THE DEGREE AND CLASS OF MULTIPLY TRANSITIVE GROUPS, III*

BY

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If a group of substitutions of class u (>3) is more than triply transitive, its degree does not exceed $2u+1$. This is Bochert's Theorem† (reduced one unit) and was the most that could be said in an entirely general way up to the present about the degree of highly transitive groups of given class. It will be proved in this paper that a t-ply transitive group of class $u(>3)$ is of degree $n<6u/5+u/t-t$ if $t>23$. It will also be shown that if $t>4$, $n\leqq 2u$; if $t>5$, $n\leqq 5u/3$; if $t>7$, $n<3u/2$; if $t>11$, $n<4u/3$; if $t>21$, $n<5u/4$.

1. On page 648 of DC2 it is proved that for 4-ply transitive groups $n\leqq 2u+1$. The method there used is now extended to 5-ply transitive groups in the proof of the following theorem.

THEOREM I. *If n is the degree and u the class of a 5-ply transitive group, not alternating or symmetric, $n\leqq 2u$.*

If there is a substitution of order 2 and degree u in the group G, $n\leqq 5u/3$ $<2u-1$ ($u\geqq 6$; DC1, p. 463). Then unless all the substitutions of degree u are of order 3, at least one of them is of order >4. Let us say that

$$S = (abcde\cdots)\cdots,$$

and let $S_1, S_2, \cdots, S_w$ be similar to S and a complete set of conjugates under H, the subgroup of G that fixes a, c, and f, where f is a letter of S not adjacent to a or c in S. Since G is triply transitive we can make

$$S_1 = (a)(f)(c\cdots)\cdots.$$

The condition on the letter f can be satisfied if S is of order >5 by a letter of the first cycle of S, while if S is of order 5 there will be in S a second cycle from which it can be taken, because $u>5$.

We now have, quite as on page 648 of DC2,

* Presented to the Society, August 30, 1932, and June 19, 1933; received by the editors October 3, 1932, and (revised and extended) February 10, 1933.

† Manning, these Transactions, vol. 31 (1929), p. 648. This paper will be referred to as DC2, and the first paper bearing the same title and which appeared in these Transactions, vol. 18 (1917), p. 463, will be called DC1.

$$(1)\quad 3w+\frac{3w(u-1)(u-3)}{n-3}-\frac{2w(u-3)}{n-3}-\frac{w(u-2)(u-3)(u-4)}{(n-3)(n-4)}-\frac{2w(u-1)}{n-3}-\frac{w(u-1)(u-2)(u-6)}{(n-3)(n-4)}\geqq wu.$$

Writing x for $n-4$ and k for $u-3$, this is

$$(2)\quad kx^2-(3k^2+k-4)x+2k^3-8k-6\leqq 0.$$

If $x=2k+1$, (2) becomes $-k^2-2$. If $x=2k+2$, it becomes $2k+2$. Therefore, unless all the substitutions of degree u of G are of order 3, $n<2u$.

If all the substitutions of degree u are of order 3, $S=(abc)(def)\cdots$ and $S_1=(b)(d)(a\cdots)\cdots$. As on page 649 of DC2 we set up

$$(3)\quad 3w+\frac{3w(u-1)(u-3)}{n-3}-\frac{2w(u-3)}{n-3}-\frac{w(u-2)(u-3)(u-4)}{(n-3)(n-4)}-\frac{w(u-1)}{n-3}-\frac{w(u-1)(u-2)(u-5)}{(n-3)(n-4)}\geqq wu+\frac{2w(n-u)}{(n-3)(n-4)}.$$

With the same x and k as before, this is

$$(4)\quad kx^2-(3k^2+2k-4)x+2k^3+k^2-7k-2\leqq 0.$$

If $x=2k+3$, the left member is $4k+10>0$, while if $x=2k+2$, it is $-k^2+k+6$, so that $x<2k+3$, or finally $n\leqq 2u$, and our theorem is proved.

2. We take up next

THEOREM II. *If n is the degree and u (>3) the class of a 6-ply transitive group, $n\leqq 5u/3$.*

In the proof of I the well known fact that there is no 5-ply transitive group of class >3 and <8 was used. That there is no 6-ply transitive group of degree <53 is an immediate consequence of the following two theorems:

A. *If a primitive group of class >3 contains a circular substitution of prime order p (>3), $n\leqq p+2$.**

B. *Let q be an integer $\geqq 2$ and <5; p any prime $>q+1$; then the degree of a primitive group which contains a substitution of order p that displaces pq letters (not including the alternating group) is $\leqq pq+q$.*†

For example, were there such a group of degree 25, with its order necessarily a multiple of $25\cdot 24\cdot 23\cdot 22\cdot 21\cdot 20$, it would contain a substitution of order 11 of degree 11 or 22. The first is impossible by A and the second by B.

By I, the 6-ply transitive group G, if of class $\leqq 26$, is of degree $\leqq 52$, an

* C. Jordan, Bulletin de la Société Mathématique de France, vol. 1 (1873), p. 40.

† Manning, these Transactions, vol. 15 (1909), p. 247.

impossibility by A and B, so that only groups of class >26 need be considered.

The theorem is known to be true if one of the substitutions of degree u is of even order (DC1).

According to Dr. Luther the 6-ply transitive groups of class $u(>3)$ in which there are substitutions of degree $u+e$ of even order $(0<e<u/9)$ have

$$n < \frac{4u}{3} + 4e.* \tag{5}$$

However, it may be that all the substitutions of degree $\leqq u+e$ are of odd order. Suppose that to be the case.

Let S be a substitution of G of degree u, and of order >3. Then its order, being an odd number, is $\geqq 5$:

$$S = (abcde \cdots j) \cdots (\alpha) \cdots .$$

Among the conjugates of S under G there is a substitution

$$S_1 = (ab\alpha \cdots) \cdots (c) \cdots ,$$

and the complete set of conjugates under H (the subgroup of G that fixes the four letters a, b, c, and α) is $S_1, S_2, \cdots, S_w$. Reasoning as in the proof of I,

$$\sum m_i = 2w + \frac{w(u-3)^2}{n-4}, \tag{6}$$

$$\sum q_i = \sum r_i = w + \frac{w(u-3)}{n-4} + \frac{w(u-3)(u-4)^2}{(n-4)(n-5)}. \tag{7}$$

Now

$$S^{-1}S_i^{-1}SS_i = (c\alpha) \cdots ,$$

a substitution of even order, so that

$$\begin{aligned} &6w + \frac{3w(u-3)^2}{n-4} - 2w - \frac{2w(u-3)}{n-4} - \frac{2w(u-3)(u-4)^2}{(n-4)(n-5)} \\ &\geqq w(u+e+1). \end{aligned} \tag{8}$$

Here we replace $n-5$ by x and $u-3$ by k, and have

$$(e+k)x^2 + (e - 3k^2 + 3k)x + 2k(k-1)^2 \leqq 0. \tag{9}$$

If $2k-4e-2$ and $2k-4e-1$ are put for x in (9) we obtain $-2ke+16e^3+12e^2+2e$ and $(k-3e)^2+16e^3-5e^2$, respectively. While the first of these numbers

* C. F. Luther, American Journal of Mathematics, vol. 55 (1933), p. 77.

may be negative for large values of k, the second is always positive. Therefore $x<2k-4e-1$, that is,

$$n \leqq 2u - 4e - 3. \tag{10}$$

If S is of order 3,

$$S = (abc)(def) \cdots (\alpha) \cdots,$$

$$S_1 = (ab\alpha) \cdots (c) \cdots,$$

and

$$\sum m_i = 2w + \frac{w(u-3)^2}{n-4} \tag{11}$$

as before. But

$$\sum q_i = \sum r_i = w + \frac{w(u-3)^2(u-4)}{(n-4)(n-5)}. \tag{12}$$

The inequality

$$6w + \frac{3w(u-3)^2}{n-4} - 2w - \frac{2w(u-3)^2(u-4)}{(n-4)(n-5)} \geqq w(u+e+1), \tag{13}$$

with x for $n-5$ and k for $u-3$, reduces to

$$(e+k)x^2 + (e - 3k^2 + k)x + 2k^3 - 2k^2 \leqq 0. \tag{14}$$

If $2k-4e$ and $2k-4e+1$ are put for x in (14) we get $-2ke+16e^3-4e^2$ and $(k-3e+1)^2+16e^3-21e^2+8e-1$, respectively. Hence $x<2k-4e+1$, or $n<2u-4e$, or

$$n \leqq 2u - 4e - 1. \tag{15}$$

Dr. Luther's limit (which for $e=0$ is that of DC1), $4u/3+4e$, increases with e, while (15) decreases with e. It is to be proved from (5) and (15) that $n \leqq 5u/3$, irrespective of the presence or absence of substitutions of order 2. Let E be the integral part of $u/12-1/8$, which is the solution for e of the equation

$$2u - 4e - 1 = \frac{4u}{3} + 4e. \tag{16}$$

Now either all the substitutions of degree $\leqq u+E+1$ of G are of odd order, or one of the substitutions of degree $\leqq u+E+1$ is of order 2. Then if we put $E+1$ for e in (5), we have a valid upper limit for the degree of 6-ply transitive groups of class $u(>3)$. Therefore

$$n < \frac{4u}{3} + 4E + 4$$

$$< \frac{4u}{3} + 4\left(\frac{u}{12} + \frac{7}{8}\right)$$

$$< \frac{5u}{3} + \frac{7}{2}\cdot \tag{17}$$

This is not what we set out to prove, but before proceeding farther, let us make use of it to revise the lower limit, $u>26$, of the class of 6-ply transitive groups under which we have been working. From $n>52$, $5u/3+7/2>52$, and therefore $u\geqq 30$.

To find the limit stated in our theorem, we return to the two cases: (1) At least one substitution of degree u is of order >3. (2) All substitutions of degree u are of order 3.

(1) In this case $n\leqq 2u-4e-3$, and the solution of

$$2u - 4e - 3 = \frac{4u}{3} + 4e \tag{18}$$

is $e=u/12-3/8$. This number is not an integer. Let E be its integral part. Since $E+1<u/9$ for $u\geqq 30$, formula (5) holds good for E and for $E+1$. When there are substitutions of order 2 of degree $\leqq u+E$ in G, $n<4u/3+4E$, but if all the substitutions of degree $\leqq u+E$ are of odd order, $n\leqq 2u-4E-3$, and of the two, the latter gives the higher limit and should be retained. But we saw that $4u/3+4E+4$ was a true limit also. Then of these two formulas, $2u-4E-3$ and $4u/3+4E+4$, we are at liberty to choose the lower.

If $u\equiv r$, mod 12 ($r=5, 6, \cdots, 16$), $E=(u-r)/12$, and the two formulas between which we may choose are $5u/3+r/3-3$ and $5u/3+4-r/3$. One or the other gives $n\leqq 5u/3$ unless $r=10$ or 11. If $r=11$, $n<5u/3+1/3$ is equivalent to $n\leqq 5u/3-1/3$. Now Dr. Luther's limit, so concisely stated, is deduced from the inequality

$$n < 5 + \frac{4(u + e - 4)^2}{3(u - 2e/3 - 4)}\cdot \tag{19}$$

When $r=10$, and therefore $E=(u-10)/12$, we substitute $E+1=u/12+1/6$ for e in (19) and have on reduction

$$n < \frac{5u}{3} - \frac{u}{102} + 0.49 + \frac{18.68}{17u - 74}\cdot \tag{20}$$

If $u=58$, $n<5u/3-0.04$; if $u=46$, $n<5u/3+0.07$; if $u=34$, $n<5u/3+0.21$.

Since $u \geqq 30$, and since our formula (10), for $e = E+1 = u/12+1/6$, becomes $5u/3-11/3$, we conclude that $n \leqq 5u/3$ when $r=10$.

(2) Every substitution of degree u is of order 3, and $u \equiv 0$, mod 3. Equating the right hand members of the two formulas

$$n \leqq 2u - 4e - 1 \tag{15}$$

and

$$n \leqq \frac{4u}{3} + 4e - 1, \tag{21}$$

we find $e = u/12$. If $u/12$ is a whole number, (15) becomes $n \leqq 5u/3-1$. Let $u \equiv r$, mod 12 ($r=3$, 6, or 9); $E=(u-r)/12$. The condition $E+1<u/9$ is satisfied. We are at liberty to choose between $n<5u/3+r/3-1$ and $n<5u/3+3-r/3$. For $r=3$ and for $r=9$, $n \leqq 5u/3$. For $r=6$ both formulas are $n<5u/3+1$. Again we have recourse to Dr. Luther's original formula (19) and in it put $e=E+1=u/12+1/2$. It reduces to

$$n < \frac{5u}{3} - \frac{u}{102} + 1.90 + \frac{51.89}{17u - 78} \cdot \tag{22}$$

When $u=102$, $n<5u/3+0.94$, and because the sum of the last three terms of (22) decreases as u increases, $n \leqq 5u/3$ for $u \geqq 102$. For $u=90$, 78, 66, 54, and 42, $5u/3+1=151$, 131, 111, 91, and 71, respectively. But with the aid of Theorems A and B it is easy to show that there are no 6-ply transitive groups of these degrees and of class >3. Therefore $n \leqq 5u/3$, for all non-alternating 6-ply transitive groups.

3. We now undertake to prove the following fundamental theorem:

Theorem III. *If n is the degree and u (>3) is the class of a t-ply transitive group ($t>6$) in which all the substitutions of degree $\leqq u+e$ are of odd order, $n<2u-4e-5t+37$.*

Here $e=0, 1, \cdots$. There is a substitution S of order $\geqq 3$ among the substitutions of degree u of G. Because G is of class u, S is a regular substitution:

$$S = (ab \cdots) \cdots (ef \cdots ijk \cdots) \cdots (\alpha) \cdots,$$

and $a, b, \cdots, j$ are the first $t-4$ letters of S. Since G is of sufficiently high transitivity, it contains a substitution $(k\alpha) \cdots (a)(b) \cdots (j) \cdots$ which transforms S into

$$S_1 = (ab \cdots) \cdots (ef \cdots ij\alpha \cdots) \cdots (k) \cdots.$$

The indicated order of the letters of S and S_1 is to be maintained unchanged. It may be that t and the order of S are so related numerically that $S_1 = \cdots (j\alpha \cdots) \cdots$ or $S_1 = \cdots (\alpha \cdots) \cdots$, in which case nothing can be said about the order of $S^{-1}S_1^{-1}SS_1$; but if S_1 has two or more of the letters $a, b, \cdots, j$ preceding α in its cycle of S_1,

$$S^{-1}S_1^{-1}SS_1 = (k\alpha) \cdots ,$$

and $S^{-1}S_1^{-1}SS_1$ fixes all the $t-4$ letters $a, b, \cdots, j$ except perhaps the first two of the cycle of S_1 in which α occurs.

Let us assume for the moment that t is such a number that $S_1 = \cdots (ij\alpha \cdots) \cdots, \quad \cdots (hij\alpha \cdots) \cdots, \quad \cdots,$ or $\cdots (ef \cdots ij\alpha) \cdots$. There is a doubly transitive subgroup H of G that fixes the $t-2$ letters $a, b, \cdots, j, k$, and α. Its degree is $n-t+2$. Under H, S_1 is one of a complete set of w conjugate substitutions, $S_1, S_2, \cdots, S_w$. Let S_i have m_i letters of H in common with S. Of these m_i common letters (of H), let S replace q_i by common letters (of H) and let S_i replace r_i by common letters (of H). Then the degree of $S^{-1}S_i^{-1}SS_i$ does not exceed $4+3m_i-q_i-r_i$. For this substitution displaces at most 4 of the $t-2$ letters fixed by H: k, α, and the first two letters of the cycle of S_i in which α occurs. It may displace all the other common letters, m_i in number. Of the letters of S_i that are new to S and are letters of H, only those m_i-r_i that follow common letters are displaced by $S_i^{-1}SS_i$ and therefore by $S^{-1}S_i^{-1}SS_i$. And a like statement holds for $S^{-1}S_i^{-1}S$.

For use in the succeeding paragraphs we note that S displaces $u-t+3$ letters of H, and has $(u-t+3)(u-t+2)$ ordered pairs of letters of H; S has $u-t+3$ sequences in letters both of which are displaced by H if k ends a cycle of S, but only $u-t+2$ if k does not end its cycle.

Now S_1 displaces $u-t+3$ letters of H. The complete set $S_1, S_2, \cdots, S_w$ displaces $w(u-t+3)$ letters of H, one as often as any other because H is transitive. Therefore

$$\sum m_i = \frac{w(u-t+3)^2}{n-t+2}. \tag{23}$$

In S_1 there are $u-t+3$ sequences of letters of H if $S_1 = (ef \cdots j\alpha) \cdots$. But if α is not the last letter of its cycle, there are $u-t+2$ such sequences. Then the total number of these sequences in the set is $w(u-t+3)$ or $w(u-t+2)$ and each, because H is doubly transitive, occurs $w(u-t+3)/[(n-t+2)\cdot(n-t+1)]$ times, or $w(u-t+2)/[(n-t+2)(n-t+1)]$ times, respectively. Hence

$$\sum q_i = \frac{w(u-t+3)^2(u-t+2)}{(n-t+2)(n-t+1)} \tag{24}$$

or

(25) $$\sum q_i = \frac{w(u-t+2)^2(u-t+3)}{(n-t+2)(n-t+1)},$$

the first, and larger, value of $\sum q_i$ holding only when $S_1 = \cdots (ef \cdots ij\alpha) \cdots$. There are $(u-t+3)(u-t+2)$ ordered pairs of letters of H in S_1 in both cases. Then

(26) $$\sum r_i = \frac{w(u-t+3)^2(u-t+2)}{(n-t+2)(n-t+1)}$$

or

(27) $$\sum r_i = \frac{w(u-t+3)(u-t+2)^2}{(n-t+2)(n-t+1)},$$

in the two cases respectively.

The substitution $S^{-1}S_1^{-1}SS_1$ is of even order and therefore is by hypothesis of degree $>u+e$. Then

(28) $$4w + \sum(3m_i - q_i - r_i) \geqq w(u+e+1),$$

or, if to $\sum q_i$ and $\sum r_i$ are given their smaller values,

(29) $$4 + \frac{3(u-t+3)^2}{n-t+2} - \frac{2(u-t+3)(u-t+2)^2}{(n-t+2)(n-t+1)} \geqq u+e+1.$$

If $S_1 = \cdots (j\alpha \cdots) \cdots (k) \cdots$, $(j\alpha \cdots)$ is not the first cycle of S_1, while there certainly is in G a substitution T_1, a transform of S by $(a)(b) \cdots (h)(i\alpha)(j\beta)(k\gamma) \cdots$, where β and γ are two letters fixed by S:

$$T_1 = \cdots (ef \cdots gh\alpha) \cdots (i)(j)(k) \cdots .$$

The substitution

$$S^{-1}T_1^{-1}ST_1 = (\alpha i)(ef) \cdots .$$

T_1 is one of w (probably numerically different from the former w) conjugates under H. There are $u-t+5$ letters of H in T_i; in the set $T_1, T_2, \cdots, T_w$ each occurs $w(u-t+5)/(n-t+2)$ times. Then

(30) $$\sum m_i = \frac{w(u-t+5)(u-t+3)}{n-t+2}.$$

In T_i there are $u-t+5$ sequences with both letters displaced by H. Therefore

(31) $$\sum q_i = \frac{w(u-t+5)(u-t+3)(u-t+2)}{(n-t+2)(n-t+1)}.$$

Also

$$\sum r_i = \frac{w(u-t+5)(u-t+4)(u-t+2)}{(n-t+2)(n-t+1)}. \tag{32}$$

Combining,

$$\begin{aligned} &4 + \frac{3(u-t+5)(u-t+3)}{n-t+2} - \frac{(u-t+5)(u-t+3)(u-t+2)}{(n-t+2)(n-t+1)} \\ &- \frac{(u-t+5)(u-t+4)(u-t+2)}{(n-t+2)(n-t+1)} \geqq u+e+1. \end{aligned} \tag{33}$$

In particular, this inequality (33) arises when $t=8$ and S is of order 3.

There remains the case in which $S_1 = \cdots (\alpha \cdots) \cdots$, and in which we actually use the transform of S by $(a)(b) \cdots (i)(j\alpha)(k\beta) \cdots$:

$$T_1 = \cdots (\cdots hi\alpha) \cdots (j)(k) \cdots .$$

We say that T_1 is one of a complete set of w conjugates under H. T_1 displaces $u-t+4$ letters of H, and therefore

$$\sum m_i = \frac{w(u-t+4)(u-t+3)}{n-t+2}. \tag{34}$$

In T_i there are $u-t+4$ sequences in letters of H. Therefore

$$\sum q_i = \sum r_i = \frac{w(u-t+4)(u-t+3)(u-t+2)}{(n-t+2)(n-t+1)}. \tag{35}$$

Hence

$$\begin{aligned} &4 + \frac{3(u-t+4)(u-t+3)}{n-t+2} \\ &- \frac{2(u-t+4)(u-t+3)(u-t+2)}{(n-t+2)(n-t+1)} \geqq u+e+1. \end{aligned} \tag{36}$$

In the three inequalities (29), (33), and (36), we put $x=n-t+1$, $k=u-3$, and $s=t-7$, and simplify. They become, in order,

$$\begin{aligned} (37)\ &(e+k)x^2 + [e-3k^2+(6s+7)k-3s^2-6s-3]x + 2k^3 - (6s+10)k^2 \\ &+ (6s^2+20s+16)k - 2s^3 - 10s^2 - 16s - 8 \leqq 0, \end{aligned}$$

$$\begin{aligned} (38)\ &(e+k)x^2 + [e-3k^2+(6s+1)k-3s^2+3]x + 2k^3 - (6s+3)k^2 \\ &+ (6s^2+6s-3)k - 2s^3 - 3s^2 + 3s + 2 \leqq 0, \end{aligned}$$

$$\begin{aligned} (39)\ &(e+k)x^2 + [e-3k^2+(6s+4)k-3s^2-3s]x + 2k^3 - (6s+6)k^2 \\ &+ (6s^2+12s+4)k - 2s^3 - 6s^2 - 4s \leqq 0. \end{aligned}$$

In (37) we put $x=2k-4e-6s-3$, and write the result:

$$(k-7e-8s-1)^2+16e^3+(48s-29)e^2 + (48s^2-58s+4)e+16s^3-29s^2+4s. \tag{40}$$

This is clearly positive for $s\geqq 2$. If $s=1$, it is positive if $e\geqq 1$, and when $e=0$ it reduces to $(u-12)^2-9$, which is positive for $t=8$ and $u\geqq 30$. A similar detailed examination shows it to be positive for all $s\geqq 0$ and $e\geqq 0$. When $x=2k-4e-6s-4$, (37) becomes

$$-10ke-10sk-2k+16e^3+(48s+28)e^2 + (48s^2+66s+24)e+16s^3+38s^2+26s+4, \tag{41}$$

which is negative for some sets of values of u, t, and e. Therefore $x<2k-4e-6s-3$, or

$$n<2u-4e-5t+32 \quad (t>6). \tag{42}$$

We next put $x=2k-4e-6s+2$ in (38) and find

$$\left(k-5e-6s+\frac{9}{2}\right)^2+16e^3+(48s-45)e^2 + (48s^2-90s+39)e+16s^3-45s^2+39s-\frac{49}{4}. \tag{43}$$

This is positive for $s\geqq 2$, in fact, is positive for all $s\geqq 0$, $e\geqq 0$. The value of (38) when $x=2k-4e-6s+1$ is

$$-6ke-6sk+5k+16e^3+(48s-12)e^2 + (48s^2-18s-10)e+16s^3-6s^2-15s+5. \tag{44}$$

Therefore

$$n<2u-4e-5t+37 \quad (t>6). \tag{45}$$

Finally, we seek the limit given by (39). Put $x=2k-4e-6s-1$. Then the left member of (39) becomes

$$(k-5e-6s)^2+k+16e^3+(48s-21)e^2+(48s^2-42s)e+16s^3-21s^2-s, \tag{46}$$

and this too is positive for $s\geqq 0$; $e\geqq 0$. The result of substituting $2k-4e-6s-2$ for x in (39) is

$$-6ke-6sk-k+16e^3+(48s+12)e^2+(48s^2+30s+2)e+16s^3+18s^2+2s. \tag{47}$$

The limit on n, deduced from (39), is

$$n<2u-4e-5t+34 \quad (t>6). \tag{48}$$

Of the three results (42), (45), and (48), (45) is to be used when $t>7$. If $t=7$, only (37) and (39) are applicable, the latter when S is of order 3, and therefore the limit for $t=7$ is $n<2u-4e-1$.

4. From this point on only groups more than 7-ply transitive will be in question. If Theorems A, B, and II are applied to 8-ply transitive groups, one quickly finds that $n>158$, and $u>94$. We are now to prove

THEOREM IV. *The degree of an 8-ply transitive group of class $u(>3)$ is less than $3u/2$.*

Dr. Luther* has proved a remarkable theorem which may be stated as follows:

C. *Let G be a more than $2^a+p_1+p_2+\cdots+p_r$ times transitive group ($a\geqq 2$; $p_1, p_2, \cdots, p_r$ distinct odd primes) of class $u(>3)$; if G contains a substitution of even order of degree $u+e$,*

$$n < u + \frac{2u}{2^a p_1 p_2 \cdots p_r - 2} + e + \frac{2e}{2^a - 2} + 1. \tag{49}$$

If G is 8-ply transitive and includes a substitution of order 2 and degree $\leqq u+e$, this theorem asserts that

$$n < \frac{6u}{5} + 2e + 1, \tag{50}$$

without any restriction upon e. Theorem III states that if G contains no substitution of even order of degree $\leqq u+e$,

$$n < 2u - 4e - 3. \tag{51}$$

Equate the right hand members of these two inequalities and solve for e:

$$e = \frac{2u}{15} - \frac{2}{3}. \tag{52}$$

We have a true upper limit for the degree of all 8-ply transitive groups that are not alternating or symmetric if we put $2u/15+1/3$ for e in (50). Thus

$$\begin{aligned} n &< \frac{22u}{15} + \frac{5}{3} \\ &= \frac{3u}{2} - \frac{u-50}{30} \\ &< \frac{3u}{2}. \end{aligned} \tag{53}$$

* American Journal of Mathematics, vol. 50 (1933).

The last step follows because $u>94$.

5. The next theorem can be disposed of very briefly.

THEOREM V. *If a group of class* $u(>3)$ *is more than* 11-*ply transitive, its degree is less than* $4u/3$.

Let $a=3$, $r=1$, and $p_1=3$ in (49). Then, if there are substitutions of order 2 and of degree $\leqq u+e$ in G,

$$n < \frac{12u}{11} + \frac{4e}{3} + 1. \tag{54}$$

Also, for $t=12$, III becomes, all the substitutions of degree $\leqq u+e$ being of odd order,

$$n < 2u - 4e - 23. \tag{55}$$

If

$$\frac{12u}{11} + \frac{4e}{3} + 1 = 2u - 4e - 23, \tag{56}$$

$e=15u/88-9/2$. Then, as before, when we put $15u/88-7/2$ for e in (54),

$$\begin{aligned} n &< \frac{29u}{22} - \frac{11}{3} \\ &< \frac{4u}{3}. \end{aligned} \tag{57}$$

6. It is possible to go one step farther in the elaboration of these extremely concise limit formulas.

THEOREM VI. *The degree of a t-ply* $(t>21)$ *transitive group of class* $u(>3)$ *is less than* $5u/4-t$.

From C, granting its hypothesis, and putting $a=4$, $p_1=5$, it follows that

$$n < \frac{40u}{39} + \frac{8e}{7} + 1. \tag{58}$$

By III, if its hypothesis is granted,

$$n < 2u - 4e - 5t + 37. \tag{45}$$

Equating, solving for e, and proceeding as before,

$$(59)\qquad \begin{aligned} n &< \frac{436u}{351} - \frac{10t}{9} + \frac{71}{7} \\ &< \frac{5u}{4} - \frac{10t}{9} - \frac{11u}{1404} + \frac{71}{7} \\ &< \frac{5u}{4} - \frac{10t}{9} - \frac{11}{1404}(u - 1295). \end{aligned}$$

From (59) it is clear than $n<5u/4$ $(t>21)$, so that if $u<1295$, $n<1619$. Since A, B, and a short list of prime numbers tell us that there is no non-alternating 22-ply transitive group of degree <1619,

$$(60)\qquad n < \frac{5u}{4} - \frac{10t}{9}$$

$$(61)\qquad < \frac{5u}{4} - t.$$

7. In what follows it is of advantage to know that the class of a 24-ply transitive group exceeds 7600. This is a consequence of VI if n exceeds 9500, which fact can easily be verified by means of A and B and a list of primes.

THEOREM VII. *Let n be the degree of a t-ply $(t>23)$ transitive group of class u (>3); then*

$$(62)\qquad n < \frac{6u}{5} + \frac{u}{t} - t.$$

Let $s=p_1+p_2+\cdots+p_r$ be the sum of r distinct odd primes, given in advance, and let p be their product. G is a t-ply transitive group, and t is large. Now let a be the largest integer such that

$$2^a < t - s \leqq 2^{a+1}.$$

The solution for e of

$$(63)\qquad \frac{2^a pu}{2^a p - 2} + \frac{2^a e}{2^a - 2} + 1 = 2u - 4e - 5t + 37,$$

where (49) has been set equal to (45), is

$$(64)\qquad e = \frac{(2^a - 2)(2^a p - 4)u}{(5\cdot 2^a - 8)(2^a p - 2)} - \frac{(2^a - 2)(5t - 36)}{5\cdot 2^a - 8}\cdot$$

Then, on the insertion of $e+1$ in (49),

$$n < \frac{2^a(6\cdot 2^a p - 8p - 4)u}{(5\cdot 2^a - 8)(2^a p - 2)} - \frac{2^a(5t - 36)}{5\cdot 2^a - 8} + \frac{2^a}{2^a - 2} + 1 \tag{65}$$

$$< \frac{6u}{5} + \frac{hu}{t} - t + j, \tag{66}$$

where

$$h = \frac{(8p + 40)t}{5(5\cdot 2^a p - 8p - 10)} \tag{67}$$

and

$$j = \frac{46\cdot 2^a - 8t - 16}{5\cdot 2^a - 8} + \frac{2}{2^a - 2}\cdot \tag{68}$$

Now let $p=35$, $s=12$. In this case, because t exceeds $12+2^a$, $j<38/5$; and because of $t\leqq 12+2^{a+1}$,

$$h = \frac{64t}{5(35\cdot 2^a - 58)} \tag{69}$$

$$\leqq \frac{128(2^a + 6)}{5(35\cdot 2^a - 58)}. \tag{70}$$

If $a\geqq 7$, we conclude from (70) that $h<4/5$. Then for groups that are more than 140 times transitive,

$$n < \frac{6u}{5} + \frac{u}{t} - t - \frac{1}{5}\left(\frac{u}{t} - 38\right). \tag{71}$$

Dr. Luther has proved in a simple way that for non-alternating t-ply transitive groups

$$n \geqq \frac{t^2 - 2t}{4}. \tag{72}$$

Now for $t>21$, we know that $n<5u/4-t$, or $u>4(n+t)/5$, by VI. Therefore

$$u > \frac{t^2 + 2t}{5}\cdot \tag{73}$$

This inequality (73) holds for all primitive non-alternating t-ply transitive groups, as can be easily seen by examining it for $t=1, 2, \cdots, 21$.

In (71), $u/t>38$ if $(t+2)/5\geqq 38$, that is, if $t\geqq 188$. We have therefore proved (62) in case $t>187$.

Let (66) be written thus:

(74) $$n < \frac{6u}{5} + \frac{u}{t} - t - \frac{1-h}{t}(u - l),$$

where $l = jt/(1-h)$. It is clear that (62) is true when $l < 7600$. Now let

$$\begin{aligned}
140 < t &\leqq 187, & p &= 35; & h &< 0.6, & j &< 7.6, & l &< 3560;\\
95 < t &\leqq 140, & p &= 1001; & h &< 0.8, & j &< 7.0, & l &< 4900;\\
76 < t &\leqq 95, & p &= 35; & h &< 0.6, & j &< 7.6, & l &< 1820;\\
44 < t &\leqq 76, & p &= 35; & h &< 0.92, & j &< 7.6, & l &< 7250;\\
37 < t &\leqq 44, & p &= 5; & h &< 0.94, & j &< 7.7, & l &< 5650;\\
30 < t &\leqq 37, & p &= 33; & h &< 0.96, & j &< 6.7, & l &< 6200;\\
24 < t &\leqq 30, & p &= 15; & h &< 0.90, & j &< 7.4, & l &< 2230;\\
t &= 24, & p &= 7; & h &< 0.94, & j &< 7.5, & l &< 3000.
\end{aligned}$$

It is proved that $n < 6u/5 + u/t - t$ if $t > 23$ and $u > 3$.

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