ON FINITE-ROWED SYSTEMS OF LINEAR INEQUALITIES IN INFINITELY MANY VARIABLES. II*

BY

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1. **Introduction.** In a previous paper on the same subject† a certain class of systems of linear inequalities in infinitely many variables were solved and applications to the theory of completely monotonic functions were derived from a particular type of such systems which were called Hausdorff systems. The present paper gives an extension of those results to a similar class of systems of linear inequalities involving a double sequence of variables (§2). This extension has already been performed by T. H. Hildebrandt and myself in the very particular case of completely monotonic double sequences.‡

In §4 Hausdorff systems involving a double sequence of variables are solved and applied to extend the results of F. Hausdorff, S. Bernstein, and D. V. Widder (see I, §11) to completely monotonic functions of two variables. The results of §3 (minimal solutions, minimal representations of solutions) though not absolutely necessary for the applications made in §5, help to present them in a more elegant manner.

2. **On a certain class of systems of linear inequalities for a double sequence of variables.** Let

$$(2.1)\qquad A = \left\| \begin{matrix} a_{01} & a_{02} & a_{03} & \cdots \\ a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \end{matrix} \right\|, \qquad B = \left\| \begin{matrix} b_{01} & b_{02} & b_{03} & \cdots \\ b_{11} & b_{12} & b_{13} & \cdots \\ b_{21} & b_{22} & b_{23} & \cdots \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \end{matrix} \right\|$$

be two given infinite matrices of real numbers. Let

$$(i_1, i_2, \cdots, i_j) = \left| a_{i_\alpha, \beta} \right|, \ [i_1, i_2, \cdots, i_j] = \left| b_{i_\alpha, \beta} \right| \quad (\alpha, \beta = 1, 2, \cdots, j).$$

Throughout this paper we shall suppose that

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† I. J. Schoenberg, *On finite-rowed systems of linear inequalities in infinitely many variables*, these Transactions, vol. 34, pp. 594–619. In the text this paper will be designated by the symbol I.

‡ See Theorem 1 of T. H. Hildebrandt and I. J. Schoenberg, *On linear functional operations and the moment problem for a finite interval in one or several dimensions*, to appear in the Annals of Mathematics. We shall frequently use the results of §3 of this paper and refer to it by the symbol HS.

$$(2.2)\qquad (i_1, i_2, \cdots, i_j) > 0,\ [i_1, i_2, \cdots, i_j] > 0 \quad (0 \leqq i_1 < i_2 < \cdots < i_j;\ j = 1, 2, 3, \cdots).$$

Let

$$(2.3)\quad D_1^k \xi_m \equiv \begin{vmatrix} \xi_m & a_{m,1} & \cdots & a_{m,k} \\ \xi_{m+1} & a_{m+1,1} & \cdots & a_{m+1,k} \\ \vdots & \vdots & & \vdots \\ \xi_{m+k} & a_{m+k,1} & \cdots & a_{m+k,k} \end{vmatrix},$$

$$D_2^h \eta_n \equiv \begin{vmatrix} \eta_n & b_{n,1} & \cdots & b_{n,h} \\ \eta_{n+1} & b_{n+1,1} & \cdots & b_{n+1,h} \\ \vdots & \vdots & & \vdots \\ \eta_{n+h} & b_{n+h,1} & \cdots & b_{n+h,h} \end{vmatrix} \qquad (k, h, m, n = 0, 1, 2, \cdots)$$

where $D_1^0\xi_m = \xi_m$ and $D_2^0\eta_n = \eta_n$.

Let $\mu_{mn}(m, n = 0, 1, 2, \cdots)$ be a double sequence of real variables. Both expressions

$$D_1^k(D_2^h\mu_{mn}),\ D_2^h(D_1^k\mu_{mn}),$$

in which the operator D_1^k applies to the subscript m and the operator D_2^h applies to the subscript n, are linear homogeneous combinations of the $(k+1)(h+1)$ variables $\mu_{m+k',n+h'}$ $(k'=0, 1, \cdots, k;\ h'=0, 1, \cdots, h)$. Writing out these expressions it is readily found that the coefficients of $\mu_{m+k',n+h'}$ in both expressions are equal to the product of the same two cofactors:

$$\frac{\partial}{\partial \xi_{m+k'}} D_1^k \xi_m \times \frac{\partial}{\partial \eta_{n+h'}} D_2^h \eta_n.$$

Hence

$$D_1^k(D_2^h\mu_{mn}) = D_2^h(D_1^k\mu_{mn}) = D_1^k D_2^h \mu_{mn}.$$

We shall be concerned with the problem of solving the system of linear inequalities

$$(2.4)\qquad D_1^k D_2^h \mu_{mn} \geqq 0 \qquad (k, h, m, n = 0, 1, 2, \cdots).$$

Without essentially restricting this problem, we shall suppose for convenience that

$$a_{i1} = b_{i1} = 1 \qquad (i = 0, 1, 2, \cdots).$$

As in I, Part II, and in HS, §4, we shall first solve the finite system

$$(2.5)\qquad D_1^k D_2^h \mu_{mn} \geqq 0 \qquad (k + m \leqq p;\ h + n \leqq p)$$

involving the $(p+1)^2$ variables μ_{mn} $(m, n=0, 1, \cdots, p)$. From the two identities (see I, p. 602)

$$D_1^k D_2^h \mu_{mn} = \frac{(m+1, \cdots, m+k)}{(m+1, \cdots, m+k+1)} D_1^{k+1} D_2^h \mu_{mn} + \frac{(m, \cdots, m+k)}{(m+1, \cdots, m+k+1)} D_1^k D_2^h \mu_{m+1,n},$$

$$D_1^k D_2^h \mu_{mn} = \frac{[n+1, \cdots, n+h]}{[n+1, \cdots, n+h+1]} D_1^k D_2^{h+1} \mu_{mn} + \frac{[n, \cdots, n+h]}{[n+1, \cdots, n+h+1]} D_1^k D_2^h \mu_{m,n+1}$$

and (2.2) it follows that (2.5) is equivalent to its partial system

$$D_1^{p-m} D_2^{p-n} \mu_{mn} \geqq 0 \qquad (m, n = 0, 1, \cdots, p), \tag{2.6}$$

which we shall now solve.

Let us define two new operators

$$O_1^k \bar{\xi}_m = \sum_{r=0}^{m+k} \frac{(m, r+1, \cdots, m+k)}{(r+1, \cdots, m+k)(r, \cdots, m+k)} \bar{\xi}_r,$$

$$O_2^h \bar{\eta}_n = \sum_{s=0}^{n+h} \frac{[n, s+1, \cdots, n+h]}{[s+1, \cdots, n+h][s, \cdots, n+h]} \bar{\eta}_s.$$

From the fact that the two linear transformations I(7.4) and I(7.6) are inverse to each other, it follows that the two linear transformations

$$\bar{\xi}_m = D_1^{p-m} \xi_m, \quad \xi_m = O_1^{p-m} \bar{\xi}_m \qquad (m = 0, 1, \cdots, p) \tag{2.7}$$

are inverse to each other and also that the same thing is true for the transformations

$$\bar{\eta}_n = D_2^{p-n} \eta_n, \ \eta_n = O_2^{p-n} \bar{\eta}_n \qquad (n = 0, 1, \cdots, p). \tag{2.8}$$

Let us now consider the linear transformation

$$\rho_{pmn} = D_1^{p-m} D_2^{p-n} \mu_{mn} \qquad (m, n = 0, 1, \cdots, p). \tag{2.9}$$

From (2.7) and (2.8) we derive successively

$$D_2^{p-n} \mu_{mn} = O_1^{p-m} \rho_{pmn} \text{ and } \mu_{mn} = O_2^{p-n} O_1^{p-m} \rho_{pmn}.$$

Hence

$$\mu_{mn} = O_1^{p-m} O_2^{p-n} \rho_{pmn} \qquad (m, n = 0, 1, \cdots, p) \tag{2.10}$$

is the linear transformation inverse to (2.9): *The system* (2.10) *gives, for* $\rho_{pmn} \geqq 0$, *the most general solution of* (2.6) *and hence of* (2.5).

The explicit form of (2.10) is

$$\mu_{mn} = \sum_{r,s=0}^{p} \frac{(m, r+1, \cdots, p)}{(r, \cdots, p)(r+1, \cdots, p)} \cdot \frac{[n, s+1, \cdots, p]}{[s, \cdots, p][s+1, \cdots, p]} \rho_{prs}.$$

Introducing the quantities

$$(2.11)\quad \lambda_{prs} = \frac{(0, r+1, \cdots, p)}{(r, \cdots, p)(r+1, \cdots, p)} \cdot \frac{[0, s+1, \cdots, p]}{[s, \cdots, p][s+1, \cdots, p]} \rho_{prs}$$

we get

$$\mu_{mn} = \sum_{r,s=0}^{p} \frac{(m, r+1, \cdots, p)}{(0, r+1, \cdots, p)} \cdot \frac{[n, s+1, \cdots, p]}{[0, s+1, \cdots, p]} \lambda_{prs}$$

which we write

$$(2.12)\qquad \mu_{mn} = \sum_{r,s=0}^{p} c_{mrp} d_{nsp} \lambda_{prs} \qquad (m, n = 0, 1, \cdots, p),$$

where

$$(2.13)\qquad c_{mrp} = \frac{(m, r+1, \cdots, p)}{(0, r+1, \cdots, p)}, \quad d_{nsp} = \frac{[n, s+1, \cdots, p]}{[0, s+1, \cdots, p]}$$

$$(m, r, n, s = 0, 1, \cdots, p;\ c_{mpp} = (m)/(0) = 1;\ d_{npp} = [n]/[0] = 1).$$

Let

$$(2.14)\qquad x_{pr} = c_{1rp},\ y_{ps} = d_{1sp} \qquad (r, s = 0, 1, \cdots, p).$$

As in I, §8, let $u = P_m^{(p)}(x)$ $(p \geqq m)$ denote the polygonal line in the plane (x, u), joining the points (x_{pr}, c_{mrp}) $(r = 0, 1, \cdots, p)$; similarly let $v = Q_n^{(p)}(y)$ $(p \geqq n)$ be the polygonal line in the plane (y, v), joining the points (y_{ps}, d_{nsp}) $(s = 0, 1, \cdots, p)$.

It has been proved in I, §8, that

$$(2.15)\quad \lim_{p\to\infty} P_m^{(p)}(x) = \phi_m(x), \quad \lim_{p\to\infty} Q_n^{(p)}(y) = \psi_n(y) \qquad (m, n = 0, 1, 2, \cdots)$$

hold uniformly in x and y respectively, in the interval $(0, 1)$; moreover $\phi_m(x)$ and $\psi_n(y)$ are the sequences of functions associated with the matrices A and B by Theorem 8.1 of I. It is shown there that the $\phi_m(x)$ are continuous, non-decreasing, convex, and

$$\phi_0(x) = 1,\ \phi_1(x) = x,\ \phi_{n+1}(0) = 0,\ \phi_n(1) = 1 \ (0 \leqq x \leqq 1;\ n = 0, 1, 2, \cdots).$$

The $\psi_n(y)$ have the same properties.

From Theorem 8.1 of I we know that the most general solutions of the two systems of linear inequalities

(2.16) $D_1^k \xi_m \geqq 0 \quad (k, m = 0, 1, 2, \cdots), \quad D_2^h \eta_n \geqq 0 \quad (h, n = 0, 1, 2, \cdots)$

are given respectively by

$$\xi_m = \int_0^1 \phi_m(x) d\chi_1(x) \qquad (m = 0, 1, 2, \cdots),$$

(2.17)

$$\eta_n = \int_0^1 \psi_n(y) d\chi_2(y) \qquad (n = 0, 1, 2, \cdots),$$

where $\chi_1(x)$ and $\chi_2(y)$ are monotonic in (0, 1). The same theorem says that the monotonic function $\chi_1(x)$ is essentially uniquely defined by the first set of equations (2.17) if and only if every function $f(x)$ which is continuous on (0, 1) can be uniformly approximated as close as we want by linear combinations of functions of the sequence $\{\phi_m(x)\}$. A sequence of continuous functions $\{\phi_m(x)\}$ with this property shall be called *a base of continuous functions on* (0, 1). The same definition will be used for functions of several variables.

For convenience we introduce the following

DEFINITION 2.1. *The first system* (2.16) *shall be called a determining system if and only if the corresponding sequence* $\{\phi_m(x)\}$ *is a base of continuous functions on* (0, 1). *Otherwise it shall be called a non-determining system.*

The following theorem is readily proved.

THEOREM 2.1. (1) *If the solutions of the systems* (2.16), *as given by Theorem* 8.1 *of* I, *are* (2.17), *then the most general solution of the system*

(2.18) $$D_1^k D_2^h \mu_{mn} \geqq 0 \qquad (k, h, m, n = 0, 1, 2, \cdots)$$

may be expressed in the form

(2.19) $$\mu_{mn} = \int_0^1 \int_0^1 \phi_m(x)\psi_n(y) d_x d_y \chi(x, y) \quad (m, n = 0, 1, 2, \cdots)$$

with $\chi(x, y)$ *monotonic in the sense of Hardy and Krause* (see HS, §3), *and conversely,* (2.19) *always represents a solution of* (2.18).

(2) *A necessary and sufficient condition that the function* $\chi(x, y)$ *be uniquely defined by the set* (2.19) *and the additional conditions*

(2.20) $$\chi(0, 0) = \chi(x, 0) = \chi(0, y) = 0, \ \chi(x, y) = \chi(x + 0, y + 0), \quad \text{for } 0 < x \leqq 1, \ 0 < y \leqq 1,$$

is that both systems (2.16) *shall be determining systems, in which case also* (2.18) *shall be called a determining system.*

Let $\mu_{mn}(m, n=0, 1, 2, \cdots)$ be a solution of (2.18). Then (2.5) holds for every value of p and therefore also all the consequences derived therefrom.

Let us define in the unit-square U $(0\leqq x\leqq 1,\ 0\leqq y\leqq 1)$ a step-function $\chi_p(x, y)$ as follows:

(a) $$\chi_p(x, 0) = \chi_p(0, y) = 0;$$

(b) $$\chi_p(x_{pr}+0, y_{ps}+0) - \chi_p(x_{pr}+0, y_{ps}-0) - \chi_p(x_{pr}-0, y_{ps}+0) + \chi_p(x_{pr}-0, y_{ps}-0) = \lambda_{prs} \quad (r, s = 0, 1, \cdots, p);^*$$

(c) $\chi_p(x, y)$ is constant in each of the rectangles

$$x_{pr} < x < x_{p,r+1}, \quad y_{ps} < y < y_{p,s+1},$$

and also on each of the line segments

$$x_{pr} < x < x_{p,r+1},\ y = 1; \quad x = 1,\ y_{ps} < y < y_{p,s+1};$$

moreover $\chi_p(x, y) = \chi_p(x+0, y+0)$ for $0<x\leqq 1$, $0<y\leqq 1$.

From (2.11), (2.9), (2.6) and (2.12) (for $m=n=0$) we conclude that

$$\lambda_{prs} \geqq 0 \quad (r, s = 0, 1, \cdots, p), \qquad \sum_{r,s=0}^{p} \lambda_{prs} = \mu_{00}, \tag{2.21}$$

which shows that $\{\chi_p(x, y)\}$ is a sequence of uniformly bounded monotonic functions in U. On the other hand, from (2.12) we derive for $p\geqq \max(m, n)$

$$\mu_{mn} = \sum_{r,s=0}^{p} P_m^{(p)}(x_{pr})Q_n^{(p)}(y_{ps})\lambda_{prs} = \int_0^1\int_0^1 P_m^{(p)}(x)Q_n^{(p)}(y)d_xd_y\chi_p(x, y)$$
$$= \int_0^1\int_0^1 \phi_m(x)\psi_n(y)d_xd_y\chi_p(x, y) + \epsilon_{pmn},$$

where $\epsilon_{pmn}\to 0$ as $p\to\infty$, on account of the uniform convergence in (2.15) and the uniform boundedness of the $\chi_p(x, y)$. From a theorem† of J. Radon, we know that there is a subsequence χ_q of χ_p converging in U to a monotonic function χ. From the same lemma and our last relation we derive (2.19) as $p=q\to\infty$.

* $\chi_p(1+0, y_{ps}\pm 0)$ means $\chi_p(1, y_{ps}\pm 0)$; $\chi_p(0-0, y_{ps}\pm 0)$ means $\chi_p(0, y_{ps}\pm 0)$, etc.

† This theorem, which is an extension to functions of two variables of a well known theorem of Helly, says: If $\{\chi_p(x, y)\}$ is a sequence of functions which are uniformly bounded and uniformly of bounded variation in U, then there is a subsequence $\{\chi_q(x, y)\}$ converging everywhere in U to a function $\chi(x, y)$ of bounded variation in U. Moreover, for every $f(x, y)$ continuous in U

$$\lim_{q\to\infty}\int_0^1\int_0^1 f(x, y)d_xd_y\chi_q(x, y) = \int_0^1\int_0^1 f(x, y)d_xd_y\chi(x, y).$$

J. Radon, Sitzungsberichte der Wiener Akademie, vol. 122 IIa (1913), pp. 1337–1342, and vol. 128 IIa (1919), pp. 1092–1094, proved this theorem in a slightly weaker form, which, however, would also suffice for our purpose. For the present statement see HS, §3, Lemma 1.

Conversely, let χ in (2.19) be a monotonic function. We have to show that (2.19) represents a solution of (2.18). Indeed

$$D_1^k D_2^h \mu_{mn} = \int_0^1 \int_0^1 D_1^k D_2^h \phi_m(x)\psi_n(y) d_x d_y \chi(x, y)$$
$$= \int_0^1 \int_0^1 [D_1^k \phi_m(x)][D_2^h \psi_n(y)] d_x d_y \chi(x, y) \geqq 0,$$

since $D_1^k \phi_m(x) \geqq 0$, $D_2^h \psi_n(y) \geqq 0$, for any x and y on (0, 1).

The second part of Theorem 2.1 follows readily from an extension to two variables of a theorem of F. Riesz.* From this theorem it follows that $\chi(x, y)$ is uniquely defined by (2.19) and (2.20) if and only if $\{\phi_m(x)\psi_n(y)\}$ is a base of continuous functions in U. However, it is readily shown that $\{\phi_m(x)\psi_n(y)\}$ is a base of continuous functions in U if and only if both sequences $\{\phi_m(x)\}$ and $\{\psi_n(y)\}$ are such bases in (0, 1). For if both sequences $\{\phi_m(x)\}$, $\{\psi_n(y)\}$ are bases, then every polynomial $P(x, y)$ can be uniformly approximated by expressions of the form $\sum_{m,n} \gamma_{mn}\phi_m(x)\psi_n(y)$, hence also any continuous $f(x, y)$. Conversely, if this is true for any $f(x, y)$, then in particular for any continuous $f(x)$ we have

$$f(x) = \sum \gamma_{mn}\phi_m(x)\psi_n(y) + \rho(x, y) \quad (|\rho| < \epsilon)$$

throughout U. An integration over (0, 1) with respect to y shows that $\{\phi_m(x)\}$ is a base in (0, 1). This completes the proof of Theorem 2.1.

3. **Minimal solutions.** We have so far solved completely the following three systems of linear inequalities:

$$D_1^k \xi_m \geqq 0 \qquad (k, m = 0, 1, 2, \cdots), \tag{3.1}$$

$$D_2^h \eta_n \geqq 0 \qquad (h, n = 0, 1, 2, \cdots), \tag{3.2}$$

$$D_1^k D_2^h \mu_{mn} \geqq 0 \qquad (k, h, m, n = 0, 1, 2, \cdots), \tag{3.3}$$

and their most general solutions were found to be

* See F. Riesz, loc. cit. in I, §1. The extended theorem says: If $\phi_{mn}(x, y)$ $(m, n = 0, 1, 2, \cdots)$ is a double sequence of continuous functions in U, then a function $\chi(x, y)$ monotonic in U is uniquely defined by the set of equations

$$\mu_{mn} = \int_0^1 \int_0^1 \phi_{mn}(x, y) d_x d_y \chi(x, y) \qquad (m, n = 0, 1, 2, \cdots)$$

and the conditions (2.20), if and only if $\{\phi_{mn}(x, y)\}$ is a base of continuous functions in U. There seems to be no proof of this theorem in the literature. However, the proof for the case of one variable given by W. Seidel, Annals of Mathematics, (2), vol. 32 (1931), pp. 777–784, can be extended immediately to prove the theorem just stated, if one applies Lemma 1 of HS, §3. The same theorem may also be derived from general results concerning linear metric spaces. See I. J. Schoenberg and W. Seidel, *On linear operations in linear metric spaces*, to appear in these Transactions.

$$(3.1') \qquad \xi_m = \int_0^1 \phi_m(x)d\chi_1(x) \qquad (m = 0, 1, 2, \cdots)$$

$$(\chi_1(0) = 0, \chi_1(x) = \chi_1(x+0) \text{ for } 0 < x < 1),$$

$$(3.2') \qquad \eta_n = \int_0^1 \psi_n(y)d\chi_2(y) \qquad (n = 0, 1, 2, \cdots)$$

$$(\chi_2(0) = 0, \chi_2(y) = \chi_2(y+0) \text{ for } 0 < y < 1),$$

$$(3.3') \qquad \mu_{mn} = \int_0^1 \int_0^1 \phi_m(x)\psi_n(y)d_xd_y\chi(x, y) \qquad (m, n = 0, 1, 2, \cdots)$$

$$(\chi(x, y) \text{ satisfies } (2.20)),$$

respectively, where $\chi_1(x)$, $\chi_2(y)$ and $\chi(x, y)$ are monotonic.

If ξ_m is a solution of (3.1), then also $\xi_m + (0)^m\gamma$ $(\gamma > 0)$ is such a solution.* We shall need the following

DEFINITION 3.1. *A solution ξ_m of* (3.1) *is called a minimal solution*† *if there is no other solution ξ_m' of* (3.1) *and a constant $\gamma > 0$ such that*

$$\xi_m = \xi_m' + (0)^m\gamma \qquad (m = 0, 1, 2, \cdots).$$

We prove now

THEOREM 3.1. *Let* (3.1) *be a determining system.*‡ *Its solution ξ_m given by* (3.1′) *is a minimal solution if and only if the monotonic function $\chi_1(x)$ is continuous at $x = 0$.*

The condition $\chi_1(0) = \chi_1(+0)$ is necessary for ξ_m to be a minimal solution of (3.1). For let us suppose that $0 = \chi_1(0) < \chi_1(+0)$, and let us define the function $\chi_{10}(x) = \chi_1(x) + (0)^x\chi_1(+0)$ which is continuous at the origin. Then

$$\xi_m = \int_0^1 \phi_m d\chi_1(x) = \int_0^1 \phi_m(x)d\chi_{10}(x) + (0)^m\chi(+0)$$

shows that ξ_m is no minimal solution of (3.1).

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* We define $(0)^m = 0$ for $m > 0$, $(0)^0 = 1$. Note that $\phi_m(0) = (0)^m$, $\phi_n(0) = (0)^n$.

† Hausdorff has already called attention to the distinction between minimal and non-minimal completely monotonic sequences. The name *minimal* is due to D. V. Widder, these Transactions, vol. 33 (1931), p. 880.

‡ That the condition of this theorem is not always sufficient to insure that ξ_m is a *minimal* solution of a *non-determining* system is shown by the following example. Let (3.1) be a non-determining Hausdorff system of the type considered in Theorem 10.2 of I. Let $a_0 = 0$, hence $c_{00} = c_{01} = c_{02} = \cdots = 1$. Take $\chi_1(x) = x$ for $0 \leqq x \leqq c_{11}$, $\chi_1(x) = c_{11}$ for $c_{11} \leqq x \leqq 1$. This function is continuous throughout (0, 1). However, the solution given by (3.1′) is readily found to be $\xi_0 = c_{11}/2 + c_{11}/2$, $\xi_1 = c_{11}^2/2$, $\xi_2 = \xi_3 = \cdots = 0$ and this is a non-minimal solution since $c_{11} > 0$, and $c_{11}/2$, $c_{11}^2/2$, 0, 0, 0, $\cdots$ is the solution of (3.1) given by Theorem 10.2 of I, for $\gamma = \mu_0 = \mu_2 = \mu_3 = \cdots = 0$, $\mu_1 = c_{11}/2$.

To show the sufficiency of our condition let us prove that

$$\xi_m = \int_0^1 \phi_m(x)d\chi_1(x) \text{ with } \chi_1(0) = \chi_1(+0)$$

is a minimal solution. Indeed, if ξ_m were not a minimal solution, then we should have

$$\int_0^1 \phi_m d\chi_1(x) = \int_0^1 \phi_m d\bar{\chi}_1(x) + (0)^m\gamma$$

with $\bar{\chi}_1$ monotonic, $\bar{\chi}_1(0)=0$, and $\gamma>0$. The function $\bar{\chi}_{11}(x)=\chi_1(x)+[1-(0)^x]\gamma$ is monotonic and

$$\int_0^1 \phi_m d\chi_1(x) = \int_0^1 \phi_m d\bar{\chi}_{11}(x) \qquad (m = 0, 1, 2, \cdots)$$

which is impossible, since (3.1) is a determining system, while

$$0 = \chi_1(0) = \bar{\chi}_{11}(0) = \chi_1(+0) < \bar{\chi}_{11}(+0) = \bar{\chi}_1(+0) + \gamma.$$

For a more thorough investigation of the solutions of the system (3.3) we shall need the following

LEMMA 3.1. *Let $\chi(x, y)$ be monotonic in U and satisfy the conditions* (2.20). *If we define a new function $\chi_0(x, y)$ as follows*:

$$\begin{aligned} \chi_0(x, y) &= \chi(x, y) \text{ for } 0 < x \leqq 1, 0 < y \leqq 1;\ \chi_0(0, 0) = \chi(+0, +0); \\ \chi_0(x, 0) &= \chi(x, +0) \text{ for } 0 < x \leqq 1;\ \chi_0(0, y) = \chi(+0, y) \text{ for } 0 < x \leqq 1; \end{aligned} \tag{3.4}$$

then the solution (3.3′) *of* (3.3) *may be written in the form*

$$\begin{aligned} \mu_{mn} &= \int_0^1 \int_0^1 \phi_m(x)\psi_n(y)d_xd_y\chi(x, y) \\ &= \int_0^1 \int_0^1 \phi_m(x)\psi_n(y)d_xd_y\chi_0(x, y) \\ &\quad + (0)^n \int_0^1 \phi_m(x)d\chi_0(x, 0) \\ &\quad + (0)^m \int_0^1 \psi_n(y)d\chi_0(0, y) + (0)^{m+n}\chi_0(0, 0). \end{aligned} \tag{3.5}$$

This follows readily from the definition of the double Stieltjes integral. We know that μ_{mn} is the limit of the following expression (see HS, §3; here we take $\xi_{ij}=\xi_i$, $\eta_{ij}=\eta_j$, with $\xi_0=\eta_0=0$)

$$\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\phi_m(\xi_i)\psi_n(\eta_j)\Delta(\chi;\ x_{i+1},\ y_{j+1},\ x_i,\ y_j)^*$$

$$= \sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\phi_m(\xi_i)\psi_n(\eta_j)\Delta(\chi_0;\ x_{i+1},\ y_{j+1},\ x_i,\ y_j)$$

$$+ (0)^n \sum_{i=0}^{p-1}\phi_m(\xi_i)\left[\chi_0(x_{i+1},\ 0) - \chi_0(x_i,\ 0)\right]$$

$$+ (0)^m \sum_{j=0}^{q-1}\psi_n(\eta_j)\left[\chi_0(0,\ y_{j+1}) - \chi_0(0,\ y_j)\right] + (0)^{m+n}\chi_0(0,\ 0),$$

as $p\to\infty$, $q\to\infty$, and all subintervals tend to zero. The last identity follows from (3.4). Passing to the limit, it goes over into (3.5) which is thus proved.

LEMMA 3.2. *If* μ_{mn} *is a solution of* (3.3) *then also*

$$\bar{\mu}_{mn} = \mu_{mn} + (0)^n\xi_m + (0)^m\eta_n + (0)^{m+n}\gamma, \tag{3.6}$$

where ξ_m *and* η_n *are solutions of* (3.1) *and* (3.2) *respectively, and* $\gamma \geqq 0$, *is a solution of* (3.3).

Let

$$\xi_m = \int_0^1 \phi_m(x)d\chi_1(x) + (0)^m\gamma_1 \quad (\chi_1(0) = \chi_1(+0) = 0,\ \gamma_1 \geqq 0), \tag{3.7}$$

$$\eta_n = \int_0^1 \psi_n(y)d\chi_2(y) + (0)^n\gamma_2 \quad (\chi_2(0) = \chi_2(+0) = 0,\ \gamma_2 \geqq 0). \tag{3.8}$$

With $\chi_0(x, y)$ defined by (3.4), we define two new functions

$$\bar{\chi}_0(x,\ y) = \chi_0(x,\ y) + \chi_1(x) + \chi_2(y) + \gamma_1 + \gamma_2 + \gamma \tag{3.9}$$

and

$$\bar{\chi}(x,\ y) = \bar{\chi}_0(x,\ y),\quad \bar{\chi}(x,\ 0) = \bar{\chi}(0,\ y) = \bar{\chi}(0,\ 0) = 0, \quad \text{for } 0 < x \leqq 1,\ 0 < y \leqq 1. \tag{3.10}$$

Then $\bar{\chi}(x, y)$ is a monotonic function of which the corresponding function given by (3.4) is precisely $\chi_0(x, y)$. From (3.6), (3.5), (3.7) and (3.8) we then derive

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* We write $\chi(x_{i+1}, y_{j+1}) - \chi(x_{i+1}, y_j) - \chi(x_i, y_{j+1}) + \chi(x_i, y_j) = \Delta(\chi; x_{i+1}, y_{j+1}, x_i, y_j)$.

$$\bar{\mu}_{mn} = \int_0^1 \int_0^1 \phi_m \psi_n d_x d_y \bar{\chi}_0(x, y) + (0)^n \int_0^1 \phi_m d\bar{\chi}_0(x, 0)$$

$$+ (0)^m \int_0^1 \psi_n d\bar{\chi}_0(0, y) + (0)^{m+n} \bar{\chi}_0(0, 0)$$

$$= \int_0^1 \int_0^1 \phi_m(x)\psi_n(y) d_x d_y \bar{\chi}(x, y).$$

Hence $\bar{\mu}_{mn}$ is a solution of (3.3).

Our last lemma justifies the following

DEFINITION 3.2. *A solution $\bar{\mu}_{mn}$ of* (3.3) *is called a minimal solution if there is no other solution μ_{mn} of* (3.3), *as well as two solutions ξ_m and η_n of* (3.1) *and* (3.2), *and a constant $\gamma \geqq 0$, with $\xi_0+\eta_0+\gamma>0$, such that* (3.6) *shall hold for $m, n=0, 1, 2, \cdots$.*

A first criterion for minimal solutions of (3.3) is given by

LEMMA 3.3. *A solution μ_{mn} of the determining system* (3.3) *is a minimal solution if and only if both sequences μ_{m0} and μ_{0n} are minimal solutions of the corresponding systems* (3.1) *and* (3.2), *in which case μ_{mn} is, for any fixed value of n, a minimal solution of* (3.1), *and similarly, for any fixed value of m, a minimal solution of* (3.2).

If μ_{mn} of (3.5) is a minimal solution of (3.3), then necessarily

$$\chi_0(x, 0) \equiv \chi_0(0, y) \equiv \chi_0(0, 0) = 0, \tag{3.11}$$

otherwise the representation (3.5) would contradict the assumption that μ_{mn} is minimal. We therefore have $\chi(x, y) \equiv \chi_0(x, y)$. Integrating by parts* we get

$$\mu_{m0} = \int_0^1 \int_0^1 \phi_m(x) d_x d_y \chi(x, y) = \int_0^1 \phi_m(x) d\chi(x, 1),$$

which shows that μ_{m0} is a minimal solution of the determining system (3.1) because $\chi(+0, 1) = \chi_0(0, 1) = \chi(0, 1) = 0$ (Theorem 3.1). A similar proof shows that μ_{0n} is a minimal solution of (3.2).

Suppose now that μ_{mn} is not a minimal solution of (3.3). Then

$$\mu_{mn} = \mu'_{mn} + (0)^n \xi_m + (0)^m \eta_n + (0)^{m+n} \gamma,$$

with $\xi_0+\eta_0+\gamma>0$. One of the quantities $\xi_0+\gamma$, $\eta_0+\gamma$ is >0. Suppose that $\eta_0+\gamma>0$. For $n=0$ we derive

* See E. W. Hobson, *Functions of a Real Variable*, vol. 1, 3d edition, 1927, §448.

$$\mu_{m0} = \mu'_{m0} + \xi_m + (0)^m(\eta_0 + \gamma),$$

and hence μ_{m0} is no minimal solution of (3.1).

Suppose again that μ_{mn} is a minimal solution of (3.3). Writing

$$\bar{\chi}(x, y) = \int_0^x \int_0^y \psi_n(y) d_x d_y \chi(x, y),$$

we get

$$\mu_{mn} = \int_0^1 \int_0^1 \phi_m(x)\psi_n(y) d_x d_y \chi(x, y) = \int_0^1 \int_0^1 \phi_m(x) d_x d_y \bar{\chi}(x, y)$$
$$= \int_0^1 \phi_m(x) d_x \bar{\chi}(x, 1),$$

while

$$\bar{\chi}(x, 1) = \int_0^x \int_0^1 \psi_n(y) d_x d_y \chi(x, y) = \int_0^1 \psi_n(y) d_y \chi(x, y)$$

is obviously continuous at $x=0$. We conclude again from Theorem 3.1 that μ_{mn} is a minimal solution of (3.1), for n fixed.

From this lemma we derive the following

THEOREM 3.2. *The solution* (3.3′) *of the determining system* (3.3) *is a minimal solution of this system if and only if* $\chi(x, y)$ *is continuous as a function of* (x, y) *along the two sides of the unit-square* U *which meet at the origin.*

For convenience, we denote by L those two sides of U. If μ_{mn} is a minimal solution of (3.3), then (3.11) holds and this obviously implies the continuity of $\chi(x, y)$ along L.

Conversely, if $\chi(x, y)$ is continuous along L, then

$$\mu_{m0} = \int_0^1 \int_0^1 \phi_m(x) d_x d_y \chi(x, y) = \int_0^1 \phi_m(x) d_x \chi(x, 1)$$

is a minimal solution of the determining system (3.1), since $\chi(+0, 1)=0$ (Theorem 3.1). Similarly μ_{0n} is a minimal solution of (3.2). It therefore follows from Lemma 3.3 that μ_{mn} is a minimal solution of (3.3) and the theorem is proved.

Of importance is the following

LEMMA 3.4. *Every solution* μ_{mn} *of the determining system* (3.3) *may be expressed in the form*

$$\mu_{mn} = \bar{\mu}_{mn} + (0)^n \xi_m + (0)^m \eta_n + (0)^{m+n}\gamma, \tag{3.12}$$

where $\bar{\mu}_{mn}$, ξ_m, η_n *are minimal solutions of the systems* (3.3), (3.1), (3.2), *and* $\gamma \geqq 0$. *This representation is unique and shall be called the minimal representation of the solution* μ_{mn} *of the system* (3.3).

Let our solution μ_{mn} be given in the form (3.5). Introducing the new function

$$\chi_{00}(x, y) = \chi_0(x, y) - \chi_0(x, 0) - \chi_0(0, y) + \chi_0(0, 0), \tag{3.13}$$

from (3.5) we derive

$$\begin{aligned} \mu_{mn} = &\int_0^1 \int_0^1 \phi_m(x)\psi_n(y) d_x d_y \chi_{00}(x, y) \\ &+ (0)^n \int_0^1 \phi_m(x) d\chi_0(x, 0) + (0)^m \int_0^1 \psi_n(y) d\chi_0(0, y) + (0)^{m+n}\chi_0(0, 0). \end{aligned} \tag{3.14}$$

This is a representation of the type (3.12). Indeed, $\chi_0(x, 0)$ and $\chi_0(0, y)$ are both continuous at the origin and $\chi_0(0, 0) \geqq 0$. Moreover, the function $\chi_{00}(x, y)$ defined by (3.13) is readily found to be continuous along L and vanishing on L. From Theorems 3.1 and 3.2 we infer the truth of our last statement.

To prove the uniqueness of (3.12), let

$$\begin{aligned} \mu_{mn} &= \bar{\mu}_{mn} + (0)^n \xi_m + (0)^m \eta_n + (0)^{m+n}\gamma \\ &= \bar{\mu}'_{mn} + (0)^n \xi'_m + (0)^m \eta'_n + (0)^{m+n}\gamma' \end{aligned} \tag{3.15}$$

be two representations of the type (3.12). For any particular value of $n > 0$, we get

$$\bar{\mu}_{mn} = \bar{\mu}'_{mn} \qquad (m = 1, 2, 3, \cdots). \tag{3.16}$$

Since $\bar{\mu}_{mn}, \bar{\mu}$ $(m = 0, 1, 2, \cdots)$ are both minimal solutions of (3.1) (Lemma 3.3), (3.16) must hold also for $m = 0$. A similar argument applied to the subscript n, shows that (3.16) holds whenever $m + n > 0$. Then necessarily $\bar{\mu}_{00} = \bar{\mu}'_{00}$, since both $\bar{\mu}_{mn}$ and $\bar{\mu}'_{mn}$ are minimal solutions of (3.3). From (3.15) we now derive

$$(0)^n \xi_m + (0)^m \eta_n + (0)^{m+n}\gamma = (0)^n \xi'_m + (0)^m \eta'_n + (0)^{m+n}\gamma',$$

from which, for $n = 0$, $m > 0$, we get $\xi_m = \xi'_m$, which holds also for $m = 0$. Similarly $\eta_n = \eta'_n$, and finally for $m = n = 0$, we obtain $\gamma = \gamma'$.

A consequence is the following

THEOREM 3.3. *Every solution* μ_{mn} *of the determining system* (3.3) *may be represented as follows*:

$$\begin{aligned} \mu_{mn} = &\int_0^1 \int_0^1 \phi_m(x)\psi_n(y) d_x d_y \chi_{00}(x, y) \\ &+ (0)^n \int_0^1 \phi_m(x) d\chi_1(x) + (0)^m \int_0^1 \psi_n(y) d\chi_2(y) + (0)^{m+n}\gamma \\ &\qquad (m, n = 0, 1, 2, \cdots; \gamma \geqq 0), \end{aligned} \tag{3.17}$$

where $\chi_{00}(x, y), \chi_1(x), \chi_2(y)$ *are monotonic and satisfy the conditions*

$$\chi_{00}(0,0)=\chi_{00}(x,0)=\chi_{00}(0,y)=0,\ \chi_{00}(x,y)=\chi_{00}(x+0,y+0)\quad \text{for } 0<x\leqq 1,\ 0<y\leqq 1;\tag{3.18}$$
$$\chi_{00}(x,y) \text{ is continuous along } L(0\leqq x\leqq 1,\ y=0;\ x=0,\ 0\leqq y\leqq 1);$$

$$\begin{aligned}\chi_1(0)&=\chi_1(+0)=0,\ \chi_1(x)=\chi_1(x+0)\ \text{for } 0<x<1,\\ \chi_2(0)&=\chi_2(+0)=0,\ \chi_2(y)=\chi_2(y+0)\ \text{for } 0<y<1.\end{aligned}\tag{3.19}$$

This is a minimal representation of the solution μ_{mn} and is unique in the sense that the three monotonic functions $\chi_{00}(x, y)$, $\chi_1(x)$, $\chi_2(y)$ and the constant γ are uniquely defined by (3.17), (3.18), *and* (3.19). *The solution μ_{mn} is minimal if and only if $\chi_1(x)\equiv\chi_2(y)\equiv\gamma=0$.*

From the minimal representation (3.12) and Theorems 3.1 and 3.2 we immediately derive (3.17). The uniqueness of (3.17) follows from the uniqueness of a minimal representation (Lemma 3.4) and from the fact that our systems (3.1), (3.2) and (3.3) are determining systems.

From (3.3′) we derive (3.17) by means of (3.4), (3.13) and

$$\chi_1(x)=\chi_0(x,0)-\chi_0(0,0),\ \chi_2(y)=\chi_0(0,y)-\chi_0(0,0),\ \gamma=\chi_0(0,0).\tag{3.20}$$

4. **Hausdorff systems for double sequences.** The system of linear inequalities (3.3) is called a *Hausdorff system* if both systems (3.1) and (3.2) are Hausdorff systems, that is to say (see I, §9), when both matrices A and B of (2.1) are of the Vandermondean type:

$$A=\left\|\begin{matrix}1 & a_0 & a_0^2 & \cdots\\ 1 & a_1 & a_1^2 & \cdots\\ 1 & a_2 & a_2^2 & \cdots\\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \end{matrix}\right\|,\quad B=\left\|\begin{matrix}1 & b_0 & b_0^2 & \cdots\\ 1 & b_1 & b_1^2 & \cdots\\ 1 & b_2 & b_2^2 & \cdots\\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \end{matrix}\right\|,\tag{4.1}$$

with

$$a_0<a_1<a_2<\cdots,\quad b_0<b_1<b_2<\cdots.\tag{4.2}$$

We shall invariably suppose that

$$\sum^{+\infty}\frac{1}{a_r}=+\infty,\quad \sum^{+\infty}\frac{1}{b_r}=+\infty,\tag{4.3}$$

whenever $a_r\to+\infty$, or $b_r\to+\infty$, and discuss the following three further possibilities:

$$\lim_{r\to+\infty} a_r=+\infty,\quad \lim_{r\to+\infty} b_r=+\infty,\tag{4.4}$$

$$\lim_{r\to+\infty} a_r=\alpha,\quad \lim_{r\to+\infty} b_r=\beta\qquad (\alpha,\beta<+\infty),\tag{4.5}$$

$$\lim_{r\to+\infty} a_r=\alpha,\quad \lim_{r\to+\infty} b_r=+\infty\qquad (\alpha<+\infty).\tag{4.6}$$

As we know from I, §§9–10, only the case (4.4) leads to a determining system (3.3). Moreover, in this case

$$\phi_m(x) = x^{(a_m - a_0)/(a_1 - a_0)}, \quad \psi_n(y) = y^{(b_n - b_0)/(b_1 - b_0)}. \tag{4.7}$$

According to Theorem 3.3, a minimal solution of the system (3.3) (with (4.1), (4.2), (4.3), and (4.4)) is given by

$$\mu_{mn} = \int_0^1 \int_0^1 x^{(a_m - a_0)/(a_1 - a_0)} y^{(b_n - b_0)/(b_1 - b_0)} d_x d_y \chi_{00}(x, y),$$

where $\chi_{00}(x, y)$ satisfies the conditions (3.18). Writing

$$\chi_{00}(x^{a_1 - a_0}, y^{b_1 - b_0}) = \chi(x, y),$$

we get

$$\mu_{mn} = \int_0^1 \int_0^1 x^{a_m - a_0} y^{b_n - b_0} d_x d_y \chi(x, y) \qquad (m, n = 0, 1, 2, \cdots), \tag{4.8}$$

where $\chi(x, y)$ satisfies the conditions (3.18) and is uniquely defined by (4.8) and (3.18).

Let us define on $U - L$ the function

$$\omega(x, y) = \int_x^1 \int_y^1 x^{-a_0} y^{-b_0} d_x d_y \chi(x, y) \qquad (0 < x \leqq 1, 0 < y \leqq 1). \tag{4.9}$$

Then, since $\chi(x, y)$ is continuous along L,

$$\mu_{mn} = \lim_{\epsilon \to 0} \int_\epsilon^1 \int_\epsilon^1 x^{a_m - a_0} y^{b_n - b_0} d_x d_y \chi(x, y) = \lim_{\epsilon \to 0} \int_\epsilon^1 \int_\epsilon^1 x^{a_m} y^{b_n} d_x d_y \omega(x, y)$$

and hence

$$\mu_{mn} = \int_0^1 \int_0^1 x^{a_m} y^{b_n} d_x d_y \omega(x, y) \quad (m, n = 0, 1, 2, \cdots), \tag{4.10}$$

where the integrals are improper and converge in the sense that $\lim_{\epsilon \to 0} \int_\epsilon^1 \int_\epsilon^1$ exists.

Conversely let $\bar{\omega}(x, y)$ be a function defined on $U - L$, with the properties

$$\begin{aligned} &\bar{\omega}(x, 1) = \bar{\omega}(1, y) = 0 \text{ for } 0 < x \leqq 1, 0 < y \leqq 1; \\ &\bar{\omega}(x', y) \geqq \bar{\omega}(x'', y), \bar{\omega}(x, y') \geqq \bar{\omega}(x, y''), \\ &\Delta(\bar{\omega}; x'', y'', x', y') = \bar{\omega}(x'', y'') - \bar{\omega}(x'', y') - \bar{\omega}(x', y'') \\ &\qquad + \bar{\omega}(x', y') \geqq 0, \end{aligned} \tag{4.11}$$

for $0 < x' < x'' \leqq 1, 0 < y' < y'' \leqq 1, 0 < x \leqq 1, 0 < y \leqq 1$, and such that in the sense that $\lim_{\epsilon\to 0}\int_\epsilon^1\int_\epsilon^1$ exists,

$$\mu_{mn} = \int_0^1 \int_0^1 x^{a_m} y^{b_n} d_x d_y \bar{\omega}(x, y) \qquad (m, n = 0, 1, 2, \cdots). \tag{4.12}$$

Let

$$\bar{\chi}(x, y) = \int_0^x \int_0^y x^{a_0} y^{b_0} d_x d_y \bar{\omega}(x, y) \text{ in } U, \tag{4.13}$$

from which we derive

$$\bar{\omega}(x, y) = \int_x^1 \int_y^1 x^{-a_0} y^{-b_0} d_x d_y \bar{\chi}(x, y) \text{ in } U - L, \tag{4.14}$$

and

$$\mu_{mn} = \int_0^1 \int_0^1 x^{a_m - a_0} y^{b_n - b_0} d_x d_y \bar{\chi}(x, y) \quad (m, n = 0, 1, 2, \cdots). \tag{4.15}$$

From (4.8), (4.15), and a theorem* of C. A. Fischer it follows that

$$\chi(x, y) = \bar{\chi}(x, y)$$

in all the points of U, except possibly for a set of points lying on two denumerable sets of vertical and horizontal line segments

$$\begin{aligned} V_i: &\quad x = \xi_i, 0 \leqq y \leqq 1 \quad (0 < \xi_i < 1;\ i = 1, 2, 3, \cdots), \\ H_j: &\quad y = \eta_j, 0 \leqq x \leqq 1 \quad (0 < \eta_j < 1;\ j = 1, 2, 3, \cdots). \end{aligned} \tag{4.16}$$

From (4.9) and (4.14) it follows that

$$\omega(x, y) = \bar{\omega}(x, y)$$

in all the points of $U - L$ outside the segments (4.16).

A consequence of these results, of Lemma 3.4, and of Corollary 9.1 of I, is the following

* The theorem referred to is equivalent to the following statement: If $g_{mn}(x, y)$ is a base of continuous functions in U, and $\chi(x, y)$, $\bar{\chi}(x, y)$ are two functions of bounded variation in U (both vanishing on L), then

$$\int_0^1 \int_0^1 g_{mn}(x, y) d_x d_y \chi(x, y) = \int_0^1 \int_0^1 g_{mn}(x, y) d_x d_y \bar{\chi}(x, y) \qquad (m, n = 0, 1, 2, \cdots),$$

if and only if $\chi(x, y) = \bar{\chi}(x, y)$ throughout U, except possibly for a set of points contained in two sets of line segments of the type (4.16). See C. A. Fischer, Annals of Mathematics, (2), vol. 19 (1917–18), pp. 39–40, and HS, §3, Lemma 2.

THEOREM 4.1: *Every solution* μ_{mn} *of the determining Hausdorff system* (3.3) *derived from the matrices* (4.1), *whose elements satisfy* (4.2), (4.3), *and* (4.4), *admits the following minimal representation*:

$$(4.17)\quad \begin{aligned}\mu_{mn} = \int_0^1\int_0^1 x^{a_m}y^{b_n}d_xd_y\omega(x, y) + (0)^n\int_0^1 x^{a_m}d\rho_1(x)\\ + (0)^m\int_0^1 y^{b_n}d\rho_2(y) + (0)^{m+n}\gamma \qquad (m, n = 0, 1, 2, \cdots; \gamma \geqq 0),\end{aligned}$$

where $\omega(x, y)$, *defined on* $U-L$, *satisfies the conditions* (4.11) *and*

$$(4.18)\qquad \omega(x, y) = \omega(x-0, y-0) \quad (0 < x < 1, 0 < y < 1),$$

while $\rho_1(x)$ *and* $\rho_2(y)$ *are monotonic on* $0<x\leqq 1$ *and* $0<y\leqq 1$, *respectively, with*

$$(4.19)\qquad \begin{aligned}\rho_1(1) = \rho_2(1) = 0,\ \rho_1(x) = \rho_1(x-0),\ \rho_2(y) = \rho_2(y-0)\\ \text{for } 0 < x < 1,\ 0 < y < 1.\end{aligned}$$

The integrals in (4.17) *are improper and convergent in the sense that* $\lim_{\epsilon\to 0}\int_\epsilon^1\int_\epsilon^1$ *and* $\lim_{\epsilon\to 0}\int_\epsilon^1$ *respectively exist. The functions* $\omega(x, y)$, $\rho_1(x)$, $\rho_2(y)$ *and the constant* γ *are uniquely defined by* (4.17) *and all their further properties described above.*

Conversely, the double sequence μ_{mn} *given by* (4.17) *is always, if* $\mu_{00}<\infty$, *a solution of* (3.3).

We conclude the consideration of this case with the following remark which will be useful in the next section: *The function* $\omega(x, y)$ *is uniquely defined by the set of equations* (4.10) *and the condition* (4.11) *and* (4.18), *even if we leave out a finite number of equations of the set* (4.10). Let the equations with $m<m'$, $n<n'$ be left out of the set (4.10). The proof of the uniqueness of $\omega(x, y)$ is exactly the same as above, with the only difference that instead of (4.13) we associate with $\omega(x, y)$ the function

$$\chi(x, y) = \int_0^x\int_0^y x^{-a_{m'}}y^{-b_{n'}}d_xd_y\omega(x, y).$$

We consider now the second assumption (4.5). We know from I, §10, that if we write

$$(4.20)\qquad x_p = \left(\frac{\alpha - a_1}{\alpha - a_0}\right)^p, \quad y_q = \left(\frac{\beta - b_1}{\beta - b_0}\right)^q \qquad (p, q = 0, 1, 2, \cdots),$$

then $u=\phi_m(x)$ is the polygonal line which joins the vertices

$$(4.20')\qquad \left(x_p, \left(\frac{\alpha - a_m}{\alpha - a_0}\right)^p\right) \qquad (p = 0, 1, 2, \cdots) \qquad \text{and } (0, (0)^m),$$

and $v=\psi_n(y)$ is the polygonal line which joins the vertices

$$(4.20'')\qquad \left(y_q, \left(\frac{\beta-b_n}{\beta-b_0}\right)^q\right) \qquad (q=0,1,2,\cdots) \qquad \text{and } (0,(0)^n).$$

Let A_p be the slope of $\phi_m(x)$ on the interval $x_{p+1}\leqq x\leqq x_p$, and similarly B_q the slope of $\psi_n(y)$ on the interval $y_{q+1}\leqq y\leqq y_q$.

Let

$$(4.21)\qquad \bar{\mu}_{mn}=\int_0^1\int_0^1 \phi_m(x)\psi_n(y)d_xd_y\chi_{00}(x,y) \qquad (m,n=0,1,2,\cdots)$$

be a solution of (3.3), with $\chi_{00}(x,y)$ monotonic in U, satisfying the conditions (3.18). Applying integration by parts in each subrectangle, we obtain

$$\int_{x_N}^1\int_{y_N}^1 \phi_m(x)\psi_n(y)d_xd_y\chi_{00}(x,y) = \sum_{p,q=0}^{N-1}A_pB_q\int_{x_{p+1}}^{x_p}\int_{y_{q+1}}^{y_q}\chi_{00}(x,y)dxdy$$

$$-\int_{x_N}^1\chi_{00}(x,1)d\phi_m(x)-\int_{y_N}^1\chi_{00}(1,y)d\psi_n(y)+\phi_m(x_N)\int_{y_N}^1\chi_{00}(x_N,y)d\psi_n(y)$$

$$+\psi_n(y_N)\int_{x_N}^1\psi_{00}(x,y_N)d\phi_m(x)+\chi_{00}(1,1)-\phi_m(x_N)\chi_{00}(x_N,1)$$

$$-\psi_n(y_N)\chi_{00}(1,y_N)+\phi_m(x_N)\psi_n(y_N)\chi_{00}(x_N,y_N).$$

As $N\to\infty$, this goes over into

$$(4.22)\qquad \begin{aligned}\bar{\mu}_{mn} = \sum_{p,q=0}^{\infty}A_pB_q\int_{x_{p+1}}^{x_p}\int_{y_{q+1}}^{y_q}\chi_{00}(x,y)dxdy - \sum_{p=0}^{\infty}A_p\int_{x_{p+1}}^{x_p}\chi_{00}(x,1)dx\\ -\sum_{q=0}^{\infty}B_q\int_{y_{q+1}}^{y_q}\chi_{00}(1,y)dy+\chi_{00}(1,1).\end{aligned}$$

Let us define in U a step-function $\bar{\chi}(x,y)$ as follows:

$$\bar{\chi}(x,y)=0 \text{ on } L;\ \bar{\chi}(1,1)=\chi_{00}(1,1);$$

$$\bar{\chi}(x,y)=\frac{1}{(x_p-x_{p+1})(y_q-y_{q+1})}\int_{x_{p+1}}^{x_p}\int_{y_{q+1}}^{y_q}\chi_{00}(x,y)dxdy$$

$$\text{for } x_{p+1}\leqq x<x_p,\ y_{q+1}\leqq y<y_q \quad (p,q=0,1,2,\cdots);$$

$$\bar{\chi}(x,1)=\frac{1}{x_p-x_{p+1}}\int_{x_{p+1}}^{x_p}\chi_{00}(x,1)dx \text{ for } x_{p+1}\leqq x<x_p \quad (p=0,1,2,\cdots);$$

$$\bar{\chi}(1,y)=\frac{1}{y_q-y_{q+1}}\int_{y_{q+1}}^{y_q}\chi_{00}(1,y)dy \text{ for } y_{q+1}\leqq y<y_q \quad (q=0,1,2,\cdots).$$

This step-function is immediately found to be monotonic in U and continuous along L. Moreover, from (4.22) we infer that

$$\bar{\mu}_{mn} = \int_0^1 \int_0^1 \phi_m(x)\psi_n(y)d_x d_y \bar{\chi}(x, y) \qquad (m, n = 0, 1, 2, \cdots). \tag{4.23}$$

If we write

$$\begin{aligned}\bar{\lambda}_{pq} = \bar{\chi}(x_p + 0, y_q + 0) - \bar{\chi}(x_p + 0, y_q - 0) - \bar{\chi}(x_p - 0, y_q + 0) \\ + \bar{\chi}(x_p - 0, y_q - 0),\end{aligned}$$

then from (4.23), (4.20′), (4.20″) and the fact that $\bar{\chi}(x, y)$ is continuous along L, we derive

$$\bar{\mu}_{mn} = \sum_{p,q=0}^{\infty} \left(\frac{\alpha - a_m}{\alpha - a_0}\right)^p \left(\frac{\beta - b_n}{\beta - b_0}\right)^q \bar{\lambda}_{pq};$$

with

$$\bar{\lambda}_{pq} = (\alpha - a_0)^p(\beta - b_0)^q \lambda_{pq}$$

we finally obtain

$$\bar{\mu}_{mn} = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \lambda_{pq}(\alpha - a_m)^p(\beta - b_n)^q \quad (m, n = 0, 1, 2, \cdots). \tag{4.24}$$

We now immediately obtain the following

THEOREM 4.2. *Every solution μ_{mn} of the non-determining Hausdorff system* (3.3), *derived from the matrices* (4.1) *whose elements satisfy the conditions* (4.2), (4.3), *and* (4.5), *admits the following minimal representation*:

$$\begin{aligned}\mu_{mn} = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \lambda_{pq}(\alpha - a_m)^p(\beta - b_n)^q + (0)^n \sum_{p=0}^{\infty} \rho_p(\alpha - a_m)^p \\ + (0)^m \sum_{q=0}^{\infty} \sigma_q(\beta - b_n)^q + (0)^{m+n}\gamma \\ (m, n = 0, 1, 2, \cdots; \lambda_{pq} \geqq 0, \rho_p \geqq 0, \sigma_q \geqq 0, \gamma \geqq 0).\end{aligned} \tag{4.25}$$

The non-negative coefficients λ_{pq}, ρ_p, σ_q and γ are all uniquely defined by the set of equations (4.25).

Conversely, the double sequence μ_{mn} given by (4.25) *is always, if $\mu_{00} < \infty$, a solution of* (3.3).

Indeed, let μ_{mn} be a solution of (3.3). From (3.3′), Lemma (3.1), and (3.13) we have

$$
\begin{aligned}
\mu_{mn} = & \int_0^1 \int_0^1 \phi_m(x)\chi_n(y)d_x d_y \chi_{00}(x, y) \\
(4.26) \qquad & + (0)^n \int_0^1 \phi_m(x) d\chi_0(x, 0) \\
& + (0)^m \int_0^1 \psi_n(y) d\chi_0(0, y) + (0)^{m+n}\chi_{00}(0, 0),
\end{aligned}
$$

where, as we know, $\chi_{00}(x, y)$ is continuous along L, and $\chi_0(x, 0)$, $\chi_0(0, y)$ are continuous at the origin.

From (4.24) and Theorem 10.1 of I, we obtain

$$
\begin{aligned}
\bar{\mu}_{mn} &= \int_0^1 \int_0^1 \phi_m \psi_n d_x d_y \chi_{00} \\
&= \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \lambda_{pq}(\alpha - a_m)^p (\beta - b_n)^q,
\end{aligned}
$$

$$
\xi_m = \int_0^1 \phi_m(x) d\chi_0(x, 0) = \sum_{p=0}^{\infty} \rho_p (\alpha - a_m)^p,
$$

$$
\eta_n = \int_0^1 \psi_n(y) d\chi_0(0, y) = \sum_{q=0}^{\infty} \sigma_q (\beta - b_n)^q,
$$

which are, as is easily seen, minimal solutions of (3.3), (3.1) and (3.2) respectively. With $\chi_0(0, 0) = \gamma$, (4.26) goes over into (4.25). The uniqueness of the coefficients λ_{pq}, ρ_p, σ_q follows from known properties of power series.

We pass now to the last assumption (4.6). In this case $\phi_m(x)$ is again the polygonal line joining the points (4.20′), while

$$
(4.27) \qquad \psi_n(y) = y^{(b_n - b_0)/(b_1 - b_0)}.
$$

Let again

$$
(4.28) \qquad \bar{\mu}_{mn} = \int_0^1 \int_0^1 \phi_m(x)\psi_n(y) d_x d_y \chi_{00}(x, y) \quad (m, n = 0, 1, 2, \cdots)
$$

be a solution of (3.3), where the monotonic function $\chi_{00}(x, y)$ has the properties (3.18).

From a theorem of Fréchet* we obtain

* M. Fréchet, Nouvelles Annales de Mathématiques, (4), vol. 10 (1910), p. 253.

$$(4.29) \qquad \bar{\mu}_{mn} = \int_0^1 \psi_n(y) d_y \int_0^1 \phi_m(x) d_x \chi_{00}(x, y).$$

Let us define in U a new function $\bar{\chi}(x, y)$ as follows:

$$\bar{\chi}(0, y) = \chi_{00}(0, y) = 0, \quad \bar{\chi}(1, y) = \chi_{00}(1, y) \text{ for } 0 \leqq y \leqq 1;$$

$$\bar{\chi}(x, y) = \frac{1}{x_p - x_{p+1}} \int_{x_{p+1}}^{x_p} \chi_{00}(x, y) dx \text{ for } x_{p+1} \leqq x < x_p, \ 0 \leqq y \leqq 1 \quad (p = 0, 1, 2, \cdots).$$

This function $\chi(x, y)$ is also monotonic in U, and from I, §10, we know that

$$(4.30) \qquad \begin{aligned} \int_0^1 \phi_m(x) d_x \chi_{00}(x, y) &= \int_0^1 \phi_m(x) d_x \bar{\chi}(x, y) \\ &= \sum_{p=0}^{\infty} \left(\frac{\alpha - a_m}{\alpha - a_0} \right)^p \bar{\lambda}_p(y) \text{ for } 0 \leqq y \leqq 1, \end{aligned}$$

where the

$$\bar{\lambda}_p(y) = \bar{\chi}(x_p + 0, y) - \bar{\chi}(x_p - 0, y)$$

are also monotonic functions which are continuous at $y=0$.

From (4.29), (4.30), and (4.27) we derive

$$\begin{aligned} \bar{\mu}_{mn} &= \sum_{p=0}^{\infty} \left(\frac{\alpha - a_m}{\alpha - a_0} \right)^p \int_0^1 \psi_n(y) d\bar{\lambda}_p(y) \\ &= \sum_{p=0}^{\infty} \left(\frac{\alpha - a_m}{\alpha - a_0} \right)^p \int_0^1 y^{(b_n - b_0)/(b_1 - b_0)} d\bar{\lambda}_p(y), \end{aligned}$$

and if we write

$$\lambda_p(y) = (\alpha - a_0)^{-p} \bar{\lambda}_p(y^{b_1 - b_0}),$$

this becomes

$$(4.31) \qquad \bar{\mu}_{mn} = \sum_{p=0}^{\infty} (\alpha - a_m)^p \int_0^1 y^{b_n - b_0} d\lambda_p(y).$$

In this last form we easily recognize that $\bar{\mu}_{mn}$ is a minimal solution of the system (3.3). Introducing the monotonic functions

$$\omega_p(y) = - \int_y^1 y^{-b_0} d\lambda_p(y) \quad (0 < y \leqq 1; p = 0, 1, 2, \cdots),$$

(4.31) becomes

$$\bar{\mu}_{mn} = \sum_{p=0}^{\infty} (\alpha - a_m)^p \int_0^1 y^{b_n} d\omega_p(y).$$

Just as above we immediately derive the following

THEOREM 4.3. *Every solution μ_{mn} of the non-determining Hausdorff system* (3.3), *derived from the matrices* (4.1) *whose elements satisfy the conditions* (4.2), (4.3), *and* (4.6), *admits the following minimal representation*:

$$
\begin{aligned}
\mu_{mn} = {} & \sum_{p=0}^{\infty}(\alpha - a_m)^p \int_0^1 y^{b_n} d\omega_p(y) \\
& + (0)^n \sum_{p=0}^{\infty}\rho_p(\alpha - a_m)^p + (0)^m \int_0^1 y^{b_n} d\sigma(y) + (0)^{m+n}\gamma \qquad (4.32)\\
& \qquad (m, n = 0, 1, 2, \cdots ; \rho_p \geqq 0, \gamma \geqq 0),
\end{aligned}
$$

where all the functions $\omega_p(y)$ and $\sigma(y)$ are monotonic for $0<y\leqq 1$ and satisfy the conditions

$$(4.33)\quad \omega_p(1) = \sigma(1) = 0,\ \omega_p(y) = \omega_p(y-0),\ \sigma(y) = \sigma(y-0)\ \textit{for}\ 0 < y < 1,$$

while all the integrals are improper and converge in the sense that $\lim_{\epsilon\to 0} \int_\epsilon^1$ *exists. The functions $\omega_p(y)$, $\sigma(y)$ and the coefficients ρ_p and γ are uniquely defined by* (4.32) *and* (4.33).

Conversely, the double sequence μ_{mn} given by (4.32) *and* (4.33) *is always, if $\mu_{00}<+\infty$, a solution of the system* (3.3).

From the results of this section we derive easily the solutions of Hausdorff systems of a somewhat different kind. Consider the two Vandermondean matrices

$$
(4.34)\qquad
\begin{aligned}
A' &= \left\| \begin{matrix} \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ 1 & a_{-1} & a_{-1}^2 & \cdots \\ 1 & a_0 & a_0^2 & \cdots \\ 1 & a_1 & a_1^2 & \cdots \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \end{matrix} \right\|, \\
B' &= \left\| \begin{matrix} \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ 1 & b_{-1} & b_{-1}^2 & \cdots \\ 1 & b_0 & b_0^2 & \cdots \\ 1 & b_1 & b_1^2 & \cdots \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \end{matrix} \right\|,
\end{aligned}
$$

where a_m and b_n are two increasing sequences for $-\infty<m<+\infty$, $-\infty<n<+\infty$, and satisfying besides (4.3) one of the conditions (4.4), (4.5) or (4.6). An immediate consequence of Theorems 4.1, 4.2, and 4.3 is the following

COROLLARY 4.1. *The most general solution of the new type of Hausdorff system*

$$D_1^k D_2^h \mu_{mn} \geqq 0 \qquad (k, h = 0, 1, 2, \cdots; m, n = 0, \pm 1, \pm 2, \cdots) \tag{4.35}$$

is given by

$$\mu_{mn} = \int_0^1 \int_0^1 x^{a_m} y^{b_n} d_x d_y \omega(x, y), \tag{4.36}$$

$$\mu_{mn} = \sum_{p=0}^{+\infty} \sum_{q=0}^{+\infty} \lambda_{pq}(\alpha - a_m)^p (\beta - b_n)^q \qquad (\lambda_{pq} \geqq 0), \tag{4.37}$$

$$\mu_{mn} = \sum_{p=0}^{+\infty} (\alpha - a_m)^p \int_0^1 y^{b_n} d\omega_p(y), \tag{4.38}$$

for $m, n = 0, \pm 1, \pm 2, \cdots$, *these three representations corresponding respectively to the possible assumptions* (4.4), (4.5), *and* (4.6). *The functions* $\omega(x, y)$, $\omega_p(y)$ *and the coefficients* λ_{pq} *enjoy the properties, in particular the uniqueness properties, described in Theorems* 4.1, 4.2, *and* 4.3.

5. **Completely monotonic functions of two variables.** A function $f(x)$ was said to be completely monotonic in an open interval (see I, §11) if it possessed derivatives of every order and

$$(-1)^p f^{(p)}(x) \geqq 0 \qquad (p = 0, 1, 2, \cdots)$$

throughout this interval.

Let $f(x, y)$ be defined in an open region R. We shall say that $f(x, y)$ is *completely monotonic* in R, if all the partial derivatives of $f(x, y)$ of every order exist and

$$(-1)^{p+q} \frac{\partial^{p+q} f(x, y)}{\partial x^p \partial y^q} \geqq 0 \qquad (p, q = 0, 1, 2, \cdots) \tag{5.1}$$

throughout the region R.

Hausdorff, Bernstein, and Widder have characterized the completely monotonic functions of one variable (see I, §11). *In this section we shall determine all the functions* $f(x, y)$ *which are completely monotonic in a rectangular region*

$$R(\alpha_0, \beta_0, \alpha, \beta): \ \alpha_0 < x < \alpha, \ \beta_0 < y < \beta \tag{5.2}$$
$$(-\infty \leqq \alpha_0 < \alpha \leqq +\infty, \ -\infty \leqq \beta_0 < \beta \leqq +\infty).$$

All possible cases will be taken care of if we consider successively the following three regions:

(5.2′) $\alpha_0 < x < +\infty, \quad \beta_0 < y < +\infty,$

(5.2″) $\alpha_0 < x < \alpha, \quad \beta_0 < y < \beta \qquad (\alpha, \beta < +\infty),$

(5.2‴) $\alpha_0 < x < \alpha, \quad \beta_0 < y < +\infty \qquad (\alpha < +\infty),$

where α_0 and β_0 are either finite or else $= -\infty$.

Let us first assume that $f(x, y)$ is completely monotonic in the region (5.2′). Consider the Hausdorff system (4.35) derived from the matrices (4.34) whose elements enjoy, besides (4.3) and (4.4), the further property

$$\lim_{r\to-\infty} a_r = \alpha_0, \quad \lim_{r\to-\infty} b_r = \beta_0. \tag{5.3}$$

It follows immediately from (5.1) that for any fixed value of $x(>\alpha_0)$ and for $p \geqq 0$, the function

$$g(y) = (-1)^p(\partial^p/\partial x^p)f(x, y)$$

is completely monotonic in y for $\beta_0 < y < +\infty$. Hence

$$G(x) = \overset{h}{D_2}f(x, b_n)$$

is completely monotonic for $\alpha_0 < x < +\infty$, since

$$(-1)^p G^{(p)}(x) = \overset{h}{D_2}g(b_n) \geqq 0$$

(see I, §11, formula (11.3)). For the same reason we have

$$\overset{k}{D_1}\overset{h}{D_2}f(a_m, b_n) = \overset{k}{D_1}[\overset{h}{D_2}f(a_m, b_n)] = \overset{k}{D_1}G(a_m) \geqq 0$$

and therefore

$$\mu_{mn} = f(a_m, b_n) \qquad (m, n = 0, \pm 1, \pm 2, \cdots) \tag{5.4}$$

is a solution of the Hausdorff system (4.35).

From Corollary 4.1 we then derive

$$f(a_m, b_n) = \int_0^1 \int_0^1 \xi^{a_m}\eta^{b_n} d_\xi d_\eta \omega(\xi, \eta) \qquad (m, n = 0, \pm 1, \pm 2, \cdots)$$

where $\omega(\xi, \eta)$ has the properties given by Theorem 4.1. From the remark following Theorem 4.1, we infer that any element of the sequences a_m, b_n may vary without affecting the function $\omega(\xi, \eta)$, hence

$$f(x, y) = \int_0^1 \int_0^1 \xi^x \eta^y d_\xi d_\eta \omega(\xi, \eta) \text{ in } R(\alpha_0, \beta_0, +\infty, +\infty).$$

The transformation

$$(5.5) \qquad \xi = e^{-u}, \ \eta = e^{-v}, \ \omega(e^{-u}, \ e^{-v}) = \tau(u, v)$$

leads to the following

THEOREM 5.1. *Every function $f(x, y)$ which is completely monotonic in the region* (5.2′) *admits in this region the following representation*:

$$(5.6) \qquad f(x, y) = \int_0^{+\infty} \int_0^{+\infty} e^{-xu-yv} d_u d_v \tau(u, v),$$

where the function $\tau(u, v)$ has the following properties:

$$(5.7) \qquad \begin{aligned} &\tau(0, 0) = \tau(u, 0) = \tau(0, v) = 0, \ \tau(u, v) = \tau(u + 0, v + 0), \\ &\tau(u', v) \leqq \tau(u'', v), \ \tau(u, v') \leqq \tau(u, v''), \\ &\Delta(\tau; u'', v'', u', v') \equiv \tau(u'', v'') - \tau(u'', v') - \tau(u', v'') + \tau(u', v') \geqq 0, \\ &\textit{for } 0 < u < +\infty, 0 < v < +\infty, 0 \leqq u' < u'' < +\infty, 0 \leqq v' < v'' < +\infty. \end{aligned}$$

The improper Stieltjes integral (5.6) *is absolutely convergent in $R(\alpha_0, \beta_0, +\infty, +\infty)$ and the function $\tau(u, v)$ is uniquely defined by* (5.6) *and* (5.7).

Conversely, every function $f(x, y)$ defined by (5.6) *and* (5.7) *is always, if finite throughout R, a completely monotonic function in $R(\alpha_0, \beta_0, +\infty, +\infty)$.*

The properties (5.7) as well as the uniqueness of $\tau(u, v)$ follow from the properties (4.11), (4.18), and the uniqueness of $\omega(\xi, \eta)$, by means of the transformation (5.5). The last sentence of the theorem follows from the relations

$$(-1)^{p+q} \frac{\partial^{p+q} f(x, y)}{\partial x^p \partial y^q} = \int_0^{+\infty} \int_0^{+\infty} u^p v^q e^{-xu-yv} d_u d_v \tau(u, v) \geqq 0,$$

which are a consequence of (5.6) and where the double integral converges throughout the region R.

A particular consequence of Theorem 5.1 is that $f(x, y)$ is a real analytic and regular function of the real variables x and y in $R(\alpha_0, \beta_0, +\infty, +\infty)$.

We pass now to the consideration of a function $f(x, y)$ which is completely monotonic in the region (5.2″). Let the Hausdorff system (4.35) be defined by two sequences a_m, b_n with the properties (4.3), (4.5), and (5.3). Just as in the previous case, we conclude that (5.4) is a solution of the system (4.35). Hence from Corollary 4.1 we derive

$$f(a_m, b_n) = \sum_{p=0}^{+\infty} \sum_{q=0}^{+\infty} \lambda_{pq} (\alpha - a_m)^p (\beta - b_n)^q \quad (m, n = 0, \pm 1, \pm 2, \cdots)$$

and since again any of the numbers a_m or b_n may vary without affecting the coefficients λ_{pq}, we derive the following

THEOREM 5.2. *Every function $f(x, y)$ which is completely monotonic in the region* (5.2″), *admits in this region the following representation*:

$$f(x, y) = \sum_{p=0}^{+\infty} \sum_{q=0}^{+\infty} \lambda_{pq}(\alpha - x)^p(\beta - y)^q, \tag{5.8}$$

where $\lambda_{pq} \geqq 0$, from which it follows that $f(x, y)$ may be analytically extended and is still represented by (5.8) *in the region*

$$\alpha_0 < x < 2\alpha - \alpha_0, \quad \beta_0 < y < 2\beta - \beta_0. \tag{5.9}$$

Conversely, every function $f(x, y)$ defined by (5.8), *with $\lambda_{pq} \geqq 0$, is always, if finite throughout R, a completely monotonic function in $R(\alpha_0, \beta_0, \alpha, \beta)$.*

The last remark follows from the relations

$$(-1)^{p+q}\frac{\partial^{p+q}f(x, y)}{\partial x^p \partial y^q} = p!q! \sum_{p'=p}^{+\infty} \sum_{q'=q}^{+\infty} \binom{p'}{p}\binom{q'}{q}\lambda_{p'q'}(\alpha - x)^{p'-p}(\beta - y)^{q'-q} \geqq 0,$$

which follows from (5.8) throughout $R(\alpha_0, \beta_0, \alpha, \beta)$.

Let us finally consider a function $f(x, y)$ which is completely monotonic in the region (5.2‴). Let the Hausdorff system (4.35) be defined by two sequences a_m, b_n with the properties (4.3), (4.6), and (5.3). As in the two previous cases we derive from Corollary 4.1 the representation

$$f(x, y) = \sum_{p=0}^{\infty}(\alpha - x)^p \int_0^1 \eta^y d\omega_p(\eta) \text{ in } R(\alpha_0, \beta_0, \alpha, +\infty),$$

and hence the following

THEOREM 5.3. *Every function $f(x, y)$ which is completely monotonic in the region* (5.2‴) *admits in this region the following representation*:

$$f(x, y) = \sum_{p=0}^{+\infty} g_p(y)(\alpha - x)^p, \tag{5.10}$$

where the functions $g_p(y)$ are completely monotonic for $\beta_0 < y < +\infty$. The representation (5.10), *which is also unique, converges and gives an analytic extension of $f(x, y)$ in the region*

$$\alpha_0 < x < 2\alpha - \alpha_0, \quad \beta_0 < y < +\infty. \tag{5.11}$$

Conversely, every function $f(x, y)$ defined by (5.10) *with the $g_p(y)$ completely monotonic for $\beta_0 < y < +\infty$, is always, if finite throughout R, a completely monotonic function in $R(\alpha_0, \beta_0, \alpha, +\infty)$.*

The last converse statement follows from the relations

$$(-1)^{p+q}\frac{\partial^{p+q}f(x,y)}{\partial x^p\partial y^q} = p!\sum_{p'=p}^{+\infty}(-1)^q g_{p'}^{(q)}(y)\binom{p'}{p}(\alpha - x)^{p'-p} \geqq 0,$$

which follows from (5.10) throughout the region $R(\alpha_0, \beta_0, \alpha, +\infty)$.

On account of the results of this section we may express the results of §4 in the following

COROLLARY 5.1. (1) *The most general solution of the Hausdorff system*

$$D_1^k D_2^h \mu_{mn} \geqq 0 \qquad (k, h = 0, 1, 2, \cdots; m, n = 0, \pm 1, \pm 2, \cdots), \tag{5.12}$$

defined by the matrices (4.34), *whose elements form increasing sequences satisfying the conditions* (4.3), *is given by*

$$\mu_{mn} = f(a_m, b_n) \qquad (m, n = 0, \pm 1, \pm 2, \cdots), \tag{5.13}$$

where $f(x, y)$ *is a function which is completely monotonic in the region*

$$\lim_{r\to-\infty} a_r < x < \lim_{r\to+\infty} a_r, \quad \lim_{r\to-\infty} b_r < y < \lim_{r\to+\infty} b_r. \tag{5.14}$$

The function $f(x, y)$ *is uniquely defined.*

(2) *The most general solution of the Hausdorff system*

$$D_1^k D_2^h \mu_{mn} \geqq 0 \qquad (k, h, m, n = 0, 1, 2, \cdots), \tag{5.15}$$

defined by the matrices (4.1), *whose elements satisfy the conditions* (4.2) *and* (4.3), *is given by*

$$\mu_{mn} = f(a_m, b_n) + (0)^n g_1(a_m) + (0)^m g_2(b_n) + (0)^{m+n}\gamma \qquad (\gamma \geqq 0), \tag{5.16}$$

where the functions $f(x, y)$, $g_1(x)$ *and* $g_2(y)$ *are completely monotonic for*

$$a_0 \leqq x < \lim_{r\to+\infty} a_r, \quad b_0 \leqq y < \lim_{r\to+\infty} b_r \tag{5.17}$$

and are uniquely defined by the relations (5.16).

In the second part of this theorem it is understood that $f(x, y)$ is completely monotonic in the interior of the region (5.17) and also continuous on the part of the boundary which belongs to this region.

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