NON-CONJUGATE OSCULATING QUADRICS
OF A CURVE ON A SURFACE*

BY

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1. **Introduction.** This paper is concerned with making a study of the projective differential geometry of a non-conjugate net of curves on a surface in three-space by means of a pair of osculating quadrics defined in the following manner. Consider a curve C on the surface S. At a point P of C and at two neighboring points P_1, P_2 on C construct the tangents of the curves of one family of the non-conjugate net. The limit of the quadric surface determined by these three lines as the points P_1, P_2 approach P along C is a *non-conjugate osculating quadric* at the point P on C. The other osculating quadric is obtained in a similar manner by drawing tangents to the other family of the net. Thus we associate with each point of the surface a pair of osculating quadrics analogous to the *asymptotic osculating quadrics*† of Bompiani and Klobouček and the *conjugate osculating quadrics*‡ of Lane.

We compute the equations of the osculating quadrics and note some results that follow rather immediately therefrom. The chief contribution of the paper is a complete discussion of the nature of the curve of intersection of the two osculating quadrics at a point of a curve on the surface. We also study the curve of intersection of corresponding osculating quadrics for two curves which are respectively members of two families of curves which form a conjugate net on the surface. As a by-product we obtain a new necessary and sufficient condition that a net of curves on the surface be a conjugate net.

2. **Analytic basis.** In this section we set up an analytic basis for the study of a non-conjugate net of curves on a surface in projective three-space following Green's method.§ We shall also list in this section certain of Green's results that we shall need for later reference. Let it be noted here that we shall assume that the surface sustaining the net is not developable, and that unless otherwise specifically stated the net in question is not the asymptotic net.

Let the surface under consideration be the analytic surface whose vector equation in homogeneous coordinates is

$$x = x(u, v).$$

A necessary and sufficient condition that the parametric curves $u = $ const., $v = $ const. shall not form a conjugate net is that x shall not satisfy an equation of Laplace of the form

$$ax_{uv} + bx_u + cx_v + dx = 0.$$

This is equivalent to saying that the fourth-order determinant

$$W = (x_{uv}, x_u, x_v, x)$$

does not vanish. Hence the functions x are solutions of a pair of partial differential equations of the form

$$(1) \qquad x_{uu} = px + \alpha x_u + \beta x_v + L x_{uv}, \qquad x_{vv} = qx + \gamma x_u + \delta x_v + N x_{uv}.$$

If we adjoin certain integrability conditions, which we shall not write down, these equations will form a completely integrable system; that is, any fundamental set of solutions can be expressed as a linear combination, with constant coefficients, of any other fundamental system. We shall base our projective theory of surfaces on this completely integrable system of partial differential equations.

If we regard the u-tangent at a point P_x as a generator of the congruence of u-tangents, its other focal point is given by

$$(2) \qquad \rho = x_u + (\beta/L)x.$$

Similarly the focal point of the v-tangent at P_x is given by

$$(3) \qquad \sigma = x_v + (\gamma/N)x.$$

The line joining ρ and σ is called by Green the *ray of the point P_x.*

Since the parametric net is not asymptotic the osculating planes of the u-curve and v-curve through P_x determine a line that passes through P_x but does not lie in the tangent plane. This line is called by Green the *axis of the point P_x.* The totality of axes of points of the surface generate the axis congruence. The point τ which is the harmonic conjugate of the point x with respect to the two focal points of the axis is given by

$$(4) \qquad \tau = K'x + (\gamma/N)x_u + (\beta/L)x_v + x_{uv},$$

where the value of K' is given by

$$(5) \qquad \begin{aligned} K' = -\tfrac{1}{2}[&b^{(21)} + c^{(12)} - (\beta/L)a^{(12)} - (\gamma/N)a^{(21)} + (\alpha\gamma)/N + (\beta\delta)/L \\ &- L\gamma^2/N^2 - N\beta^2/L^2 - 2\beta\gamma/(LN) + (N\gamma_u - N_u\gamma)/N^2 \\ &+ (L\beta_v - L_v\beta)/L^2], \end{aligned}$$

wherein

$$
\begin{aligned}
a^{(21)} &= [L_v + \alpha + \beta N + L(N_u + \gamma L + \delta)]/(1 - LN), \\
a^{(12)} &= [N(L_v + \alpha + \beta N) + N_u + \gamma L + \delta]/(1 - LN), \\
b^{(21)} &= [\alpha_v + \beta\gamma + L(\alpha\gamma + \gamma_u + q)]/(1 - LN), \\
c^{(12)} &= [N(\beta\delta + \beta_v + p) + \beta\gamma + \delta_u]/(1 - LN).
\end{aligned}
$$

(6)

The points ρ, σ, τ are covariant points with respect to all transformations of proportionality factor and independent variable in equations (1) that preserve the net.

The curvilinear differential equation defining the asymptotic net on the surface is

(7) $$Ldu^2 + 2dudv + Ndv^2 = 0.$$

Evidently $L = N = 0$ is a necessary and sufficient condition that the parametric net be asymptotic. We also see that the surface is developable if, and only if, $1 - LN = 0$.

In order that the net defined by the equation

$$(dv - \lambda du)(dv - \mu du) = 0 \qquad\qquad (\lambda \neq \mu)$$

shall be a conjugate net the two directions λ and μ must separate harmonically the asymptotic directions. A necessary and sufficient condition for this is

(8) $$\mu = -(L + \lambda)/(N\lambda + 1).$$

The conjugate net whose directions separate harmonically the parametric directions is called by Green the *associate conjugate net*, and it is defined by the following equation:

(9) $$Ldu^2 - Ndv^2 = 0.$$

We list for reference the following invariants due to Green and Grove:[*]

$$
\begin{aligned}
r &= L + \lambda, \quad s = N\lambda + 1, \\
t &= L\lambda' - L'\lambda, \quad \pi = c_v - b_{1u}, \\
h' &= q + b_1(\delta - b_1) - b_{1v} + Nb_{1u}, \\
k &= p + c(\alpha - c) - c_u + Lc_v, \\
\mathfrak{B} &= Nc - \tfrac{1}{2}(a^{(12)} - \delta), \\
\mathfrak{C}' &= Lb_1 - \tfrac{1}{2}(a^{(21)} - \alpha), \\
G &= c^{(21)} + ca^{(21)} - c_u - c^2, \\
G' &= b^{(12)} + b_1a^{(12)} - b_{1v} - b_1{}^2, \\
F &= c^{(12)} + ca^{(12)} - c\delta - Nc^2 - 2c_v, \\
F' &= b^{(21)} + b_1a^{(21)} - b_1\alpha - Lb_1{}^2 - 2b_{1u}, \\
W^{(u)} &= F - NG,
\end{aligned}
$$

(10)

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    * V. G. Grove, *A theory of a general net on a surface*, these Transactions, vol. 28 (1926), p. 496.

wherein $a^{(21)}$, $a^{(12)}$, $b^{(21)}$, $c^{(12)}$ are given by (6) and where

$$a = 1/L, \quad b = -\alpha/L, \quad c = -\beta/L, \quad d = -p/L,$$

$$a_1 = 1/N, \quad b_1 = -\gamma/N, \quad c_1 = -\delta/N, \quad d_1 = -q/N,$$

$$b^{(12)} = [N(\alpha_v + \beta\gamma) + \alpha\gamma + \gamma_u + q]/(1 - LN),$$

$$c^{(21)} = [\beta\delta + \beta_v + p + L(\beta\gamma + \delta_u)]/(1 - LN).$$

3. The non-conjugate osculating quadrics. We now compute the equations of the quadrics defined in §1. Let us consider a non-conjugate net N on a surface S of the kind specified in §2, and suppose S is an integral surface of the system (1). On this surface let us consider a curve C_λ which is a member of the family defined by the equation

$$(11) \qquad\qquad dv - \lambda du = 0,$$

and suppose this family of curves is not conjugate to either family of the net N. We shall first find the equation of the osculating quadric Q_u determined by the tangents to three consecutive u-curves at a point P_x of C_λ. The equation will first be referred to the local tetrahedron of reference whose vertices are the points x, x_u, x_v, x_{uv}. We may define any point X on C_λ near P_x by the following power series in Δu:

$$X = x + x'\Delta u + \frac{x''\Delta u^2}{2} \cdots \qquad\qquad \left(x' = \frac{dx}{du}\right).$$

By making use of the differential equations (1) and expressions obtained from them by differentiation this series can be expressed in the form

$$X = x_1 x + x_2 x_u + x_3 x_v + x_4 x_{uv},$$

where, for a suitably chosen unit point, x_1, x_2, x_3, x_4 are local coordinates of the point X referred to the local tetrahedron x, x_u, x_v, x_{uv}, and the power series defining them sufficiently far for the purposes of this paper are

$$(12) \quad \begin{aligned} x_1 &= 1 + \cdots, \\ x_2 &= \Delta u + \cdots, \\ x_3 &= \lambda\Delta u + (\beta + \lambda' + \lambda^2\delta)\Delta u^2/2 + \cdots, \\ x_4 &= (L + 2\lambda + N\lambda^2)\Delta u^2/2 + \cdots. \end{aligned}$$

Similarly for a point X_u on the u-tangent near P_x we get the local power series expansions

$$
\begin{aligned}
(13) \quad y_1 &= p\Delta u + \cdots, \\
y_2 &= 1 + \alpha\Delta u + \cdots, \\
y_3 &= \beta\Delta u + [\beta_u + \alpha\beta + (L + 2\lambda)c^{(21)} \\
&\qquad + \lambda^2 c^{(12)}]\Delta u^2/2 + \cdots, \\
y_4 &= (L + \lambda)\Delta u + [\lambda' + L_u + \beta + \alpha L \\
&\qquad + (L + 2\lambda)a^{(21)} + \lambda^2 a^{(12)}]\Delta u^2/2 + \cdots.
\end{aligned}
$$

Now let us set

$$
\begin{aligned}
e_1 &= (\beta + \lambda' + \lambda^2\delta)/2, \\
e_2 &= (L + 2\lambda + N\lambda^2)/2, \\
e_3 &= [\beta_u + \alpha\beta + (L + 2\lambda)c^{(21)} + \lambda^2 c^{(12)}]/2, \\
e_4 &= [\lambda' + L_u + \beta + \alpha L + (L + 2\lambda)a^{(21)} + \lambda^2 a^{(12)}]/2.
\end{aligned}
$$

Any point on the line XX_u is given by the linear combination

$$
mX + nX_u \qquad\qquad (m, n \text{ scalars}).
$$

On making use of equations (12) and (13) the power series expansions for the local coordinates of the point are found to be

$$
\begin{aligned}
(14) \quad z_1 &= m + pn\Delta u + \cdots, \\
z_2 &= n + (m + \alpha n)\Delta u + \cdots, \\
z_3 &= (\lambda m + \beta n)\Delta u + (e_1 m + e_3 n)\Delta u^2 + \cdots, \\
z_4 &= rn\Delta u + (e_2 m + e_4 n)\Delta u^2 + \cdots.
\end{aligned}
$$

Now demand that the power series (14) satisfy the most general equation of a quadric surface identically in m, n and identically in Δu as far as terms of the second degree. The computation involved in doing this is rather laborious, and the details will be omitted here. The equation of the quadric Q_u is found to be

$$
(15) \quad (L - N\lambda^2)x_3^2 + Rx_3x_4 + S'x_4^2 + 2\lambda(\lambda x_1 x_4 - r x_2 x_3 + \beta x_2 x_4) = 0,
$$

where r was given in equations (10) and R and S' are given by

$$
\begin{aligned}
R &= 2\lambda(\mathfrak{C}' - Lb_1) + 2\lambda^2(\mathfrak{B} - Nc) + [\lambda' - 2\beta - \lambda(\log r)'], \\
S' &= (\beta^2 + \lambda\beta' - \alpha\beta\lambda - p\lambda^2 - \beta\lambda')/r + \lambda[c^{(21)} + \lambda c^{(12)}].
\end{aligned}
$$

The equation of the osculating quadric Q_v may be obtained immediately from that of Q_u by performing in equation (15) the following substitution:

(16)
$$\begin{pmatrix} u & x_2 & \alpha & \beta & p & L & \lambda & \lambda' \\ v & x_3 & \delta & \gamma & q & N & 1/\lambda & -\lambda'/\lambda^3 \end{pmatrix}.$$

The resulting equation of Q_v is

$$(17) \quad (L - N\lambda^2)x_2^2 + Px_2x_4 + Qx_4^2 - 2(x_1x_4 - sx_2x_3 + \gamma\lambda x_3x_4) = 0,$$

where s was given in equations (10) and P and Q are defined by

$$P = 2\gamma\lambda^2 + (\log s)' - 2(\mathfrak{C}' - Lb_1) - 2\lambda(\mathfrak{B} - Nc),$$
$$Q = \lambda[\delta\gamma\lambda - \gamma' + q - \gamma(\log \lambda)' - \gamma^2\lambda^2]/s - (b^{(21)} + \lambda b^{(12)}).$$

If we specialize our net to be the asymptotic net by putting $L = N = 0$, equations (15) and (17) reduce to the equations for the asymptotic osculating quadrics of Bompiani and Klobouček.

4. A covariant tetrahedron of reference. The tetrahedron to which the equations of the osculating quadrics were referred in the last section was not a covariant tetrahedron, and hence we were unable to express the coefficients in the equations completely in terms of invariants. We find it advantageous to change our tetrahedron of reference to a new covariant tetrahedron whose vertices are the points x, ρ, σ, τ, of which the last three were defined by equations (2), (3), (4) of §2. The transformation that effects this change of reference system is

$$x_1 = y_1 + (\beta/L)y_2 + (\gamma/N)y_3 + K'y_4,$$
$$x_2 = y_2 + (\gamma/N)y_4,$$
$$x_3 = y_3 + (\beta/L)y_4,$$
$$x_4 = y_4,$$

where the x's are the old variables and the y's are the new. After renaming the variables so that the x's are the new local coordinates, the transformed equations of Q_u and Q_v are respectively

(18)
$$(L - N\lambda^2)x_3^2 + (2\lambda^2\mathfrak{B} + 2\lambda\mathfrak{C}' + t/r)x_3x_4 + \lambda(G - \lambda F'$$
$$+ \lambda\pi - \lambda k/r)x_4^2 + 2\lambda^2x_1x_4 - 2\lambda rx_2x_3 = 0,$$
$$(L - N\lambda^2)x_2^2 - [2\lambda\mathfrak{B} + 2\mathfrak{C}' - (\log s)']x_2x_4 + (F - \lambda G'$$
$$+ \lambda\pi + \lambda h'/s)x_4^2 - 2x_1x_4 + 2sx_2x_3 = 0.$$

The coefficients in the equations of the osculating quadrics are now expressed completely in terms of invariants defined in (10).

5. Some immediate results. In this section we establish some theorems

analogous to those of Lane for a conjugate net and in addition obtain an interesting reciprocity property.

The osculating quadric Q_u intersects the tangent plane, $x_4 = 0$, in the u-tangent, $x_3 = x_4 = 0$, and the residual line

$$(19) \qquad\qquad x_4 = (L - N\lambda^2)x_3 - 2\lambda r x_2 = 0.$$

This line is the v-tangent if, and only if, $L - N\lambda^2 = 0$. Likewise Q_v intersects the tangent plane in the v-tangent, $x_2 = x_4 = 0$, and the residual line

$$(20) \qquad\qquad x_4 = (L - N\lambda^2)x_2 + 2s x_3 = 0.$$

On referring to equation (9) we conclude the following result:

A necessary and sufficient condition that a curve on our surface belong to the associate conjugate net is that either of the two osculating quadrics have the tangents of the net as generators at every point of the curve.

Now suppose $L - N\lambda^2 \neq 0$. Then the two lines (19) and (20) coincide at every point of C_λ if, and only if, equation (7) holds. Thus we have the following theorem:

The residual intersections of the osculating quadrics with the tangent plane coincide at every point of a curve if, and only if, the curve is a member of one family of the asymptotic net on the surface. The line of coincidence is the tangent to the curve.

The coordinates ξ of the polar plane of any point x with respect to the osculating quadric Q_u are given by

$$
\begin{aligned}
\xi_1 &= 2\lambda^2 x_4, \quad \xi_2 = -2\lambda r x_3, \\
\xi_3 &= -2\lambda r x_2 + 2(L - N\lambda^2)x_3 \\
&\quad + (2\lambda^2\mathfrak{B} + 2\lambda\mathfrak{C}' + t/r)x_4, \\
\xi_4 &= 2\lambda^2 x_1 + (2\lambda^2\mathfrak{B} + 2\lambda\mathfrak{C}' + t/r)x_3 \\
&\quad + 2\lambda(G - \lambda F' + \lambda\pi - \lambda k/r)x_4.
\end{aligned}
$$

(21)

The polar plane of the point ρ, whose local coordinates are $(0, 1, 0, 0)$, with respect to Q_u is the plane $x_3 = 0$, which is the osculating plane to the u-curve C_u at P_x. But we see from the first of equations (18) that ρ lies on Q_u, hence we conclude that *the osculating plane to the curve C_u at a point of a curve C_λ is tangent to the corresponding osculating quadric Q_u at the point ρ.* In a similar manner it may be shown that *the osculating plane to C_v at P_x is tangent to Q_v at the point σ.*

By referring to (21) we see that the polar plane of the point σ with respect to Q_u passes through the point P_x, and it passes also through the point τ if, and only if,

(22) $$2\lambda^2\mathfrak{B} + 2\lambda\mathfrak{C}' + t/r = 0.$$

We readily draw the following conclusion:

Condition (22) *is necessary and sufficient for the ray and axis to be reciprocal polar lines with respect to the quadric* Q_u.

Similar results hold for the quadric Q_v.

6. Nature of the intersection of the osculating quadrics. We now make a complete study of the nature of the curve of intersection of the osculating quadrics Q_u and Q_v. This study is made by means of the elementary divisors of the matrix of the pencil of quadrics based on Q_u and Q_v. The method is to compute the elementary divisors, write down the characteristic, and ascertain the general nature of the intersection by referring to results tabulated by Snyder and Sisam.*

For the sake of brevity let us write equations (18) for the osculating quadrics Q_u and Q_v in the following form:

(23)
$$A x_3^2 + 2Bx_3x_4 + Cx_4^2 + 2\lambda^2x_1x_4 - 2\lambda rx_2x_3 = 0,$$
$$A x_2^2 - 2Dx_2x_4 + Ex_4^2 - 2x_1x_4 + 2sx_2x_3 = 0,$$

where A, B, C, D, E are new notations introduced here, and their values may be readily read off from (18). We now give the results obtained for the various cases that arise in our problem.

Case I. If $A \neq 0$, $B - D\lambda \neq 0$, $L + 2\lambda + N\lambda^2 \neq 0$, we find that the characteristic is [13]. Hence the intersection of the quadrics Q_u and Q_v at each point of a curve C_λ satisfying the conditions stated is a quartic curve with a cusp. But if $B - D\lambda = 0$ we get by using the values of B and D

(24) $$t/r + \lambda(\log s)' = 0,$$

and

$$t/r = (L\lambda' - L'\lambda)/(L + \lambda) = \lambda' - \lambda[\log (L + \lambda)]'.$$

Substituting this in (24), dividing by λ, and integrating we get

$$\lambda(N\lambda + 1)/(L + \lambda) = e \qquad (e = \text{const.}).$$

Now by replacing λ by dv/du we see that if C_λ is a curve such that $B - D\lambda = 0$ then C_λ belongs to the pencil of families of curves defined by the curvilinear differential equation

(25) $$eLdu^2 + (e + 1)dudv + Ndv^2 = 0.$$

But let us note that for $e = -1$ this becomes $L - N\lambda^2 = 0$, and for $e = +1$ it

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* V. Snyder and C. H. Sisam, *Analytic Geometry of Space*, New York, 1914, p. 163.

becomes $L+2\lambda+N\lambda^2=0$ after dividing through by du^2. Thus the three conditions imposed at the beginning of this case are all implied by the second. Since in later work no other case arises in which the curve of intersection of the two quadrics is non-degenerate, we state the following theorem:

The two osculating quadrics at each point of a curve on our net intersect in a non-composite quartic curve if, and only if, the curve does not belong to the pencil of families of curves defined by equation (25). *The quartic has a cusp at the point* P_x.

In our general treatment *no case arises in which at each point of a curve on our net the total intersection of the pair of osculating quadrics contains as a part of it a non-degenerate cubic.*

Case II (a). Suppose $B-D\lambda=0$, $A\neq0$, $L+2\lambda+N\lambda^2\neq0$, $A(C+E\lambda^2)$ $-B^2\neq0$. We get the characteristic $[1(21)]$, which shows that the curve of intersection is composed of two conics which touch each other. *The conics touch at P_x and lie in the planes whose equations are*

$$(26) \qquad A\lambda(x_3 - \lambda x_2) + [B\lambda \pm (B^2 - A(C + E\lambda^2))^{1/2}]x_4 = 0.$$

Case II (b). If in II(a) we have $A(C+E\lambda^2)-B^2=0$, then the characteristic is $[1(111)]$, and hence the intersection is a conic counted twice. *The equations of this conic are found to be*

$$(27) \qquad \begin{aligned} A(x_3 - \lambda x_2) + Bx_4 &= 0, \\ A(L + 2\lambda + N\lambda^2)x_3^2 + 2B\lambda s x_3 x_4 + ACx_4^2 + 2A\lambda^2 x_1 x_4 &= 0. \end{aligned}$$

Case II (c). If $L+2\lambda+N\lambda^2=0$, $A(C+E\lambda^2)-B^2\neq0$, the characteristic is $[(22)]$, and the intersection is three generators, one counted twice. *The generator that is counted twice is the line* $x_4=x_3-\lambda x_2=0$, *which is tangent to the curve C_λ at P_x. The other two generators are the lines whose equations are as follows:*

$$(28) \qquad \begin{aligned} A\lambda(x_3 - \lambda x_2) + [B\lambda \pm (B^2 - A(C + E\lambda^2))^{1/2}]x_4 &= 0, \\ A\lambda(Cx_4 + 2\lambda^2 x_1) - [B\lambda^2 s \mp (B^2 - A(C + E\lambda^2))^{1/2}]x_3 &= 0, \end{aligned}$$

where the upper signs in the two equations are paired to give one generator and the lower signs the other.

Case II (d). If now in II(c) we let $A(C+E\lambda^2)-B^2=0$, then the characteristic is $[(211)]$, and the intersection is two intersecting generators, each counted twice. *The equations of these generators are respectively*

$$(29) \qquad \begin{aligned} x_4 = x_3 - \lambda x_2 &= 0, \\ A(x_3 - \lambda x_2) + Bx_4 = 2A\lambda^2 x_1 + 2B\lambda s x_3 + ACx_4 &= 0. \end{aligned}$$

From the above results we note the following theorem:

At each point of an asymptotic curve on the net the intersection of the osculating quadrics degenerates completely into generators.

Case III. Now let $A = 0$, $B \neq 0$. The characteristic is then $[(31)]$, and the intersection is two intersecting generators, *the tangents of our net at P_x,* and a residual conic. We thus obtain the following result by using (22):

If the ray and axis are not reciprocal polar lines with respect to the osculating quadric Q_u, and if the curve C_λ belongs to the associate conjugate net $(A = 0)$, then the quadrics Q_u and Q_v intersect in the tangents of the net and a residual non-degenerate conic. The equations of this conic are

(30)
$$2D\lambda^2 x_2 - 2Bx_3 - (C + E\lambda^2)x_4 = 0,$$
$$CDx_4^2 + [2BD - s(C + E\lambda^2)]x_3 x_4 - 2Bsx_3^2 + 2D\lambda^2 x_1 x_4 = 0.$$

Case IV. If $A = B = 0$, $C + E\lambda^2 \neq 0$, we find that the characteristic is $[(211)]$, and *the intersection is composed of the tangents of the net counted twice. The quadrics touch at each point of each tangent.*

Case V. If $A = B = C + E\lambda^2 = 0$, the characteristic is $[(1111)]$, and *the two quadrics Q_u and Q_v coincide. Their equations both become*

(31)
$$Cx_4^2 + 2\lambda^2 x_1 x_4 - 2\lambda r x_2 x_3 = 0.$$

This completes the discussion of all cases that arise.

7. **Conjugate nets.** Let us now consider two curves C_λ and C_μ on our net, and let us suppose that they are respectively embedded in two conjugate one-parameter families of curves on the net. By equation (8) we see that a necessary and sufficient condition for this is $\mu = -r/s$. Let P_x denote the point where C_λ crosses C_μ. We shall denote by $Q_{u\lambda}$ the osculating quadric Q_u of C_λ at P_x and by $Q_{u\mu}$ the osculating quadric Q_u of C_μ at P_x. The equation of $Q_{u\mu}$ may be obtained from that of $Q_{u\lambda}$ simply by replacing λ by $-r/s$. The equations of the osculating quadrics $Q_{u\lambda}$ and $Q_{u\mu}$ are thus as follows:

(32)
$$A x_3^2 + 2Bx_3 x_4 + Cx_4^2 + 2\lambda^2 x_1 x_4 - 2\lambda r x_2 x_3 = 0,$$
$$A(1 - LN)x_3^2 + 2Hx_3 x_4 + Kx_4^2 + 2r^2 x_1 x_4 - 2\lambda r(1 - LN)x_2 x_3 = 0,$$

where the first of these is the same as the first of equations (18) and H and K are defined by

$$H = r^2 \mathfrak{B} - rs \mathfrak{C}' + s^2[L(rs' - r's) \\ + r(L_u s - L_v r)]/[2\lambda s(LN - 1)],$$
$$K = r\{\pi r - rF' - sG - (rsk)/[\lambda(LN - 1)]\}.$$

The cone projecting the curve of intersection of the two quadrics (32) *from the second vertex of the tetrahedron of reference is given by*

(33)
$$2[B(1 - LN) - H]x_3x_4 + [C(1 - LN) - K]x_4^2$$
$$+ 2[\lambda^2(1 - LN) - r^2]x_1x_4 = 0.$$

This cone evidently consists of two planes, one of which is the tangent plane, $x_4 = 0$. But the tangent plane intersects each of the quadrics (32) in the two lines

(34)
$$x_3 = x_4 = 0,$$
$$x_4 = Ax_3 - 2\lambda rx_2 = 0.$$

Hence we draw the following conclusion:

At any point on the net the osculating quadric $Q_{u\lambda}$ of a curve of one family of a conjugate net always intersects the corresponding quadric $Q_{u\mu}$ of the curve of the other family in two conics, one of which is degenerate and is represented by equations (34). *The other conic lies in the plane*

(35) $2L(L + 2\lambda + N\lambda^2)x_1 - 2[B(1 - LN) - H]x_3 - [C(1 - LN) - K]x_4 = 0.$

Let us note that since C_λ and C_μ are assumed distinct it follows as a consequence that $L + 2\lambda + N\lambda^2 \neq 0$, and previous restrictions insure that $L\lambda r \neq 0$. Now let us project the curve of intersection of the quadrics (32) from the first vertex, P_x, of the tetrahedron of reference. *The equation of the projecting cone is*

(36)
$$A[r^2 - \lambda^2(1 - LN)]x_3^2 + 2(r^2B - \lambda^2H)x_3x_4 + \{[L\lambda r(L + 2\lambda$$
$$+ N\lambda^2)W^{(u)}]/(1 - LN)\}x_4^2 - 2\lambda r[r^2 - \lambda^2(1 - LN)]x_2x_3 = 0.$$

Evidently the lines (34) are generators of this cone; so this is really the cone projecting from P_x the conic lying in the plane (35), which does not pass through P_x. Hence the condition for the cone (36) to degenerate is just the condition for the residual conic to degenerate. But evidently $W^{(u)} = 0$ is necessary and sufficient for this, and, as Green has shown, this is just the condition that insures that the u-tangents of the net form a W-congruence; that is, that the asymptotic curves on the two focal sheets of the congruence correspond. Hence we have proved the following theorem:

A necessary and sufficient condition that the congruence of tangents to the curves C_u of the net be generators of a W-congruence is that the residual conic of intersection of the two quadrics $Q_{u\lambda}$ and $Q_{u\mu}$ degenerate at every point of the net.

We have already seen that if the curve C_μ is embedded in a one-parameter family of curves that is conjugate to the family to which C_λ belongs, then the

osculating quadrics $Q_{u\lambda}$ and $Q_{u\mu}$ each intersect the tangent plane, $x_4 = 0$, in the lines (34); that is, in a u-tangent and a residual line. Now let us suppose that C_λ and C_μ are any two curves which are respectively members of two distinct one-parameter families of curves, neither of which coincides with either family of the net itself. Then the corresponding osculating quadrics Q_u for these two curves ordinarily will intersect the tangent plane in the u-tangent, $x_3 = x_4 = 0$, and the two residual lines

$$(37) \qquad \begin{aligned} x_4 &= (L - N\lambda^2)x_3 - 2\lambda(L + \lambda)x_2 = 0, \\ x_4 &= (L - N\mu^2)x_3 - 2\mu(L + \mu)x_2 = 0. \end{aligned}$$

Now if we demand that these two residual lines coincide we get $\mu = -r/s$. Hence we draw the following conclusion:

A necessary and sufficient condition that a net on our surface be a conjugate net is that the residual intersections (37) of the quadrics $Q_{u\lambda}$ and $Q_{u\mu}$ with the tangent plane coincide at every point of the net.

8. **Metric considerations.** It is the purpose of this section to indicate a method of studying the analogue of the preceding problem in metric three-space and to give a few results that have been obtained by the author for the metric situation. Throughout this section we shall employ the notation and vector methods of Blaschke* for studying the geometry of a surface in ordinary space.

The equations of the osculating quadrics at the point x of a curve on the surface are here referred to a local orthogonal cartesian coordinate system, which is defined by the convention that any point X given by an expression of the form

$$(38) \qquad X - x = y_1 x_u / E^{1/2} + y_2 x_v / G^{1/2} + y_3 \xi,$$

where ξ is the unit normal at the point x, shall have local coordinates y_1, y_2, y_3.

If we now let the partial differential equations of Gauss and Weingarten play the role of equations (1) and compute the equation of Q_u in a manner similar to that of §3, we find that *the equation of Q_u is*

$$(39) \qquad \begin{aligned} E^{1/2}(L - N\lambda^2)y_2^2 + SGE^{1/2}y_3^2 &- 2QG^{1/2}y_1y_2 \\ &+ 2PGy_1y_3 + 2S(EG)^{1/2}y_2y_3 + 2\lambda GE^{1/2}y_3 = 0, \end{aligned}$$

where

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* W. Blaschke, *Vorlesungen über Differentialgeometrie*, 3d edition, vol. 1, Berlin, 1930, pp. 85–118.

$$P = \begin{Bmatrix}11\\2\end{Bmatrix} + \lambda\begin{Bmatrix}12\\2\end{Bmatrix}, \quad Q = L + M\lambda,$$

$$\lambda QS = \begin{Bmatrix}11\\2\end{Bmatrix}^2 - \lambda'\begin{Bmatrix}11\\2\end{Bmatrix} + \lambda\left(2\begin{Bmatrix}11\\2\end{Bmatrix}\begin{Bmatrix}12\\2\end{Bmatrix} - \begin{Bmatrix}11\\1\end{Bmatrix}\begin{Bmatrix}11\\2\end{Bmatrix} + \begin{Bmatrix}11\\2\end{Bmatrix}_u\right.$$

$$+ L(FL - EM)/W^2 \Big) + \lambda^2\left(\begin{Bmatrix}11\\2\end{Bmatrix}\begin{Bmatrix}22\\2\end{Bmatrix} - 2\begin{Bmatrix}11\\2\end{Bmatrix}\begin{Bmatrix}12\\1\end{Bmatrix} + 2\begin{Bmatrix}11\\2\end{Bmatrix}_v\right.$$

$$+ 2L(FM - EN)/W^2 \Big) + \lambda^3\left(\begin{Bmatrix}11\\2\end{Bmatrix}\begin{Bmatrix}22\\1\end{Bmatrix} - 2\begin{Bmatrix}12\\1\end{Bmatrix}\begin{Bmatrix}12\\2\end{Bmatrix} + \begin{Bmatrix}22\\2\end{Bmatrix}_u\right.$$

$$+ N(FL - EM)/W^2 \Big),$$

$$2\lambda QT = L\lambda' - 2L\begin{Bmatrix}11\\2\end{Bmatrix} + \lambda\left(L\begin{Bmatrix}11\\1\end{Bmatrix} - L\begin{Bmatrix}12\\2\end{Bmatrix} - 2M\begin{Bmatrix}11\\2\end{Bmatrix} - L_u\right)$$

$$+ \lambda^2\left(L\begin{Bmatrix}22\\2\end{Bmatrix} + 2L\begin{Bmatrix}12\\1\end{Bmatrix} - N\begin{Bmatrix}11\\2\end{Bmatrix} - 2L_v\right)$$

$$+ \lambda^3\left(N\begin{Bmatrix}12\\2\end{Bmatrix} + 2M\begin{Bmatrix}12\\1\end{Bmatrix} - L\begin{Bmatrix}22\\1\end{Bmatrix} - N_u\right).$$

The equation of the osculating quadric Q_v may now be written down by symmetry, or it may be obtained by direct calculation.

The discriminant Δ of the quadric (39) is

$$\Delta = \lambda^2 EG^3 Q^2 \neq 0,$$

for E, G are positive on a real surface, $\lambda \neq 0$ since C_λ was assumed not a u-curve, and $Q \neq 0$ since C_λ was supposed not a curve of the family conjugate to the family of u-curves of the net. *Thus the osculating quadric Q_u is always non-singular for a real analytic surface.*

If we simplify the analysis by taking the parametric net to be orthogonal we get the following interesting result. The determinant D' of the matrix of the second degree terms in the resulting equation of the quadric Q_u then becomes

(40) $$D' = (E^{1/2}/\lambda)(A_0\lambda' + A_1 + A_2\lambda + A_3\lambda^2 + A_4\lambda^3),$$

wherein

$A_0 = - G(LG_u + ME_v)/2,$

$A_1 = (4L^2MG + 2LGE_{uv} + ME_v^2 - LE_vG_u - 2L_uE_vG)/4,$

$$A_2 \doteq (2GL_uG_u + 8L^2NG + 4LGE_{vv} - 2LE_vG_v - MGE_uE_v/E + 4LM^2G$$
$$+ 2NE_v^2 + 2MGE_{uv} - 2E_vG_u - 4GL_vE_v - LGE_uG_u/E)/4,$$

$$A_3 = (12LMNG - 2LGG_{uv} + 4MGE_{vv} - 3ME_vG_v - 2GN_uE_v + LG_uG_v$$
$$- 2NE_vG_u)/4,$$

$$A_4 = (4M^2NG - MGE_vG_u/E - 2MGG_{uv} + 2MG_uG_v - LGG_u^2/E + 2GN_uG_u)/4.$$

Now suppose C_λ is a curve such that $A_0 \neq 0$. Since by definition the rulings of the quadric Q_u are real, we find that *the osculating quadric Q_u at every point of a curve C_λ on an orthogonal net is a hyperbolic paraboloid if, and only if, C_λ belongs to the family of hypergeodesics defined by $D' = 0$. Otherwise Q_u is a hyperboloid.*

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