THE FIRST AND SECOND VARIATIONS OF AN n-TUPLE INTEGRAL IN THE CASE OF VARIABLE LIMITS*

BY

H. A. SIMMONS

1. **Introduction.** The main purpose of this paper is to generalize except in one detail all of the results which we obtained relative to a double integral in a previous article.† The first eight sections here would constitute a complete generalization of those results if the integrand of the $(n-1)$-tuple integral in equation (6.9) below could be expressed as a linear homogeneous function of ζ and not involve the ζ_{x_i} (cf. equation (49) of the previous article). We have not been able to do this. A special device that was used in obtaining equation (49) of the previous article does not seem capable of generalization here. In §9 we suggest for the fundamental formulas in (4.8), for the integrals $J'(0)$ and $J''(0)$, certain applications that are not made in §§1–8.

In view of certain papers of Lichtenstein‡ and Reid§, in which Jacobi's condition is stated in terms of characteristic numbers of boundary value problems somewhat like the problem of §7 below, and also because of recent advances in the theory of elliptic partial differential equations‖ our generalization seems desirable.

The theses of Bates¶ and Powell** are useful in studying curvilinear coordinate systems of the type that we employ here. We take the lines of curvature as the parameter lines.

The legitimacy of the use that we make of an extended Green's theorem is well known.††

In this paper the variables $x_1, \cdots, x_n$ are the coordinates of a euclidean space X of n ($n \geq 2$) dimensions in the euclidean space XZ of $(n+1)$ dimensions of coordinates $x_1, \cdots, x_n, z$. An equation $z = z(x)$ or $\phi(x, z) = 0$, where

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$x \equiv (x_1, \cdots, x_n)$, defines an n-dimensional hypersurface in XZ. We let $p_i \equiv \partial z/\partial x_i$ $(i = 1, \cdots, n)$ and let W stand for a $(2n+1)$-dimensional open region in the space XZP of the variables x, z, and $p \equiv (p_1, \cdots, p_n)$. Then we define an *admissible hypersurface* $z = z(x)$ to be one with its elements in W and having the following four properties: (i) z is a single-valued function of the x's; (ii) z is of class C''; (iii) it has a real, *simply closed* $(n-1)$-dimensional intersection (that is, a connected $(n-1)$-dimensional intersection that is bounded, closed, and does not intersect itself) L_0', with a fixed hypersurface $\phi(x, z) = 0$, which is of class C'' and has no singular point for x and z in W; (iv) it is such that the projection, L_0, of L_0' on X is met by a line parallel to any one of the coordinate axes, x_i, in a finite number of points and segments.

Property (iii) indicates the sense in which we use the term *variable limits*. The manifold L_0' is the boundary of the portion of the admissible hypersurface $z = z(x)$ that we consider. On account of property (i), the correspondence between points of L_0 and L_0' is one-to-one; and so L_0 is also a simply closed $(n-1)$-dimensional manifold (cf. (iii)). It bounds a simply connected portion of X space. This we call S_0.

Property (iv) is required to insure that our application to L_0 and S_0 of an extended Green's theorem in §§5, 7 below shall be legitimate.

We consider here the n-tuple integral $(n \geq 2)$

$$(1.1) \qquad\qquad I = \int_{S_0}^{n} f(x, z, p)dx,$$

where x, z, p, S_0 are as defined above, and f is of class C''' in W. This integral I is our generalization of the double integral

$$\iint_{A_0} f(x, y, z, p, q)dxdy$$

of the previous article.

Assuming that $z = z(x)$ is a minimizing admissible hypersurface for the integral I, we let $\zeta(x)$ be any function of the x's with properties (i), (ii), and such that if a is a real parameter sufficiently small numerically, then $z = z(x) + a\zeta(x)$ is admissible. Let L_a' denote the $(n-1)$-dimensional manifold

$$\phi(x, z(x) + a\zeta(x)) = 0, \quad z = z(x) + a\zeta(x)$$

when x is sufficiently near the x's that determine points of S_0^*; let L_a denote the projection on X space of L_a'; and designate by S_a the hyperarea in the X space that is bounded by L_a. Then in place of the integral (1.1), we have $I(a)$:

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* So near that the correspondence between points x and points uv in §2 is $(1,1)$, reversible.

$$(1.2) \qquad I(a) = \int_{S_a}^n f(x, z + a\zeta, p + a\zeta_x)dx.$$

Our main problem is to obtain the first and second derivatives $I'(0)$ and $I''(0)$, which are analogous to corresponding integrals of the previous article. We assume that $f \neq 0$ on the hypersurface $z = z(x)$ along its intersection L_0' with the fixed hypersurface $\phi(x, z) = 0$ for a reason analogous to that which made a similar assumption desirable in the previous article.

In §2, we set up a normal curvilinear coordinate system which plays an important rôle in later sections of this paper; in §3, the equations of Rodriguez* are generalized and the result is used to obtain a simple expansion of a functional determinant of §2; in §4, Theorem 1 of the previous article is generalized to the case of an n-tuple integral; in §5, the results relative to the first variation in the previous article are generalized; in §6, two expressions for $I''(0)$ are given; in §§7 and 8, the boundary value problem and the discussion of the minimal surface, respectively, of the previous article are generalized; and the object of §9 is as we stated above.

We wish to thank Professor L. P. Eisenhart for numerous suggestions that he has given relative to §§1, 2 of this paper.

2. The normal coordinate system.† Let L_0 be a simply closed $(n-1)$-dimensional manifold with equations

$$x_i = \xi_i(u) \qquad (i = 1, \cdots, n),$$

where u is $(n-1)$-partite and the ξ's are defined for all real values of the u's, are of class C'', have

$$\sum_{i=1}^n \xi_{iu_j}^2 \neq 0 \qquad (j = 1, \cdots, n-1),$$

where the subscript u_j indicates partial differentiation of the ξ_i with respect to u_j (similar subscript notation is used throughout the sequel) and each ξ_i has in u_j a period, say t_j, which is passed through once (exactly) when a point x passes once-around the u_j-curve on L_0. We agree further, as stated above, to take the lines of curvature on L_0 as our parameter lines, so that the u-curves on L_0 are mutually orthogonal.‡

We now introduce near L_0 a uv-coordinate system determined by the equations

$$(2.1) \qquad x_i = \xi_i(u) + vA_i \qquad (i = 1, \cdots, n), \ 0 \leq u_j \leq t_j, \ v_1 \leq v \leq v_2,$$

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* Cf. Eisenhart's *Differential Geometry*, p. 122.

† Some of the ideas of this section are also expressed in §6 of Powell's thesis, which was referred to in §1.

‡ Cf., for example, Bates, loc. cit., p. 25, Theorem 1.

where

$$(2.2) \qquad A_i = (-1)^{i-1} \frac{\partial(\xi_1, \cdots, \xi_{i-1}, \xi_{i+1}, \cdots, \xi_n)}{\partial(u_1, \cdots, u_{n-1})};$$

$\sum_{i=1}^{\infty} A_i^2 = 1$, a condition that can be realized by a suitable choice of parameters; v is one-partite; and $v_1 < 0$, $v_2 > 0$ are sufficiently small numerically that there exist unique functions

$$u_j = U_j(x_1, \cdots, x_n), \quad v = V(x_1, \cdots, x_n)$$

of class C' satisfying equations (2.1). This is possible* since for x on L_0, $(u, v) = (u, 0)$ and the functional determinant

$$\Delta(v)\big|^{v=0} = \begin{vmatrix} \xi_{1u_1} + vA_{1u_1} & , \cdots , & \xi_{nu_1} + vA_{nu_1} \\ \cdot & \cdot & \cdot & \cdot \\ \xi_{1u_{n-1}} + vA_{1u_{n-1}}, & \cdots & \xi_{nu_{n-1}} + vA_{nu_{n-1}} \\ A_1 & , \cdots , & A_n \end{vmatrix}^{v=0}$$

$$(2.3)$$

$$= \begin{vmatrix} \xi_{1u_1}, & \cdots , & \xi_{nu_1} \\ \cdot & \cdot & \cdot \\ \xi_{1u_{n-1}}, & \cdots , & \xi_{nu_{n-1}} \\ A_1, & \cdots , & A_n \end{vmatrix} = (-1)^{n-1} \sum_{i=1}^{n} A_i^2 = (-1)^{n-1} \neq 0.$$

We call our coordinate system *normal* because the $(n-1)$ lines of curvature u_j are mutually orthogonal and, at every point P of L_0, v measures along the unique normal in the X space to L_0 through P for all values of $n \geq 2$.

After obtaining, in §3, a simple expansion of the determinant $\Delta(v)$, we shall employ, in §4, the coordinate system of this section in differentiating an n-tuple integral with respect to a parameter.

3. **Use of generalized equations of Rodriguez†in expanding $\Delta(v)$.** The generalization of the equations of Rodriguez may be obtained from the first set of equations that Bates displays on page 24 of his article referred to above. In these equations we let n be the $(n-1)$, number of u's, of the present paper, and we take his x_j, ζ_j^i as our u_j, A_{iu_j}, respectively. Since we are taking the lines u_j to be lines of curvature (mutually orthogonal), we thus obtain the following generalized equations of Rodriguez:

$$(3.1) \qquad\qquad \rho_j A_{iu_j} = \xi_{iu_j} \qquad\qquad (i = 1, \cdots, n),$$

where ρ_j is, except possibly for sign, the radius of curvature of the u_j-line of

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* Cf. G. A. Bliss, Princeton Colloquium Lectures, p. 20.

† Cf. Eisenhart's *Differential Geometry*, p. 122.

curvature. As a consequence of (3.1), the determinant $\Delta(v)$ in (2.3) may be written

$$
\begin{vmatrix}
\xi_{1u_1}(1+v/\rho_1) & , \cdots , & \xi_{nu_1}(1+v/\rho_1) \\
\cdot & \cdot \quad \cdot \quad \cdot & \cdot \\
\xi_{1u_{n-1}}(1+v/\rho_{n-1}), & \cdots , & \xi_{nu_{n-1}}(1+v/\rho_{n-1}) \\
A_1 & , \cdots , & A_n
\end{vmatrix}
$$

$$
= \pm \, (1+v/\rho_1) \cdots (1+v/\rho_{n-1}) \sum_{i=1}^{n} A_i^2 \; ;
$$

or

(3.2) $$\Delta(v) = \pm \, (1 + \pi_1 v + \cdots + \pi_{n-1} v^{n-1}),^{*}$$

where π_t is the elementary symmetric function of tth order of the ρ_j^{-1} $(j = 1, \cdots, n-1)$.

4. **The derivatives of an n-tuple integral with respect to a parameter.** Consider a family of $(n-1)$-dimensional manifolds, one of which, L_a (cf. the equation of L_0 in §2), is given by the equations

(4.1) $$x_i = \xi_i + v(u, a)A_i \qquad (i = 1, \cdots, n),$$

where $v(u, a)$ is defined and of class C'' for all (u, a) having each u_j as it was defined in §2 and a sufficiently near zero $(v_1 \leqq v(u, a) \leqq v_2)$, where $v(u, a)$ has for each u_j a period $T_j(a)$ that reduces to $T_j(0) = t_j$ for $a = 0$; and $v(u, 0) \equiv 0$. All of the manifolds L_a are closed on account of this periodicity and each L_a is also simply closed for all values of a sufficiently near zero since L_0 is simply closed. We let S_a denote the hyperarea bounded by L_a with the special understanding that when a satisfies the equation $v(u, a) = v_1$, we are to designate S_a and L_a as S_1 and L_1, respectively.

Let $g(x, a)$ be a function of a and the $x_i(i = 1, \cdots, n)$ which is of class C'' for all sets (x, a) having x in a sufficiently small neighborhood of the hyperarea S_0 bounded by L_0 and having a such that $v_1 \leqq v(u, a) \leqq v_2$. Define $J(a)$ by the formula

(4.2) $$J(a) = \int_{S_a}^{n} g(x, a)dx.$$

We desire the derivatives $J'(0)$ and $J''(0)$. To obtain them, we first express the integral (4.2) as a sum of two integrals:

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* The relative simplicity of this expansion may be observed by comparing it with the one that results if $\Delta(v)$, in (2.3), is expanded by minors in the notation that Bates used in a similar expansion (cf. Bates, loc. cit., equation (34)).

$$(4.3) \qquad J(a) = \int_{S_1}^{n} g(x, a)dx + \int_{\Delta S}^{n} g(x, a)dx,$$

where S_1 is as it was defined in §2 and ΔS is the hyperarea in the X space that is bounded by the $(n-1)$-dimensional manifolds L_1 and L_a. The derivative of the first integral in (4.3) has the value

$$(4.4) \qquad \int_{S_1}^{n} g_a(x, a)dx.$$

To differentiate the last integral in (4.3), we first transform it to the uv-coordinate system by means of (2.1). *Letting $\Delta(v)^{+}$ stand for the value of $\Delta(v)$ when the $+$ sign is used before the parenthesis in (3.2),* we find

$$(4.5) \qquad \int_{\Delta S}^{n} g(x, a)dx = \int_{L_0}^{n-1} \left[\int_{v_1}^{v(u, a)} g(\xi + vA, a)\Delta(v)^{+}dv \right] du,$$

where $\xi+vA$ stands for the n expressions $\xi_1+vA_1, \cdots, \xi_n+vA_n$. Since a occurs only in the upper limit of the inner integral of (4.5) and explicitly in g, we find the derivative in question to be

$$\int_{L_0}^{n-1} g v_a \Delta(v)^{+}du + \int_{L_0}^{n-1} \left[\int_{v_1}^{v(u, a)} g_a \Delta(v)^{+}dv \right] du,$$

where in the first integral $v = v(u, a)$. Adding this result to the expression (4.4), we obtain

$$J'(a) = \int_{S_1}^{n} g_a dx + \int_{L_0}^{n-1} \left[\int_{v_1}^{v(u,a)} g_a \Delta(v)^{+}dv \right] du$$
$$+ \int_{L_0}^{n-1} v_a g(\xi + vA, a)\Delta(v)^{+}du.$$

Hence, after transforming the second integral to x-coordinates, we find

$$(4.6) \qquad J'(a) = \int_{S_a}^{n} g_a dx + \int_{L_0}^{n-1} v_a g(\xi + vA, a)\Delta(v)^{+}du.$$

From the above procedure (perhaps with reference to the previous article), we now find without difficulty that

$$(4.7) \quad J''(a) = \int_{S_a}^{n} g_{aa}dx + \int_{L_0}^{n-1} \left[(g v_{aa} + 2 g_a v_a + g_v v_a^2)\Delta(v)^{+} + g v_a \Delta_a(v)^{+} \right]du.$$

Putting $a=0$ in (4.6) and (4.7) and recalling that $v(u, 0)\equiv 0$, we find the desired results, which we express as follows.

THEOREM 4.1. *The derivatives $J'(0)$ and $J''(0)$ of the n-tuple integral $J(a)$ defined by* (4.2), *taken over the n-dimensional region S_a bounded by the manifold L_a, defined by equations* (4.1), *have the values*

(4.8)
$$J'(0) = \int_{S_0}^n g_a dx + \int_{L_0}^{n-1} g v_a du,$$

$$J''(0) = \int_{S_0}^n g_{aa} dx + \int_{L_0}^{n-1} (g v_{aa} + \pi_1 g v_a^2 + 2 g_a v_a + g_v v_a^2) du.*$$

The derivatives (4.8) have been computed for the family of variations (4.1) of L_0. However we can obtain from (4.8) analogous formulas for a more general family of variations of L_0 of the form

(4.9) $x_i = X_i(\tau, a)$ $(i = 1, \cdots, n)$,

where τ is $(n-1)$-partite and $\tau_j = \tau_j(u_1, \cdots, u_{n-1}, a)$ $(j = 1, \cdots, n-1)$, with $\tau_j(u_1, \cdots, u_{n-1}, 0) = u_j$. We suppose that (4.9) represents a one-parameter family of simply closed $(n-1)$-dimensional manifolds containing L_0 for $a = 0$. The functions X_i are supposed to be of class C'' for all values of (τ, a) having each τ_j real and a sufficiently near zero. They have a period $\Gamma_j(a)$ for every a that we consider, with $\Gamma_j(0) = t_j$. Such a family is representable in the form (4.1) if we can solve the equations

(4.10) $X_i(\tau, a) - \xi_i(u) - v A_i(u) = 0$

for v and the τ_j as functions of a and the u_j. According to the implicit function theorem used in §2, this can be done since the equations (4.10) have the particular solution $(v, \tau, u, a) = (0, u, u, 0)$ for $0 \leqq u_j \leqq t_j$, on which the functional determinant

$$\begin{vmatrix} X_{1\tau_1}, & \cdots, & X_{1\tau_{n-1}}, & X_{1v} \\ \cdot & \cdot & \cdot & \cdot \\ X_{n\tau_1}, & \cdots, & X_{n\tau_{n-1}}, & X_{nv} \end{vmatrix} \begin{vmatrix} \xi_{1u_1}, & \cdots, & \xi_{1u_{n-1}}, & -A_1 \\ \cdot & \cdot & \cdot & \cdot \\ \xi_{nu_1}, & \cdots, & \xi_{nu_{n-1}}, & -A_n \end{vmatrix} = -\Delta(0) = \mp 1 \neq 0.$$

Hence we can obtain v_a and v_{aa} for the general family (4.9). Differentiating (4.10) once, twice, and agreeing that a term in which j appears as a repeated index (even though it be a subscript of a subscript) is to be summed for all integral values of j from 1 to $(n-1)$, and setting $a = 0$, we obtain (4.11), (4.12), respectively,

(4.11) $X_{i\tau_j}\tau_{ja} - A_i v_a + X_{ia} = 0,$

(4.12) $X_{i\tau_j}\tau_{jaa} - A_i v_{aa} + X_{iaa} + 2X_{ia\tau_j}\tau_{ja} + X_{i\tau_j\tau_k}\tau_{ja}\tau_{ka} = 0.$

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* In the second of equations (9) of the previous article there is a misprint. The last term of the integrand of the line integral of $J''(0)$ there should be $g_v v_a^2$.

The determinant of the n equations (4.11) in the τ_{ja} and v_a, like that of the n equations (4.12) in the τ_{jaa} and v_{aa}, is $-\Delta(0) = \mp 1$. After a sense is assigned to L_0, so as to give $\Delta(0)$ a definite value (cf. (2.2)), say $+1$, equations (4.11) define v_a as a polynomial in the X_{ia} and the $X_{i\tau_j}$ ($= \xi_{iu_j}$ on L_0), while (4.12) define v_{aa} similarly in the X_{iaa}, $X_{ia\tau_j}$, $X_{i\tau_j\tau_k}$, $X_{i\tau_j}$, and τ_{ja}. But the τ_{ja} in v_{aa} can be eliminated by means of (4.11); the τ_{ja} are then polynomials in the X_{ia}, $X_{i\tau_j}$ since the A_i are polynomials in the $X_{i\tau_j}$ (cf. the functional determinant last displayed above). Hence we have the following corollary of Theorem 4.1.

COROLLARY. *The derivatives $J'(0)$ and $J''(0)$ in equations (4.8) can be generalized to the case where (4.1) is replaced by (4.9). When this is done, v_a is a polynomial in the X_{ia} and the $X_{i\tau_j}$, while v_{aa} is a polynomial in the X_{ia}, $X_{i\tau_j}$, X_{iaa}, $X_{ia\tau_j}$, $X_{i\tau_j\tau_k}$.*

5. **The first variation.** In the sequel, any term that contains a repeated index other than a, v, z, *however it may appear*, is to be summed for all integral values of the index from 1 to n. Thus we write $f_{p_i}(\phi_{z_i} + p_i\phi_z)$ for the sum of the n terms that one obtains from this expression by taking $i = 1, \cdots, n$. Further, when we use Kronecker δ's with $\delta_i{}^k = 1$ or 0 according as $k = i$ or $k \neq i$, respectively, we extend customary convention by admitting subscripts of subscripts as summation indices; thus we would write $\zeta_{z_k}\delta_i{}^k = \zeta_{z_i}$.

Proceeding now as we did in §3 of the previous article, we find without difficulty the following equations, of which we number only those that are to be referred to later:

$$I'(0) = \int_{S_0}^n f_a dx + \int_{L_0}^{n-1} f v_a du, \quad f_a = f_z\zeta + f_{p_i}\zeta_{z_i};$$

$$\phi(\xi + vA, z(\xi + vA) + a\zeta(\xi + vA)) = 0,$$

$\xi + v(u, a)A$ being as in (4.5), so that ϕ contains the variables u, v, a;

(5.1) $v_a = -\phi_a/\phi_v$ $(\phi_v \neq 0, \text{ cf. } (5.8))$;

(5.2) $\phi_a = \phi_z\zeta, \quad \phi_v = (\phi_{z_i} + p_i\phi_z)A_i$;

(5.3) $v_a = -\phi_z\zeta/(\phi_{z_i} + p_i\phi_z)A_i$;

(5.4) $I'(0) = \int_{S_0}^n (f_z\zeta + f_{p_i}\zeta_{z_i})dx + \int_{L_0}^{n-1} \dfrac{-\zeta f\phi_z du}{(\phi_{z_i} + p_i\phi_z)A_i}.$

Hence we have the following theorem:

THEOREM 5.1. *The first derivative $I'(0)$ of the n-tuple integral (1.1), taken over the portion of the hypersurface $z = z(x) + a\zeta(x)$ bounded by its intersection with the hypersurface $\phi(x, z) = 0$, has the value given by (5.4).*

From the point of view of the calculus of variations it is desirable to perform an integration by parts on the terms $f_{p_i}\zeta_{x_i}$ of (5.4). Since

$$f_{p_i}\zeta_{x_i} = \frac{\partial}{\partial x_i}(f_{p_i}\zeta) - \zeta\frac{\partial}{\partial x_i}f_{p_i},$$

we can replace the n-tuple integral in (5.4) by

$$\int_{S_0}^n \left[\zeta\left(f_z - \frac{\partial}{\partial x_i}f_{p_i}\right) + \frac{\partial}{\partial x_i}f_{p_i}\zeta\right]dx$$

$$= \int_{S_0}^n \zeta\left(f_z - \frac{\partial}{\partial x_i}f_{p_i}\right)dx + \int_{L_0}^{n-1}\zeta f_{p_i}A_i du.*$$

Using this result in (5.4), we obtain

$$(5.5)\quad I'(0) = \int_{S_0}^n \zeta\left(f_z - \frac{\partial}{\partial x_i}f_{p_i}\right)dx + \int_{L_0}^{n-1}\zeta\left(\frac{\phi_v A_i f_{p_i} - f\phi_z}{\phi_v}\right)du.$$

Since, along L_0', $\phi(\xi_1(u), \cdots, \xi_n(u), z(\xi_1, \cdots, \xi_n))\equiv 0$ in the u_j, we can replace (5.5) by an equivalent equation analogous to equation (32) of the previous article. Differentiating this identity, $\phi\equiv 0$, with respect to each u_j, we obtain the $(n-1)$ equations

$$(5.6)\qquad (\phi_{x_i} + p_i\phi_z)\xi_{iu_j} = 0 \qquad (j = 1, \cdots, n-1).$$

Hence with (5.6) and the last equation in (5.2), we have n equations which determine the $\phi_{x_i}+p_i\phi_z$:

$$(5.7)\qquad \phi_{x_i} + p_i\phi_z = A_i\phi_v \text{ (cf. (2.2))},$$

where

$$(5.8)\qquad \phi_v^2 = \sum_{i=1}^n(\phi_{x_i} + p_i\phi_z)^2 \neq 0.$$

That $\phi_v^2\neq 0$ may be proved as follows. Suppose $\phi_v^2=0$, so that

$$\phi_{x_i} + p_i\phi_z = 0 \qquad (i = 1, \cdots, n).$$

Then the hypersurfaces $z=z(x)$ and $\phi(x, z)=0$ are tangent to each other, which is impossible when $f\neq 0$ along L_0', as we shall see just below Corollary 5.2. In view of (5.7), (5.5) is equivalent to

$$(5.9)\quad I'(0) = \int_{S_0}^n \zeta\left(f_z - \frac{\partial}{\partial x_i}f_{p_i}\right)dx + \int_{L_0}^{n-1}\zeta\frac{[f_{p_i}(\phi_{x_i} + p_i\phi_z) - f\phi_z]du}{(\phi_{x_i} + p_i\phi_z)A_i}.$$

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* In obtaining this term we have used the extended Green's theorem referred to above with the A_i as direction cosines of the outer normal to L_0.

The Euler necessary condition for a minimum value of I in the case of fixed limits (where the hypersurface $\phi(x, z) = 0$ is replaced by a bounded, closed, connected $(n-1)$-dimensional manifold, such as L_0) is

$$(5.10) \qquad f_z - \frac{\partial}{\partial x_i} f_{p_i} = 0^*$$

at every point of S_0. This is surely a necessary condition for the case of variable limits. Since $I'(0) = 0$ is a necessary condition for a minimum value of I, it now follows that if $z = z(x)$ minimizes I the second integral in (5.9) vanishes, and indeed that the numerator, N, of the integrand of this integral is zero at every point of L_0, $0 \leq u_j \leq t_j$, as we presently prove. Suppose N does not so vanish. Either N has one sign on the entire manifold L_0 or there is a non-zero $(n-1)$-dimensional subregion of L_0 on which N has one sign. We may take $\zeta(u_1, \cdots, u_{n-1})$ to be of the same sign as N on one such subregion and zero elsewhere. Then since $\phi_v \neq 0$, $I'(0) \neq 0$ (contradiction). Hence we have the transversality condition

$$(5.11) \qquad f_{p_i}(\phi_{x_i} + p_i \phi_z) - f \phi_z = 0$$

at every point of L_0.

We now have the following corollaries of Theorem 5.1.

COROLLARY 5.1. *The first derivative of the n-tuple integral $I(a)$, of (1.2), taken over the portion of the hypersurface $z = z(x) + a\zeta(x)$ bounded by its intersection with the fixed hypersurface $\phi(x, z) = 0$, has the value given by (5.9).*

COROLLARY 5.2. *In case $z = z(x)$ is a minimizing hypersurface for the n-tuple integral (1.1), the Euler equation (5.10) must hold at every point of the portion of the hypersurface $z = z(x)$ inside L_0', and the transversality condition (5.11) must hold at every point of the boundary, L_0', which is the manifold of intersection of the hypersurfaces $z = z(x)$ and $\phi(x, z) = 0$.*

Since we have assumed in §1 that $f \neq 0$ along L_0', it follows from (5.11) that the hypersurface $z = z(x)$ is not tangent to the hypersurface $\phi(x, z) = 0$ at any point of L_0'. In the case of the minimal hypersurface for which $f = (1 + p_i p_i)^{1/2}$, (5.11) reduces to $p_i \phi_{x_i} - \phi_z = 0$, which shows that the hypersurfaces $z = z(x)$ and $\phi(x, z) = 0$ meet at right angles.

6. **The second variation.** To get $I''(0)$, we apply to (1.2) the result in the second of equations (4.8). Replacing g in that equation by f, we obtain

$$(6.1) \qquad I''(0) = \int_{S_0}^{n} f_{aa} dx + \int_{L_0}^{n-1} M \, du,$$

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* Cf. page 5 of Powell's thesis, loc. cit.

where $M \equiv f v_{aa} + f \pi_1 v_a{}^2 + f_v v_a{}^2 + 2 f_a v_a$ and

(6.2) $f_{aa} = f_{zz} \zeta^2 + 2 f_{z p_i} \zeta \zeta x_i + f_{p_i p_j} \zeta x_i \zeta x_j.$

By differentiating the equation $\phi_v v_a + \phi_z = 0$ (cf. (5.1)), with respect to a, we find, as in the previous article, that

(6.3) $v_{aa} = -\dfrac{1}{\phi_v} (\phi_{vv} v_a{}^2 + 2\phi_{av} v_a + \phi_{aa}),$

where ϕ_v is defined in (5.2), and ϕ_{vv}, ϕ_{av}, ϕ_{aa}, at $a = 0$, are obtained by differentiating ϕ as a function of the arguments

$$\xi_i + v A_i \ (i = 1, \cdots, n), \quad z(\xi + vA) + a \zeta (\xi + vA),$$

$\xi + vA$ standing for the set of n expressions $\xi_i + vA_i$. We find, for $a = 0$,

$$\phi_{vv} = (\phi_{x_i x_j} + 2\phi_{x_i z} p_j + \phi_{zz} p_i p_j + \phi_z r_{ij}) A_i A_j, \quad r_{ij} \equiv z_{x_i x_j};$$
$$\phi_{av} = (\phi_{x_i z} \zeta + \phi_{zz} \zeta p_i + \phi_z \zeta_{x_i}) A_i; \quad \phi_{aa} = \phi_{zz} \zeta^2.$$

Using these derivatives together with (5.2) and (5.3), we find (cf. (6.3))

(6.4)
$$v_{aa} = -\frac{1}{\phi_v{}^3} \{ \phi_z{}^2 \zeta^2 [\phi_{x_i x_j} + 2\phi_{x_i z} p_j + \phi_{zz} p_i p_j + \phi_z r_{ij}]$$
$$- 2\phi_z \zeta [\zeta(\phi_{x_i z} + p_i \phi_{zz})(\phi_{x_j} + p_j \phi_z)$$
$$+ \phi_z(\phi_{x_j} + p_j \phi_z)\zeta_{x_i}] + \phi_{zz} \zeta^2 (\phi_{x_i} + p_i \phi_z)(\phi_{x_j} + p_j \phi_z) \} A_i A_j.$$

Hence if we collect in (6.4) the terms involving the second derivatives of ϕ, those involving the second derivatives of z, and those free of second derivatives, we find

(6.5)
$$v_{aa} = -\frac{\zeta^2}{\phi_v{}^3} (\phi_z{}^2 \phi_{x_i x_j} - 2\phi_z \phi_{x_j} \phi_{x_i z} + \phi_{zz} \phi_{x_i} \phi_{x_j}) A_i A_j - \frac{\phi_z{}^3 \zeta^2}{\phi_v{}^3} \cdot r_{ij} A_i A_j$$
$$+ \frac{2 \zeta \zeta_{x_i}}{\phi_v{}^3} (\phi_z{}^2 \phi_{x_j} + \phi_z{}^3 p_j) A_i A_j.$$

Now using the notation

$$q_i = -\phi_{x_i}/\phi_z, \quad \Delta = [(p_i - q_i)(p_i - q_i)]^{1/2},$$
$$s_{ij} = -\frac{\phi_z{}^2 \phi_{x_i x_j} - 2\phi_z \phi_{x_j} \phi_{x_i z} + \phi_{zz} \phi_{x_i} \phi_{x_j}}{\phi_z{}^3},$$

so that $(\partial q_i/\partial x_j) = (s_{ij} + s_{ji})/2$, we find from (6.5) that

(6.6) $v_{aa} = \dfrac{1}{\Delta^3} [\zeta^2(s_{ij} - r_{ij}) + 2\zeta \zeta_{x_i}(p_j - q_j)] A_i A_j.$

The introduction of the q_i and the s_{ij} (whose denominators involve $\phi_z{}^3$) does not require that $\phi(x, z) = 0$ be representable in the form $z = z_1(x)$ (cf. the $(n-1)$-tuple integral in (6.9), in which $\Delta \neq 0$ since $0 \neq \phi_v = \phi_z \Delta$).

The other three terms of M are

$$f v_a{}^2 \pi_1 = \frac{f \zeta^2}{\Delta^2} \left(\frac{1}{\rho_1} + \cdots + \frac{1}{\rho_{n-1}} \right),$$

(6.7)

$$f_v v_a{}^2 = \frac{\zeta^2}{\Delta^3} (f)_{z_i} (p_i - q_i),$$

$$2 f_a v_a = - (2\zeta/\Delta)(f_z \zeta + f_{p_i} \zeta_{z_i}),$$

where

(6.8) $$(f)_{z_i} = f_{z_i} + f_z p_i + f_{p_j} r_{ij}.$$

Collecting the terms of M, as they are given by (6.6), (6.7), (6.8), and using the value (6.2) of f_{aa}, we obtain the following theorem.

THEOREM 6.1. *The second derivative $I''(0)$ of the n-tuple integral $I(a)$ of equation (1.2), taken over the portion of the hypersurface $z = z(x) + a\zeta(x)$ bounded by its intersection with the hypersurface $\phi(x, z) = 0$, has the value*

(6.9) $$I''(0) = \int_{S_0}^{n} 2\Omega \, dx + \int_{L_0}^{n-1} (\zeta/\Delta^3)(B\zeta + C_i \zeta_{z_i}) du,$$

where

$$2\Omega \equiv f_{zz} \zeta^2 + 2 f_{zp_i} \zeta \zeta_{z_i} + f_{p_i p_j} \zeta_{z_i} \zeta_{z_j},$$

(6.10) $$B \equiv f(s_{ij} - r_{ij}) A_i A_j + f \Delta \pi_1 + (f)_{z_i}(p_i - q_i) - 2 f_z \Delta^2,$$

$$C_i \equiv 2 [f(p_i - q_i) A_j A_i - f_{p_i} \Delta^2].$$

7. A boundary value problem associated with the second variation. We generalize the boundary value problem of the previous article. By Euler's theorem on homogeneous functions, the n-tuple integral of (6.9) can be written in the form

$$\int_{S_0}^{n} (\zeta \Omega_\zeta + \zeta_{z_i} \Omega_i) dx; \qquad \Omega_i \equiv \Omega_{\zeta_{z_i}}.$$

Then, after performing a customary integration by parts, we find

$$\int_{S_0}^{n} 2\Omega \, dx = \int_{S_0}^{n} \left[\zeta \psi(\zeta) + \frac{\partial}{\partial x_i} \zeta \Omega_i \right] dx, \quad \psi(\zeta) \equiv \Omega_\zeta - \frac{\partial}{\partial x_i} \Omega_i.$$

Applying the extended Green's theorem heretofore used, we now find

$$(7.1) \qquad \int_{S_0}^{n} 2\Omega dx = \int_{S_0}^{n} \zeta\psi(\zeta)dx + \int_{L_0}^{n-1} \zeta A_i\Omega_i du.$$

From (6.9) and (7.1), we now obtain

$$I''(0) = \int_{S_0}^{n} \zeta\psi(\zeta)dx + \int_{L_0}^{n-1} \zeta(B\zeta + C_i\zeta_{x_i} + A_i\Omega_i)du;$$

or since

$$\Omega_i = \zeta f_{x p_i} + \zeta_{x_j} f_{p_i p_j},$$

we have

$$(7.2) \qquad I''(0) = \int_{S_0}^{n} \zeta\psi(\zeta)dx + \int_{L_0}^{n-1} \zeta(D\zeta + E_i\zeta_{x_i})du,$$

where

$$(7.3) \qquad D \equiv B + A_i f_{x p_i}, \quad E_i \equiv C_i + A_j f_{p_j p_i} \quad (\text{cf. } (6.10)).$$

From (7.2) we can now state a new necessary condition in order that the hypersurface $z = z(x)$ shall minimize the n-tuple integral (1.1).

THEOREM 7.1. *In order that the hypersurface $z = z(x)$ shall minimize the n-tuple integral (1.1), it is necessary that for negative values of λ the boundary value problem*

$$\psi(\zeta) - \lambda\zeta = 0 \text{ in the region } S_0,$$
$$D\zeta + E_i\zeta_{x_i} = 0 \text{ on the boundary } L_0 \text{ of } S_0$$

have no solution except $\zeta \equiv 0$, D and the E_i being defined through (6.10) and (7.3).

8. The minimal hypersurface. Here we define $f = (1 + p_i p_i)^{1/2}$. We shall compute the $I''(0)$ of (6.9) for the present case. Since $f_z = 0$, the only derivatives needed here are

$$f_{p_i} = p_i/f, \quad f_{p_i p_j} = (f^2\delta_i^j - p_i p_j)/f^3, \quad (f)_{x_i} = p_j r_{ij}/f \quad (\text{cf. } (6.8)),$$
$$f_{aa} = 2\Omega = \zeta\Omega_\zeta + f_{p_i p_j}\zeta_{x_i}\zeta_{x_j} = \zeta_{x_i}\zeta_{x_j}(f^2\delta_i^j - p_i p_j)/f^3.$$

One now finds that the B, C_i of (6.10) reduce to B', C_i', respectively, where

$$(8.1) \qquad
\begin{aligned}
B' &= f(s_{ij} - r_{ij})A_i A_j + f\left(\frac{1}{\rho_1} + \cdots + \frac{1}{\rho_{n-1}}\right)\Delta + p_j r_{ij}(p_i - q_i)/f, \\
C_i' &= (2/f)[f^2(p_j - q_j)A_i A_i - p_i\Delta^2].
\end{aligned}$$

Hence we have the following corollary of Theorem 6.1.

COROLLARY 8.1. *In the case of the minimal hypersurface, the $I''(0)$ of Theorem 6.1 reduces to*

$$I''(0) = \int_{S_0}^{n} (\zeta_{x_i}\zeta_{x_j}/f^3)(f^2\delta_i{}^j - p_i p_j)dx + \int_{L_0}^{n-1} \zeta(B'\zeta + C_i'\zeta_{x_i})du,$$

where B' and the C_i' are as defined in (8.1).

To make a similar specialization of Theorem 7.1, we observe that in the present case

$$\psi(\zeta) = -\frac{\partial}{\partial x_i}\Omega_i = -\frac{\partial}{\partial x_i}\zeta_{x_j}(f^2\delta_i{}^j - p_i p_j)/f^3,$$

(8.2)

$$D = B', \text{ of } (8.1),$$

$$E_i = C_i' + A_j(f^2\delta_i{}^j - p_i p_j)/f^3 \text{ (cf. } (7.3) \text{ and } (8.1)).$$

Consequently we have the following corollary to Theorem 7.1.

COROLLARY 8.2. *In order that the hypersurface $z = z(x)$ shall minimize the n-tuple integral* (1.1) *in the case where $f = (1 + p_i p_i)^{1/2}$, it is necessary that for negative values of λ the following boundary value problem have no solution except $\zeta \equiv 0$:*

$$\frac{\partial}{\partial x_i}\Omega_i + \lambda\zeta = 0 \text{ on } S_0,$$

$$B'\zeta + E_i\zeta_{x_i} = 0 \text{ on } L_0,$$

where

$$-\frac{\partial}{\partial x_i}\Omega_i = \psi(\zeta)$$

by (8.2), *B' is defined by* (8.1), *and the E_i are given in* (8.2).

9. **Further applications of Theorem 4.1.** Since g in §4 is merely required to be a function of class C'' in the x's and the parameter a, there may be numerous applications of Theorem 4.1, even to more complicated variation problems than the problem associated with §§1–8 above. We have made one such application. We have used the first equation of (4.8) to compute the first variation of the integral

$$K = \int_{S_0}^{n} f(x, z, p, r)dx$$

where S_0, n, x, z, p have the meaning relative to K which they had for I in §§1–8; r is the set of all of the derivatives

$$\frac{\partial p_i}{\partial x_j} = r_{ij} \qquad\qquad (i, j = 1, \cdots, n) ;$$

and f is supposed to have suitable continuity in a region W of the space $XZPR$ in which a minimum value of K is desired. A fixed hypersurface $\phi(x, z) = 0$, with suitable continuity, is employed as in §§1–8.

We state without proof that the analogs of (5.4) and (5.9) here are (9.1) and (9.2) below, respectively:

$$(9.1) \quad K'(0) = \int_{S_0}^{n} (f_z \zeta + f_{p_i} \zeta_{x_i} + f_{r_{ij}} \zeta_{x_i x_j}) dx - \int_{L_0}^{n-1} \frac{f \zeta \phi_z du}{(\phi_{x_i} + p_i \phi_z) A_i} ;$$

$$(9.2) \quad K'(0) = \int_{S_0}^{n} \zeta \left(f_z - \frac{\partial}{\partial x_i} f_{p_i} + \frac{\partial^2 f_{r_{ij}}}{\partial x_i \partial x_j} \right) dx$$

$$+ \int_{L_0}^{n-1} \left[(f_{p_i} \zeta + f_{r_{ij}} \zeta_{x_j})(\phi_{x_i} + p_i \phi_z) - \zeta(\phi_{x_j} + p_j \phi_z) \frac{\partial}{\partial x_i} f_{r_{ij}} \right.$$

$$\left. - \phi_z \zeta f \right] \cdot \frac{du}{(\phi_{x_i} + p_i \phi_z) A_i} .$$

By methods that were used in preceding sections, one could compute $K''(0)$.

NORTHWESTERN UNIVERSITY,
    EVANSTON, ILL.