TRAJECTORIES AND LINES OF FORCE*

BY

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In this paper we generalize certain theorems of Kasner† relative to the geometry of arbitrary fields of force in the plane.

Consider the motion of a particle which starts from rest in a positional field of force at a point where the force does not vanish. It begins to move along the line of force on which it is situated. However, due to the effect of inertia, it does not remain on this line of force, but travels in a somewhat straighter path. In general, the line of force and the trajectory will have the same initial direction but different initial curvatures. Kasner has shown that *the curvature of the trajectory is always one-third the curvature of the line of force.* If the initial curvature of the line of force vanishes, this result, while still valid, is not significant. In this case Kasner studies the ratio between the infinitesimal departures of the path and the line of force from their common tangent line. He proves the following theorem:

THEOREM. *If the line of force has contact of nth order with the tangent line, the trajectory produced by starting a particle from rest will also have contact of nth order; and the limiting ratio of the departure of the trajectory to the departure of the line of force from the common tangent will be* $1:(2n+1)$.

We extend this result to the more general cases in which the contact between the line of force and its tangent is of any order, finite or infinite, as well as to some cases in which no definite order of contact exists.‡ The theo-

* Presented to the Society, March 30, 1934; received by the editors October 8, 1934.

† For a complete report of Kasner's work, see Proceedings of the National Academy of Sciences, vol. 20 (1934), pp. 130–136. Some results also appear in these Transactions, 1906–1910; Bulletin of the American Mathematical Society, vol. 16 (1909–1910), p. 172; Princeton Colloquium Lectures, *Differential Geometric Aspects of Dynamics*, 1913, p. 9; Science, vol. 75 (1932), p. 671; Zurich Congress Proceedings, 1932, vol. 2, p. 180.

‡ A curve, $y=f(x)$, where $f(x)$ is single-valued, continuous and

$$\lim_{x\to+0}\frac{f(x)}{x}=0,$$

has contact of finite order α with the x-axis if $\lim_{x\to+0}f(x)/x^{\alpha+1}$ is a non-zero constant. If

$$\lim_{x\to+0}\frac{f(x)}{x^{\alpha+1}}=0$$

for all values of α, $f(x)$ has contact of infinite order. In all other cases, $f(x)$ has no definite order of contact.

rems are stated more simply in terms of the inverse of Kasner's ratio, i.e., the ratio of the departure of the line of force to the departure of the trajectory from their common tangent. For brevity, we call the limits of this ratio the *ratio set*. The trajectory produced by starting a particle from rest will be referred to simply as "the trajectory." In general the ratio set will not be a single number but will consist of a set of numbers. An easy application of a theorem of Hardy leads to the result that for certain simple types of fields the ratio set is a unique number. In the course of the work, we give an indication of the extent to which the ratio set determines the field. The theorems which Kasner obtains when friction is allowed or when the particle is projected with non-zero velocity in the direction of the force are also generalized.

We proceed to obtain a formula for the ratio set. The components of the field of force are assumed to be continuous and to possess continuous first partial derivatives. Furthermore we assume that the direction of the force at each point of some neighborhood of the initial point differs from that at the initial point. In fact, it is sufficient for this property to hold in a sufficiently small portion of a neighborhood of the initial point, containing some first part of the trajectory and the tangent in its interior, and having the initial point on its boundary. In all that follows, we choose the initial point as the origin of coordinates and the tangent to the line of force as the x-axis; we assume unit mass and we write f for the force at the origin. It is clear that this causes no loss in generality. An equation in x and y in which the variables are referred to the above set of axes will be called *normal*. The formula for the ratio set is given by

THEOREM I. *Let $y=g(x)$ and $y=h(x)$ be the normal equations of the trajectory and the line of force respectively. Then the ratio set is identical with the set of limits of the expression*

$$\frac{2x\,\dfrac{dg(x)}{dx}}{g(x)}-1$$

or of the equivalent expression

$$\frac{2h(x)}{x^{1/2}\displaystyle\int_0^x \frac{h(x)}{x^{3/2}}\,dx}$$

as x approaches zero.

In the course of the proof of this theorem, we shall also discover a sufficient condition that two different fields of force have the same ratio set at a point. For this purpose, we introduce the notion of the *direction function* of a field of force. Through a fixed point in the plane, there passes a single line of force. The slope of the force at each point of the tangent to this line of force is a function of x. This function approaches zero with x and is the direction function of the field at the fixed point.

THEOREM II. *Two fields of force have the same ratio set at a fixed point if the quotient of the direction functions of the fields approaches a finite non-zero limit at the given point.*

Thus the ratio set at a point is completely determined merely by the limiting behavior of the direction function. The two fields need not have the same direction at the point. The proofs of these theorems follow.

By the hypothesis of Theorem I, the equation of the trajectory is

$$y = g(x). \tag{1}$$

The components of the field are $\phi(x, y)$ and $\psi(x, y)$ where

$$\phi(0, 0) = f \quad (f \neq 0), \qquad \psi(0, 0) = 0. \tag{2}$$

By the theorem of the mean*

$$\frac{\psi(x, y)}{\phi(x, y)} = \frac{\psi(x, 0)}{\phi(x, 0)} + \left[\frac{\psi(x, \theta y)}{\phi(x, \theta y)}\right]_y \cdot y, \qquad 0 < \theta < 1.$$

By hypothesis, ϕ_y and ψ_y, and hence also $[\psi/\phi]_y$, are continuous in a sufficiently small neighborhood of the origin. We may write the last equation as

$$\frac{\psi(x, y)}{\phi(x, y)} = D(x) + A(x, y) \cdot y, \tag{3}$$

where, by definition, $D(x)$ is the direction function; $D(0) = 0$ and $A(x, y)$ is bounded in a neighborhood of the origin.

Now the trajectory is a solution† of

* The only partial derivatives whose existence we require in our work are those with respect to y. Since we do not assume the existence of ϕ_x and ψ_x, the mathematics may allow more than one trajectory through a point. Of course, this theory would have physical application only when a unique trajectory existed.

† For this solution dg/dx which appears in Theorem I exists and is a continuous function for a sufficiently small neighborhood of the origin. d^2g/dx^2 which also appears in the work is equal to

$$\frac{\dfrac{d^2y}{dt^2} \cdot \dfrac{dx}{dt} - \dfrac{d^2x}{dt^2} \cdot \dfrac{dy}{dt}}{\left(\dfrac{dx}{dt}\right)^3}$$

and hence is continuous wherever $dx/dt \neq 0$. Since $(d^2x/dt^2)_0 \neq 0$, it follows from Rolle's Theorem that d^2g/dx^2 is continuous in a sufficiently small positive neighborhood of the origin.

$$\ddot{x} = \phi(x, y),^* \qquad \ddot{y} = \psi(x, y). \tag{4}$$

Since the initial velocity is zero, the parametric equations of the trajectory (1) in terms of the time are

$$x = \tfrac{1}{2}ft^2 + k(t), \qquad y = y(t),$$

where $k(t)$ and $y(t)$ and their first two derivatives vanish at the origin. If we eliminate t from these two equations we obtain (1).† Now $\dot{y} = g' \cdot \dot{x}$ and $\ddot{y} = g''\dot{x}^2 + g'\ddot{x}$. Hence

$$\frac{\ddot{y}}{\ddot{x}} = g''\frac{\dot{x}^2}{\ddot{x}} + g' = g''\left[\frac{f^2t^2 + 2ft\dot{k}(t) + \dot{k}(t)^2}{f + \ddot{k}(t)}\right] + g',$$

$$\frac{\ddot{y}}{\ddot{x}} = [2x + m(x)]g'' + g', \tag{5}$$

where $m(x)/x \to 0$ as $x \to 0$. Comparing (3), (4), and (5), we have

$$[2x + m(x)]g'' + g' = D(x) + A(x, g) \cdot g. \tag{6}$$

Thus (6) is a differential equation whose solution through the element (0, 0, 0) is the trajectory.

We now obtain a similar differential equation for the corresponding line of force. The equation of the line of force is

$$y = h(x). \tag{7}$$

By definition of a line of force, (7) must satisfy the equation

$$h' = \frac{\psi(x, h)}{\phi(x, h)},$$

or

$$h' = D(x) + A(x, h) \cdot h. \tag{8}$$

Since the ratio set consists of the limiting values of $h(x)/g(x)$ as $x \to 0$, we now proceed to compare the solutions of (6) and (8) through the element (0, 0, 0). For this purpose we prove two lemmas.

LEMMA I. *Let $h = h(x)$ be a solution of* (8) *through the element* (0, 0, 0). *Then*

$$\lim_{x \to 0} h(x) \Big/ \int_0^x D(x)dx = 1.$$

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* In all that follows, primes denote differentiation with respect to x, and dots differentiation with respect to time.

† It is an easy consequence of the theorem on implicit functions that the elimination of t gives a unique solution for $t>0$ and a unique solution for $t<0$. In each case $x>0$ for small values of t. In what follows, $y = g(x)$ signifies either branch of the trajectory.

Let $A(x, h(x)) = B(x)$. Then a solution of

$$w' = D(x) + B(x)\cdot w \tag{9}$$

is $w = h(x)$. Since (9) is a linear differential equation, its solution through the origin is

$$h(x) = \exp\left[\int_0^x B(x)dx\right]\int_0^x \exp\left[-\int_0^x B(x)dx\right]D(x)dx.$$

Hence

$$h(x) = (1 + E_1(x))\int_0^x (1 + E_2(x))D(x)dx,$$

where $E_1(x)$ and $E_2(x)$ approach zero with x, since $B(x)$ is bounded as $x\to 0$. Since, by hypothesis, $D(x)\neq 0$ in a positive neighborhood of the origin, we may apply L'Hospital's Rule. Therefore

$$\lim_{x\to 0} h(x)\Big/\int_0^x D(x)dx = 1\cdot\lim_{x\to 0}(1 + E_2(x))D(x)/D(x) = 1,$$

which proves Lemma I. As a consequence of this lemma, the line of force must be on one side of its tangent in a neighborhood of the origin.

Lemma II. *Let $g = g(x)$ be a solution of* (6) *through the element* $(0, 0, 0)$. *Then*

$$\lim_{x\to 0}\frac{2xg'(x) - g(x)}{\int_0^x D(x)dx} = 1.$$

Let $A(x, g(x)) = C(x)$. Then

$$[2x + m(x)]z'' + z' = D(x) + C(x)\cdot z \tag{10}$$

has the solution $z = g(x)$. Consider the equation

$$w' = D(x) + C(x)\cdot w. \tag{11}$$

Let $w = w(x)$ be the solution of (11) through the origin. Subtracting (10) from (11),

$$w' - z' - [2x + m(x)]z' = C(x)\cdot[w - z]. \tag{12}$$

For the solution of (12) through the origin,

$$u' - C(x)u = [2x + m(x)]g''(x),$$

where $u(x) = w(x) - g(x)$. Since $m(x)/x\to 0$ as $x\to 0$, $[2x + m(x)]g'' = 2xg''\cdot(1 + E_3(x))$. Hence, as in Lemma I,

$$\lim_{x\to 0} \frac{\int_0^x 2xg''(1+E_4(x))\,dx}{u(x)} = 1,$$

where $E_4(x)$ approaches zero with x.

We now show that $E_4(x)$ may be neglected. By hypothesis $\psi(x, y)/\phi(x, y) \neq 0$ in some initial neighborhood of the trajectory and the x-axis. From (4), this is also true for $\dddot{y}(t)$. By Rolle's Theorem the same obtains for $\dot{y}(t)$ and consequently for $g'(x)$ and $g(x)$. It is easy to show that the line of force and the trajectory are on the same side of the tangent line near the origin.

Now, from (6),

$$(2xg'' + g')(1 + E_3(x)) = \frac{\ddot{y}}{\ddot{x}} + E_3(x)\,\frac{\ddot{y}}{\ddot{x}}.$$

When this expression is zero,

$$E_3(x) = -\frac{\dot{x}\dddot{y}}{\ddot{x}\dot{y}} = -\frac{t\dddot{y}}{\dot{y}}\,(1 + E_5(t))$$

where $E_5(t)$ approaches zero with t. As will be shown later,

$$\lim_{t\to 0} \frac{t\dddot{y}}{\dot{y}} \neq 0.$$

Hence $2xg''+g'$ does not change sign in some deleted neighborhood of the origin. Hence, by L'Hospital's Rule,

$$\lim_{x\to 0} \frac{\int_0^x (2xg'' + g')(1 + E_4(x))dx}{\int_0^x (2xg'' + g')dx} = 1.$$

Furthermore

$$\lim_{x\to 0} \frac{\int_0^x g'(x)(1 + E_4(x))dx}{\int_0^x g'(x)dx} = 1$$

and therefore

$$\lim_{x\to 0} \frac{\int_0^x (2xg'' + g')(1 + E_4(x))dx}{u + g} = 1.$$

It follows easily that

$$\lim_{x\to 0} \frac{\int_0^x (2xg'' + g')dx}{u + g} = 1.$$

Performing the indicated integration in the numerator and using $w(x) = u(x) + g(x)$ and

$$\lim_{x\to 0} \frac{w(x)}{\int_0^x D(x)dx} = 1$$

(Lemma I), we have

$$\lim_{x\to 0} \frac{\int_0^x (2xg'' - g')dx}{\int_0^x D(x)dx} = 1.$$

A comparison of the results of Lemmas 1 and 2 shows that

$$2xg'(x) - g(x) = h(x)[1 + E_6(x)] \tag{13}$$

where $E_6(x)\to 0$ as $x\to 0$. From this it follows that the limits of $h(x)/g(x)$ and of $2xg'(x)/g(x) - 1$ as x approaches zero are identical. This proves the first part of Theorem I.

Solving (13) for $g(x)$,

$$g(x) = \frac{x^{1/2}}{2}\int_0^x \frac{h(x)[1 + E_6(x)]}{x^{3/2}}\,dx. \tag{14}$$

We now show that a suitable approximation for $g(x)$ in terms of $h(x)$ may be derived by neglecting $E_6(x)$ in (14). Now

$$\lim_{x\to 0} \frac{h(x)}{x} = 0.$$

Hence $|h(x)/x^{3/2}| < x^{-1/2}$ for x sufficiently small. Also

$$\left|\int_0^x \frac{h(x)}{x^{3/2}}\,dx\right| < \int_0^x x^{-1/2}dx = 2x^{1/2}.$$

Hence this integral approaches zero with x, as does the similar integral in (14), and we may apply L'Hospital's Rule to their quotient. We have

$$\lim_{x\to 0} \frac{\displaystyle\int_0^x \frac{h(x)}{x^{3/2}}\,dx}{\displaystyle\int_0^x \frac{h(x)[1 + E_6(x)]}{x^{3/2}}\,dx} = \lim_{x\to 0} \frac{\dfrac{h(x)}{x^{3/2}}}{\dfrac{h(x)[1 + E_6(x)]}{x^{3/2}}} = 1.$$

Hence*

$$\lim_{x\to 0} \frac{g(x)}{\dfrac{x^{1/2}}{2}\displaystyle\int_0^x \frac{h(x)}{x^{3/2}}\,dx} = 1. \tag{15}$$

* Therefore the limiting ratio of both branches of the trajectory is unity and the same ratio set is obtained by using either branch.

The second part of Theorem I is an easy consequence of (15) and the definition of the ratio set. The ratio set is therefore determined by the complete equation of the trajectory or line of force.

To establish the truth of Theorem II, we show that the ratio set is identical with the limits of yet a third expression, depending only upon the direction function at the point. By Lemma I,

$$\lim_{x\to 0} \frac{h(x)}{\int_0^x D(x)dx} = 1.$$

On the basis of this result and the second part of Theorem I, it follows by a proof analogous to that used in deriving (15) that *the ratio set is identical with the limits of the expression*

$$\frac{2\int_0^x D(x)dx}{x^{1/2}\displaystyle\int_0^x \frac{\int_0^x D(x)dx}{x^{3/2}}\,dx}$$

as x approaches zero. By means of this formula the ratio set may be calculated directly from the components of the force without integrating the equations of motion.

Now consider two different fields of force whose direction functions at a fixed point are $D(x)$ and $D_1(x)$. It is easy to show that if

$$\lim_{x\to 0} \frac{D(x)}{D_1(x)} = c,$$

where c is a non-zero constant, the ratio set for each field, computed from the above expression, will be the same. For, by the argument used to derive (15), a suitable approximation for $D(x)$ in the formula for the ratio set is $cD_1(x)$ which obviously gives the same values for the ratio set as does $D_1(x)$. This proves Theorem II.

We now apply Theorem I to the case in which the line of force has any finite contact, integral, fractional, or irrational, with its tangent. This is a first generalization of Kasner's theorem and includes it as a special case.

THEOREM III. *If the line of force has contact of order α with the tangent line, the trajectory will also have contact of order α; and the ratio set will be $2\alpha+1$.*

By the hypothesis of the theorem,

$$\lim_{x\to 0} \frac{h(x)}{x^{\alpha+1}} = c\,(\neq 0).$$

Hence, as shown in the derivation of (15), $h(x)$ may be replaced by $cx^{\alpha+1}$ in taking the limit of the second expression in Theorem I:

$$\lim_{x\to 0}\frac{2h(x)}{x^{1/2}\int_0^x \frac{h(x)}{x^{3/2}}\,dx}=\lim_{x\to 0}\frac{2cx^{\alpha+1}}{x^{1/2}\int_0^x cx^{\alpha-1/2}dx}=2\alpha+1.$$

Since

$$\lim_{x\to 0}\frac{g(x)}{h(x)}=\frac{1}{2\alpha+1},\qquad \text{we have}\qquad \lim_{x\to 0}\frac{g(x)}{x^{\alpha+1}}=\frac{c}{2\alpha+1},$$

which shows that the trajectory also has contact of order α with the common tangent.

If we consider the case in which the line of force has infinite contact with its tangent, the corresponding theorem is

THEOREM IV. *If the line of force has contact of infinite order with the tangent line, the trajectory will also have contact of infinite order; and the ratio set will be* $+\infty$ *or all numbers in some non-negative closed interval including* $+\infty$. *Furthermore, any given interval of this kind will be the ratio set of some field of force for which the line of force has contact of infinite order with its tangent.*

We first prove that the trajectory has contact of infinite order with its tangent line. From (14), for every $k>\frac{1}{2}$,

$$\lim_{x\to 0}\frac{g(x)}{x^k}=\lim_{x\to 0}\frac{\int_0^x \frac{h(x)[1+E_6(x)]}{2x^{3/2}}\,dx}{x^{k-1/2}}=\frac{1}{2k-1}\lim_{x\to 0}\frac{h(x)}{x^k}.$$

By the hypothesis this last limit is zero, which proves the preliminary result of Theorem IV. For the rest of the theorem, we consider the expression

$$G(x)=\frac{xg'(x)}{g(x)} \tag{16}$$

which appears in the formula for the ratio set. The possible limits of $G(x)$ are investigated under the assumption that

$$\lim_{x\to 0}\frac{g(k)}{x^k}=0$$

for every k.

As already shown the origin is an isolated point of the zeros of $g(x)$. Then $G(x)$ is a continuous function in a sufficiently small positive neighborhood of the origin. The limiting values of $G(x)$ for $x>0$ must be a closed interval (which may degenerate into a single point). For if a and b $(>a)$ are lower and upper limits of $G(x)$, then $G(x)$ assumes values which lie in the bands

$[a-\epsilon, a+\epsilon]$ and $[b-\epsilon, b+\epsilon]$, E arbitrarily small, an infinite number of times in every neighborhood of the origin. Since $G(x)$ is continuous, it assumes each value between these bands an infinite number of times. Therefore the closed interval $[a, b]$ is the set of limits of $G(x)$.

We now indicate which closed intervals may actually appear as limits of $G(x)$. In the following, we suppose that $g(x)\neq 0$ in the interval $0 < x \leqq 1$. Integrating (15),

$$g(x) = c \exp\left[\int_1^x (G(x)/x)dx\right]. \tag{17}$$

From (17),

$$\frac{g(x)}{cx^k} = \exp\left[\int_1^x \{(G(x) - k)/x\}dx\right].$$

Since $g(x)$ has contact of infinite order, $G(x)$ cannot remain less than $+k$ in any neighborhood of the origin. For suppose $G(x)\leqq k-\epsilon$ when $0<x\leqq\delta$:

$$\frac{g(x)}{cx^k} = d + \exp\left[\int_\delta^x \{(G(x) - k)/x\}dx\right]$$

where

$$d = \exp\left[\int_1^\delta \{(G(x) - k)/x\}dx\right].$$

Then

$$\lim_{x\to 0}\left|\frac{g(x)}{cx^k}\right| \geqq d + \lim_{x\to 0} \exp\left[\int_\delta^x (-\epsilon/x)dx\right] = +\infty$$

contrary to hypothesis. Therefore $G(x)$ assumes values as great as any fixed k an infinite number of times in every neighborhood of the origin. Since k is any positive number, lim sup $G(x)=+\infty$. Hence the only closed intervals which may occur as limits of $G(x)$ are those which include $+\infty$. No negative ratio may appear since the line of force and trajectory are both on the same side of their common tangent near the origin.

Furthermore, any interval of this kind will be the limit of $G(x)$ for some field of force. To prove this last statement, it will suffice to present a trajectory, $y=g(x)$, having contact of infinite order with its tangent, such that the limit of the associated function $G(x)$ is a given closed interval $[a, +\infty]$. For then the field of force mentioned above surely exists. For example, a field which generates the trajectory, $y=g(x)$, is

$$\phi(x, y) = 1, \qquad \psi(x, y) = 2xg''(x) + g'(x). \tag{18}$$

We list the possible limits of $G(x)$ together with the corresponding trajectories:

$$(19_1)\quad \lim_{x\to 0} G(x) = +\infty, \qquad y = \exp\left[-1/x^2\right],$$

$$(19_2)\quad \lim_{x\to 0} G(x) = [a, +\infty], \quad y = \exp\left[\int_1^x \{(\sin(1/x)+1)/x^2 + a)/x\}\,dx\right].$$

It is easily seen that in each case $G(x)$, calculated from (16), has the prescribed limit. It only remains to show that the line of force corresponding to each case actually has contact of infinite order. We first prove that each trajectory has contact of infinite order. This is immediate for (19_1). For (19_2), we must show that

$$\int_1^0 \frac{\dfrac{1}{x^2}\left(\sin\dfrac{1}{x}+1\right) + a - n}{x}\,dx = -\infty$$

for all values of n. The substitution $y = 1/x$ makes it possible to perform the integration in finite terms and establishes the required result.

Now by Theorem I

$$\lim_{x\to 0}\frac{h(x)}{g(x)} = \lim_{x\to 0}\,(2G(x) - 1)$$

or

$$\lim_{x\to 0}\frac{h(x)}{x^k} = \lim_{x\to 0}\,(2G(x) - 1)\cdot\frac{g(x)}{x^k}. \tag{20}$$

Since $g(x)$ in each case has contact of infinite order and $G(x)$ involves only powers of x, the right hand member of (20) approaches zero. Hence the corresponding line of force has contact of infinite order with its tangent.

There still remains the case in which no definite order of contact exists. We make the following definition:

A curve, $y = f(x)$, where $f(x)$ is single-valued, continuous and

$$\lim_{x\to +0}\frac{f(x)}{x} = 0,$$

has generalized contact of order α with the x-axis if α is the upper bound of all numbers k such that

$$\lim_{x\to +0}\frac{f(x)}{x^{k+1}} = 0.$$

Note that if a curve has ordinary contact of order α, it also has generalized contact of order α. For infinite contact, the two definitions coincide. To every curve there is assigned some generalized contact $\alpha \geqq 0$.

THEOREM V. *If the line of force has generalized contact of order α with the tangent line, the trajectory will have generalized contact $\geqq \alpha$; and the ratio set will be a non-negative closed interval containing at least one of the numbers $2\alpha+1$, $+\infty$. This interval may degenerate into a single point. Furthermore, any given interval of this kind will be the ratio set of some field for which the line of force has generalized contact of order α.*

The case $\alpha = +\infty$ has been treated in Theorem IV. We therefore assume α is finite. As shown in the beginning of the proof of Theorem IV, since

$$\lim_{x\to 0} \frac{h(x)}{x^{k+1}} = 0, \quad \text{we have} \quad \lim_{x\to 0} \frac{g(x)}{x^{k+1}} = 0.$$

This proves the first part of Theorem V.

We proceed to study the possible limits of $G(x)$ defined by (16). The proof is parallel with that of Theorem IV and is outlined in what follows. We note again that the limiting values of $G(x)$ for $x>0$ form a non-negative closed interval. Repeating the proof following (17), we conclude that $G(x)$ assumes values as great as any fixed $k+1<\alpha+1$ an infinite number of times in every neighborhood of the origin. Therefore $\limsup G(x) \geqq \alpha+1$.

We now show that either $\alpha+1$ or $+\infty$ is a limit of $G(x)$. For suppose $\alpha+1$ is not a limit of $G(x)$. Then $\liminf G(x) = \gamma > \beta > \alpha+1$. If, in addition, $+\infty$ is not a limit of $G(x)$, we shall prove that

$$\lim_{x\to 0} \frac{h(x)}{x^{\beta}} = 0,$$

which contradicts the hypothesis. From (17),

$$\begin{aligned} \lim_{x\to 0} \frac{g(x)}{x^{\beta}} &= \lim_{x\to 0} c \exp\left[\int_1^x \{(G(x)-\beta)/x\} dx\right] \\ &< \lim_{x\to 0} c \exp\left[\int_1^x \{(\gamma-\beta)/x\} dx\right] = 0. \end{aligned} \tag{21}$$

In (19), replace k by β. Now since $+\infty$ is not a limit of $G(x)$, $[2G(x)-1]$ remains bounded. Hence, from (20) and (21), it follows that

$$\lim_{x\to 0} \frac{h(x)}{x^{\beta}} = 0$$

which is the predicted contradiction. Therefore either $\alpha+1$ or $+\infty$ is a limit of $G(x)$.

It remains to show that any interval of this kind is the limit of $G(x)$ for some field of force. As in the proof of Theorem IV, it will suffice to present suitable trajectories.

We first introduce several auxiliary functions. Let

$$\phi_1(x) = \sin^2 \frac{1}{x} \qquad \text{when } \frac{1}{2^m\pi} \geqq x \geqq \frac{1}{(2^m+1)\pi} \qquad (m = 0, 1, 2, \cdots),$$

$$\phi_1(x) = 0 \qquad \text{for all other values of } x.$$

Then $\phi_1(x)$ oscillates between 0 and $+1$ and $\int_0^1[\phi_1(x)/x]dx$ converges to a negative constant. For

$$\left|\int_0^1 \frac{\phi_1(x)}{x}\,dx\right| = \left|\int_1^\infty \frac{\phi_1\left(\frac{1}{y}\right)}{y}\,dy\right| = \left|\sum_{m=0}^{\infty}\int_{2^m\pi}^{(2^m+1)\pi} \frac{\phi_1\left(\frac{1}{y}\right)}{y}\,dy\right|$$
$$< \sum_{m=0}^{\infty} \frac{1}{2^m} = 2.$$

Note also that $\phi_1(x)$ is continuous and has a continuous first derivative. It is clear that similar functions having any finite number of continuous derivatives may be constructed by using sufficiently high powers of $\sin^2(1/x)$. Let

$$\phi_2(x) = \sin^2 \frac{1}{x} \qquad \text{when } \frac{1}{2^{2m}\pi} \geqq x \geqq \frac{1}{(2^{2m}+1)\pi} \qquad (m = 0, 1, 2, \cdots),$$

$$\phi_2(x) = 0 \qquad \text{for all other values of } x.$$

Obviously $\phi_2(x)$ has the same properties as $\phi_1(x)$. Similarly let $\phi_3(x)$ and $\phi_4(x)$ be continuous differentiable functions which oscillate between 0 and $+1$ in the neighborhood of the origin and such that

$$\int_1^0 x^{\alpha+1}e^{1/x}\phi_3(x)dx \quad \text{and} \quad \int_1^0 x^{\alpha+1-c}\phi_4(x)dx \qquad (c > \alpha)$$

converge to negative constants.

We now list the possible limits of $G(x)$ together with the corresponding trajectories*:

* In (22_2), if $b=+\infty$, it is replaced by $-\log x$. If $\alpha \leqq 1$, the expression $2xg''(x)+g'(x)$ in (18) will not be zero at the origin. In this case, more complicated ϕ functions must be used.

$$(22_1)\quad \lim_{x\to 0} G(x) = \alpha + 1, \qquad y = x^{\alpha+1} \text{ or } y = \frac{x^{\alpha+1}}{\log x};$$

$$(22_2)\quad \begin{aligned} &\lim_{x\to 0} G(x) = [a, b], \qquad a \leqq \alpha + 1 \leqq b, \\ &y = \exp\left[\int_1^x \{((a - \alpha - 1)\phi_1(x) + (b - a)\phi_2(x) + \alpha + 1)/x\}\,dx\right]; \end{aligned}$$

$$(22_3)\quad \lim_{x\to 0} G(x) = +\infty, \qquad y = \exp\left[\int_1^x \{(1/x + x^{\alpha+1}e^{1/x}\phi_3(x))/x\}\,dx\right];$$

$$(22_4)\quad \begin{aligned} &\lim_{x\to 0} G(x) = [c, +\infty], \qquad \alpha + 1 < c, \\ &y = \exp\left[\int_1^x \{(c + x^{\alpha+1-c}\phi_4(x))/x\}\,dx\right]. \end{aligned}$$

It is easy to verify that in each case $G(x)$ has the prescribed limit. It is necessary to show that the corresponding line of force has generalized contact of order α. This follows at once for the first line of force from (20) and (22_1). The trajectory (22_2) has contact of order α. Hence, from (20), the corresponding line of force has generalized contact of order α, if $\lim_{x\to 0}G(x)$ is bounded. This remains true even if $b = -\log x$, since $\log x$ is greater than any power of x in the neighborhood of the origin. The trajectory (22_3) has the same contact as $e^{-1/x}$ and (22_4) as x^c. By substituting these values in (20), we find that the corresponding lines of force have generalized contact of order α. This completes the proof.

These theorems indicate how the field of force determines the ratio set. Indeed, as proved in Theorem II, the ratio set depends only upon the limiting behavior of the direction function. The converse question arises: To what extent does the ratio set determine the field? The simplest answer seems to be in terms of the trajectory, although there are similar statements about the line of force and the direction function.

THEOREM VI. *Let the ratio set be the closed interval* $[a, b]$. *Let* $y = g(x)$ *be the normal equation of the trajectory. For a sufficiently small neighborhood of the origin, let* e_1 *be the upper bound of* e *such that* $x^{-e}|g(x)|$ *is an increasing function and let* e_2 *be the lower bound of* e *such that* $x^{-e}|g(x)|$ *is a decreasing function. Then* $e_1 = (a+1)/2$ *and* $e_2 = (b+1)/2$.

As shown in the proof of Lemma II, $g(x)$ is either an increasing or a decreasing function in a sufficiently small neighborhood of the origin. Hence $|g(x)|$ is an increasing function in this neighborhood. From (17),

$$p_e(x) = x^{-e} | g(x) | = | c | \exp\left[\int_1^x \{(G(x) - e)/x\} dx \right],$$
$$p_e'(x) = |c| \frac{G(x) - e}{x} p_e(x).$$

Hence $p_e(x)$ is an increasing function as long as $(G(x)-e)>0$ for small values of x. Then e_1 is the lower limit of $G(x)$. The first part of the theorem follows from (20). Since $p_e(x)$ is a decreasing function if $(G(x)-e)<0$ for small values of x, we may prove the remainder of the theorem in a similar manner.

THEOREM VII. *Let $y=g(x)$ be the normal equation of the trajectory. Let $g(x)$ be an L-function of x. Then the ratio set will be a unique number.*

Since $xg'(x)/g(x)$ is also an L-function, this follows from Theorem I and a theorem of Hardy on L-functions.*

All these theorems are derived upon the assumption that the particle encounters no resistance. For those cases in which resistance is allowed, we have

THEOREM VIII. *Let a particle start from rest in a continuous resisting medium. Let f be the intensity of the force at the initial point and let R_0 be the resistance due to zero speed. Let α be a number of the ratio set of the field of force which gives the same trajectory when the resistance is neglected. Then the ratio set will consist of the numbers $\alpha(1-R_0/f)+R_0/f$ when the motion takes place in the resisting medium.*

If $R_0=0$, as in a gas, the resisting medium may be entirely disregarded in calculating the ratio set. If the initial point is not a point of inflection of the line of force, $\alpha=3$, and we obtain Kasner's result: $3-2R_0/f$.

The proof follows. Let (1) and (7) be the equations of the trajectory and line of force respectively and (2) the components of the field of force. Then the trajectory is a solution of

$$\ddot{x} = \phi(x, y) - R(v)\cdot\cos\theta,$$
$$\ddot{y} = \psi(x, y) - R(v)\cdot\sin\theta,$$

where $\tan\theta$ is the slope of the tangent of the trajectory. Hence

$$\frac{\psi(x, y)}{\phi(x, y)} = \frac{\dfrac{\ddot{y}}{\ddot{x}} + \dfrac{R_0}{f - R_0}(1 + E_1(x))\cdot g'(x)\cdot\cos\theta}{1 + \dfrac{R_0}{f - R_0}(1 + E_1(x))\cdot\cos\theta}, \tag{23}$$

* G. H. Hardy, *Orders of Infinity*, 1924. An L-function is a real one-valued function defined by a finite combination of the ordinary algebraic symbols and the function symbols log ( ) and exp ( ) operating on the variable x and on real constants. The theorem referred to above is as follows: An L-function is ultimately continuous, of constant sign, and monotonic, and tends as $x\to+\infty$ to infinity or to zero or to some other definite limit. This applies also if $x\to 0$ through positive values.

where $E_1(x)\to 0$ as $x\to 0$ since $R(v)$ is continuous and $R(0)=R_0$ (of course, $R_0<f$). Now $\cos\theta\to 1$ as $x\to 0$. Therefore, from (3), (5), and (23), we obtain

$$(2x+n(x))\left(1-\frac{R_0}{f}\right)g''+g' = D(x)+A(x,g)\cdot g, \tag{24}$$

where $n(x)/x\to 0$ as $x\to 0$. Thus (24) is a differential equation whose solution through the element (0, 0, 0) is the trajectory. Similarly a differential equation for the line of force is (8). Proceeding as in Lemmas 1 and 2 of Theorem I, we find that

$$\lim_{x\to 0}\frac{h(x)}{g(x)} = \lim_{x\to 0}\left(\frac{2xg'}{g}-1\right)\left(1-\frac{R_0}{f}\right)+\frac{R_0}{f},$$

which, together with Theorem I, completes the proof of the theorem.

We now consider the case in which the particle is projected with a non-zero velocity in the direction of the force. Kasner has obtained a theorem, assuming that the line of force has integral order of contact with its tangent, which we generalize.

THEOREM IX. *If the line of force has generalized contact of order α with the tangent line, any trajectory obtained by projecting a particle with a non-zero speed in the direction of the force will have generalized contact of order $\alpha+1$; and the departure from the common tangent of these trajectories will vary inversely as the square of the speed. If the line of force has ordinary contact of order α with the tangent line, any trajectory obtained by projection will have ordinary contact of order $\alpha+1$.*

The proof is similar to that of the preceding theorem. Again let (1) and (7) be the equations of the trajectory and the line of force respectively and (2) the components of the force. Then the trajectory is a solution of (4), having initial velocity $v\neq 0$. Its equation may be written in the parametric form

$$x = vt+\tfrac{1}{2}ft^2+k(t), \qquad y = y(t),$$

where $k(t)$ and $y(t)$ and their first two derivatives vanish at the origin. Proceeding as in the derivation of (5),

$$\frac{\ddot{y}}{\ddot{x}} = \left[\frac{v^2}{f}+m(x)\right]g''+g', \tag{25}$$

where $m(x)\to 0$ as $x\to 0$. From (3), (4), and (25)

$$\left[\frac{v^2}{f}+m(x)\right]g''+g' = D(x)+A(x,g)\cdot g. \tag{26}$$

We apply the method in Lemmas 1 and 2 of Theorem I to (8) and (26) and find that

$$\lim_{x \to 0} \frac{\frac{v^2}{f} g'(x) + g(x)}{h(x)} = 1.$$

By a proof analogous to that used in the derivation of (15) from (13), it can be shown that

$$\lim_{x \to 0} \frac{\int_0^x h(x)dx}{\frac{v^2}{f} g(x)} = 1. \tag{27}$$

By hypothesis, $h(x)$ has generalized contact of order α. Hence, by L'Hospital's Rule, $\int_0^x h(x)dx$ has generalized contact of order $\alpha+1$. From (27), it follows that $g(x)$ has generalized contact of order $\alpha+1$ and that its departure from the common tangent varies inversely as the square of the speed of projection. The case of ordinary contact is treated similarly. This completes the proof of Theorem IX.

In a later paper, we shall extend these results to fields of force which fluctuate with the time.

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