A SET OF FOUR POSTULATES FOR BOOLEAN ALGEBRA IN TERMS OF THE "IMPLICATIVE" OPERATION*

BY

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1. Introduction. Whitehead and Russell's *Principia Mathematica* makes fundamental the notion " $\supset$ " of "implication," defined by

$$p \supset q \cdot = \cdot \sim p \vee q, \text{ Df.}$$

The main object of my paper is to present in terms of this "implicative" operation† $\supset$ a set of four postulates for Boolean algebra. This will secure for Boolean algebra, for the first time, a set of postulates expressed in terms of an operation other than "rejection" having as few postulates as the present minimum sets.‡ Of course, by the principle of duality in Boolean algebra, my postulates will also be a set in terms of the dual of $p \supset q$, namely $\sim pq$.

I prove for my postulates (a) their *consistency*, (b) their mutual *independence*, (c) their *sufficiency* for Boolean algebra, (d) their *necessariness* for Boolean algebra.§ The consistency and independence systems are all Boolean

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† The *Principia* calls $\supset$ a *relation*.

‡ For the present minimum sets, see B. A. Bernstein, (I) *A set of four independent postulates for Boolean algebras*, these Transactions, vol. 17 (1916), pp. 50–51; (II) *Simplification of the set of four postulates for Boolean algebras in terms of rejection*, Bulletin of the American Mathematical Society, vol. 39 (1933), pp. 783–787. For another set of postulates in terms of $\supset$, the *first* set in terms of $\supset$, see E. V. Huntington, (I) *A new set of independent postulates for the algebra of logic, with special reference to Whitehead and Russell's Principia Mathematica*, Proceedings of the National Academy of Sciences, vol. 18 (1932), pp. 179–180. Huntington's postulates are eight in number (including an inadvertently omitted existence postulate).

§ I offer (a)–(d) as a set of *defining* properties of a set of postulates: a system of propositions S is a set of *postulates* for a (consistent) system Σ if and only if the propositions of S are (a) consistent, (b) mutually independent, (c) sufficient for Σ, (d) necessary for Σ. More simply and less formally stated, a set of postulates for a system is a set of propositions of the system which cannot be derived from one another but from which all the other propositions of the system *can* be derived. This view of postulates is opposed to the view, seemingly held widely, that demands of postulates only sufficiency and necessariness (hence also consistency). The latter view would have to accept as a set of postulates for euclidean geometry *all* of Euclid's *Elements*, and would violate the generally accepted distinction between *postulate* and *theorem*. My view of postulates is of course opposed to the view, seemingly held by some, that demands of postulates only sufficiency. This view would have to accept S as a set of postulates for Σ, not only when S is the *whole* of Σ, but also when S is *inconsistent* (since "a false proposition implies any proposition") and when S is only a *special case* of Σ (when S, for example, is the theory of Abelian groups and Σ the theory of groups in general). If I am correct in my view, the term "independent postulates," found in the literature, must be understood to mean "postulates whose independence has been proved," and the term "postulates" applied to S when only (c) and (d) have and very simple. The proof of sufficiency consists in deriving from the postulates my second set of postulates in terms of rejection (see my Paper II, loc. cit.); the proof of necessariness consists in the converse derivation.

I shall derive from my postulates the "theory of deduction" of the *Principia*. This will verify the fact, obtained elsewhere* less directly and from another point of view, that the theory of deduction is derivable from the general logic of classes.

There is a close relation between $\supset$ and the operation "$-$" of "exception" used by me†, and later by Taylor‡, in postulates for Boolean algebra. I shall bring out this relation.

If in a set of independent postulates for the logic of classes there is a proposition demanding that the number of elements be *at least* two, and if this proposition be replaced by a proposition demanding that the number of elements be *just* two, then the propositions resulting from the change will be sufficient for the logic of propositions as a two-element Boolean algebra. But these propositions will, in general, not be independent. I have so chosen my postulates that the change in question will render them a set of *independent* postulates for the logic of propositions.§

2. **The postulates.** The postulates with which we are mainly concerned have as primitive ideas a *class* K and a *binary operation* $\supset$, and are the propositions A_1-A_4 below.|| In Postulates A_2 and A_3, there is to be understood the supposition that *the elements involved and their indicated combinations belong to* K. This must especially be borne in mind when the independence of the postulates is considered. The postulates follow.

POSTULATE A_1. $a \supset b$ is an element of K whenever a and b are elements of K.

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been proved for S, must be understood to mean "*provisional* postulates" for Σ. If desired, "provisional postulates" might have a distinctive name, say *basic propositions* of Σ, or *defining conditions* for Σ.

* B. A. Bernstein, (III) *On section A of Principia Mathematica*, Bulletin of the American Mathematical Society, vol. 39 (1933), pp. 788–792.

† B. A. Bernstein, (IV) *A complete set of postulates for the logic of classes expressed in terms of the operation "exception," and a proof of the independence of a set of postulates due to Del Re*, University of California Publications in Mathematics, vol. 1, pp. 87–96.

‡ J. S. Taylor, *A set of five postulates for Boolean algebras in terms of the operation "exception,"* University of California Publications in Mathematics, vol. 1, pp. 242–248.

§ For a discussion of the nature of the logic of propositions, see B. A. Bernstein, (V) *Sets of postulates for the logic of propositions*, these Transactions, vol. 28 (1926), pp. 472–478.

|| The symbol "$=$" used in the postulates is taken as an idea *outside* the system. Compare my (VI) *Whitehead and Russell's theory of deduction as a mathematical science*, Bulletin of the American Mathematical Society, vol. 37 (1931), pp. 480–488. For sets of postulates for Boolean algebra in which "$=$" is taken as an idea *within* the system, see E. V. Huntington, (II) *New sets of independent postulates for the algebra of logic, with special reference to Whitehead and Russell's Principia Mathematica*, these Transactions, vol. 35 (1933), pp. 274–304.

POSTULATE A_2. $(a \supset b) \supset a = a$.

POSTULATE A_3. There is an element z in K such that

$$(d \supset d) \supset [(a \supset b) \supset c] = \{[(c \supset z) \supset a] \supset [(b \supset c) \supset z)]\} \supset z.$$

POSTULATE A_4. K consists of at least two distinct elements.

3. **Consistency and independence of the postulates.** The consistency and the independence of Postulates A_1-A_4 are given by the following systems S_0-S_4, in which S_0 is the consistency system, and S_1, S_2, S_3, S_4 are the independence systems for A_1, A_2, A_3, A_4 respectively. The systems are all Boolean.

System	K	$a \supset b$
S_0	0,1	$a'+b$
S_1	0,1	0/0*
S_2	0,1	0
S_3	0,1	a
S_4	0	0

4. **Theorems.** The proof of the sufficiency of Postulates A_1-A_4 for Boolean algebra, and for the theory of deduction of the *Principia*, will be effected with the help of the following theorems 1a-26a.

1a. $(a \supset z) \supset z = a.$

2a. $(a \supset a) \supset b = b.$

3a. $(a \supset b) \supset c = \{[(c \supset z) \supset a] \supset [(b \supset c) \supset z]\} \supset z.$

4a. $(a \supset z) \supset b = (b \supset z) \supset a.$

5a. $a \supset b = (b \supset z) \supset (a \supset z).$

6a. $a \supset (b \supset z) = b \supset (a \supset z).$

7a. *The element z of Postulate A_3 is unique.*

DEFINITION 1a. $a_1 = a \supset z$.

8a. $a_{11} = a, \quad where \quad a_{11} = (a_1)_1.$

9a. $(a \supset b) \supset c = [(c_1 \supset a) \supset (b \supset c)_1]_1.$

10a. $a_1 \supset b = b_1 \supset a.$

11a. $a \supset b = b_1 \supset a_1.$

12a. $a \supset b_1 = b \supset a_1.$

13a. $a_1 = a \supset a_1.$

14a. $a \supset a = b \supset b.$

* In a two-element Boolean algebra, we may define the *quotient* precisely as in the case of the algebra of number.

DEFINITION 2a. $u = a \supset a$.

15a. $$z \supset a = u.$$

16a. $$u \supset a = a.$$

17a. $$z_1 = u; \quad u_1 = z.$$

18a. $$a \supset b = (b \supset a_1) \supset a_1.$$

19a. $$(a \supset b) \supset b = b_1 \supset a.$$

20a. $$a \supset b = b_1 \supset (a \supset b).$$

21a. $$a \supset b = a \supset (a \supset b).$$

22a. $$a \supset b = (b \supset a) \supset (a \supset b).$$

23a. $$a \supset (b \supset a) = u.$$

24a. $$a \supset (b \supset c) = (b \supset a) \supset (b \supset c).$$

DEFINITION 3a. $a | b = b \supset a_1$.
DEFINITION 4a. $a' = a | a$.

25a. $$a' = a_1.$$

DEFINITION 5a. $\sim a = a_1$.
DEFINITION 6a. $a \vee b = a_1 \supset b$.

26a. $$\sim a \vee b = a \supset b.$$

DEFINITION 7a. $\vdash a = (a = u)$.

5. **Proofs of the theorems.** The proofs of the theorems 1a-26a follow.

Proof of 1a. $a = (a \supset a) \supset a = (a \supset a) \supset [(a \supset z) \supset a] = \{[(a \supset z) \supset a] \supset [(z \supset a) \supset z]\} \supset z = (a \supset z) \supset z$, by A_2, A_2, A_3, A_2.

Proof of 2a. $(a \supset a) \supset b = (a \supset a) \supset [(b \supset z) \supset b] = \{[(b \supset z) \supset b] \supset [(z \supset b) \supset z]\} \supset z = (b \supset z) \supset z = b$, by A_2, A_3, A_2, 1a.

Proof of 3a. $(a \supset b) \supset c = (d \supset d) \supset [(a \supset b) \supset c] = \{[(c \supset z) \supset a] \supset [(b \supset c) \supset z]\} \supset z$, by 2a, A_3.

Proof of 4a. $(a \supset z) \supset b = \{[(b \supset z) \supset a] \supset [(z \supset b) \supset z]\} \supset z = \{[(b \supset z) \supset a] \supset z\} \supset z = (b \supset z) \supset a$, by 3a, A_2, 1a.

Proof of 5a. $a \supset b = [(a \supset z) \supset z] \supset b = (b \supset z) \supset (a \supset z)$, by 1a, 4a.

Proof of 6a. $a \supset (b \supset z) = [(b \supset z) \supset z] \supset (a \supset z) = b \supset (a \supset z)$, by 5a, 1a.

Proof of 7a. Suppose that two elements, y and z, have the property of z. Then $y = (y \supset z) \supset z = \{[(z \supset y) \supset y] \supset [(z \supset z) \supset y]\} \supset y = \{z \supset [(z \supset z) \supset y]\} \supset y = (z \supset y) \supset y = z$, by 1a, 3a, 1a, 2a, 1a.

Proof of 8a. By def. 1a, 1a.

Proof of 9a. By def. 1a, 3a.

Proof of 10a. By def. 1a, 4a.

Proof of 11a. By def. 1a, 5a.

Proof of 12a. By def. 1a, 6a.

Proof of 13a. $a_1=(a_1\supset z)\supset a_1=[(a\supset z)\supset z]\supset a_1=a\supset a_1$, by A_2, def. 1a, 1a.

Proof of 14a. $a\supset a=[(a\supset a)\supset z]\supset z=z\supset z=[(b\supset b)\supset z]\supset z=b\supset b$, by 1a, 2a, 2a, 1a.

Proof of 15a. $z\supset a=[(z\supset a)\supset z]\supset z=z\supset z=u$, by 1a, A_2, def. 2a.

Proof of 16a. $u\supset a=(a\supset a)\supset a=a$, by def. 2a, A_2.

Proof of 17a. $z_1=z\supset z=u$, by def. 1a, def. 2a; $u_1=(z\supset z)_1=(z\supset z)\supset z=z$, by def. 2a, def. 1a, A_2.

Proof of 18a. $(b\supset a_1)\supset a_1=[(a_{11}\supset b)\supset(a_1\supset a_1)_1]_1=[(a_1\supset a_1)\supset(a_{11}\supset b)_1]_1=(a_{11}\supset b)_{11}=a\supset b$, by 9a, 12a, 2a, 8a.

Proof of 19a. $b_1\supset a=(a\supset b_{11})\supset b_{11}=(a\supset b)\supset b$, by 18a, 8a.

Proof of 20a. $b_1\supset(a\supset b)=[(a\supset b)\supset b]\supset b=(b_1\supset a)\supset b=(a_1\supset b)\supset b=b_1\supset a_1=a\supset b$, by 19a, 19a, 10a, 19a, 11a.

Proof of 21a. $a\supset(a\supset b)=[(a\supset b)\supset a_1]\supset a_1=[(a_{11}\supset a)\supset(b\supset a_1)_1]_1\supset a_1=[(a\supset a)\supset(b\supset a_1)_1]_1\supset a_1=(b\supset a_1)_{11}\supset a_1=(b\supset a_1)\supset a_1=a\supset b$, by 18a, 9a, 8a, 2a, 8a, 18a.

Proof of 22a. $(b\supset a)\supset(a\supset b)=\{[(a\supset b)_1\supset b]\supset[a\supset(a\supset b)]_1\}_1=\{[(a\supset b)_1\supset b]\supset(a\supset b)_1\}_1=(a\supset b)_{11}=a\supset b$, by 9a, 21a, A_2, 8a.

Proof of 23a. $a\supset(b\supset a)=[(b\supset a)\supset a_1]\supset a_1=[(a_{11}\supset b)\supset(a\supset a_1)_1]_1\supset a_1=[(a_{11}\supset b)\supset a_{11}]_1\supset a_1=[(a\supset b)\supset a]_1\supset a_1=a_1\supset a_1=u$, by 18a, 9a, 13a, 8a, A_2, def. 2a.

Proof of 24a. $(b\supset a)\supset(b\supset c)=\{[(b\supset c)_1\supset b]\supset[a\supset(b\supset c)]_1\}_1=\{[b_1\supset(b\supset c)]\supset[a\supset(b\supset c)]_1\}_1=\{[b_1\supset(c_1\supset b_1)]\supset[a\supset(b\supset c)]_1\}_1=\{u\supset[a\supset(b\supset c)]_1\}_1=[a\supset(b\supset c)]_{11}=a\supset(b\supset c)$, by 9a, 10a, 11a, 23a, 16a, 8a.

Proof of 25a. $a'=a|a=a\supset a_1=a_1$, by def. 4a, def. 3a, 13a.

Proof of 26a. $\sim a\vee b=a_1\vee b=a_{11}\supset b=a\supset b$, by def. 5a, def. 6a, 8a.

6. **Sufficiency of the postulates.** I shall now prove the sufficiency of postulates A_1-A_4 by deriving from them my second set of postulates for Boolean algebra in terms of rejection.* This set has as primitive ideas K and "$|$," and as postulates the propositions B_1-B_4 following (in postulates B_3 and B_4 there is to be understood the supposition that *the elements involved and their indicated combinations belong to* K).

B_1. K contains at least two distinct elements.

B_2. If a and b are elements of K, $a|b$ is an element of K.

Definition 1b. $a'=a|a$.

B_3. $$a=(b\mid a)\mid(b'\mid a).$$

B_4. $$a\mid(b\mid c)=[(c'\mid a)\mid(b'\mid a)]'.$$

* See my Paper II, loc. cit.

The derivations of B_1-B_4 from A_1-A_4 follow.

Proof of B_1. By A_4.

Proof of B_2. By def. 3a, def. 1a, A_1.

Proof of B_3. $a=(a_1)_1=[(b\supset b)\supset a_1]_1=[(a_{11}\supset b)\supset(b\supset a_1)_1]_{11}=(a\supset b)\supset(b\supset a_1)_1=(b\supset a_1)\supset(a\supset b)_1=(b\supset a_1)\supset(b_1\supset a_1)_1=(b|a)|(b_1|a)=(b|a)|(b'|a)$, by 8a, 2a, 9a, 8a, 12a, 11a, def. 3a, 25a.

Proof of B_4. $a|(b|c)=a\supset(b\supset c_1)_1=(b\supset c_1)\supset a_1=[(a_{11}\supset b)\supset(c_1\supset a_1)_1]_1=[(b_1\supset a_1)\supset(c_1\supset a_1)_1]_1=[(c_1\supset a_1)\supset(b_1\supset a_1)_1]_1=[(c'\supset a_1)\supset(b'\supset a_1)_1]'=[(c'|a)|(b'|a)]'$, by def. 3a, 12a, 9a, 10a, 12a, 25a, def. 3a.

7. **Necessariness of the postulates.** I shall prove that A_1-A_4 are necessary for Boolean algebra by deriving A_1-A_4 from the rejection postulates B_1-B_4 above. For this derivation I shall use as auxiliary theorems propositions 1b-8b following, derivable from B_1-B_4.

1b. $$a''=a, \quad \text{where} \quad a''=(a')'.$$

2b. $$a|b=b|a.$$

3b. $$a|(b|b')=a'.$$

4b. $$(a|c)|(b|c)=[c|(a'|b')]'.$$

5b. $$a|a'=b|b'.$$

Definition 2b. $u=a|a'$.

6b. $$a|u=a'.$$

7b. $$a|u'=u.$$

Definition 3b. $a\supset b=a|b'$.

8b. $$a\supset u'=a'.$$

Propositions 1b, 2b, 3b, 5b are respectively Sheffer's Postulate 3, Theorem A, Postulate 4, Theorem B.* The proofs of 4b, 6b, 7b, and 8b follow.

Proof of 4b. $[c|(a'|b')]'=[(b''|c)|(a''|c)]''=(b|c)|(a|c)$, by B_4, 1b.

Proof of 6b. $a|u=a|(a|a')=a'$, by def. 2b, 3b.

Proof of 7b. $a|u'=[(a|u')']'=[(a|u')|(a|a')]'=[(u'|a)|(a'|a)]'=[a|(u''|a'')]''=a|(u|a)=a|(a|u)=a|a'=u$, by 1b, 3b, 2b, 4b, 1b, 2b, 6b, def. 2b.

Proof of 8b. $a\supset u'=a|u''=a|u=a'$, by def. 3b, 1b, 6b.

The derivations of A_1-A_4 from B_1-B_4 now follow.

Proof of A_1. By def. 3b, def. 1b, B_2.

Proof of A_2. $(a\supset b)\supset a=(a|b')|a'=(a|b')|[a|(b|b')]=(b'|a)|[(b'|b)|a]$

* See H. M. Sheffer, *A set of five independent postulates for Boolean algebras, with application to logical constants*, these Transactions, vol. 14 (1913), pp. 481–488.

$= \{a|[b''|(b'|b)']\}' = \{a|[b|(b'|b)']\}' = \{a|[b|(b|b')']\}' = [a|(b|u')]' = (a|u)' = a'' = a$, by def. 3b, 3b, 2b, 4b, 1b, 2b, def. 2b, 7b, 6b, 1b.

Proof of A_3. The element u' will serve as the required element z. For, $\{[(c \supset u') \supset a] \supset [(b \supset c) \supset u']\} \supset u' = [(c' \supset a) \supset (b \supset c)']' = [(c'|a')|(b|c')'']' = [(c'|a')|(b|c')]' = [(a'|c')|(b|c')]' = \{[c'|(a''|b')]'\}' = [c'|(a''|b')]' |(d|d') = [c'|(a|b')]'|(d|d') = (d|d')|[(a|b')|c']' = (d \supset d) \supset [(a \supset b) \supset c]$, by 8b, def. 3b, 1b, 2b, 4b, 3b, 1b, 2b, def. 3b.

Proof of A_4. By B_1.

8. **Derivation of the theory of deduction.** I now come to the derivation from A_1-A_4 of the theory of deduction of *Principia Mathematica*. The primitive ideas of this theory are a *class* K, a *unary operation* "$\sim$," a *binary operation* "$\vee$," and a notion "$\vdash$," which may perhaps be termed a *predicative relation*. The postulates of the theory are the propositions C_1-C_7 below. These postulates are expressed in terms of K, $\sim$, $\vee$, $\vdash$, and an operation "$\supset$" defined by

DEFINITION 1c. $a \supset b = \sim a \vee b$.

By 26a, the "$\supset$" of Definition 1c is seen to be the same as the "$\supset$" of postulates A_1-A_4. This fact will be used hereafter without further mention. The postulates C_1-C_7 follow.†

C_1[*1·1]. If $\vdash a$ and $\vdash (a \supset b)$ then $\vdash b$.

C_2[*1·2]. $\vdash [(a \vee a) \supset a]$.

C_3[*1·3]. $\vdash [a \supset (b \vee a)]$.

C_4[*1·4]. $\vdash [(a \vee b) \supset (b \vee a)]$.

C_5[*1·6]. $\vdash \{(a \supset b) \supset [(c \vee a) \supset (c \vee b)]\}$.

C_6[*1·7]. If a is in K, then $\sim a$ is in K.

C_7[*1·71]. If a and b are in K, then $a \vee b$ is in K.

The derivations of C_1-C_7 from A_1-A_4 follow.

Proof of C_1. Let $\vdash a$ and $\vdash (a \supset b)$. Then $a = u$ and $a \supset b = u$, by def. 7a. Hence $u \supset b = u$. Hence $b = u$, by 16a; hence $\vdash b$, by def. 7a.

Proof of C_2. $(a \vee a) \supset a = (a_1 \supset a) \supset a = a_1 \supset a_1 = u$, by def. 6a, 19a, def. 2a. Hence the theorem, by def. 7a.

Proof of C_3. $a \supset (b \vee a) = a \supset (b_1 \supset a) = u$, by def. 6a, 23a. Hence the theorem, by def. 7a.

Proof of C_4. $(a \vee b) \supset (b \vee a) = (a_1 \supset b) \supset (b_1 \supset a) = (a_1 \supset b) \supset (a_1 \supset b) = u$, by def. 6a, 10a, def. 2a. Hence the theorem, by def. 7a.

Proof of C_5. $(a \supset b) \supset [(c \vee a) \supset (c \vee b)] = (a \supset b) \supset [(c_1 \supset a) \supset (c_1 \supset b)] =$

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† The numbers associated with C_1-C_7 are those of the *Principia*. For the form of *1·1, see my (VII) *Remarks on propositions* *1·1 *and* *3·35 *of Principia Mathematica*, Bulletin of the American Mathematical Society, vol. 39 (1933), pp. 111–114. The *Principia* proposition *1·5 has been omitted, since *1·5 is redundant (see P. Bernays, Mathematische Zeitschrift, vol. 25 (1926), pp. 305–320).

$(a \supset b) \supset [a \supset (c_1 \supset b)] = b \supset [a \supset (c_1 \supset b)] = \{[a \supset (c_1 \supset b)] \supset b_1\} \supset b_1 = \{(b_{11} \supset a) \supset [(c_1 \supset b) \supset b_1]_1\}_1 \supset b_1 = \{(b \supset a) \supset [(c_1 \supset b) \supset b_1]_1\}_1 \supset b_1 = \{(b \supset a) \supset [(b_1 \supset c) \supset b_1]_1\}_1 \supset b_1 = [(b \supset a) \supset b_{11}]_1 \supset b_1 = [(b \supset a) \supset b]_1 \supset b_1 = b_1 \supset b_1 = u$, by def. 6a, 24a, 24a, 18a, 9a, 8a, 10a, A_2, 8a, A_2, def. 2a. Hence the theorem, by def. 7a.

Proof of C_6. By def. 5a, def. 1a, A_1.

Proof of C_7. By def. 6a, def. 1a, A_1.

9. **Relation between the implicative operation and the operation exception.** I shall now bring out the relation existing between the implicative operation $\supset$ and the operation "$-$" of "exception."

The considerations are simple. The element $a-b$ is, in the usual Boolean notation, the element ab'. Since $a \supset b$ is the element $a'+b$, the elements $a \supset b$ and $b-a$ *are the duals of each other*. Hence, *a postulate-set in terms of* $\supset$ *is essentially also a set in terms of* "$-$," *and vice versa.*

Let me actually transform Postulates A_1-A_4 into a set in terms of "$-$." To do this, it will be convenient to re-letter the formulas in A_1-A_4. If we write $b-a$ for $a \supset b$, z for u (the dual of z), and re-letter, Postulates A_1-A_4 become the following postulates D_1-D_4 in terms of "exception."

D_1. $a-b$ is an element of K whenever a and b are elements of K.

D_2. $a-(b-a)=a$.

D_3. There is an element u in K such that

$$[a-(b-c)]-(d-d)=u-\{[u-(a-b)]-[c-(u-a)]\}.$$

D_4. K consists of at least two distinct elements.

To actually transform a set of "exception" postulates into a set of "implication" postulates, let me take a set due to Taylor.* This set consists of the postulates E_1-E_5 following.†

E_1. K contains at least two distinct elements.

E_2. If a and b are elements of K, $a-b$ is an element of K.

E_3. $a-(b-b)=a$.

E_4. There exists a unique element u in K such that $a-(u-b)=b-(u-a)$.

DEFINITION 1e. $a_1=u-a$.

E_5. $a-(b-c)=[(a-b)_1-(a-c_1)]_1$.

If we replace $a-b$ by $b \supset a$ and u by z, and re-letter, E_1-E_5 become the following postulates F_1-F_5 in terms of implication.

F_1. K contains at least two distinct elements.

F_2. If a and b are elements of K, $a \supset b$ is an element of K.

F_3. $(a \supset a) \supset b=b$.

* J. S. Taylor, loc. cit.

† In E_3, E_4, E_5 there is to be understood the supposition that *the elements involved and their indicated combinations belong to* K. In E_5 there is to be understood the further supposition that E_4 *holds*.

F_4. There exists a unique element z such that $(a \supset z) \supset b = (b \supset z) \supset a$.

DEFINITION 1f. $a_1 = a \supset z$.

F_5. $(a \supset b) \supset c = [(a_1 \supset c) \supset (b \supset c)_1]_1$.

10. **Postulates for the logic of propositions.** I take up finally the last item of my paper: to show that a simple change in one of the postulates A_1-A_4 will make these postulates a set of *independent* postulates for the logic of propositions as a two-element Boolean algebra. The change consists in replacing Postulate A_4 by Postulate A_4' following:

POSTULATE A_4'. K consists of two distinct elements.

That A_1, A_2, A_3, A_4' are *necessary* and *sufficient* for a two-element Boolean algebra, is obvious. That A_1, A_2, A_3, A_4' are mutually *independent*, is seen from the table of §3: in that table *each of the independence systems for* A_1, A_2, A_3 *consists of only two elements.**

* In the same way, and for the same reasons, one can transform into independent postulate-sets for the logic of propositions my two sets for Boolean algebra in terms of rejection (see my Papers I, II, loc. cit.). A like remark applies to Huntington's first set of postulates for Boolean algebra (E. V. Huntington, (III) *Sets of independent postulates for the algebra of logic*, these Transactions, vol. 5 (1904), pp. 288–309).

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