INTER-RELATIONS AMONG THE FOUR PRINCIPAL TYPES OF ORDER*

BY

EDWARD V. HUNTINGTON

The four types of order whose inter-relations are considered in this paper may be called, for brevity, (1) serial order; (2) betweenness; (3) cyclic order; and (4) separation.

We first recapitulate the known sets of postulates which define each of these types as an abstract system, and recall the usual geometric interpretation of each type; we then develop the way in which each of these four types may be defined in terms of each of the other three. (For convenience of reference, the numbering of the postulates in earlier publications has been retained.)

1. **Serial order.** A system (K, R), where K is a class of elements A, B, $C, \cdots$, and $R(AB)$, or simply AB, is a dyadic relation, is called a "system of serial order" when and only when the following four postulates are satisfied.

In each of these postulates it is understood that distinct letters represent distinct elements of K. [The notation " $=0$" means "is false"; the "horseshoe", $\supset$, means "If . . . then"; the "wedge," $\vee$, means "or" (in the sense of "at least one"); and the "dot," . , means "and." Dots, singly or in groups, serve also as punctuation marks.]

POSTULATE D. $AA = 0$.   ("Irreflexiveness.")
POSTULATE I. $AB \vee BA$.   ("Connexity.")
POSTULATE II. $AB . BA : = 0$.   ("Asymmetry.")
POSTULATE IV. $AB . \supset . XB \vee AX$. ("Inclusiveness.")

From properties II and IV, the following property is deducible as a theorem:

POSTULATE III. $AB . BC . \supset . AC$.   ("Transitivity for distinct elements.")

(Proof. By IV, $AB . \supset . CB \vee AC$. But CB conflicts with BC, by II. Hence AC.)

Also, IV is a consequence of III and I.

(Proof. By I, $XB \vee BX$. But if BX, then by III, $AB . BX . \supset . AX$. Hence $XB \vee AX$.)

Hence the following alternative sets of postulates are equivalent, and

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either of them may be taken as a set of independent postulates for serial order*:

$$(1) \quad D, I, II, III; \qquad\qquad (2) \quad D, I, II, IV.$$

Geometrically speaking, this "abstract" system (K, R) of serial order may be represented by a "concrete" system in which K is the class of points on a directed straight line (that is, a straight line having a definite "sense" indicated by an arrow), and $R(AB)$ means "the point A precedes the point B" when the line is traversed in the direction of the arrow. Or briefly: serial order is the order of points on a *directed straight line*.

2. **Betweenness.** A system (K, R), where K is a class of elements A, B, $C, \cdots$, and $R(ABC)$, or simply ABC, is a triadic relation, is called a system of "betweenness" when and only when the following five postulates are satisfied. (In each of these postulates, beyond D, it is understood that distinct letters represent distinct elements of K.)

POSTULATE D. If ABC is true, then A, B, C are distinct.
POSTULATE B. $BAC \lor CAB \lor ABC \lor CBA \lor ACB \lor BCA$.
POSTULATE A. $ABC . \supset .CBA$.
POSTULATE C. $ABC . ACB := 0$.
POSTULATE 9. $ABC . \supset .ABX \lor XBC$.

From these five properties, the following eight properties are deducible as theorems:

POSTULATE 1. $XAB . ABY . \supset .XAY$.
POSTULATE 2. $XAB . AYB . \supset .XAY$.
POSTULATE 3. $XAB . AYB . \supset .XYB$.

POSTULATE 4. $AXB . AYB . \supset .AXY \lor AYX$.
POSTULATE 5. $AXB . AYB . \supset .(AXY \lor YXB).(AYX \lor XYB)$.

POSTULATE 6. $XAB . YAB . \supset .XYB \lor YXB$.
POSTULATE 7. $XAB . YAB . \supset .XYA \lor YXA$.
POSTULATE 8. $XAB . YAB . \supset .(XYA \lor YXB).(YXA \lor XYB)$.

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* Sets of postulates for serial order date back at least as far as the early work of G. Peano and B. Russell. [Partial references will be found in E. V. Huntington's *A complete set of postulates for the theory of absolute continuous magnitude*, these Transactions, vol. 3 (1902), pp. 264–279; or, *Complete existential theory of the postulates for serial order*, Bulletin of the American Mathematical Society, vol. 23 (1917), pp. 276–280; or *The Continuum and Other Types of Serial Order*, second edition, Harvard University Press, 1917.] In Set 1, if Postulate III (the law of transitivity for distinct elements) is replaced by a law of transitivity for *all* elements, then Postulate II becomes redundant, being a consequence of this extended law of transitivity and Postulate D. In the present paper, Postulate IV is introduced for the sake of its analogy with Postulates 9 and 10, below.

The following twelve sets of postulates are equivalent, and any one of them may be taken as a set of independent postulates for betweenness*:

(1) A, B, C, D, 1, 2.   (5) A, B, C, D, 1, 8.   (9) A, B, C, D, 3, 4, 6.
(2) A, B, C, D, 1, 5.   (6) A, B, C, D, 2, 4.   (10) A, B, C, D, 3, 4, 7.
(3) A, B, C, D, 1, 6.   (7) A, B, C, D, 2, 5.   (11) A, B, C, D, 3, 4, 8.
(4) A, B, C, D, 1, 7.   (8) A, B, C, D, 3, 5.   (12) A, B, C, D, 9.

The most familiar concrete example of this abstract system of betweenness is the system (K, R) in which K is the class of points on an undirected straight line, and $R(ABC)$ means "B is between A and C" in the geometric sense. (This is the concrete example from which the abstract system takes its name.) In brief, "betweenness" is the order of points on an *undirected straight line*.

3. **Cyclic order.** A system (K, S), where K is a class of elements A, B, C, $\cdots$, and $S(ABC)$, or simply ABC, is a triadic relation, is called a system of "cyclic order" when and only when the following five postulates are satisfied. (In each of these postulates, beyond D, it is understood that distinct letters represent distinct elements of K.)

POSTULATE D. If ABC is true, then A, B, C are distinct.
POSTULATE B′. The system contains at least one true triad, say XYZ.
POSTULATE E. $ABC . \supset . BCA$.
POSTULATE C. $ABC . ACB : = 0.$
POSTULATE 9. $ABC . \supset . ABX \vee XBC$.

From these five properties, the following three properties are deducible as theorems:

POSTULATE B. $ABC \vee BCA \vee CAB \vee CBA \vee BAC \vee ACB$.
POSTULATE 2. $XAB . AYB . \supset . XAY$.
POSTULATE 3. $XAB . AYB . \supset . XYB$.

The following four sets of postulates are equivalent, and any one of them

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may be taken as a set of independent postulates for cyclic order:*

(1) B, C, D, E, 2.	(3) B, C, D, E, 9.
(2) B, C, D, E, 3.	(4) B′, C, D, E, 9.

Cyclic order is represented geometrically by a class K of points on a directed closed line, with $S(ABC)$ meaning "the arc running from A through B to C, in the direction of the arrow, is less than one complete circuit." In brief, cyclic order is the order of points on a *directed closed line.*

When necessary to distinguish between the two triadic relations, $R(ABC)$ for betweenness, and $S(ABC)$ for cyclic order, the prefixes R and S will be retained.

4. **Separation.** A system (K, R), where K is a class of elements A, B, $C, \cdots$, and $R(ABCD)$, or simply $ABCD$, is a tetradic relation, is called a system of "separation of pairs," or simply a system of "separation," when and only when the following six postulates are satisfied. (In each of these postulates, beyond D, it is understood that distinct letters represent distinct elements of K.)

POSTULATE D. If $ABCD$ is true, then A, B, C, D are distinct elements of K.

POSTULATE F′. The system contains at least one true tetrad, say $XYZW$.

POSTULATE G. $ABCD . \supset .BCDA$.

POSTULATE H. $ABCD.ABDC: = 0$.

POSTULATE R′. At least one true tetrad is reversible; that is, if there is any true tetrad, then there is at least one true tetrad $ABCD$ such that $DCBA$ is also true.

POSTULATE 10. $ABCD . \supset .AXCD \vee ABCX$.

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* See E. V. Huntington, *Sets of completely independent postulates for cyclic order*, Proceedings of the National Academy of Sciences [Washington], vol. 10 (1924), pp. 74–78. Another definition of cyclic order (in terms of a tetradic relation) is referred to in the next footnote.

Set (4) is new, and requires a proof of B′, E, C, 9, which proceeds as follows:

By B′, there is one true triad, say UVW. By 9, $UVW . \supset .AVW \vee UVA$.

If UVA, then by E, VAU, whence by 9, $VAU . \supset .WAU \vee VAW$, whence by E, $UWA \vee AWV$. Therefore $AVW \vee UWA \vee AWV$.

But if UWA, then by 9, $UWA . \supset .VWA \vee UWV$, where UWV conflicts with UVW by C, and $VWA . \supset .AVW$ by E. Therefore $AVW \vee AWV$, whence by E, $VWA \vee WVA$. We have thus proved

(a) $$UVW . \supset .VWA \vee WVA,$$

where UVW represents any true triad.

Case 1. If VWA, then by (a), $VWA . \supset .WAB \vee AWB$, whence by E, $WAB \vee WBA$. But if WAB, then by (a), $WAB . \supset .ABC \vee BAC$, and if WBA, then by (a), $WBA . \supset .BAC \vee ABC$.

Case 2. If WVA, then by (a), $WVA . \supset .VAB \vee AVB$, whence by E, $VAB \vee VBA$. But if VAB, then by (a), $VAB . \supset .ABC \vee BAC$, and if VBA, then by (a), $VBA . \supset .BAC \vee ABC$.

Hence in any case, $ABC \vee BAC$. Hence by E, $ABC.BCA.CAB: \vee :BAC.ACB.CBA$, from which B follows at once.

From these six properties the following eleven properties are deducible as theorems:

POSTULATE F.   If A, B, C, D are distinct elements of K, then at least one of the twenty-four tetrads $ABCD$, $ABDC$, $\cdots$, $DCBA$ is true.

POSTULATE R.   Every true tetrad is reversible; that is, $ABCD . \supset . DCBA$.

POSTULATE 11. $ABXC . ABCY . \supset . ABXY$.

POSTULATE 12. $ABXC . ABCY . \supset . BXCY$.

POSTULATE 13. $ABXC . ABCY . \supset . AXCY$.

POSTULATE 14. $ABCX . ABCY . \supset . ABXY \lor ABYX$.

POSTULATE 15. $ABCX . ABCY . \supset . ACXY \lor ACYX$.

POSTULATE 16. $ABCX . ABCY . \supset . BCXY \lor BCYX$.

POSTULATE 17. $ABCX . ABCY . \supset . (ABXY \lor ACYX) . (ABYX \lor ACXY)$.

POSTULATE 18. $ABCX . ABCY . \supset . (ABXY \lor BCYX) . (ABYX \lor BCXY)$.

POSTULATE 19. $ABCX . ABCY . \supset . (ACXY \lor BCYX) . (ACYX \lor BCXY)$.

The following ten sets of postulates are equivalent, and any one of these may be taken as a set of independent postulates for separation*:

(1) D, F, G, H, R, 10.   (6) D, F, G, H, R, 11, 16.

(2) D, F, G, H, R, 12.   (7) D, F, G, H, R, 11, 17.

(3) D, F, G, H, R, 13.   (8) D, F, G, H, R, 11, 18.

(4) D, F, G, H, R, 11, 14.   (9) D, F, G, H, R, 11, 19.

(5) D, F, G, H, R, 11, 15.   (10) D, F', G, H, R', 10.

A geometrical example of a system of "separation" is the system (K, R) in which K is a class of points on an undirected closed line, and $R(ABCD)$ means "the pair of points A, C is separated by the pair B, D." In brief, "separation" treats of the order of points on an *undirected closed line*.

In the language of modern geometry, "separation" is the theory of order on the "projective line" (the so-called "straight line" of projective geometry). It may be suggested, in passing, that the concept of the "fourth harmonic point" determined by three given points on such a projective line is a concept which it would be interesting to define by the postulational method.

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* See E. V. Huntington and K. E. Rosinger, *Postulates for separation of point-pairs (reversible order on a closed line)*, Proceedings of the American Academy of Arts and Sciences [Boston], vol. 67 (1932), pp. 61–145. On p. 70 of this paper the following corollary is established: *In every system which satisfies Postulates* D, F', G, H, 10, *we have either* (R) *Every true tetrad is reversible*; *or else* (S) *Every true tetrad is non-reversible*; and on p. 63 it is noted that if we introduce

POSTULATE S'. At least one true tetrad is non-reversible,

then *Postulates* D, F', G, H, 10, S' *will define the theory of non-reversible order on a closed line* (just as Postulates D, F', G, H, 10, R' define the theory of reversible order on a closed line). This theory of non-reversible order on a closed line is essentially the same as the theory of cyclic order, expressed in terms of a tetradic instead of a triadic relation.

The inter-relations among these four types of order may be classified under four headings.

<div align="center">§§1.1–1.5</div>

Under the first heading, we show that each of the other three types may be defined directly in terms of serial order; and also that separation may be defined directly in terms of each of the other three types. The details of the proofs require nothing more than a checking up of all the possible cases for each of the postulates involved, and will be left to the reader.

1.1. Betweenness defined in terms of serial order. In a given system of serial order, three elements A, B, C will stand in the "betweenness" relation ABC when AB and BC are true, and also when CB and BA are true, but not otherwise. That is, in the system of serial order we may define the relation of betweenness as follows:

$$ABC :: = :: AB.BC : \vee : CB.BA.$$

The triadic relation thus defined is readily shown to satisfy all the postulates A, B, C, D, 9 for betweenness.

1.2. Cyclic order defined in terms of serial order. Similarly, in a given system of serial order the relation of cyclic order may be defined as follows:

$$ABC :: = :: AB.BC : \vee : BC.CA : \vee : CA.AB.$$

The triadic relation thus defined satisfies all the postulates B, C, D, E, 9 for cyclic order.

1.3. Separation defined in terms of serial order. Again, in a given system of serial order, four elements A, B, C, D will stand in the "separation" relation, $ABCD$, under conditions expressed by the following definition:

$$\begin{aligned}
ABCD :: = :: & AB.BC.CD : \vee : BC.CD.DA : \vee : CD.DA.AB : \vee \\
& : DA.AB.BC : \vee : DC.CB.BA : \vee : AD.DC.CB : \vee \\
& : BA.AD.DC : \vee : CB.BA.AD.
\end{aligned}$$

The tetradic relation thus defined satisfies all the postulates D, F', G, H, R', 10 for separation.

1.4. Separation defined in terms of betweenness. Suppose now we have a given system of betweenness. The relation of separation may be defined in this system as follows:

$$ABCD :: = :: ABC.BCD : \vee : BCD.CDA : \vee : CDA.DAB : \vee : DAB.ABC.$$

1.5. Separation defined in terms of cyclic order. Again, if we have a given system of cyclic order, the relation of separation may be defined in that system as follows:

$$ABCD :: = ::: ABC.CDA . \vee .ADC.CBA.$$

§§2.1–2.3

Under the second heading we consider definitions which are not absolute, but involve a reference to an arbitrarily selected element of the given system, say Z.

2.1. **Serial order defined in terms of cyclic order, with respect to Z.** In a given system of cyclic order, if we exclude any arbitrarily chosen element Z, the remaining elements may be arranged in serial order (with respect to Z) by the following definition:

$$AB: = :ZAB.$$

The element Z itself may then be brought into the series, if desired, by defining AZ as true and ZA as false.

2.2. **Betweenness, $R(ABC)$, defined in terms of cyclic order, with respect to Z.** In a given system of cyclic order, if we exclude any arbitrarily chosen element Z, we may define the betweenness relation $R(ABC)$ among the remaining elements as follows:

$$R(ABC):: = ::ZAB.BCZ: \vee :ZCB.BAZ.$$

The element Z itself may then be brought into the betweenness system, if desired, by defining $R(ABZ)$ and $R(ZBC)$ as true and $R(AZC)$ as false.

2.3. **Betweenness defined in terms of separation, with respect to Z.** Suppose now the given system is a system of separation. Then if we exclude an arbitrary element Z, we may define the betweenness relation among the remaining elements as follows:

$$ABC: = :ZABC.$$

The element Z itself may then be brought into the betweenness system, if desired, by defining ABZ and ZBC as true and AZC as false.

§§3.1–3.2

Under the third heading, the definitions are also not absolute, but involve a reference to an arbitrarily chosen pair of elements, say U and V, in the given system.

3.1. **Serial order defined in terms of betweenness, with respect to U, V.** In a given betweenness system, let U, V be any two selected elements. Then all the elements may be arranged in serial order (with respect to U, V) by the following definitions (where, as usual, distinct letters denote distinct elements):

UV true; and VU false.

$AU: = :AUV.$ $AV: = :AUV \vee UAV.$ $UA: = :UAV \vee UVA.$
$VA: = :UVA.$
$AB:: = ::AUV.ABV: \vee :AUV.AVB: \vee :UAV.UAB: \vee :UVA.UAB.$

3.2. Cyclic order, $S(ABC)$, defined in terms of betweenness, with respect to U, V. In a given betweenness system, let U, V again be any two arbitrarily chosen elements. Then we may define the relation of cyclic order among the elements of this system (with respect to U, V) as follows*:

$S(AUV).S(UVA).S(VAU):: = ::AUV \vee UVA.$

$S(AVU).S(VUA).S(UAV):: = ::UAV.$

$S(ABU).S(BUA).S(UAB)::$
$= ::ABU.BUV: \vee :BUV.BUA: \vee :UAV.UAB: \vee :UVA.UAB.$

$S(AUB).S(UBA).S(BAU)::$
$= ::AUV.AUB: \vee :UBV.UBA: \vee :UVB.UBA: \vee :UAB.AUV.$

$S(ABV).S(BVA).S(VAB)::$
$= ::AUV.ABV: \vee :UAB.ABV: \vee :AVB.AVU: \vee :UVA.UAB.$

$S(AVB).S(VBA).S(BAV)::$
$= ::AVB.UVB: \vee :UVB.UBA: \vee :BUV.BAV: \vee :UBA.BAV.$

$S(ABC).S(BCA).S(CAB)::$
$= ::BCA.BCU.BUV: \vee :BCA.BUC.BUV: \vee :UCA.UBC.UBV:$
$\vee :UCA.UAB.UCV: \vee :UAB.UBC.UAV: \vee :AUB.ABC.AUV:$
$\vee :ABU.AUV.ABC: \vee :CUA.CAB.CUV: \vee :CAU.CUV.CAB:$
$\vee :UVB.VBC.BCA: \vee :UVA.VAB.ABC: \vee :UVC.VCA.CAB.$

<center>§§4.1–4.2</center>

Under the fourth heading, the definitions involve reference to three arbitrary elements of the given system. In §4.1, one of the three reference elements, say Z, is distinguished from the other two, say U, V. In §4.2, the three reference elements, say U, V, W, are coordinate.

4.1. Serial order defined in terms of separation, with respect to Z and U, V. Given a separation system, in which two elements U and V are arbitrarily selected as reference elements. Then if a third arbitrary element Z is excluded from the system, all the remaining elements may be arranged in serial order (with respect to U, V) by the following definitions:

UV true; and VU false.

$AU: = :ZAUV. \quad UA: = :ZUAV \vee ZUVA.$

$AV: = :ZAUV \vee ZAUV. \quad VA: = :ZUVA.$

$AB:: = ::ZAUV.ZABU: \vee :ZAUV.ZAUB: \vee :ZUAV.ZUAB:$
$\vee :ZUVA.ZUAB.$

The element Z itself may then be brought into the series, if desired, by defining UZ, VZ, AZ as true and ZU, ZV, ZA as false.

* For assistance in formulating and verifying the definitions under §§3.2 and 4.2, I am indebted to Mr. B. Notcutt.

4.2. **Cyclic order defined in terms of separation, with respect to** U, V, W. Given a separation system, in which three elements, U, V, W, are arbitrarily selected as reference elements ("anchorage points"). The relation of cyclic order (with respect to U, V, W) may be thus defined in this system:

$UVW.VWU.WUV$ true; and $WVU.UWV.VUW$ false.

$AUV.UVA.VAU := :AUVW \lor AWUV.$

$AVW.VWA.WAV := :AVWU \lor AUVW.$

$AWU.WUA.UAW := :AWUV \lor AVWU.$

$AVU.VUA.UAV := :AVWU.$

$AWV.WVA.VAW := :AWUV.$

$AUW.UWA.WAU := :AUVW.$

$UAB.ABU.BUA := :AVWU.ABVU: \lor :AVWU.AVBU:$
$\lor :AWUV.ABUV: \lor :AUVW.ABUV.$

$VAB.ABV.BVA := :AWUV.ABWV: \lor :AWUV.AWBV:$
$\lor :AUVW.ABVW: \lor :AVWU.ABVW.$

$WAB.ABW.BWA := :AUVW.ABUW: \lor :AUVW.AUBW:$
$\lor :AVWU.ABWU: \lor :AWUV.ABWU.$

$ABC:: = ::AUVW.ABVW.ABCW: \lor :BUVW.BCVW.BCAW:$
$\lor :CUVW.CAVW.CABW: \lor :AUVW.ABVW.ABWC:$
$\lor :BUVW.BCVW.BCWA: \lor :CUVW.CAVW.CAWB.$

Summary. These results may be summarized in the following table, in which Z, U, V, W denote arbitrarily selected elements of the given system.

	The relation of	may be defined within a given system of
1.1	Betweenness	serial order.
1.2	Cyclic order	serial order.
1.3	Separation	serial order.
1.4	Separation	betweenness.
1.5	Separation	cyclic order.
2.1	Serial order	cyclic order, with respect to Z.
2.2	Betweenness	cyclic order, with respect to Z.
2.3	Betweenness	separation, with respect to Z.
3.1	Serial order	betweenness, with respect to U, V.
3.2	Cyclic order	betweenness, with respect to U, V.
4.1	Serial order	separation, with respect to Z and U, V.
4.2	Cyclic order	separation, with respect to U, V, W.

HARVARD UNIVERSITY,
CAMBRIDGE, MASS.