ON ANALYTICAL COMPLEXES*

BY

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1. In his Colloquium Lectures† one of us outlined a proof of an important theorem regarding the covering of analytic loci by complexes. A proof for algebraic varieties had previously been given by B. van der Waerden‡ and B. O. Koopman and A. B. Brown§ have recently proved the theorem for analytic loci. The object of this paper is to give a detailed proof along the lines indicated in *Topology*.

2. We begin with certain general observations‖ concerning the nature of a configuration ξ (at first complex) represented by an analytic system

$$F_h(x_1, \cdots, x_n) \equiv F_h(x) = 0 \qquad (h = 1, 2, \cdots, r), \tag{2.1}$$

in the vicinity of a given point O of ξ which we take as the origin throughout for the complex euclidean space S_n containing ξ. There is a neighborhood of O relative to ξ consisting of a finite number of algebroid elements, any one of them, say w_p, having about its center O, in a suitable coordinate system y_i, a canonical representation

$$(2.2)\quad \begin{aligned} &\text{(a)} \qquad H(y_1, \cdots, y_p, y_{p+1}) = 0, \\ &\text{(b)} \qquad \frac{\partial H}{\partial y_{p+1}} \cdot y_{p+1+i} + G_i(y_1, \cdots, y_{p+1}) = 0, \end{aligned}$$

where H, G_i are *pseudopolynomials* in y_{p+1}, i.e. polynomials with coefficients analytic in $y_1, \cdots, y_p$ at $(y) = (0)$, and where moreover H is algebraically irreducible and *special*, i.e. its leading coefficient is unity and its other coefficients are zero at (0). p is the *complex dimension* of w_p (dim w_p), and also of ξ at O ($\dim_O \xi$) when dim $w = p$ for some w component of ξ at O, and $\leqq p$ for all others. When O is not on ξ we agree to take $\dim_O \xi = -1$.

We have the following basic *irreducibility property*: if ξ does not contain w_p, then the intersection $\xi \cdot w_p$ is a ξ, whose dimension $< p$ at O. For the case

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† S. Lefschetz, *Topology*, Colloquium Publications, vol. XII, New York, 1930, p. 364. Except as introduced here the same notation and terminology will be used as in *Topology*.

‡ Mathematische Annalen, vol. 102, pp. 337–362.

§ These Transactions, vol. 34 (1932), pp. 231–252.

‖ Based on Osgood's *Lehrbuch der Funktionentheorie*, vol. II, chapter II.

where ξ is defined by a single relation (2.1) see Osgood's proof (loc. cit., p. 133), and the extension to any ξ is obvious.

We shall now recall a series of properties most of them direct consequences of the preceding.

I. The solution of an infinite system (2.1) about any point O is of the same type as for a finite system.

II. A point of w_p is *singular* if the rank of the Jacobian matrix J of (2.2) is $<n-p$ at the point; it is an *ordinary* point otherwise. The locus α of the singular points is the *singular* locus of w_p. Since J contains a minor of order $n-p$ equal to $(\partial H/\partial y_{p+1})^{n-p}\not\equiv 0$ when $H=0$, the conditions that the rank be $<n-p$ define a ξ not containing w_p. Hence $\alpha\cdot w_p=\alpha$ is a ξ and $\dim_O \alpha<p$.

The characteristic property of an ordinary point (a) is to have relative to w_p a neighborhood which is a $2p$-cell E_{2p} with a parametric representation

$$x_i - a_i = \phi_i(u_1, \cdots, u_p), \tag{2.3}$$

where at $(u)=(0)$ the ϕ's are analytic, vanish and have a Jacobian matrix of rank p. Every point of w_p is a limit-point of ordinary points.

III. It is impossible to decompose w_p about O into a sum of $r>1$ sets $w^i_{q_i}$. For otherwise $w^i=w^i\cdot w_p\neq w_p$, hence $q_i<p$. Therefore w_p would have points about which the coordinates depend upon $q_i<p$ parameters, which is untrue. As a noteworthy consequence the resolution of ξ into w components about O is unique and hence $\dim_O \xi$ depends solely upon O and ξ.

IV. Given a fixed coordinate system x_i we shall call *vertical* the direction of its x_n axis and denote by $P(\lambda)$ the projection of the locus λ on $x_n=0$. If the center O of w_p is an isolated intersection with the vertical through O, then $P(w_p)$ is a w_p of center $P(O)$. This does not require that the coordinates x_i be canonical for w_p. We may of course assume that O is the origin so that $P(O)=O$. Under the assumption w_p may be represented by a system (2.1) such that no $F_h(0, \cdots, 0, x_n)\equiv 0$, hence we may replace all the F's by pseudopolynomials in x_n. The algebraic elimination of x_n yields then a system such as (2.1) without x_n, representing $P(w_p)$; hence $P(w_p)$ is a ξ. If this ξ had $r>1$ components about O the vertical cylinders erected on them would decompose w_p into a ξ having at least r components w about O. Therefore $r=1$ and $P(w_p)$ is a w of center O. If a point Q varies on w_p, $x_n(Q)$ is a finite-valued function of $P(Q)$, hence $P(Q)$ depends on p parameters and $P(w_p)$ is a w_p.

Since x_n is a finite-valued function on $P(w_p)$ we have for w_p a representation (Osgood, loc. cit., p. 114)

$$\begin{aligned} &\text{(a)} \qquad G_i(x_1, \cdots, x_{n-1}) = 0, \\ &\text{(b)} \qquad H(x_1, \cdots, x_n) = 0, \end{aligned} \tag{2.4}$$

where H is a pseudopolynomial in x_n and (2.4a) represents $P(w_p)$ in $x_n=0$. Since no true subset of w_p is a w_p, H is irreducible.

The branch locus β of w_p is its intersection with $\partial H/\partial x_n=0$. Just as for the singular locus we have $\dim_O\beta<p$. Hence w_p possesses ordinary points not on β.

3. We shall now consider a real analytic variety η. It is a real locus represented by a real system (2.1) or system with F's all real.* The same system represents a ξ to be denoted by (η). Let O be a point of η. On following up Osgood's resolution of (η) into w components about O we find that their canonical coordinates y_i may be chosen real. This being assumed done we have for a component w_p three possibilities: (a) The canonical system of w_p is real and w_p possesses real ordinary points. The real subset of w_p (real algebroid element) will be denoted by v_p, so that $w_p=(v_p)$; incidentally the form of (2.2) shows that when $p=0$, O is an ordinary point, i.e., it is a v_0. (b) This case is the same as the preceding except that the real points of w_p are all singular. (c) The canonical system of w_p cannot be chosen real. When $w_p=w_p$, H and G_i in (2.2) may be replaced by $H+\overline{H}$, $G_i+\overline{G}_i$, both real and of the same form, hence we have cases (a) or (b). Therefore in case (c) necessarily $w_p\neq\widehat{w}_p$.

We shall now show that η may be decomposed about O into a finite sum of v's. Let $p=\dim_O(\eta)$. Since the required result holds when $p=0$ we use induction on p. The real points of a w_p of type (b) are on the singular locus of w_p which is an (η) whose dimension at O is $<p$. As regards the real points of a w_p of type (c), let $f_i=0$ be the canonical equations of w_p. Since (2.1) is real, $\bar{f}_i=0$ are the canonical equations of another component of (η) which is $\widehat{w}_p$. Hence the real points in question are on $w_p\cdot\widehat{w}_p$ and since w_p does not contain $\widehat{w}_p$, this is a ξ whose dimension at O is $<p$. But this ξ, being represented by the real system $f_i+\bar{f}_i=0$, $-i(f_i-\bar{f}_i)=0$, is also an (η). The real points of components not of type (a) being thus on varieties (η) whose dimensions at O are $<p$, the required result is a consequence of the hypothesis of the induction.

The meaning of $\dim v_p$, $\dim_O\eta$ is as before. As it happens they are precisely the Urysohn-Menger dimensions, but this does not matter for our purpose.

The *irreducibility property* holds for η: if η does not contain v_p, $\dim_O\eta\cdot v_p<p$. Its proof is as follows. Under the hypothesis (η) does not contain (v_p), hence $p>\dim_O(\eta\cdot v_p)=\dim_O\eta\cdot v_p$.

Properties I, $\cdots$, IV hold with v in place of w and with these modifications: (a) (2.3) represents a real analytic E_p; (b) (2.4) still represents v_p in the

* The condition $\overline{F}(\bar{x})=\overline{F(x)}$ defines an analytic function $\overline{F}$, the *conjugate* of F, and F is *real* whenever $\overline{F}\equiv F$. The set of the conjugate points of the points of a locus λ will be denoted by $\widehat{\lambda}$, the usual "bar" notation being reserved for the closure.

real and (v_p) in the complex domains, but (2.4a) represents a real v_p' which may be $\neq P(v_p)$, since in addition it may contain points which are the projections of pairs of conjugate points of (v_p). Thus we can only assert that $P(v_p)$ is a subset of a v_p'. Here again v_p contains an ordinary point Q not on the branch locus β. Q possesses then relative to v_p a neighborhood which is an analytic E_p homeomorphic with $P(E_p)$. This implies that in (2.3) the Jacobian matrix of $\phi_1, \cdots, \phi_{n-1}$ is of rank p at $(u)=(0)$. Hence $P(Q)$ has a neighborhood relative to $P(v_p)$, and not merely relative to v_p', which is an analytic E_p. We may think of $P(Q)$ as an ordinary point of $P(v_p)$.

Henceforth we shall deal exclusively with the real domain.

4. The segments on v_p. Let α_h be direction cosines for S_n, so that (α) is a point of the unit-sphere H_{n-1} of S_n. A point (x) of v_p $(p<n)$ will not be an isolated intersection of v_p with the line $x_h+s\alpha_h$ (s variable) when and only when the MacLaurin series for s of the functions $f_i(x+s\alpha)$ are $\equiv 0$, where the f's are the left-hand sides of a representation (2.1) for v_p. There results a real analytic system

$$\Phi_j(x;\alpha) = 0. \tag{4.1}$$

Its solutions for $(x;\alpha)$ in the vicinity of any solution $(x^0;\alpha^0)$ make up a finite number of sets v_q. On such a v_q we shall then have a parametric representation

$$\text{(a)}\quad x_i = \phi_i(y_1, \cdots, y_q); \quad \text{(b)}\quad \alpha_i = \psi_i(y_1, \cdots, y_q), \tag{4.2}$$

where ϕ_i, ψ_i are analytic on v_q. The system (4.2b) represents on H_{n-1} the directions near (α^0) corresponding to segments on our given v_p associated with (4.2).

Since (4.2) represents a v_q,

$$\left\|\frac{\partial\phi_i}{\partial y_j};\ \frac{\partial\psi_i}{\partial y_j}\right\| \tag{4.3}$$

is of rank q at some points as near as we please to (y^0). On the other hand, for $y_1, \cdots, y_q$ near $y_1^0, \cdots, y_q^0$ and s arbitrary but small, $\phi_i+s\psi_i$ represents a point of our initial v_p, and hence among these functions at most p are functionally independent, or

$$\left\|\frac{\partial\phi_i}{\partial y_j}+s\frac{\partial\psi_i}{\partial y_j};\ \psi_i\right\| \tag{4.4}$$

is of rank $\leqq p$, and this must hold for s small but arbitrary. Now any determinant of this matrix containing s is a polynomial in s whose leading coefficient is the corresponding determinant of

$$\left\| \frac{\partial \psi_i}{\partial y_j}; \psi_i \right\| \tag{4.5}$$

which must therefore be of rank $\leqq p$. Owing to the relations

$$\sum \psi_i^2 = 1, \quad \sum \psi_i \frac{\partial \psi_i}{\partial y_j} = 0,$$

the new matrix may be bordered with a row $0, \cdots, 0, 1$ without changing its rank. It follows that the rank of

$$\left\| \frac{\partial \psi_i}{\partial y_j} \right\|$$

is at most $p-1 \leqq n-2$. Therefore the directions of segments meeting v_p in an infinite set are represented on H_{n-1} by a variety η whose dimension at any point $<n-1$, and hence they are nowhere dense on the sphere.†

5. Analytic complexes. By an *analytic structure* ζ we shall mean a real point set in a real S_n which constitutes a topological space with varieties η as the neighborhoods. Each point Q of ζ has then a neighborhood relative to ζ made up of a finite set $v_{q_1}{}^1, \cdots, v_{q_r}{}^r$, where the v's have Q as their common center. The largest q_r is the *dimension of* ζ *at* Q ($\dim_Q \zeta$), and the largest value p of $\dim_Q \zeta$ for Q on ζ is the *dimension* of ζ ($\dim \zeta$) which is then designated by ζ_p.

We now define the point Q of ζ, whose neighborhood is $v_{q_1}{}^1 + \cdots + v_{q_r}{}^r$, as *singular* when $r>1$, or when $r=1$ and $\dim_Q \zeta < p$, or else $\dim_Q \zeta = p$ and Q is singular for its unique v. From property II of §2 for an η, we have that the set of all singular points or *singular locus* is a ζ_r, $r<p$. A point of $\zeta_p - \zeta_r$ is an *ordinary* point of ζ_p. Its characteristic property is that it possesses relative to ζ_p a neighborhood which is an analytic E_p.

By an *analytic p-element*, or merely p-element, ϵ_p, we shall mean a relatively open subset of a structure ζ_p, containing at least one ordinary point, and such that $\bar{\epsilon}_p \subset \zeta_p$. Under these conditions we shall describe ϵ_p and ζ_p as *associated* with each other. By an *analytic p-complex*, κ_p, we shall mean a finite set of non-intersecting elements ϵ, of dimension up to and including p, which constitute a closed bounded point set in S_n. By convention the empty set is to be a ζ_{-1}, an ϵ_{-1} or a κ_{-1}. We shall write $F(\zeta) = \bar{\zeta} - \zeta$. We do not consider here infinite κ's, since they may be taken care of as in *Topology*.

The intersection of two or more ζ's or ϵ's is respectively a ζ or an ϵ. If $\kappa = \sum \epsilon$, $\kappa^* = \sum \epsilon^*$ are complexes, so is $\kappa \cdot \kappa^* = \sum \epsilon \cdot \epsilon^*$. Similarly when $\kappa \cdot \zeta$ is closed then $\kappa \cdot \zeta$ is the complex $\sum \epsilon \cdot \zeta$.

† Cf. Koopman and Brown, loc. cit., p. 242.

A complex κ' will be called a *subdivision* of κ if the two coincide as point sets and if each element of κ' is contained in one of κ. If κ^* is any complex on κ it is clear from the preceding paragraph that κ has a subdivision with one of κ^* as a subcomplex. It follows that $\kappa+\kappa^*$ can be covered by a complex having subdivisions of κ and κ^* as subcomplexes. For κ and κ^* can each be subdivided to form a complex having a common subcomplex covering $\kappa\cdot\kappa.^*$

Whenever throughout κ we have $\epsilon_p\cdot\bar{\epsilon}_q=0$ for $q\leqq p$, κ is said to be *normal.* When κ is normal, it remains closed, and hence a κ (moreover a normal κ) when one or more p-elements are removed from it.

Every complex has a normal subdivision. Given any ζ_p, we shall denote its singular locus by $\zeta_{p'}$ $(p'<p)$. Let then ϵ_p, ϵ_q be two elements of κ_p and ζ_p, ζ_q associated structures. We shall first show that there exists a $\zeta_r\supset\epsilon_p\cdot\bar{\epsilon}_q$ such that $r<p$ and that the distance from $\epsilon_p\cdot\bar{\epsilon}_q$ to $F(\zeta_r)>0$. In any case $\epsilon_p\cdot\bar{\epsilon}_q\subset\bar{\epsilon}_p\cdot\bar{\epsilon}_q\subset\zeta_p\cdot\zeta_q=\zeta_s$. Also $F(\zeta_s)\subset F(\zeta_p)+F(\zeta_q)$. Since no $\bar{\epsilon}$ meets its $F(\zeta)$, $\bar{\epsilon}_p\cdot\bar{\epsilon}_q$ does not meet $F(\zeta_s)$, and as $\bar{\epsilon}_p\cdot\bar{\epsilon}_q$ is self-compact the distance of the two sets >0. Therefore when $s<p$ we may take $\zeta_r=\zeta_s$.

Let now $s=p$ and let Q be a point of $\epsilon_p\cdot\bar{\epsilon}_q$ not on $\zeta_{s'}$, so that $Q\subset\zeta_s-\zeta_{s'}$ and $\dim_Q\zeta_s=p$. This implies $q=p$ and that the nieghborhoods of Q relative to ζ_p and ζ_q have a common v_p which is then wholly on ϵ_p near Q. In that case necessarily $Q\subset\zeta_{q'}$. For otherwise v_p would be a complete neighborhood of Q relative to ζ_q, hence it would contain points of ϵ_q infinitely near Q, and we should have $\epsilon_p\cdot\epsilon_q\neq 0$, which is ruled out. It follows $\epsilon_p\cdot\bar{\epsilon}_q\subset\zeta_{q'}+\zeta_{s'}$.

Since a singular locus is closed relative to its ζ, and since $F(\zeta_s)\subset F(\zeta_p)+F(\zeta_q)$, we find that $\zeta_r=\zeta_{q'}+\zeta_{s'}-F(\zeta_p)-F(\zeta_q)$ satisfies the condition for a structure, with $F(\zeta_r)\subset F(\zeta_p)+F(\zeta_q)$. Since the last two F's do not meet $\bar{\epsilon}_p\cdot\bar{\epsilon}_q$, this is likewise the case as regards $F(\zeta_r)$, which implies also $\zeta_r\supset\bar{\epsilon}_p\cdot\bar{\epsilon}_q\supset\epsilon_p\cdot\bar{\epsilon}_q$, and that the distance condition holds. Since $r=p'$ or q', both $<p$, ζ_r has all the properties that we require.

We can find a closed polyhedral neighborhood of $\bar{\epsilon}_p\cdot\bar{\epsilon}_q$ not meeting $F(\zeta_r)$, and its intersection with ζ_r is a κ_r. The sum of these complexes for all p-elements is a κ_t, $t<p$, and $\epsilon_p'=\epsilon_p-\kappa_t$ is an element. Replacing ϵ_p by ϵ_p' together with the sum of the elements $\epsilon_p\cdot\kappa_t$, we obtain a subdivision κ_p', such that $\epsilon_p'\cdot\bar{\epsilon}_q'=0$ if $\epsilon_q'\neq\epsilon_p'$. Hence $\kappa_p'-\sum\epsilon_p'$ is a κ whose dimension $<p$. The required result follows then by induction on p.

6. **The covering theorem.** Let κ be any complex and let "vertical" direction or projection have the same meaning as in §2, IV. Every point of κ has a neighborhood relative to κ made up of a finite number of v's. Since κ is self-compact it can be covered with a finite number of v's. It follows then from §4 that the axes may be so chosen that no vertical meets κ in an infinite set.

From §§3, 5, we conclude that κ has a subdivision (obtained as its intersection with a suitable polyhedral complex) whose elements are each on a v represented by a system (2.4). The subdivision can then be normalized so that at each step in the process the preceding property is preserved. Ultimately we turn the complex into a normal complex, still called κ, whose elements all have the property just described.

Let ϵ_p be any element of our new κ with its ζ_p given by (2.4). The branch locus ζ^* is of dimension $<p$. We verify at once that $\epsilon^*=\epsilon_p\cdot\zeta^*$ and $\epsilon_p-\epsilon^*$ are elements with ζ^* and ζ_p as associated structures. Referring to the end of §3 we find also that, when $p<n$, $P(\epsilon_p-\epsilon^*)$ is an ϵ_p. Moreover when $(x_1, \cdots, x_{n-1})$ ranges over $P(\epsilon_p-\epsilon^*)$, to certain real roots x_n' of $H=0$ there will correspond points $(x_1, \cdots, x_{n-1}, x_n')$ which generate elements ϵ_p' whose sum is $\epsilon_p-\epsilon^*$.

Assuming that our complex is a κ_p, $p<n$, we decompose every ϵ_p of κ_p in the set of p-elements ϵ_p' plus ϵ^* (whose dimension $<p$), and repeat the operation for the elements of next lower dimension of the new complex, etc. Ultimately then we have in place of κ_p a new normal complex, still to be called κ_p, such that every ϵ of κ_p has for projection $P(\epsilon)$ an element, and ϵ is represented by an analytic relation $x_n=f(Q)$, $Q\subset P(\epsilon)$ (analytic homeomorphism).

A final subdivision $\kappa_p'=\sum\epsilon'$ of κ_p will now be made, such that $P(\kappa_p')$ can be covered with a $\kappa_p''=\sum\epsilon''$ having the property that every $P(\epsilon')$ is an exact sum of elements ϵ''. For $p=0$ this is trivial, hence we use induction on p. Taking κ_p in the reduced form just obtained, $\kappa_q=\kappa_p-\sum\epsilon_p$ is a complex with $q<p$. Under the hypothesis of the induction it has a subdivision $\kappa_q'=\sum\epsilon'$ of the desired type. Let δ be a positive number such that $-\delta<x_n<\delta$ on κ_p, and let $C(\kappa_q')$ be the $(q+1)$-complex whose elements are the parts of the vertical cylinders based on the ϵ''s lying between the spaces $x_n=\pm\delta$, together with their intersections with these spaces. Let ϵ_r be an element of the original κ_p. Since an ϵ_r carries no vertical segment, the intersection $\epsilon_r\cdot C(\kappa_q')$ consists of elements of dimension $\leqq q$, some being of dimension q when $r=q$. Therefore $\kappa_p\cdot C(\kappa_q')$ is a q-complex, and since $q<p$, it has a subdivision $\kappa_q^{*\prime}$ such that $P(\kappa_q^{*\prime})$ is covered by a κ_q'' of the required type. Given any ϵ_p of κ_p we form a new element $\epsilon_p'=\epsilon_p-\kappa_q^{*\prime}$. Then $\kappa_p'=\kappa_q^{*\prime}+\sum\epsilon_p'$ is the required subdivision of κ_p. For let κ_p' contain m p-elements $\epsilon_p'^{\alpha}$ and let $\eta^\alpha=P(\epsilon_p'^{\alpha})$. When $m=1$, we can take $\kappa_p''=\kappa_q''+\eta'$. Therefore we may use induction on m. Removing ϵ'^m from κ_p' we have a complex $\kappa_p^{*\prime}$ which, under the hypothesis of the induction, possesses an associated $\kappa_p^{*\prime\prime}$ covering $P(\kappa_p^{*\prime})$. Now $\eta'=\eta^m-\kappa_p^{*\prime\prime}$ is also an element and $\kappa_p''=\eta'+\kappa_p^{*\prime\prime}$ is a covering of $P(\kappa_p')$ such as we are seeking.

Observe that every ϵ' is still analytically homeomorphic with its projection $P(\epsilon')$ since this holds as regards the ϵ of κ_p on which it lies. We are now ready for the

THEOREM. *Every analytic complex has a simplicial subdivision.*

We first assume $p<n$, and κ_p in its ultimate reduced form, κ_p', κ_p'' having the same meaning as above. The theorem being trivial for $n=0$ we use induction on n. Since κ_p'' is on an S_{n-1} it has then a simplicial subdivision $K_p{}^0=\sum\sigma$. Let QQ' be any vertical with $Q\subset\sigma_q$. It intersects κ_p' in points $Q^1, Q^2, \cdots, Q^r$ (r finite) each on a different ϵ' of κ_p', say $Q^i\subset\epsilon'^i$. When Q ranges continuously over σ_q, by the above Q^i remains on ϵ'^i and generates a homeomorph of σ_q, a cell $E_q^i\subset\epsilon'^i$, and no two of these cells intersect. As a consequence $K_p=\sum E_q^i$ is a cellular subdivision of κ_p' and hence of κ_p. We shall now show that the homeomorphism between E_q and E_q^i can be extended to their boundaries. This merely requires that we prove that when $Q\subset F(E_q)$, it has a unique image on $F(E_q^i)$. Suppose that it has s images Q'^j. We may choose for each Q'^j a neighborhood relative to $E_q{}^i$ consisting of a cell $E_q{}^{ij}$ whose projection is a simplex σ_q^j (in the straightness of σ_q), no two of the cells $E_q{}^{ij}$ intersecting. As a consequence $\sigma^j\cdot\sigma^h=0$ $(j\neq h)$ and Q has for neighborhood relative to σ_q a set of s non-intersecting q-simplexes, which can only be if $s=1$, as asserted. Since $\overline{E}_q{}^i$ and $\bar{\sigma}_q$ are homeomorphic, E^i is simplicial and so is K_p.

If we have a κ_n, on removing its n-elements we have a κ_p, $p<n$, which we identify with the κ_p just considered. When $Q\subset\sigma_q$, the points of κ_n projected on Q may include some of the segments Q^iQ^{i+1} and we observe that, since r is fixed throughout any σ, if the segment is zero anywhere on a face of σ_q it is zero throughout that face. As a consequence we find by an elementary induction that when Q ranges over σ_q the segments $\neq 0$ generate $(q+1)$-cells E_{q+1}^i whose structure is that of a truncated simplicial prism. Since these cells are convex, the covering K_n thus obtained for κ_n is convex, and its first derived, which is simplicial, answers the question.

COROLLARY. *If $\kappa_q\subset\kappa_p$, κ_p has a simplicial subdivision with a subcomplex covering κ_q.*

For κ_p has a subdivision κ_p' having a subdivision of κ_q as a subcomplex. In particular κ_p may be a closed polyhedral region of S_n containing κ_q. This is substantially the theorem of *Topology*, p. 364.

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