). It will be seen that this particular case has important applications.

THEOREM 3.3. *Let $f_1(z), f_2(z), \cdots$ be a uniformly bounded sequence of functions analytic for $|z|<1$ with Fatou boundary functions $F_1(z), F_2(z), \cdots$ respectively. Suppose that $f_n(z)\neq 0$, $n=1, 2, \cdots$.*

(a) *If there is a sequence of points $\{z_n\}$ such that*

(3.301) $$\limsup_{n\to\infty} |\,z_n\,| < 1, \quad \lim_{n\to\infty} f_n(z_n) = 0,$$

it follows that $\lim_{n\to\infty} f_n(z)=0$ uniformly in every closed subregion of $|z|<1$ and that

(3.302) $$\lim_{n\to\infty} \frac{\underline{B}\{\,|\,F_n(z)\,|\,, E_n\}}{1 + O\{\operatorname{arc} F_n(z), E_n\}} = 0^*$$

for every sequence $\{E_n\}$ of measurable point sets on $|z|=1$ satisfying

(3.303) $$\liminf_{n\to\infty} mE_n > 0.$$

(b) *If there is a sequence of measurable point sets $\{\mathcal{E}_n\}$ on $|z|=1$ satisfying* (3.303) *and such that every sequence $\{E_n\}$ of measurable point sets on $|z|=1$ such that $E_n\subset \mathcal{E}_n$, $n=1, 2, \cdots$, which satisfies* (3.303) *also satisfies* (3.302), *then $\lim_{n\to\infty} f_n(z)=0$ uniformly in every closed subregion of $|z|<1$.*

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* Cf. the first note on p.420. Arc $F_n(z)$ can be defined as the imaginary part of $\log F_n(z)$: $\mathfrak{I}\log F_n(z)$. Its oscillation on E_n is independent of the branch of $\log F_n(z)$ chosen.

The statement (b) is stronger than the converse of (a). This theorem shows what happens if $\lim_{n\to\infty} f_n(z)=0$ under the above circumstances and if the sequence of boundary functions does not converge in measure to 0. The condition (3.302) is only slightly stronger than (3.26) as is to be expected.

(a) Suppose that (3.301) is satisfied. By the previous theorem

$$\lim_{n\to\infty} f_n(z) = 0$$

uniformly in every closed subregion of $|z|<1$. Unless (3.302) is true, there is a subsequence $\{F_{a_n}(z)\}$, a positive number λ and a sequence $\{E_{a_n}\}$ of measurable point sets on $|z|=1$ such that $\liminf_{n\to\infty} mE_{a_n}>0$ and such that

$$(3.304) \qquad \frac{\underline{B}\{\,|F_{a_n}(z)|\,,\,E_{a_n}\}}{1+O\{\operatorname{arc} F_{a_n}(z),\,E_{a_n}\}} \geqq \lambda \qquad (n = 1, 2, \cdots).$$

Let P_{a_n} be a point of E_{a_n} and choose $\log F_{a_n}(z)$ so that

$$(3.305) \qquad |\,\mathfrak{I}\log F_{a_n}(P_{a_n})\,| \leqq \pi \qquad (n = 1, 2, \cdots).$$

Then, since $O\{\operatorname{arc} F_{a_n}(z),\,E_{a_n}\}\leqq M/\lambda$ by (3.304), where $|F_n(z)|\leqq M$, $n=1, 2, \cdots$,

$$(3.306) \qquad |\,\mathfrak{I}\log F_{a_n}(z)\,| \leqq \pi + M/\lambda \qquad (n = 1, 2, \cdots),$$

on E_{a_n}. Now by (3.304), $|F_{a_n}(z)|\geqq\lambda$ on E_{a_n}. Then

$$(3.307) \qquad |\log F_{a_n}(z)\,| \leqq |\log \lambda| + |\log M| + \pi + M/\lambda \qquad (n = 1, 2, \cdots),$$

on E_{a_n}. But by Theorem 3.2 the sequence $\{1/\log F_{a_n}(z)\}$ converges in measure to 0 on $|z|=1$, so this inequality is impossible.

(b) Suppose that the hypotheses of (b) are satisfied. Determine $\log F(z)$ from a branch of $\log f_n(z)$ for which

$$(3.308) \qquad |\,\mathfrak{I}\log f_n(0)\,| \leqq \pi \qquad (n = 1, 2, \cdots).$$

Then it is sufficient to show that the measure of the subset of $\mathcal{E}_n$ on which $|\log F_n(z)|\leqq K$ approaches 0 with $1/n$ for every value of K. For then, by Theorem 3.2 (b) in its generalized form, $\lim_{n\to\infty} 1/\log f_n(0)=0$, which implies, by (3.308), that $\lim_{n\to\infty} f_n(0)=0$. This is sufficient that $\lim_{n\to\infty} f_n(z)=0$ uniformly in every closed subregion of $|z|<1$, by Theorem 3.2 (a). Thus if (b) were not true there would be a number K, and a subsequence $\{F_{a_n}(z)\}$, such that

$$(3.309) \qquad |\log F_{a_n}(z)\,| \leqq K \qquad (n = 1, 2, \cdots),$$

on a subset E_{a_n} of $\mathcal{E}_{a_n}$, where $\liminf_{n\to\infty} mE_{a_n}>0$. Then

$$|F_{a_n}(z)| \geqq e^{-K} \tag{3.310}$$

on E_{a_n}, i.e.

$$\underline{B}\{ |F_{a_n}(z)|, E_{a_n}\} \geqq e^{-K}$$

so, by (3.302),

$$\lim_{n\to\infty} O\{\operatorname{arc} F_{a_n}(z), E_{a_n}\} = +\infty. \tag{3.311}$$

Then there are points P_{a_n}, Q_{a_n} on E_{a_n} for large values of n such that

$$|\operatorname{arc} F_{a_n}(P_{a_n}) - \operatorname{arc} F_{a_n}(Q_{a_n})| \geqq 4K. \tag{3.312}$$

But then either $|\log F_{a_n}(P_{a_n})| \geqq 2K$ or $|\log F_{a_n}(Q_{a_n})| \geqq 2K$, which contradicts (3.309). The theorem is thus completely proved.

Now consider the general problem proposed at the beginning of the section. Necessary and sufficient conditions are to be found on the sequence (3.02) that the sequence (3.01) converge uniformly to 0 in every closed subregion of $|z|<1$. The problem has just been solved if $f_n(z)\neq 0$ except for a finite number of values of n. The following theorem gives the general solution.

THEOREM 3.4. *Let $f_1(z), f_2(z), \cdots$ be a uniformly bounded sequence of functions analytic for $|z|<1$, with Fatou boundary functions $F_1(z), F_2(z), \cdots$ respectively. Let $\{A_n\}$, $\{A_n'\}$ be sequences of arcs on $|z|=1$ such that A_n, A_n' have no points in common and such that*

$$\liminf_{n\to\infty} mA_n > 0, \quad \liminf_{n\to\infty} mA_n' > 0. \tag{3.401}$$

Then there exist two sequences of uniformly bounded functions $g_1(z), g_2(z), \cdots$, $h_1(z), h_2(z), \cdots$ analytic for $|z|<1$ with Fatou boundary functions $G_1(z)$, $G_2(z), \cdots, H_1(z), H_2(z), \cdots$ respectively such that

$$f_n(z) = g_n(z)\cdot h_n(z) \qquad (n = 1, 2, \cdots), \tag{3.402}$$

and such that $g_n(z)\neq 0$ in $\mathfrak{S}(A_n)$, $h_n(z)\neq 0$ in $\mathfrak{S}(A_n')$. A necessary and sufficient condition that*

$$\lim_{n\to\infty} f_n(z) = 0 \tag{3.403}$$

uniformly in every closed subregion of $|z|<1$ is that

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* It will be convenient in this and the following sections if A is an arc of the unit circle, to denote by $\mathfrak{S}(A)$ the interior of the segment bounded by A and by its chord.

$$(3.404)\qquad \lim_{n\to\infty} \frac{\underline{B}\{|G_n(z)|, E_n\}}{1+O\{\operatorname{arc} G_n(z), E_n\}}\cdot\frac{\underline{B}\{|H_n(z)|, E_n'\}}{1+O\{\operatorname{arc} H_n(z), E_n'\}} = 0^*$$

for every pair of sequences $\{E_n\}$, $\{E_n'\}$ *of measurable point sets on* $|z|=1$ *such that* $E_n\subset A_n$, $E_n'\subset A_n'$, $n=1, 2, \cdots$, *and such that*

$$(3.405)\qquad \liminf_{n\to\infty} mE_n > 0,\quad \liminf_{n\to\infty} mE_n' > 0.$$

We consider only the case in which A_n and A_n' are the same arcs A and A' respectively for all values of n. The general case can be proved by a slight modification of the following proof.

Let $\phi_1(w)$, $\phi_2(w)$ be functions mapping $|w|<1$ in a one-to-one and conformal way on $S(A)$ and $S(A')$ respectively. These functions can easily be determined explicitly. Let

$$(3.406)\qquad \alpha_1^{(n)}, \alpha_2^{(n)}, \cdots, |\alpha_1^{(n)}| \leqq |\alpha_2^{(n)}| \leqq \cdots,$$

be the zeros of $f_n(z)$ in the interior of $S(A)$, where the non-simple zeros appear in the list a number of times equal to their multiplicity. By a theorem of Blaschke†

$$(3.407)\qquad h_n(z) = \prod_{j=1}^{\infty} \frac{z-\alpha_j^{(n)}}{\bar{\alpha}_j^{(n)}z-1}\bar{\alpha}_j^{(n)}$$

defines a bounded function, analytic for $|z|<1$, where the product converges uniformly in every closed subregion of $|z|<1$. Define the function $g_n(z)$ by

$$(3.408)\qquad g_n(z) = f_n(z)/h_n(z).$$

Then it is readily seen that $g_n(z)$ is a bounded function, analytic in the interior of the unit circle. The zeros of the functions $g_n(z)$, $h_n(z)$ have the required properties.

Equation (3.404) is equivalent to the following:

$$(3.409)\qquad \lim_{n\to\infty} g_n[\phi_1(w)]\cdot h_n[\phi_2(w)] = 0,$$

uniformly in every closed subregion of $|w|<1$. For suppose that (3.409) is true. If (3.404) is not also true there are subsequences $\{g_{a_n}(z)\}$, $\{h_{a_n}(z)\}$ (which we can suppose convergent in $|z|<1$ since the sequences $\{g_n(z)\}$,

* Each branch of arc $g_n(z)$ is a single-valued function in $S(A_n)$, thus determining a single-valued branch of arc $G_n(z)$. There are single-valued branches of arc $H_n(z)$ on A_n' by the same argument.

† See, for example, Montel, *Leçons sur les Familles Normales*, Paris, 1927, p. 180. If $S(A)$ includes $z=0$ and if $f_n(z)$ has a zero of order λ_n there, the product is taken from $j=\lambda_n+1$ to ∞ and the factor z^{λ_n} replaces the first λ_n factors.

$\{h_n(z)\}$ are normal families) and sets $\{E_{a_n}\}$, $\{E_{a_n}'\}$ satisfying (3.405) such that

$$\liminf_{n\to\infty} \frac{\underline{B}\{|G_{a_n}(z)|, E_{a_n}\}}{1+O\{\text{arc } G_n(z), E_{a_n}\}} \cdot \frac{\underline{B}\{|H_{a_n}(z)|, E_{a_n}'\}}{1+O\{\text{arc } H_{a_n}(z), E_{a_n}'\}} > 0. \tag{3.410}$$

Let the point sets on $|w|=1$ transformed into E_{a_n} and E_{a_n}' on $|z|=1$ by the transformations $z=\phi_1(w)$, $z=\phi_2(w)$ be $\mathcal{E}_{a_n}$, $\mathcal{E}_{a_n}'$ respectively. Then it is easily seen that

$$\liminf_{n\to\infty} m\,\mathcal{E}_{a_n} > 0, \quad \liminf_{n\to\infty} m\,\mathcal{E}_{a_n}' > 0. \tag{3.411}$$

Inequality (3.410) is the same as

$$\liminf_{n\to\infty} \frac{\underline{B}\{|G_{a_n}[\phi_1(w)]|, \mathcal{E}_{a_n}\}}{1+O\{\text{arc } G_{a_n}[\phi_1(w)], \mathcal{E}_{a_n}\}} \cdot \frac{\underline{B}\{|H_{a_n}[\phi_2(w)]|, \mathcal{E}_{a_n}'\}}{1+O\{\text{arc } H_{a_n}[\phi_2(w)], \mathcal{E}_{a_n}'\}} > 0. \tag{3.412}$$

But this means, by Theorem 3.3, that neither of the (convergent) sequences $\{g_{a_n}[\phi_1(w)]\}$, $\{h_{a_n}[\phi_2(w)]\}$ can converge to the function 0, which contradicts (3.409). Conversely suppose that (3.404) is true. If (3.409) is not true, there are subsequences $\{g_{a_n}[\phi_1(w)]\}$, $\{h_{a_n}[\phi_2(w)]\}$ which are convergent to functions which do not vanish identically. Then by Theorem 3.3 there are sequences of sets $\{\mathcal{E}_{a_n}\}$, $\{\mathcal{E}_{a_n}'\}$ on $|w|=1$ such that (3.411) and (3.412) are satisfied. Letting the sets $\{E_{a_n}\}$, $\{E_{a_n}'\}$ on $|z|=1$ correspond as above to the sets $\{\mathcal{E}_{a_n}\}$, $\{\mathcal{E}_{a_n}'\}$, (3.404) is contradicted. Equations (3.409) and (3.404) are thus equivalent.

Equations (3.409) and (3.403) are also equivalent. For suppose that (3.403) is true. We have to show that every limit function of the sequence $\{g_n[\phi_1(w)]\cdot h_{a_n}[\phi_2(w)]\}$ is the function 0. Suppose that this is not the case. Let $\{g_{a_n}[\phi_1(w)]\cdot h_{a_n}[\phi_2(w)]\}$ be a convergent subsequence, not converging to the function 0. We can suppose further that the sequences $\{g_{a_n}[\phi_1(w)]\}$, $\{h_{a_n}[\phi_2(w)]\}$ are also convergent, say to $g(w)$, $h(w)$ respectively. Since $g(w)h(w)\not\equiv 0$, $g(w)\not\equiv 0$, and $h(w)\not\equiv 0$. Then the sequences $\{g_{a_n}(z)\}$, $\{h_{a_n}(z)\}$ converge in $S(A)$, $S(A')$ respectively to functions which do not vanish identically. By a theorem of Stieltjes* these sequences are convergent throughout $|z|<1$ (to functions which cannot vanish identically). Then the sequence $\{f_{a_n}(z)\}$ cannot converge to 0, contrary to the hypothesis that (3.403) is true. Conversely suppose that (3.409) is true. We must show that (3.403) is true, i.e. that the only limit function of the sequence $\{f_n(z)\}$ is the function 0. Suppose $\{f_{a_n}(z)\}$ were a subsequence of $\{f_n(z)\}$ not converging to

* See for example P. Montel, loc. cit., pp. 28–30.

0. We can suppose that the sequences $\{g_{a_n}(z)\}$, $\{h_{a_n}(z)\}$ are both convergent. They converge to functions not vanishing identically, so the sequences $\{g_{a_n}[\phi_1(w)]\}$, $\{h_{a_n}[\phi_2(w)]\}$ have the same property, contradicting (3.409).

It has thus been shown that (3.409), (3.404), and (3.409), (3.403), are pairs of equivalent statements, proving the theorem.

4. The properties of the cluster boundary functions of a sequence of meromorphic functions

Let

$$f_1(z), f_2(z), \cdots \tag{4.01}$$

be a sequence of functions meromorphic for $|z|<1$, with cluster boundary functions

$$F_1(z), F_2(z), \cdots \tag{4.02}$$

respectively, as defined in §2. The problem to be attacked in this section is that of finding the relations between these two sequences.

Theorem 4.1. *Let $f_1(z), f_2(z), \cdots$ be a sequence of functions meromorphic for $|z|<1$. Let the cluster sets of the sequence in $|z|<1$ and on $|z|=1$ be s and S respectively. Then if there is a point α belonging to s but not to S, no point of the domain D containing α and bounded only by points of S* is omitted by the sequence.*

This theorem is proved by an application of the maximum principle for analytic functions which is fairly obvious, so the proof will be omitted. The theorem is stated only to allow ready comparison with Theorem 4.2, the principal result of this section. To prove Theorem 4.2, which generalizes Theorem 4.1, we need a succession of lemmas.

Lemma 4.1. *Let $\{f_n(z)\}$ be a uniformly bounded sequence of functions analytic for $|z|<1$. Let $\{A_n\}$ be a sequence of arcs on $|z|=1$,*

$$\liminf_{n\to\infty} mA_n > 0,$$

and let the cluster set of the sequence on $|z|=1$ on these arcs be S. Then if there is a point α omitted by the sequence, not belonging to S, and such that

$$\lim_{n\to\infty} f_n(0) = \alpha, \tag{4.11}$$

every point except α of the domain D containing α and bounded only by points of S is assumed by the sequence.

* The frontier points of a point set are the points every neighborhood of which contains a point both of the set and of its complement. If every frontier point of a domain belongs to a point set S, the domain will be said to be bounded only by points of S.

The fact that, in the case considered, a subset of S necessarily bounds a finite domain is not surprising in view of the information given by Theorem 3.3 about the oscillation of arc $[F_n(z)-\alpha]$ (where $F_n(z)$ is the Fatou boundary function of $f_n(z)$). The theorem will be proved first under the hypothesis that $f_n(z)$ is continuous on A_n.

Suppose that $\beta\neq\alpha$ were a point of D not assumed by the sequence. Then there would be a subsequence $\{f_{a_n}(z)\}$ omitting the value β. Let D' be a domain which, together with all its frontier points, is contained in D, and which contains the points α and β. Then $f_{a_n}(z)$ on A_{a_n} is outside D' for large values of n. Now if $\psi(w)$ is defined by

$$\psi(w) = \mathfrak{I} \log\left(\frac{w-\alpha}{w-\beta}\right) = \text{arc } (w-\alpha) - \text{arc } (w-\beta), \tag{4.12}$$

it is seen that $\psi(w)$ is single-valued and continuous in the complementary set of D', choosing that branch for which $\psi(\infty)=0$. Then $\psi(w)$ must be bounded in the complement of D'. In particular, there is a number M such that

$$\left|\psi[f_{a_n}(z)]\right| = \left|\text{arc } [f_{a_n}(z)-\alpha] - \text{arc } [f_{a_n}(z)-\beta]\right| \leqq M \tag{4.13}$$

for z on A_{a_n} and for values of n so large that $f_{a_n}(z)$ is outside D' on A_{a_n}: $n\geqq N$. Then if $E_{a_n}\subset A_{a_n}$ is a measurable point set on $|z|=1$,

$$\left|O\{\text{arc } [f_{a_n}(z)-\alpha], E_{a_n}\} - O\{\text{arc } [f_{a_n}(z)-\beta], E_{a_n}\}\right| \leqq 2M,\ n\geqq N. \tag{4.14}$$

Now by Theorem 3.3, if $\liminf_{n\to\infty} mE_{a_n}>0$,

$$\lim_{n\to\infty} \frac{\underline{B}\{|f_{a_n}(z)-\alpha|, E_{a_n}\}}{1+O\{\text{arc } [f_{a_n}(z)-\alpha], E_{a_n}\}} = 0. \tag{4.15}$$

This implies that

$$\lim_{n\to\infty} O\{\text{arc } [f_{a_n}(z)-\alpha], E_{a_n}\} = +\infty, \tag{4.16}$$

since $\liminf_{n\to\infty} \underline{B}\{|f_{a_n}(z)-\alpha|, E_{a_n}\}$ is not less than the minimum of the distance from α to a point of S, which is positive. But (4.16) implies, by (4.14), that

$$\lim_{n\to\infty} \frac{\underline{B}\{|f_{a_n}(z)-\beta|, E_{a_n}\}}{1+O\{\text{arc } [f_{a_n}(z)-\beta], E_{a_n}\}} = 0, \tag{4.17}$$

since the denominator becomes infinite while the numerator is bounded uniformly for all values of n. By Theorem 3.3, (4.17) implies that $\lim_{n\to\infty} f_{a_n}(0) =\beta$, which is impossible since by hypothesis $\lim_{n\to\infty} f_{a_n}(0)=\alpha\neq\beta$. The hypothesis that $\beta \neq \alpha$ was a point of D not assumed by the sequence $\{f_n(z)\}$ has thus led to a contradiction. The proof will now be given without the restriction that $f_n(z)$ be continuous on A_n. Let $\mathcal{J}_n(z)$ be the cluster boundary function of $f_n(z)$ on $|z|=1$. Let the arc A_n' have the same midpoint as A_n but be of half the length. Let $\mathcal{A}_n'$ be that arc on $|z|=r_n<1$ (where r_n will be determined below), which is cut off on $|z|=r_n$ by the sector of the unit circle intercepting A_n'. Then r_n can be so chosen that $1-r_n<1/n$ and that if $\mathfrak{z}$ is any point on $\mathcal{A}_n'$, there is a point z on A_n such that

$$|\mathcal{J}_n(z) - f_n(\mathfrak{z})| \leqq 1/n, \tag{4.18}$$

for some determination of $\mathcal{J}_n(z)$ at z. Now consider the sequence $\{f_n(r_n z)\}$. This sequence evidently has a subset S' of S as a cluster set on $|z|=1$ on the arcs $\{A_n'\}$. The function $f_n(r_n z)$ is continuous on A_n', the sequence omits the value α (not belonging to S') and $\lim_{n\to\infty} f_n(0)=\alpha$. Then by what has been proved already, the sequence assumes every value except α in the domain D' containing α and bounded only by points of S'. Since $S' \subset S$, $D' \supset D$, and the lemma is proved.

Lemma 4.2. *Let $\{f_n(z)\}$ be a sequence of functions meromorphic for $|z|<1$. Let $\{A_n\}$ be a sequence of arcs on $|z|=1$, and let the cluster set of the sequence in $|z|<1$ with respect to these arcs and on $|z|=1$ on these arcs be s and S, respectively. Then if there is a point α belonging to s but not to S and if α is omitted by the sequence $\{f_n(z)\}$, there is at most one other point in the domain D containing α and bounded only by points of S which is omitted by the sequence. If there are two points in D which are omitted by the sequence, no other point of the extended plane can be omitted by the sequence.*

(a) By hypothesis there is a subsequence $\{f_{a_n}(z)\}$ and a sequence $\{z_{a_n}\}$ of points in $|z|<1$, such that (2.05) is true and such that if A_{a_n} is transformed into A_{a_n}' by (2.09), (2.10) is true. We shall suppose that $f_n(z)\neq\alpha$, except for a finite number of values of n, that

$$\lim_{n\to\infty} f_n(0) = \alpha, \tag{4.21}$$

and that

$$\liminf_{n\to\infty} mA_n > 0. \tag{4.22}$$

If this is not true already, we can use the sequence $\{g_n(z)\}$, where

$$g_n(z) = f_{a_n}\left(\frac{z - z_{a_n}}{\bar{z}_{a_n} z - 1}\right), \tag{4.23}$$

in conjunction with the arcs $\{B_n\}$ where $B_n = A_{a_n}'$. In this form the connection between this and the previous lemma is obvious.

(b) Suppose that besides α, a, b, c are also values omitted by the sequence $\{f_n(z)\}$, where α, a, b, c are supposed distinct and where a, b, c do not necessarily belong to D. We can suppose that a, b, c are 0, 1, ∞ respectively. For if this were not so we should prove the corresponding theorem for the sequence $\{h_n(z)\}$, which has 0, 1, ∞ as exceptional points:

$$h_n(z) = \frac{f_n(z) - a}{f_n(z) - b} \cdot \frac{c - b}{c - a} \tag{4.24}$$

if a, b, c are all finite. If one of the points is ∞, we can suppose that it is the point c, and we define $h_n(z)$ by

$$h_n(z) = \frac{f_n(z) - a}{f_n(z) - b}. \tag{4.25}$$

Let $\xi = \lambda_1(\xi')$ be a single-valued analytic function mapping $|\xi'| < 1$ on the extended ξ-plane less the points 0, 1, ∞ (the elliptic modular function defined in the circle instead of in the half-plane). The function $\lambda_1(\xi')$ maps $|\xi'| < 1$ in a one-to-one way on an infinitely many sheeted Riemann surface with branch points at 0, 1, ∞. Let $\lambda(\xi)$ be the inverse of $\lambda_1(\xi')$, and form the sequence $\{\phi_n(z)\}$ where

$$\phi_n(z) = \lambda[f_n(z)]. \tag{4.26}$$

Since $f_n(z)$ omits the values 0, 1, ∞, $\phi_n(z)$ can be taken as any one of an infinite set of single-valued analytic functions defined in $|z| < 1$, by the monodromic theorem, and $|\phi_n(z)| < 1$. Choose some determination of $\lambda(\alpha):\alpha'$. Then a branch of $\phi_n(z)$ can be chosen for each value of n so that

$$\lim_{n\to\infty} \phi_n(0) = \alpha'. \tag{4.27}$$

Since $f_n(z) \neq \alpha$, $\phi_n(z) \neq \alpha'$. Let the cluster set of the sequence $\{\phi_n(z)\}$ on $|z| = 1$ on the arcs $\{A_n\}$ be S'. If ξ' is a cluster value of the function $\phi_n(z)$ at a point of A_n, and if $|\xi'| < 1$, $\lambda_1(\xi')$ is a cluster value of $f_n(z)$ at that point of A_n. Then if ξ' is a point of S' and if $|\xi'| < 1$, $\lambda_1(\xi')$ is a point of S. Then α' cannot belong to S' or α would belong to S, since $|\alpha'| < 1$. Let D' be the domain containing α' and bounded only by points of S'. Then by Lemma 4.1 every point in D' except α' is assumed by the sequence $\{\phi_n(z)\}$.

Now suppose that β is a point of D, $\beta \neq \alpha$, 0, 1, ∞. Let J be a Jordan arc joining α to β and lying wholly in D. It can be so chosen that it does not pass through 0, 1 or ∞. Choose the branch of $\lambda(\xi)$ for which $\lambda(\alpha) = \alpha'$ and using that branch determine J', the image of J in the ξ'-plane. Then we shall prove that J' lies wholly in D'. One end point, α', belongs to D'. If J' is not wholly in D', there is a point of J' on the boundary S' of D'. We can suppose that ξ' is the first such point on J', tracing J' from α'. If $|\xi'|<1$, $\xi=\lambda_1(\xi')$ belongs to S, as was noted above. But ξ is on J, belongs to D, and so cannot belong to S. If $|\xi'|=1$, the arc J must spiral infinitely often about 0, 1, or ∞, since it remains at positive distance from the first two and remains in some circle about the origin. But this is impossible, since J is a Jordan arc. Then no point on J' can be on the boundary of D', so J' lies entirely in D'. This means that $\beta'=\lambda(\beta)$ belongs to D' and is therefore assumed by the sequence $\{\phi_n(z)\}$. Then β is assumed by the sequence $\{f_n(z)\}$.

It has thus been proved that if three points a, b, c are exceptional, besides α, every point of D save α and the points of a, b, c belonging to D, is assumed by the sequence (a subsequence of the original sequence).

(c) The result of (b) will now be sharpened. Suppose that $\beta\neq\alpha$ belongs to D and is an exceptional value of the sequence $\{f_n(z)\}$ for which (4.21) and (4.22) are true. Suppose that γ is a third exceptional value of the sequence, not necessarily in D. We can suppose that $\alpha=0$, $\beta=\infty$, $\gamma=1$. Consider the sequence $\{\psi_n(z)\}$ where

$$\psi_n(z) = (f_n(z))^{1/3}. \tag{4.28}$$

Choosing any one of three branches, $\psi_n(z)$ is single-valued and analytic in $|z|<1$, by the monodromic theorem, since $f_n(z)\neq 0, \infty$. Moreover the sequence $\{\psi_n(z)\}$ has the exceptional values

$$\alpha_1 = 0,\ \beta_1 = \infty,\ a = e^{2\pi i/3},\ b = a^2,\ c = 1,$$

and

$$\lim_{n\to\infty} \psi_n(0) = 0. \tag{4.29}$$

Let S_1 be the cluster set of the sequence $\{\psi_n(z)\}$ on $|z|=1$ on the arcs $\{A_n\}$. If ξ_1 is a point of S_1, $\xi=\xi_1^3$ is a point of S. The point $\alpha_1=0$ therefore does not belong to S_1 or $\alpha=0$ would belong to S. Let D_1 be the domain containing α_1 and bounded only by points of S_1. It will be shown that D_1 contains β_1. Let J be a Jordan arc in D with end points $\alpha=0$, $\beta=\infty$.* A point ξ_0 can be chosen on J, so near $\alpha=0$ that each determination of $\xi^{1/3}$ lies in D_1—since D_1 includes some neighborhood of the origin. Then continue one determination of $\xi^{1/3}$ from ξ_0 to β along J, thus determining a Jordan arc J_1, which we shall prove lies entirely in D_1. For if it did not, a point of J_1 would also be a point of S_1,

* This means that to J corresponds a Jordan arc on the sphere.

the boundary of D_1, which would imply that a point of J was a point of S. Since this is not true, $\beta_1=\infty$ must be a point of D_1. Now the sequence $\{\psi_n(z)\}$ omits three values a, b, c besides α_1, so that every point in D_1 (not one of these values) must be assumed by the sequence, as was proved in (b). Therefore $\beta_1=\infty$ must be assumed. The sequence $\{f_n(z)\}$ must therefore assume the value $\beta=\infty$, contrary to hypothesis.

It has thus been shown that if there is an exceptional value in D besides α, no other point in the extended plane is omitted by the sequence. This proves the lemma.

LEMMA 4.3. *Let A be an arc of $|z|=1$ and let $f(z)$ be a bounded function, analytic for z in the segment $\mathfrak{S}(A)$ of the unit circle, with Fatou boundary function $F(z)$ on A. Suppose that*

$$|f(z)| \leqq K \text{ in } \mathfrak{S}(A) \tag{4.31}$$

and that

$$|F(z)| \leqq k < K \text{ on } A. \tag{4.32}$$

Then if $k'>k$,

$$|f(z)| \leqq k' \text{ in } \mathfrak{S}(A'). \tag{4.33}$$

where $A'\subset A$ and where mA' is a function of mA, k'/k, K/k only,

$$mA' = \tau(mA, k'/k, K/k), \tag{4.34}$$

$\tau>0$ increasing with mA, k'/k, k/K.

This lemma is easily proved using the maximum principle.*

LEMMA 4.4. *Let $\{f_n(z)\}$ be a sequence of functions meromorphic for $|z|<1$. Let the cluster set and the strong cluster set of the sequence in $|z|<1$ with respect to a set of arcs $\{A_n\}$ on $|z|=1$ be s, s_1 respectively, and let the cluster set of the sequence on $|z|=1$ on these arcs be S.*

(a) *$S\subset s_1\subset s$.*

(b) *The points of s not belonging to S form an open set† consisting of non-overlapping domains every one of which has at least one frontier point belonging to S.*

(c) *If one point α of one of these domains belongs to s_1, every point of the domain containing α and bounded only by points of S belongs to s_1.*

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* Cf. for example the proof of a related fact obtained by W. Seidel, these Transactions, vol. 34 (1932), pp. 3–4.

† This set may be empty.

(a) Let P belong to S. By hypothesis there is a sequence of points $\{P_{a_n}\}$, where P_{a_n} is a point of A_{a_n}, such that $\lim_{n\to\infty} \mathcal{J}_{a_n}(P_{a_n})=P$, $\mathcal{J}_{a_n}(P_{a_n})$ representing one of the values of $\mathcal{J}_{a_n}(z)$, the cluster boundary function of $f_{a_n}(z)$, at P_{a_n}. We can suppose that $\liminf_{n\to\infty} mA_{a_n}>0$. There is a sequence of points in $|z|<1$ approaching P_{a_n}, on which $f_{a_n}(z)$ approaches $\mathcal{J}_{a_n}(P_{a_n})$. Then let z_{a_n} be one of these points so close to P_{a_n} that

$$|\mathcal{J}_{a_n}(P_{a_n}) - f_{a_n}(z_{a_n})| < 1/n$$

and that

$$z' = \frac{z - z_{a_n}}{\bar{z}_{a_n}z - 1}$$

transforms A_{a_n} into an arc of length not less than $2\pi-1/n$. The existence of the sequence $\{z_{a_n}\}$ is the condition that P belong to s_1. Then $S\subset s_1$, and, by definition, $s_1\subset s$.

(b) The first part of (b) is equivalent to the statement that the frontier points of s which belong to s also belong to S. Suppose the contrary, that there is a point α, a frontier point of s belonging to s but not to S. We can suppose that α is finite, substituting the sequence $\{1/f_n(z)\}$ for $\{f_n(z)\}$ if $\alpha=\infty$. Making, if necessary, linear transformations taking $|z|<1$ into itself for all values of n, we can suppose that

$$\lim_{n\to\infty} f_{a_n}(0) = \alpha \tag{4.41}$$

and that

$$\liminf_{n\to\infty} mA_{a_n} > 0. \tag{4.42}$$

The sequence $\{f_{a_n}(z)\}$ is normal. For otherwise there would be a point z_0 with the property that in any neighborhood of z_0 at most two values are omitted by the sequence.* This would mean that every value of the plane belonged to s, contradicting the fact that α is a frontier point of s. If there were a limit function $f(z)\not\equiv\alpha$, $f(0)=\alpha$ necessarily and $f(z)$ would assume every value in some neighborhood of α, for $|z|<\frac{1}{2}$. Then the cluster set of $\{f_{a_n}(z)\}$ in $|z|<\frac{1}{2}$, which is a subset of s, would include this neighborhood of α contrary to the hypothesis that α was a frontier point of s. The sequence $\{f_{a_n}(z)\}$ is thus a normal family with the single limit function α, which implies, by an argument similar to that used in the proof of Lemma 3.1, that

$$\lim_{n\to\infty} f_{a_n}(z) = \alpha \tag{4.43}$$

uniformly in every closed subregion of $|z|<1$.

* P. Montel, *Leçons sur les Familles Normales*, Paris, 1927, p. 126.

Let every point of S be at distance greater than $3d>0$ from α. Then

(4.44) $$|\mathcal{J}_{a_n}(z) - \alpha| > 2d \text{ on } A_{a_n}$$

for large values of $n:n\geqq N$, and every determination of $\mathcal{J}_{a_n}(z)$. Since α is a frontier point of s there is a point β not belonging to s such that

(4.45) $$|\alpha - \beta| < d.$$

Then

(4.46) $$|F_{a_n}(z) - \beta| > d \text{ on } A_{a_n},\ n \geqq N.$$

There is no sequence of points $\{\xi_{b_n}\}$ such that ξ_{b_n} belongs to $\mathfrak{S}(A_{b_n})$ and such that $\lim_{n\to\infty} f_{b_n}(\xi_{b_n}) - \beta = 0$, where $\{f_{b_n}(z)\}$ is a subsequence of $\{f_{a_n}(z)\}$. For then β would have to be a point of s, contrary to hypothesis. Then there is a number K with the property that

(4.47) $$1/|f_{a_n}(z) - \beta| \leqq K \text{ in } \mathfrak{S}(A_{a_n})$$

for large values of n: $n\geqq N_1$. But then if $d'<d$, by the previous lemma there is a sequence of arcs $\{A_{a_n}'\}$, $A_{a_n}' \subset A_{a_n}$, such that $\liminf_{n\to\infty} mA_{a_n}' > 0$ and such that

(4.48) $$1/|f_{a_n}(z) - \beta| \leqq 1/d' \text{ in } \mathfrak{S}(A_{a_n}),\ n \geqq N, N_1.$$

Then by (4.3)

(4.49) $$1/|\alpha - \beta| \leqq 1/d'.$$

Since d' was arbitrary, $d'<d$, (4.49) implies that $|\alpha-\beta|\geqq d$ which contradicts (4.45). The hypothesis that there was a frontier point of s belonging to s but not to S has thus led to a contradiction.

The points common to s and the complement of S thus form an open set. This set is the sum of non-overlapping domains. At least one frontier point of each domain belongs to S. For if D is one of these domains and if α is a point of D, we can suppose that (4.41) and (4.42) are true. Consider the set E of all limit values of sequences of the form $\{f_{a_n}(\xi_{a_n})\}$ where ξ_{a_n} is a point of $|z|<1$ on the radius from $z=0$ to Q_{a_n}, the midpoint of A_{a_n}. This set E, a subset of s, is readily seen to be closed and connected and to contain α and also at least one point of S (in fact one of the limit values of the sequence $\{\mathcal{J}_{a_n}(Q_{a_n})\}$). By a well known theorem, since both a point in D and a point not in D belong to E, a frontier point of D must belong to E. We shall prove that this point P belongs to S, thus completing the proof of (b). If P did not belong to S, it would be a frontier point of s which belonged to s, since $E\subset s$. This is impossible by what has been proved already. Then P belongs to S.

(c) Statement (c) is equivalent to the statement that the frontier points of s_1 (which belong to s_1 since s_1 is closed) are points of S. The proof is similar to that of (b).

We now combine Lemma 4.4 with Lemma 4.2 to get the final result of this section.

THEOREM 4.2. *Let* $\{f_n(z)\}$ *be a sequence of functions meromorphic for* $|z|<1$. *Let the cluster set and the strong cluster set of the sequence in* $|z|<1$ *with respect to a set of arcs* $\{A_n\}$ *on* $|z|=1$ *be* s *and* s_1 *respectively and let* S *be the cluster set of the sequence on* $|z|=1$ *on these arcs. Let there be a point* α *belonging to* s *but not to* S.

(a) *Suppose that no point of the domain* D *containing* α *and bounded only by points of* S *belongs to* s_1. *Then if one point of the set* $s\cdot D$, *consisting of non-overlapping domains each with at least one frontier point belonging to* S, *is omitted by the sequence* $\{f_n(z)\}$, *only one other point of the extended plane can be omitted and every point of the extended plane belongs to* s.

(b) *If a point of* s_1 *is in* D, $D\subset s_1$, *and at most two points of* D *are omitted by the sequence* $\{f_n(z)\}$. *If two points of* D *are omitted no other point of the extended plane is omitted and every point of the extended plane belongs to* s.

(a) In (a) if a point α of $s\cdot D$, which was described in Lemma 4.4 (b), is exceptional to the sequence $\{f_n(z)\}$, α cannot be a convergence value of the sequence with respect to the arcs $\{A_n\}$, or α would belong to s_1 (cf. §2). Then if we suppose, as we can, that a subsequence $\{f_{a_n}(z)\}$ exists for which

$$\lim_{n\to\infty} f_{a_n}(0)=\alpha,\ \lim\ \inf_{n\to\infty}\ mA_{a_n}>0,$$

the sequence $\{f_{a_n}(z)\}$ cannot be normal or α would be a convergence value by an argument used in the proof of Theorem 3.2 (a). Then by a theorem used above there must be a point in $|z|<1$ in every neighborhood of which the sequence $\{f_{a_n}(z)\}$ can have at most two exceptional values, i.e. one besides α. This proves (a) completely.

(b) If a point α of s_1 is in D, $D\subset s_1\subset s$ by Lemma 4.4 (c). Then any exceptional value of the sequence in D is an exceptional cluster value with respect to the arcs $\{A_n\}$, and Lemma 4.2 can be applied. There only remains the proof that if two points of D are exceptional, s is the entire extended plane. We can suppose that there is a subsequence $\{f_{a_n}(z)\}$ such that

$$\lim_{n\to\infty} f_{a_n}(0)=\alpha,\ \text{and}\ \lim_{n\to\infty}\ mA_{a_n}=2\pi.$$

We consider $f_{a_n}(z)$ for n so large that $mA_{a_n}\geqq 3\pi/2$ considering the values of $f_{a_n}(z)$ for z in the interior of a segment $\mathrm{S}(A_{a_n}')$ determined by a subarc A_{a_n}' of A_{a_n} of length $3\pi/2$. Then it is immediate that if β is arbitrary except that β is not one of the two values in D exceptional to the sequence $\{f_n(z)\}$ by hypothesis, $f_{a_n}(z)-\beta=0$ has a root in $\mathrm{S}(A_{a_n}')$ for an infinite set of values of n (proof by mapping $\mathcal{S}(A_{a_n}')$ on a new unit circle and applying Lemma 4.2). Then β is a point of s, as was to be proved.

5. The neighborhood properties of the boundary function of a bounded analytic function

Let $f(z)$ be a bounded analytic function, defined for $|z|<1$, with Fatou boundary function $F(z)$. We shall discuss the following two questions. Let $P:e^{it}$ be a point on $|z|=1$. What are necessary and sufficient conditions on $F(z)$ in a neighborhood of P that $F(z)$ be defined at P: $\lim_{r\to 1} f(re^{it})=F(P)$? What are necessary and sufficient conditions on $F(z)$ in a neighborhood of P that $f(z)$ have the cluster value α at P? The latter case can be divided into two parts, according as α is or is not a non-tangential cluster value. The most stress in this section will be laid on conditions which are both necessary and sufficient and for this reason and for reasons of simplicity the sufficient conditions will not be stated with the full generality possible.

Theorem 5.1. *Let $f(z)$ be a bounded function analytic for $|z|<1$ with Fatou boundary function $F(z)$, $|F(z)|\leqq 1$. Let P be a point on $|z|=1$.*

(a) *If $\lim_{z\to P}|f(z)|=1$ when z approaches P on a continuous curve C, lying on one side of some chord through P, $|F(z)|$ is approximately continuous at P on that side, if $|F(P)|$ is defined as 1. In particular, if C is a non-tangential path, $\lim_{z\to P}|f(z)|=1$ when z approaches P on every non-tangential path and $|F(z)|$ is approximately continuous at P if $|F(P)|$ is defined as 1.*

(b) *If $|f(z)|$ has the cluster value 1 at P, $E(|F(z)|\geqq 1-\epsilon)$* is metrically dense at P for all positive values of ϵ. If 1 is a cluster value on some non-tangential path it is a cluster value on every continuous non-tangential or tangential curve to P and $|F(z)|$ is quasi-approximately continuous at P with limit value 1 there.*

(a) It is convenient to prove the second part of (a) first. Define $f_1(\xi)$, analytic in the upper half-plane with boundary function $F_1(\xi)$ by

$$f_1(\xi)=f\left(\frac{1+i\xi}{1-i\xi}\right),\quad F_1(\xi)=F\left(\frac{1+i\xi}{1-i\xi}\right). \tag{5.101}$$

Using Lemma 1.1, we see that it is sufficient to prove the result corresponding to the second part of (a) for $f_1(\xi)$ and its boundary function $F_1(\xi)$. We can suppose that P is the point $|z|=1$. Non-tangential paths to a point of the

* If a function $F(z)$ is defined almost everywhere on $|z|=1$, it will be convenient to denote the set of points on $|z|=1$ at which $F(z)$ satisfies a given inequality by $E(\quad)$, where the inequality is enclosed by the parentheses.

real axis are defined as paths which remain within some angle with vertex at the point whose sides are rays in the half-plane under consideration.

By hypothesis, then, $\lim_{\xi\to 0} |f_1(\xi)| = 1$, when ξ approaches $\xi = 0$ on C_1, a non-tangential continuous curve. C_1 is included in the angle determined by two rays, L', L'', meeting at $O: \xi = 0$. We can suppose that L', L'' are symmetric in the imaginary axis. Let R_λ be the interior of the rectangle having one side, of length λ, on the real axis and opposite side with end points on L' and L''. The rectangle is symmetric in the imaginary axis. Let P_λ be the intersection of the diagonals of R_λ and let $\phi(w)$ be the function mapping $|w| < 1$ in a one-to-one and conformal way on R_1, so that $\phi(0) = P_1$ and so that $\phi'(0)$ is real and positive.* We consider the family $\{g_\lambda(w)\}$ where $g_\lambda(w) = f_1[\lambda\phi(w)]$, $0 < \lambda \leqq 1$. The function $g_\lambda(w)$ takes on those values in $|w| < 1$ which $f_1(\xi)$ takes on in R_λ. Let $G_\lambda(w)$ be the Fatou boundary function of $g_\lambda(w)$.

(i) $\lim_{\lambda\to 0} |g_\lambda(w)| = 1$ uniformly in every closed subregion of $|w| < 1$. For there is a value of $\rho < 1$ such that $|w| = \rho$ corresponds to a simple closed analytic curve J in R_1 (by means of the transformation $\xi = \phi(w)$) which intersects both L' and L'' and therefore C_1.† Then for each positive value of λ, there is a point w_λ, $|w_\lambda| = \rho$, such that $\lambda\phi(w_\lambda)$ is a point of C_1. Then

$$\lim_{\lambda\to 0} | g_\lambda(w_\lambda) | = 1, \quad | w_\lambda | = \rho < 1, \tag{5.102}$$

and Lemma 3.1 proves the desired statement.‡

(ii) It follows from (i) that $\lim_{\xi\to 0} |f_1(\xi)| = 1$ uniformly in the angle considered.

(iii) It follows from (5.102), by Lemma 3.1, that the family $\{|G_\lambda(w)|\}$ converges in measure to 1 ($\lambda\to 0$). If $E_\lambda(\epsilon)$ is the set of those points on $|w| = 1$ for which $|G_\lambda(w)| \leqq 1-\epsilon$, $E_\lambda(\epsilon)$ corresponds to a set $E_\lambda'(\epsilon)$ on R_λ by the transformation $\xi = \lambda\phi(w)$ which is continuous on $|w| = 1$. The set $E_\lambda'(\epsilon)$ consists of those points on the perimeter of R_λ for which $|F_1(\xi)| \leqq 1-\epsilon$. The set $E_\lambda(\epsilon)$ corresponds to a set of measure $mE_\lambda'(\epsilon)/\lambda$ on R_1 by the transformation $\xi = \phi(w)$. We have

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* The function $\phi(w)$ can be given by means of elliptic integrals: see for example W. F. Osgood, *Lehrbuch der Funktionentheorie*, 5th edition, 1928, p. 437.

† This follows from the fact that if $\{w_n\}$ is a sequence of points in $|w| < 1$ and if

$$\lim_{n\to\infty} |w_n| = 1,$$

the sequence $\{\phi(w_n)\}$ has no limit point in R_1, for which see, for example, L. Bieberbach, *Lehrbuch der Funktionentheorie*, 2d edition, vol. 2, 1931, pp. 24–25.

‡ The theorems of the preceding section can be stated for a family of functions depending on a parameter running through a continuous set of values to a limiting value just as well as for a family depending on the parameter n taking on only integral values.

(5.103) $$\lim_{\lambda \to 0} mE_\lambda(\epsilon) = 0,$$

which implies that

(5.104) $$\lim_{\lambda \to 0} mE_\lambda'(\epsilon)/\lambda = 0^*.$$

But the measure of the set $\mathcal{E}_\lambda(\epsilon)$ of those points on the real axis on the interval $|\xi| \leqq \lambda/2$ for which $|F_1(\xi)| \leqq 1-\epsilon$ being less than $mE_\lambda'(\epsilon)$, it follows that

(5.105) $$\lim_{\lambda \to 0} m\mathcal{E}_\lambda(\epsilon)/\lambda = 0,$$

i.e. that $\mathcal{E}_\lambda(\epsilon)$ has density 0 at $\xi = 0$. Since $\epsilon > 0$ was arbitrary, it has been proved that $|F_1(\xi)|$ is approximately continuous at $\xi = 0$, if $|F_1(0)|$ is defined as 1.

To prove the first part of the statement (a) of the theorem we change the domain of definition of the function again. Define $f_2(\xi)$ analytic in the first quadrant, with boundary function $F_2(\xi)$, by

(5.106) $$f_2(\xi) = f\left(\frac{1+i\xi^2}{1-i\xi^2}\right), \quad F_2(\xi) = F_2\left(\frac{1+i\xi^2}{1-i\xi^2}\right).$$

By the corollary to Lemma 1.1 it is sufficient to prove the results desired for the function $f_2(\xi)$ and its boundary function $F_2(\xi)$ (we suppose that P is the point $z=1$, and the point in question in the ξ-plane is then the origin). Suppose then that $\lim_{\xi \to 0} |f_2(\xi)| = 1$ when ξ approaches $O: \xi = 0$ on a continuous curve C_2 which lies in the angle formed by a ray L in the first quadrant with one of the rays which bound the first quadrant. We can suppose that C_2 lies in the angle between L and the (positive) real axis. Let the angle between L and the positive real axis be θ and let L' be a ray through O into the first quadrant making an angle $\theta' > \theta$, $\theta' < \pi/2$, with the positive real axis. Let P_1 be a point of C_2 and let R_1 be the rectangle whose diagonals L_1', L_1'' intersect at P_1, and which has two vertices Q_1, Q_2 on the real axis. We suppose R_1 so chosen that $L_1' \| L'$. Then Q_1, Q_2 are both on the positive real axis; take $OQ_1 > OQ_2$. The ray through Q_2 parallel to L_1'' must meet C_2 in at least one point. Let P_2 be the intersection of the ray and C_2 which is nearest Q_2, and let R_2 be the rectangle whose diagonals L_2', L_2'' intersect at P_2, $L_2' \| L_1'$, $L_2'' \| L_1''$, and with vertices Q_2, Q_3 on the real axis. The point Q_3 must be on the positive real axis since $L_2' \| L'$. In this way we get a sequence of rectangles $R_1, R_2, \cdots$ whose diagonals intersect at $P_1, P_2, \cdots$ and a sequence

* This can be proved most readily using the fact that $\phi(w)$ on $|w|=1$ is continuous and has a continuous derivative except at four points in the neighborhood of which $\phi(w)$ has the same character as $w^{1/2}$ at $w=0$.

of vertices $Q_1, Q_2, \cdots$ on the positive real axis, $OQ_1 > OQ_2 \cdots$. The monotone sequence $\{Q_n\}$ has a unique limiting point which must be O, for if it were not O, the sequence $\{P_n\}$ would have as unique limit that same point on the positive real axis, which is impossible since P_n is on C_2 for all values of n. Let I_n be the closed interval with end points Q_n, Q_{n+1}. Then it is easily seen that

$$\frac{mI_n}{OQ_{n+1}} = \frac{mI_n}{\sum_{j+1}^{\infty} mI_j} \leqq \frac{2\tan\theta}{\tan\theta' - \tan\theta} = M. \tag{5.107}$$

Let $\phi_n(w)$ map $|w| < 1$ in a one-to-one and conformal way on the interior of R_n so that $\phi_n(0) = P_n$, $\phi_n'(0) > 0$. Then

$$\phi_n'(w) = \frac{mI_n}{mI_j}\phi_j'(w). \tag{5.108}$$

Consider the sequence $\{f_2[\phi_n(w)]\}$. Then

$$|f_2[\phi_n(w)]| \leqq 1, \quad \lim_{n\to\infty} |f_2[\phi_n(0)]| = 1. \tag{5.109}$$

Therefore, by Lemma 3.1, the sequence of the absolute values of the boundary functions is convergent in measure to 1 on $|z| = 1$. Repeating an argument used above, if $E(\epsilon)$ is the set of those points on the positive real axis for which $|F_2(\xi)| \leqq 1 - \epsilon$, $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{m[I_n \cdot E(\epsilon)]}{mI_n} = 0. \tag{5.110}$$

Now let I be a variable closed interval with one end point at O and the other on the positive real axis. Let $\lambda = \lambda(I)$ be the smallest value of j for which Q_j belongs to I. Then

$$\sum_{j=\lambda}^{\infty} mI_j \leqq mI \leqq \sum_{j=\lambda-1}^{\infty} mI_j, \quad mI_n \leqq M \sum_{j=n+1}^{\infty} mI_j, \tag{5.111}$$

and

$$m[I \cdot E(\epsilon)] \leqq \sum_{j=\lambda-1}^{\infty} m[I_j \cdot E(\epsilon)]. \tag{5.112}$$

Choose $\delta > 0$ so that

$$\frac{m[I_j \cdot E(\epsilon)]}{mI_j} \leqq \frac{\eta}{1+M} \quad \text{for } j \geqq \lambda - 1, \text{ if } mI \leqq \delta, \tag{5.113}$$

for some fixed positive number η. Then

$$(5.114)\qquad \frac{m[I\cdot E(\epsilon)]}{mI} \leqq \frac{\eta}{(1+M)mI}\sum_{j=\lambda}^{\infty} mI_j + \frac{\eta}{1+M}\,\frac{mI_{\lambda-1}}{mI} \leqq \eta.$$

Since η was an arbitrary positive number, $|F_2(\xi)|$ must be approximately continuous at O on the positive real axis, if $|F_2(0)|$ is defined as 1, as was to be proved.

(b) If $|f(z)|$ has the cluster value 1 at P (which we suppose to be the point $z=1$) we go again to the function $f_1(\xi)$ defined by (5.101). We have a sequence of points $\{P_n\}$, $P_n \to O: \xi=0$, such that

$$(5.115)\qquad \lim_{n\to\infty} |f_1(P_n)| = 1.$$

Let R_1 be a rectangle with one side on the positive real axis and whose diagonals intersect at P_1. Let R_n, $n>1$, be the rectangle with one side on the real axis, whose diagonals are parallel to those of R_1 and intersect at P_n. The rectangles are all similar and by considering $f_1(\xi)$ defined in R_n, we find by reasoning similar to that used above that there is a sequence of intervals on the real axis (the bases of the rectangles) such that, with self-explanatory notation,

$$(5.116)\qquad \lim_{n\to\infty} \frac{mI_n\cdot E(|F_1(\xi)| \leqq 1-\epsilon)}{mI_n} = 0,$$

which shows that $E(|F_1(\xi)| \geqq 1-\epsilon)$ is metrically dense at $\xi=0$ for all values of $\epsilon>0$, implying the same for $E(|F(z)| \geqq 1-\epsilon)$ at P on $|z|=1$.

If the sequence $\{\xi_n\}$ is non-tangential, the first proof given in (a) can be used, choosing a suitable sequence from the family $\{g_\lambda(w)\}$, to show that 1 is a cluster value of $|f(z)|$ on every continuous non-tangential path to P and that $|F(z)|$ is quasi-approximately continuous at P with limit value 1 there. There remains the proof that 1 is a cluster value on a continuous curve C which is tangent to $|z|=1$ at P. It is not difficult to reduce this to the results already proved, by the use of conformal mapping, and the details will not be given. If C is an arc of a circle, the preceding results show that $|f(z)|$ will be even quasi-approximately continuous on C at P, with limit value 1 there.

THEOREM 5.2. *Let $f(z)$ be a bounded function analytic for $|z|<1$ with Fatou boundary function $F(z)$, $|F(z)| \leqq 1$. Let P be a point on $|z|=1$.*

(a) *If $F(z)$ is defined at P and if $|F(P)|=1$, $F(z)$ is approximately continuous at P.**

* The converse is empty, strictly speaking, since in the definition of approximate continuity at P, $F(z)$ is supposed defined at P; cf. however Theorem 3 of the previous paper referred to above, and a note below.

(b) *A necessary and sufficient condition that $f(z)$ have the cluster value α at P, if $|\alpha| =1$, is that $E(|F(z)-\alpha| \leqq \epsilon)$ be metrically dense at P for all positive values of ϵ. A necessary and sufficient condition that $f(z)$ have α as a non-tangential cluster value at P if $|\alpha| =1$ is that $F(z)$ be quasi-approximately continuous at P with limit value α there. If α is a non-tangential cluster value at P, $|\alpha| =1$, it is a cluster value on every continuous non-tangential or tangential path to P.*

(c) *In* (a) *if $w=f(z)$, for z on some continuous non-tangential curve to P, defines a curve in the w-plane more closely tangent to $|w| =1$ at $w=F(P)$ than any circle C_ρ of radius $\rho<1$, in $|w| \leqq 1$ and tangent to $|w| =1$ at $w=F(P)$, the same is true for the curve defined by $w=f(z)$ when z is on any non-tangential curve to P and the metric density of the set E_ρ of those points at which $F(z)$ is outside C_ρ is* 1 *for all values of $\rho<1$. In* (b) *if α is a cluster value at P:*

$$\lim_{n\to\infty} f(z_n) = \alpha, \quad \lim_{n\to\infty} z_n = P, \quad |\alpha| = 1, \tag{5.21}$$

and if the sequence $\{w_n\}$, $w_n=f(z_n)$, is more closely tangent to $|w| =1$ at $w=\alpha$ than any circle C_ρ, $\rho<1$, the set E_ρ is metrically dense at P for all values of $\rho<1$. If the sequence $\{z_n\}$ in (5.21) *is non-tangential and if the sequence $\{w_n\}$ has the same property as above, E_ρ has upper mean metric density* 1 *at P for all values of $\rho<1$.*

The statements (a), (b) can be proved by an argument similar to that in the previous theorem, referring the result back to Theorem 3.1. The necessary conditions are simply Theorem 5.1 applied to

$$e^{[f(z)/\alpha]-1}.$$

We note that there are not two cases in (a) as there were in Theorem 5.1 because by a theorem of Lindelöf,* if $f(z)$ has a unique limit on a continuous curve to P, $f(z)$ will have that same limit on every non-tangential path.

The sufficient conditions are independent of the fact that $|\alpha| =1$ (cf. Theorem 3.1).†

The statement (c) can be deduced from the corollary to Theorem 3.1 or (a), (b) of Theorem 5.1 can be applied to the function

$$e^{(f(z)+1)/(f(z)-1)}.$$

This result is a sort of complement to a well known theorem of Julia, Wolff and Carathéodory.‡

* E. Lindelöf, Acta Societatis Scientiarum Fennicae, vol. 46 (1915), No. 4, p. 10.

† By using the criterion for convergence of a sequence given in the generalization of Theorem 3.1 (b), we can deduce again Theorem 3 of the previous paper, referred to above.

‡ See for example L. Bieberbach, *Lehrbuch der Funktionentheorie*, 2d edition, vol. 2, 1931, pp. 112–121.

THEOREM 5.3. *Let $f(z)$ be a bounded function, analytic for $|z|<1$ with Fatou boundary function $F(z)$. Suppose that $f(z)\neq\alpha$ in some neighborhood of a point P on $|z|=1$.*

(a) *A necessary and sufficient condition that*

$$\lim 1/\log\,[f(z)-\alpha] = 0^*$$

when z approaches P on continuous non-tangential paths is that $1/\log\,[F(z)-\alpha]$ *be approximately continuous at P if* $1/\log\,[F(P)-\alpha]$ *is defined as* 0.

(b) *A necessary and sufficient condition that* $1/\log\,[f(z)-\alpha]$ *have the cluster value* 0 *at P is that* $E\{|\log\,[F(z)-\alpha]|\geqq M\}$ *be metrically dense at P for all values of M. A necessary and sufficient condition that* $1/\log\,[f(z)-\alpha]$ *have the non-tangential cluster value* 0 *at P is that* $1/\log\,[F(z)-\alpha]$ *be quasi-approximately continuous at P with limit value* 0 *there. If* 0 *is a non-tangential cluster value it is a cluster value on every continuous non-tangential or tangential path to P.*

This theorem can be deduced by means of Theorem 3.2 or by applying Theorem 5.2 to the function

$$\frac{\log f(z)+1}{\log f(z)-1}$$

where we suppose that $\alpha=0$ and that $|f(z)|<1$ (cf. the proof of Theorem 3.2).

THEOREM 5.4. *Let $f(z)$ be a bounded function analytic for $|z|<1$, with Fatou boundary function $F(z)$. Suppose that $f(z)\neq\alpha$ in some neighborhood of a point P on $|z|=1$.*

(a) *A necessary and sufficient condition that $F(z)$ be defined at P and that $F(P)=\alpha$ is that*

$$\lim_{n\to\infty}\frac{\underline{B}\{\,|F(z)-\alpha|\,,E_n\}}{1+O\{\text{arc}\,[F(z)-\alpha],E_n\}}=0 \tag{5.41}$$

for every sequence $\{E_n\}$ of measurable point sets on $|z|=1$ with the property that there exists a sequence of arcs $\{A_n\}$ on $|z|=1$ with midpoint P such that

$$E_n\subset A_n,\ \lim_{n\to\infty} mA_n=0,\ \liminf_{n\to\infty}\frac{mE_n}{mA_n}>0. \tag{5.42}$$

(b) *A necessary and sufficient condition that $f(z)$ have α as a cluster value at P is that there be a sequence of arcs $\{A_n\}$ on $|z|=1$ whose end points approach P with the property that*

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* Cf. the note on p. 427.

$$\lim_{n\to\infty} \frac{\underline{B}\{\,|F(z) - \alpha|, E\cdot A_n\}}{1 + O\{\text{arc } [F(z) - \alpha], E\cdot A_n\}} = 0 \tag{5.43}$$

for every measurable set E such that

$$\liminf_{n\to\infty} \frac{mE\cdot A_n}{mA_n} > 0. \tag{5.44}$$

(c) *In* (b) *the conditions are necessary and sufficient that* $f(z)$ *have* α *as a non-tangential cluster value if the arcs* $\{A_n\}$ *are all taken to have the midpoint* P.

The corresponding statement for $f(z)$ defined in a half-plane is obvious, and it is convenient to prove it in this case. This is equivalent to proving the theorem as stated, as is shown by a slight extension of Lemma 1.1. The theorem is then easily deduced from Theorem 3.3 by considering $f(z)$ (defined in the half-plane $\Im(z) > 0$) in suitable rectangles with bases on the real axis. The discussion is analogous to that in the proof of Theorem 5.1.

COROLLARY. *Theorem* 5.3 *gives necessary and sufficient conditions* (a) *that* $F(z)$ *be defined at* P, (b) *that* $f(z)$ *have a given cluster value at* P, (c) *that* $f(z)$ *have a given non-tangential cluster value at* P, *if* $f(z)$ *is supposed univalent.**

For a univalent function $f(z)$ defined for $|z| < 1$ does not assume a cluster value at a point P on $|z| = 1$ in some neighborhood of P.

The problem set at the beginning of the section has thus been solved in a special case. If $f(z)$ is bounded and analytic in the interior of the unit circle, with Fatou boundary function $F(z)$, necessary and sufficient conditions have been found on $F(z)$ in a neighborhood of a point P on $|z| = 1$ that $F(z)$ be defined at P, and that $f(z)$ have the cluster value α at P, if $f(z) \neq F(P)$, $f(z) \neq \alpha$, respectively, in some neighborhood of P. In the first case the conditions need be modified only slightly if $f(z) = F(P)$ only at points of $|z| < 1$ on one side of some chord through P; we need only consider $F(z)$ on one side of P on $|z| = 1$. In both cases, by using Theorem 3.4, the general case can be solved, but the statement becomes so complicated that it is of no interest.

6. THE NEIGHBORHOOD PROPERTIES OF THE CLUSTER BOUNDARY FUNCTION OF A MEROMORPHIC FUNCTION

Let $f(z)$ be a function meromorphic for $|z| < 1$, with cluster boundary function $\mathcal{J}(z)$.† Let P be a point on $|z| = 1$. What are the relations between $f(z)$ and $\mathcal{J}(z)$ in a neighborhood of P? A partial answer to this has been given in §5, since if $f(z)$ is bounded, the value of its Fatou boundary function at a

* A function $f(z)$ is called univalent if $f(z_1) \neq f(z_2)$ unless $z_1 = z_2$.

† Cf. §2.

point P on $|z|=1$ where it is defined is also one of the values of $\mathcal{J}(P)$. It will be seen that the results of this section are generalizations of the following theorem.

THEOREM 6.1. *Let $f(z)$ be meromorphic for $|z|<1$, and let S be the sum of the cluster sets of $f(z)$ in $|z|<1$ at all the points of $|z|=1$. If $f(z)$ takes on a value α not in S, $f(z)$ assumes every value in the domain containing α and bounded only by points of S.**

Let $f_n(z)=f(z)$, $n=1, 2, \cdots$. Then this theorem is simply Theorem 4.1 for the sequence $\{f_n(z)\}$. A simple direct proof is the following. The set of values s assumed by $f(z)$ in $|z|<1$ is open. If s does not contain the domain D considered, there is a frontier point P of s in D. The point P is a limit point of assumed points:

$$\lim_{n\to\infty} f(z_n) = P$$

for some sequence $\{z_n\}$ in $|z|<1$. Since P is not assumed in $|z|<1$,

$$\lim_{n\to\infty}|z_n| = 1,$$

and so P belongs to S, contrary to the hypothesis that P was in D.

The theorem corresponding to Theorem 4.2 in this development is an important theorem first proved by W. Gross and F. Iversen.† This theorem can be proved most easily directly,‡ although the greater part can be proved without difficulty by the methods of this paper. The following is a generalization of the Gross-Iversen theorem.

THEOREM 6.2. *Let $f(z)$ be meromorphic in γ: $\mathfrak{I}(z)>0$, and let the cluster set of $f(z)$ on the real axis at $P: z=0$ on a given point set E be denoted by $S(E)$. Let α be a point of the cluster set s of $f(z)$ in γ at P:*

$$\lim_{n\to\infty} z_n = 0, \quad \lim_{n\to\infty} f(z_n) = \alpha, \tag{6.21}$$

and let $z_n=x_n+iy_n$ where x_n and y_n are real.

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* This theorem, a generalization of a well known theorem of Darboux, was obtained recently by Persidskij, Bulletin de la Société Physico-Mathématique de Kazan, vol. 3, No. 4 (1931), pp. 89–91.

† W. Gross, Monatshefte für Mathematik und Physik, vol. 29 (1918), pp. 3–47; Mathematische Zeitschrift, vol. 2 (1918), pp. 242–294.

F. Iversen, *Recherches sur les Fonctions Inverses des Fonctions Méromorphes*, Thesis, Helsingfors, 1914; Översikt av Finska Vetenskaps-Societetens Förhandlingar, vol. 58 (1915–1916), Section A, No. 25; ibid., vol. 64 (1921–1922), Section A, No. 4.

‡ Cf. a paper by the author in the Annals of Mathematics, (2), vol. 33 (1932), pp. 753–757.

(a) *Suppose that the sequence* $\{z_n\}$ *is tangential:*

$$\lim_{n\to\infty} \frac{y_n}{x_n} = 0.$$

Let R_1 *be the interior of a rectangle one of whose sides is on the real axis, and whose diagonals intersect at* z_1. *Let* R_n, $n > 1$, *be the interior of a rectangle, one of whose sides is on the real axis and whose diagonals, parallel to those of* R_1, *intersect at* $x_n + i\eta_n$ *where* η_n *is chosen so that*

$$\eta_n > 0, \lim_{n\to\infty} \frac{\eta_n}{x_n} = 0, \ \lim_{n\to\infty} \frac{y_n}{\eta_n} = 0.$$

Let E_N *be the set of all those points on a base of* R_n *for some value of* $n \geqq N$. *Then if* α *does not belong to* S: *the product of all the sets* $\{S(E_n)\}$, $f(z)$ *assumes in* R_n *each value in the domain* D *containing* α *and bounded only by points of* S, *for all except perhaps a finite set of values of* n, *with two possible exceptions; if there are two exceptions, they are the only ones in the extended plane for* $f(z)$ *in the rectangles* $\{R_n\}$.

(b) *Let* R_1 *be the interior of a rectangle one of whose sides is on the real axis and whose diagonals intersect at* z_1. *Let* R_n, $n > 1$, *be the interior of the rectangle one of whose sides is on the real axis and whose diagonals, parallel to those of* R_1, *intersect at* z_n. *Suppose that* α *is omitted by* $f(z)$ *in the rectangles* $\{R_n\}$. *Let* E_N *be the set of all those points on a base of* R_n *for some value of* $n \geqq N$. *Then if* α *does not belong to* S: *the product of all the sets* $\{S(E_n)\}$, $f(z)$ *assumes in* R_n *each value in the domain* D *containing* α *and bounded only by points of* S *for all except perhaps a finite set of values of* n, *with one possible exception, besides* α. *If there is one other such exceptional value, it is the only other one in the extended plane for* $f(z)$ *in the rectangles* $\{R_n\}$.

In (a) the sets $S(E_n)$ are identical for large values of n. If the bases of the rectangles $\{R_n\}$ do not cover the origin in (b), $S(E_n)$ can be used in place of S, for any value of n. The results (a) and (b) are consequences of Theorem 4.2 (a) and (c) respectively, applied to $f(z)$ defined in the rectangles described, and their proof presents no difficulty.

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