A PROJECTIVE GENERALIZATION OF METRICALLY DEFINED ASSOCIATE SURFACES*

BY

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1. Introduction

In the metric differential geometry of surfaces in ordinary space, two surfaces are said by Bianchi to be *associate*† if the tangent planes at corresponding points are parallel and if the asymptotic curves on either surface correspond to a conjugate net on the other.

It is the purpose of this paper to develop a projective generalization of the relation of associateness of surfaces. Since associate surfaces are parallel in the metric sense, it will first be necessary to provide a projectively defined substitute for the property of metric parallelism. We shall employ as the basis of our study in this paper a projective generalization of euclidean parallelism of surfaces which the author has developed in his Chicago doctoral dissertation.

In §2, after stating a definition of projective parallelism of surfaces and briefly explaining this idea, we introduce a canonical form of our system of differential equations employed in the study of projectively parallel surfaces in ordinary space. In §3 we formulate a definition of projectively associate surfaces and investigate to some extent their properties and relations. A more general type of associateness which may be conveniently termed ***modified projective associateness*** is introduced in §4, and a somewhat different canonical form of our system of differential equations is employed in its study. Finally, in §5, we consider a rather general completely integrable system of partial differential equations, namely, the system for two surfaces in the general analytic one-to-one point correspondence in ordinary projective space S_3, and a group of transformations that leaves this configuration invariant. We then reduce this system of equations to a new canonical form, and employ it to continue briefly the study of modified projective associateness introduced in the preceding section.

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† Eisenhart, *Differential Geometry*, Ginn and Company, 1909, p. 378. Hereinafter cited as Eisenhart. See also Bianchi, *Lezioni di Geometria Differenziale* (3d edition), vol. 2, p. 10.

2. Projective parallelism of surfaces

In formulating a projective generalization of metric parallelism of surfaces,* we begin by replacing the metric normal congruence by the *projective normal congruence*, and so consider two surfaces S_x, S_y, in ordinary projective space S_3, with a common projective normal congruence. The developables of this congruence intersect both surfaces in the projective lines of curvature, which form conjugate nets. We then demand, in analogy to the metric parallelism of the tangent planes, that the tangent planes at corresponding points of the two surfaces intersect on a fixed plane. Two surfaces so related are said to be *parallel in the projective sense.*

For the basis of our study of projective parallelism we employ one of the well known transformations of surfaces, namely, the fundamental transformation. Two surfaces are said to be in the relation of a fundamental transformation, or transformation F†, in case their points are in a one-to-one correspondence such that the lines joining corresponding points form a congruence whose developables intersect both surfaces in conjugate nets, neither surface being a focal surface of the congruence. The congruence is called the conjugate congruence of the transformation because it is *conjugate* to both nets. The tangent planes at corresponding points of the two surfaces intersect in the lines of the harmonic congruence of the transformation, which is *harmonic* to both nets. By choosing the projective normal congruence as the common conjugate congruence of the transformation F, we provide, as previously stated, a projective substitute for the metric normal congruence. Furthermore, we assume that the developables of the harmonic congruence are indeterminate, that is, that corresponding tangent planes of the two surfaces intersect in the lines of a fixed plane. This assumption affords us a projective substitute for the metric parallelism of the tangent planes. Our definition of projective parallelism may now be stated in the following way:

Two surfaces S_x, S_y are said to be projectively parallel in case they are in the relation of a fundamental transformation with the projective normal congruence as the conjugate congruence and with the developables of the harmonic congruence indeterminate.

In order to represent analytically the definition which we have introduced, let us consider two projectively parallel surfaces S_x, S_y with the respective parametric vector equations

$$x = x(u, v), \quad y = y(u, v).$$

* M. L. MacQueen, *A Projective Generalization of Euclidean Parallelism of Surfaces*, University of Chicago, December, 1933; unpublished doctoral dissertation. Hereinafter cited as Thesis.

† L. P. Eisenhart, *Transformations of Surfaces*, Princeton University Press, 1923, p. 34 et seq.

The four coordinates x and the four coordinates y form four pairs of solutions of a completely integrable system of differential equations of the form

$$
\begin{aligned}
x_{uu} &= px + \alpha x_u + \beta x_v + Ly, \\
x_{uv} &= cx + ax_u + bx_v, \\
x_{vv} &= qx + \gamma x_u + \delta x_v + Ny, \\
y_u &= fx + mx_u + Ay, \\
y_v &= gx + nx_v + By \qquad (mnLN \neq 0),
\end{aligned} \tag{1}
$$

where the notation here employed is similar to that used by Lane in his recent book.*

Before stating the conditions which characterize this system we remark that in order to treat S_x, S_y in a symmetrical manner we see that x, y satisfy a system of equations of the form (1), but with the roles of x and y interchanged. The coefficients of such a system will be indicated by dashes and will be given later. In order that S_y may be non-developable we shall suppose that $\overline{L}\overline{N} \neq 0$.

System (1) is characterized analytically by the following conditions:

$$
\begin{aligned}
&\text{(a)}\ \alpha + b + A + (\log N)_u - 3(\log r)_u/2 - 2(\log R)_u = 0, \\
&\text{(b)}\ \gamma/r + \alpha + (\log r)_u/2 = 0, \\
&\text{(c)}\ \bar{r} = nr/m, \\
&\text{(d)}\ f/m = -[\log (mn)^{1/2}R/L]_u, \\
&\text{(e)}\ m(1-n)\mathfrak{B}'^2 + nr(1-m)\mathfrak{C}'^2 + m_v(\mathfrak{B}' + m_v/(4m)) \\
&\qquad\qquad + n_u r(\mathfrak{C}' + n_u/(4n)) = 0
\end{aligned} \tag{2}
$$

and by the counterpart of (a), (b), and (d) in the substitution

$$
\begin{pmatrix} u & a & c & f & m & p & s & \alpha & \beta & A & L & M & r & R \\ v & b & c & g & n & q & t & \delta & \gamma & B & N & M & 1/r & rR \end{pmatrix}. \tag{3}
$$

The invariants $\mathfrak{B}'$, $\mathfrak{C}'$, R of Green, and the invariant r of Eisenhart, appearing in equations (2), are expressed for the projective lines of curvature on S_x in terms of the coefficients of system (1) by the following formulas:

$$
\begin{aligned}
8\mathfrak{B}' &= 4a + 2N\beta/L - 2\delta + (\log N/L)_v, \\
8\mathfrak{C}' &= 4b + 2L\gamma/N - 2\alpha + (\log L/N)_u, \\
R &= L\mathfrak{B}'^2/N + \mathfrak{C}'^2, \\
r &= N/L.
\end{aligned} \tag{4}
$$

* Lane, *Projective Differential Geometry of Curves and Surfaces*, University of Chicago Press, 1932, p. 183. Hereinafter cited as Lane.

Conditions (2) (a) imply that the line $x_u x_v$ is the reciprocal of the projective normal of S_x at P_x, and conditions (b) and (a) imply that the line xy is the projective normal of S_x at P_x; condition (c) implies that the tangent planes at corresponding points of S_x, S_y intersect in the lines of a fixed plane; conditions (c) and (d) imply that the line $y_u y_v$ is the reciprocal of the projective normal of S_y at P_y, and conditions (c), (d), and (e) imply that the line xy is the projective normal of S_y at P_y.

It may be remarked that the choice of the proportionality factors which leads to our canonical form is precisely that which gives Fubini's normal coordinates.

The integrability conditions for system (1) are given by the following equations and those obtainable therefrom by the substitution (3):

$$
\begin{aligned}
&a_u + ab + c = \alpha_v + \beta\gamma,\\
&b_u + b^2 + a\beta = \beta_v + b\alpha + \beta\delta + p + nL,\\
&c_u + bc + ap = p_v + c\alpha + q\beta + gL,\\
&g_u + cn + fB = f_v + cm + gA,\\
&an + mB + g = m_v + am,\\
&aL = L_v + BL + \beta N, \qquad A_v = B_u.
\end{aligned}
\tag{5}
$$

The coefficients of the equations corresponding to (1) when the roles of x and y are interchanged are given by

$$
\begin{aligned}
\bar{p} &= A_u + mL - A(m_u/m + f/m + \alpha) - m\beta B/n,\\
\bar{\alpha} &= \alpha + f/m + m_u/m + A, \qquad \bar{\beta} = m\beta/n,\\
\bar{L} &= -\, m(\alpha f/m + \beta g/n + (f/m)^2 - p - (f/m)_u),\\
\bar{c} &= A_v - A(m_v/m + a) - B(f/n + mb/n),\\
&= B_u - B(n_u/n + b) - A(g/m + na/m),\\
\bar{a} &= a + m_v/m = B + g/m + na/m,\\
\bar{b} &= b + n_u/n = A + f/n + mb/n,\\
\bar{q} &= B_v + nN - B(n_v/n + g/n + \delta) - n\gamma A/m,\\
\bar{\gamma} &= n\gamma/m, \qquad \bar{\delta} = \delta + g/n + n_v/n + B,\\
\bar{N} &= -\, n(\delta g/n + \gamma f/m + (g/n)^2 - q - (g/n)_v),\\
\bar{f} &= -\, A/m, \qquad \bar{m} = 1/m, \qquad \bar{A} = -\, f/m,\\
\bar{g} &= -\, B/n, \qquad \bar{n} = 1/n, \qquad \bar{B} = -\, g/n.
\end{aligned}
\tag{6}
$$

The developables of the projective normal congruence intersect S_x and S_y in parametric conjugate nets which are the projective lines of curvature thereon, the foci P_η, P_ζ of a projective normal being given by

$$\eta = y - mx, \qquad \zeta = y - nx. \tag{7}$$

The differential equation of the asymptotic curves on S_x is

$$L du^2 + N dv^2 = 0, \tag{8}$$

and the asymptotic curves on S_y are given by the equation

$$\bar{L} du^2 + \bar{N} dv^2 = 0. \tag{9}$$

3. Projectively associate surfaces

The projective generalization of metric parallelism of surfaces summarized in the preceding section will now be employed in formulating a definition of projectively associate surfaces. In analogy to the metric definition of associate surfaces, *two surfaces S_x, S_y, in ordinary projective space, will be called projectively associate if they are projectively parallel and if the asymptotic curves on either surface correspond to a conjugate net on the other.*

A necessary and sufficient condition that the asymptotic curves on one of two projectively parallel surfaces S_x, S_y correspond to a conjugate net on the other is

$$L\bar{N} + \bar{L}N = 0, \tag{10}$$

i.e., the harmonic invariant of the asymptotic curves on the two surfaces vanishes. With the aid of (2) (c) this condition is seen to be equivalent to

$$m = -n. \tag{11}$$

By means of (7), condition (11) shows that P_y is the harmonic conjugate of P_x with respect to the two focal points of a projective normal. Thus we reach the following conclusion:

If two surfaces S_x, S_y are projectively parallel, a necessary and sufficient condition that they be projectively associate is that corresponding points on a projective normal separate harmonically the foci thereon.

The Laplace-Darboux point invariants, H, K, the Weingarten invariants $W^{(u)}$, $W^{(v)}$, and the tangential invariants $\mathfrak{H}$, $\mathfrak{K}$ are given for the projective lines of curvature on S_x in terms of the coefficients of system (1) by the formulas

$$\begin{aligned} H &= c + ab - a_u, \qquad K = c + ab - b_v, \\ W^{(u)} &= 2b_v + a_u - \delta_u - B_u - (\log L)_{uv}, \\ W^{(v)} &= 2a_u + b_v - \alpha_v - A_v - (\log N)_{uv}, \\ \mathfrak{H} &= K + W^{(u)} = a_u + \beta\gamma - B_u - (\log L)_{uv}, \\ &= N(\beta_u + \beta b - \beta A - \beta(\log L)_u)/L, \\ \mathfrak{K} &= H + W^{(v)} = b_v + \beta\gamma - A_v - (\log N)_{uv}, \\ &= L(\gamma_v + \gamma a - \gamma B - \gamma(\log N)_v)/N. \end{aligned} \tag{12}$$

The corresponding invariants, indicated by dashes, for the projective lines of curvature on S_y, projectively parallel to S_x, are found* to have the following expressions:

$$
\begin{aligned}
&\bar{H} = H - (\log m^3 n)_{uv}/2, &\qquad \bar{K} = K - (\log mn^3)_{uv}/2,\\
(13)\qquad &\bar{W}^{(u)} = W^{(u)} + (\log mn^3)_{uv}/2, &\qquad \bar{W}^{(v)} = W^{(v)} + (\log m^3 n)_{uv}/2\\
&\bar{\mathfrak{H}} = \mathfrak{H}, &\qquad \bar{\mathfrak{K}} = \mathfrak{K}.
\end{aligned}
$$

It is evident from (2) (d) that

$$(14)\qquad (f/m)_v = (g/n)_u.$$

By using (14) and the integrability conditions (5) a simple calculation is made which shows that in case $m=-n$ it follows that $a_u=b_v$, and the projective lines of curvature on S_x have equal point invariants. Moreover, in this case equations (13) show that the projective lines of curvature on S_y also have equal point invariants. We therefore reach the following conclusion:

If two surfaces S_x, S_y are projectively associate, the projective lines of curvature on each surface have equal point invariants.

We shall now investigate the conjugate nets on each of two projectively associate surfaces to which correspond the asymptotic curves on the other. When use is made of (10), equation (8), which defines the asymptotic curves on S_x, may be written

$$(15)\qquad \bar{L}du^2 - \bar{N}dv^2 = 0.$$

This is the differential equation of the associate conjugate net of the projective lines of curvature on S_y, that is, the conjugate net whose tangents at each point of the surface S_y separate harmonically the tangents to the projective lines of curvature.

Similarly, by use of (10), we may write equation (9) in the form

$$Ldu^2 - Ndv^2 = 0,$$

which shows that the asymptotic curves on S_y correspond to the associate conjugate net of the projective lines of curvature on S_x. We may therefore state the following theorem:

If two surfaces S_x, S_y are projectively associate, and if the parametric net on each is the projective lines of curvature, then the asymptotic curves on either surface correspond to the associate conjugate net of the parametric conjugate net on the other.

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* Thesis.

An interesting property of a conjugate net is isothermal conjugacy, the condition for which is $W^{(u)}=W^{(v)}$ or $(\log r)_{uv}=0$. Let the projective lines of curvature on S_x be an isothermally conjugate net, and let S_y be projectively associate to S_x. From (10) or (12) it is then easy to obtain the following result:

If the projective lines of curvature are isothermally conjugate on one of two projectively associate surfaces, they are also isothermally conjugate on the other.

In this case the projective lines of curvature on S_x and S_y are called J nets, since they are isothermally conjugate and have equal point invariants.

4. Modified projective associateness of surfaces

In this section we shall drop the assumption that the common conjugate congruence of the transformation F is the projective normal congruence, and shall employ in its place a general conjugate congruence. The configuration composed of two surfaces in ordinary space in the relation of a fundamental transformation having a general conjugate congruence and with the developables of the harmonic congruence indeterminate leads us to a characterization of surfaces which are projectively parallel in a modified* sense. We shall use this type of parallelism in formulating our definition of modified projectively associate surfaces.

For the analytic basis of our work a somewhat different canonical form of the basic system of differential equations is employed. If S_x, S_y are a pair of surfaces projectively parallel in the modified sense, then the four coordinates x and the four coordinates y form four pairs of solutions of a completely integrable system of differential equations* of the form

$$
(16)\qquad \begin{aligned}
x_{uu} &= L(x+y) + \alpha x_u + \beta x_v,\\
x_{uv} &= a x_u + b x_v,\\
x_{vv} &= N(x+y) + \gamma x_u + \delta x_v,\\
y_u &= m x_u, \qquad y_v = n x_v \qquad (mnLN \neq 0).
\end{aligned}
$$

The integrability conditions for this system are found to be

$$
(17)\qquad \begin{aligned}
&a_u + ab = \alpha_v + \beta\gamma, \qquad b_v + ab = \delta_u + \beta\gamma,\\
&b_u + b^2 + a\beta = \beta_v + b\alpha + nL + \beta\delta + L,\\
&a_v + a^2 + b\gamma = \gamma_u + a\delta + mN + \alpha\gamma + N,\\
&L_v = aL - \beta N, \qquad N_u = bN - \gamma L,\\
&m_v = a(n-m), \qquad n_u = b(m-n).
\end{aligned}
$$

* Thesis.

The coefficients of the equations corresponding to (16), when the roles of x and y are interchanged, are indicated by dashes and are given by the following expressions:

$$\begin{aligned} &\bar{L} = mL, && \bar{\alpha} = \alpha + m_u/m, && \bar{\beta} = m\beta/n \\ (18)\quad &\bar{a} = na/m, && \bar{b} = mb/n, && \bar{m} = 1/m, \quad \bar{n} = 1/n, \\ &\bar{N} = nN, && \bar{\gamma} = n\gamma/m, && \bar{\delta} = \delta + n_v/n. \end{aligned}$$

We shall assume that $\bar{L}\bar{N} \neq 0$ in order that S_y may be non-developable.

The focal points P_η, P_ζ of a line xy joining corresponding points P_x, P_y of two surfaces S_x, S_y are defined by

$$(19)\qquad \eta = y - mx, \qquad \zeta = y - nx.$$

Several results similar to those in the preceding section will now be given. Inasmuch as the proofs of these results run parallel to those in §3 they will be omitted in this section. Precisely as in the preceding section, a necessary and sufficient condition that the asymptotic curves on either of two modified projectively parallel surfaces S_x, S_y correspond to a conjugate net on the other is found to be given by the condition $m = -n$. Hence the following result is readily obtained.

If two surfaces S_x, S_y are projectively parallel in the modified sense, a necessary and sufficient condition that they be projectively associate in the same sense is that corresponding points P_x, P_y of each line xy separate harmonically the foci thereon.

Some of the invariants of the parametric conjugate net N_x are found to have in our notation the following formulas:

$$\begin{aligned} H &= ab - a_u, \qquad K = ab - b_v, \\ W^{(u)} &= 2b_v + a_u - \delta_u - (\log L)_{uv}, \\ W^{(v)} &= 2a_u + b_v - \alpha_v - (\log N)_{uv}, \\ \mathfrak{H} &= K + W^{(u)} = a_u + \beta\gamma - (\log L)_{uv} \\ (20)\qquad &= N(\beta_u + b\beta - \beta(\log L)_u)/L, \\ \mathfrak{K} &= H + W^{(v)} = b_v + \beta\gamma - (\log N)_{uv} \\ &= L(\gamma_v + a\gamma - \gamma(\log N)_v)/N, \\ 8\mathfrak{B}' &= 6a - 2\delta - 3(\log L)_v + (\log N)_v, \\ 8\mathfrak{C}' &= 6b - 2\alpha - 3(\log N)_u + (\log L)_u. \end{aligned}$$

By use of (17) and (18), the corresponding invariants for N_y, indicated by dashes, are given by the following expressions:

$$
(21)\quad
\begin{aligned}
\overline{H} &= H - (\log m)_{uv}, & \overline{K} &= K - (\log n)_{uv},\\
\overline{W}^{(u)} &= W^{(u)} + (\log n)_{uv}, & \overline{W}^{(v)} &= W^{(v)} + (\log m)_{uv},\\
\overline{\mathfrak{H}} &= \mathfrak{H}, & \overline{\mathfrak{K}} &= \mathfrak{K},\\
8\overline{\mathfrak{B}}' &= 8\mathfrak{B}' + (\log m^3/n)_v, & 8\overline{\mathfrak{C}}' &= 8\mathfrak{C}' + (\log n^3/m)_u.
\end{aligned}
$$

The following result can be easily established.

If two surfaces S_x, S_y are projectively associate in the modified sense, the parametric conjugate net on each surface has equal point invariants.

The asymptotic curves on S_x and S_y are determined by the same equations as in the preceding section. Precisely as in §3 we arrive at the following result:

If two surfaces S_x, S_y are projectively associate in the modified sense, and if the parametric net on each surface is conjugate, then the asymptotic curves on either surface correspond to the associate conjugate net of the parametric net on the other.

The Laplace transformed points, or ray points, of the point P_x with respect to N_x are given by

$$x_1 = x_v - ax, \qquad x_{-1} = x_u - bx,$$

and the ray points of P_y are defined by

$$y_1 = n(x_1 - a\eta/m), \qquad y_{-1} = m(x_{-1} - b\zeta/n).$$

The ray points of the points P_η, P_ζ, defined by (19), are found to be

$$
\begin{aligned}
m_u\eta_1 &= (n - m)(H\eta + m_u x_1), & (n - m)\eta_{-1} &= m_u\zeta,\\
(m - n)\zeta_1 &= n_v\eta, & n_v\zeta_{-1} &= (m - n)(K\zeta + n_v x_{-1}),
\end{aligned}
$$

where H, K are the point invariants of the net N_x. The points x_1, y_1, η, η_1 are collinear, as are also the points x_{-1}, y_{-1}, ζ, ζ_{-1}. The cross ratio* of the four points x_{-1}, y_{-1}, ζ, ζ_{-1} is $-bn_v/(nK)$, and that of the points x_1, y_1, η, η_1 is $-am_u/(mH)$. Hence we have the following theorem:

If two surfaces S_x, S_y are projectively associate in the modified sense, the cross ratio of the four points x_1, y_1, η, η_1 is equal to that of the points x_{-1}, y_{-1}, ζ, ζ_{-1}, and the common value may be written $2ab/H$.

5. A canonical form with particular parametric curves

In this section we place as fundamental the well known system of differential equations used in the study of the configuration composed of two surfaces S_x, S_y in ordinary projective space S_3, with their points in a one-to-one

* Lane, op. cit., p. 214, exercise 11.

correspondence. We then reduce this system of equations to a canonical form so that every pair of integral surfaces is projectively associate in the modified sense, the surface S_x being referred to its asymptotic net as parametric, and the curves on S_y, corresponding to the asymptotic curves on S_x, forming the parametric conjugate net.

Let

$$x = x(u, v), \qquad y = y(u, v)$$

be the parametric vector equations of two surfaces S_x, S_y in ordinary projective space. If these surfaces have their points in a one-to-one correspondence, such that corresponding points P_x, P_y have the same curvilinear coordinates u, v, and such that each point P_y does not lie in the tangent plane of S_x at the corresponding point P_x, then S_x, S_y are a pair of integral surfaces of a system of differential equations* of the form

$$\begin{aligned} x_{uu} &= px + \alpha x_u + \beta x_v + Ly, \\ x_{uv} &= cx + ax_u + bx_v + My, \\ x_{vv} &= qx + \gamma x_u + \delta x_v + Ny, \\ y_u &= fx + mx_u + sx_v + Ay, \\ y_v &= gx + tx_u + nx_v + By. \end{aligned} \tag{22}$$

The integrability conditions for this system are given by the following equations:

$$\begin{aligned} a_u + ab + c + mM &= \alpha_v + \beta\gamma + tL, \\ b_u + b^2 + a\beta + sM &= \beta_v + b\alpha + \beta\delta + p + nL, \\ c_u + bc + ap + fM &= p_v + c\alpha + q\beta + gL, \\ M_u + aL + (b + A)M &= L_v + BL + \alpha M + \beta N, \\ t_u + t\alpha + an + mB + g &= m_v + am + s\gamma + tA, \\ g_u + pt + cn + fB &= f_v + qs + cm + gA, \\ B_u + tL + nM &= A_v + sN + mM \end{aligned} \tag{23}$$

and those obtainable therefrom by the substitution (3).

The lines xy joining pairs of corresponding points P_x, P_y of the surfaces S_x, S_y form a congruence, the developables† of which are given by

$$s\,du^2 - (m - n)du\,dv - t\,dv^2 = 0. \tag{24}$$

The focal points of a line xy are the points η, ζ given by

$$\eta = y + k_1x, \qquad \zeta = y + k_2x,$$

* Lane, op. cit., p. 183 et seq.

† Ibid., p. 181.

where k_1, k_2 are the roots of the equation

$$k^2 + (m + n)k + mn - st = 0. \tag{25}$$

It is known that the asymptotic curves on S_x are parametric in case $L = N = 0$. Let us suppose from now on that this condition is satisfied, and in order that the developables of the congruence of lines xy be determinate and intersect S_x in a conjugate net we shall suppose $m = n$, $st \neq 0$.

It is not difficult to calculate the system of equations corresponding to (22) when the roles of x and y are interchanged. We shall compute all of the coefficients of such a system later, but at the moment the only coefficients that are needed are those corresponding to L, M, and N which are indicated by dashes and given* by the following formulas:

$$\begin{aligned} \overline{L} &= f_u + np + cs + \overline{A}c_{11} + \overline{B}c_{12}, \\ \overline{M} &= f_v + cn + qs + \overline{A}c_{31} + \overline{B}c_{32} \\ &= g_u + cn + pt + \overline{B}c_{41} + \overline{A}c_{42}, \\ \overline{N} &= g_v + nq + ct + \overline{B}c_{21} + \overline{A}c_{22}, \end{aligned} \tag{26}$$

wherein the coefficients c_{ij} are defined by placing

$$\begin{aligned} c_{11} &= n_u + f + n\alpha + as, & c_{12} &= s_u + n\beta + bs, \\ c_{21} &= n_v + g + n\delta + bt, & c_{22} &= t_v + n\gamma + at, \\ c_{31} &= n_v + an + s\gamma, & c_{32} &= s_v + f + bn + s\delta, \\ c_{41} &= n_u + bn + t\beta, & c_{42} &= t_u + g + an + t\alpha, \end{aligned}$$

and where

$$\Delta\overline{A} = sg - nf, \qquad \Delta\overline{B} = tf - ng, \qquad \Delta = n^2 - st \neq 0.$$

The parametric curves on S_y form a conjugate net N_y in case $\overline{M} = 0$. We shall suppose from now on that this condition is satisfied, and in order that S_y may be non-developable we shall suppose $\overline{L}\overline{N} \neq 0$. The developables of the congruence of lines xy intersect S_y in a conjugate net in case

$$t\overline{L} - s\overline{N} = 0, \tag{27}$$

a condition which we shall suppose from now on to be satisfied.

It is possible to simplify system (22) still more by a transformation of the form

$$x = \lambda\bar{x}, \qquad y = \mu\bar{y}. \tag{28}$$

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* Lane, op. cit., p. 185.

The effect of this transformation on the coefficients f, g, A, B is found to be given by the formulas

$$\mu\bar{f} = \lambda(f + s\lambda_u/\lambda), \qquad \mu\bar{g} = \lambda(g + t\lambda_v/\lambda),$$
$$\bar{A} = A - \mu_u/\mu, \qquad \bar{B} = B - \mu_v/\mu. \tag{29}$$

The last of (23) shows that μ can be chosen so that $\bar{A} = \bar{B} = 0$. We shall suppose from now on that this choice has been made. A condition necessary and sufficient that λ can be chosen so that $\bar{f} = \bar{g} = 0$ is

$$(f/s)_u = (g/t)_v.$$

By means of (23) and (26) this condition can be shown to be equivalent to (27). We shall suppose from now on that this choice of λ has been made.

When $f = g = A = B = 0$, the line h of intersection of the tangent planes at two corresponding points P_x, P_y of the surfaces S_x, S_y joins the points P_ρ, P_σ defined by

$$\rho = y_u = x_v + nx_u/s$$
$$\sigma = y_v = x_u + nx_v/t, \tag{30}$$

as is seen on inspecting the last two of equations (22).

When u, v vary, the line h generates a congruence $\rho\sigma$, whose developables will now be determined. If, as the point P_x describes a curve of the family $dv - \lambda du = 0$ on the surface S_x, the line h generates a developable of the congruence $\rho\sigma$, and if the point P_ζ defined by

$$\zeta = \rho + k\sigma \qquad (k \text{ scalar})$$

is the corresponding focal point of the line h, then h is tangent to the locus of the point P_ζ; consequently the derivative ζ' may be expressed as a linear combination of ρ, σ only. But by actual calculation it is found that ζ' appears as a linear combination of x, ρ, σ, y. Setting equal to zero the coefficients of x, y therein, we obtain conditions on the functions k, λ necessary and sufficient that the line h may generate a developable of the congruence $\rho\sigma$ and have P_ζ for focal point, namely,

$$(cst + npt) + (cns + pst)k + (qst + nct)\lambda + (cst + nqs)k\lambda = 0,$$
$$st + nsk + nt\lambda + stk\lambda = 0. \tag{31}$$

Elimination of k and substitution of dv/du for λ give the differential equation of the developables of the congruence $\rho\sigma$, namely,

$$p\,du^2 - q\,dv^2 = 0. \tag{32}$$

A necessary and sufficient condition that the developables of the congruence $\rho\sigma$ be indeterminate is seen from (32) to be

$$p = q = 0. \tag{33}$$

We shall suppose from now on that this condition is satisfied. As a result of the conditions which we have thus far imposed we find from (26) that

$$\bar{L} = cs, \quad \bar{M} = cn = 0, \quad \bar{N} = ct. \tag{34}$$

In view of the previous assumptions it is therefore evident from (34) that $m=n=0$, $c\neq 0$.

The most general transformation of the form (28) which leaves the form of system (22) invariant, has λ and μ constant. The only coefficients not absolutely invariant under such a transformation are s, t, M, for which we find

$$\bar{s} = \lambda s/\mu, \qquad \bar{t} = \lambda t/\mu, \qquad \bar{M} = \mu M/\lambda. \tag{35}$$

It is not difficult to show by means of the integrability conditions (23) that

$$M_u/M = c_u/c, \qquad M_v/M = c_v/c,$$

and from these results it is evident that

$$M = kc \qquad (k = \text{const.}). \tag{36}$$

It is therefore possible by a suitable choice of λ and μ to make the constant appearing in (36) equal to unity. We thus reach the following conclusion.

Any system (22) *such that every pair of integral surfaces is projectively parallel in the modified sense can be reduced to the form*

$$\begin{aligned} x_{uu} &= \alpha x_u + \beta x_v, \\ x_{uv} &= M(x + y) + a x_u + b x_v, \\ x_{vv} &= \gamma x_u + \delta x_v, \\ y_u &= s x_v, \qquad y_v = t x_u \qquad (stM \neq 0). \end{aligned} \tag{37}$$

The parametric net N_x is the asymptotic net and the parametric net N_y is conjugate.

The integrability conditions for system (37) are found to be

$$\begin{aligned} a_u + ab + c &= \alpha_v + \beta\gamma, \\ b_u + b^2 + a\beta + sM &= \beta_v + b\alpha + \beta\delta, \\ M_u + bM &= \alpha M, \\ t_u + t\alpha &= s\gamma, \end{aligned} \tag{38}$$

and the formulas obtainable from these by the substitution (3).

The system of equations corresponding to (37) when the roles of x and y are interchanged is found to be

$$
(39) \qquad \begin{aligned}
y_{uu} &= \bar{L}(x+y) + \bar{\alpha} y_u + \bar{\beta} y_v, \\
y_{uv} &= \bar{a} y_u + \bar{b} y_v, \\
y_{vv} &= \bar{N}(x+y) + \bar{\gamma} y_u + \bar{\delta} y_v, \\
x_u &= \bar{s} y_v, \qquad x_v = \bar{t} y_u,
\end{aligned}
$$

where

$$
(40) \qquad \begin{aligned}
&\bar{L} = cs, && \bar{\alpha} = b + s_u/s, && \bar{\beta} = as/t, \\
&\bar{a} = \beta t/s, && \bar{b} = \gamma s/t, && \\
&\bar{N} = ct, && \bar{\gamma} = bt/s, && \bar{\delta} = a + t_v/t, \\
&\bar{s} = 1/t, && \bar{t} = 1/s. &&
\end{aligned}
$$

It is evident that the asymptotic curves which are parametric on S_x correspond to the curves of the parametric conjugate net on S_y. The asymptotic curves on S_y are given by

$$
(41) \qquad \bar{L}du^2 + \bar{N}dv^2 = 0.
$$

By use of (40) this equation becomes

$$
sdu^2 + tdv^2 = 0
$$

which defines the associate conjugate net of the net in which the developables of the congruence of lines xy intersect S_x. Thus since the asymptotic curves on each of the two surfaces S_x, S_y correspond to a conjugate net on the other, we therefore reach the following conclusion:

Any system (22) *such that every pair of integral surfaces is projectively associate in the modified sense, can be reduced to the form* (37).

The last two equations of (37) show that the tangent to an asymptotic u-curve (v-curve) through P_x on S_x intersects the tangent to the v-curve (u-curve) of the parametric conjugate net through P_y on S_y in a point which lies in a fixed plane. Since statements similar to the preceding can be made by choosing a conjugate net as parametric on S_x and the asymptotic net as parametric on S_y, we therefore have a projective generalization of the theorem* of Eisenhart:

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* Eisenhart, op. cit., p. 381.

The tangents to the asymptotic curves on one of two projectively associate surfaces meet the tangents to the curves conjugate to the corresponding curves on the other surface in points of a fixed plane.

Inspection of (25) now shows that the two focal points P_η, P_ζ of a line xy are given by

$$\eta = y + (st)^{1/2}x, \quad \zeta = y - (st)^{1/2}x. \tag{42}$$

The cross ratio of the points P_x, P_y and the two focal points of the generator of the conjugate congruence is given by

$$(x, y, \eta, \zeta) = (\infty, 0, (st)^{1/2}, -(st)^{1/2}) = -1.$$

Corresponding points P_x, P_y of two projectively associate surfaces S_x, S_y are separated harmonically by the focal points of the line joining them.

If local coordinates $x_1, \cdots, x_4$ based on the tetrahedron x, x_u, x_v, y with suitably chosen unit point are introduced, the first and second focal planes of a line xy are found to have the local equations

$$x_2 - (t/s)^{1/2}x_3 = 0, \qquad x_2 + (t/s)^{1/2}x_3 = 0.$$

Therefore *the planes $x_2 = 0$, $x_3 = 0$ containing a line xy and the asymptotic tangents through P_x on S_x separate the first and second focal planes of the line xy harmonically.*

The developables of the congruence of lines xy intersect S_x in a conjugate net whose differential equation may be written

$$s\,du^2 - t\,dv^2 = 0.$$

Similarly, the developables intersect S_y in the conjugate net given by

$$\bar{s}\,du^2 - \bar{t}\,dv^2 = 0.$$

When reference is made to (40) it is evident that these curves on S_x and S_y correspond.

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