ON THE EQUIVALENCE OF QUADRICS IN m-AFFINE n-SPACE AND ITS RELATION TO THE EQUIVALENCE OF $2m$-POLE NETWORKS*

BY

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1. **Introduction.** The recent work of Cauer† and others† in the study of equivalent $2m$-pole networks has given considerable importance to the matric study of quadratic forms under the real m-affine non-singular group of linear transformations T.

It is the purpose of this paper to exhibit a system of integer, matric, and algebraic invariants of the matrix A of the n-ary quadratic form F, under the m-affine non-singular group of linear transformations T, by means of which necessary and sufficient conditions for the m-affine congruence with respect to T of two matrices A and B as well as the equivalence of the two corresponding forms F and G may be given, where the elements of A and T belong to a field D.

The reduction of A (and F) to canonical forms is indicated, the case where $m=2$ being exhibited in detail, because of its interest in connection with the 4-pole equivalence problem in network theory. The application of these results to the geometry of the locus $F=0$ is also indicated. If $m=0$, T is projective. In case $m=1$, and the field is real, the results of a previous paper‡ appear, in which the matrix A of the real quadric F was shown to have 4 integer invariants (arithmetic invariants) under the real 1-affine group, which are sufficient to give a complete separation of quadrics into types.

In the closing paragraph, the relation of the present paper to the theory of $2m$-pole linear electrical networks is discussed. Each invariant of the network matrix A may be given a physical interpretation and the various theorems of the present paper become theorems relating to the network. So that this paper deals essentially with the mathematical structure underlying the theory relating to electrical networks.

2. **Invariants.** Consider§ the symmetric matrix

$$A \equiv (a_{ij}) \qquad (i, j = 1, \cdots, n), \tag{2.1}$$

of the quadric

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† See references at end of paper, under (1) and (2).

‡ See references at end of paper, under (3).

§ See references, under (3).

$$F \equiv \sum_{i,j=1}^{n} a_{ij} x_i x_j, \tag{2.2}$$

under the non-singular m-affine transformations

$$\begin{aligned} x_i &= x_i' & &(i = 1, \cdots, m), \\ x_j &= \sum_{k=1}^{n} b_{jk} x_k' & &(j = m+1, \cdots, n), \end{aligned} \tag{2.3}$$

of matrix

$$T \equiv \left(\begin{array}{ccc|ccc} 1 & \cdots & 0 & 0 & \cdots & 0 \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ 0 & \cdots & 1 & 0 & \cdots & 0 \\ \hline b_{m+1,1} & \cdots & b_{m+1,m} & b_{m+1,m+1} & \cdots & b_{m+1,n} \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ b_{n1} & \cdots & b_{nm} & b_{n,m+1} & \cdots & b_{nn} \end{array} \right), \quad d(T) \neq 0, \tag{2.4}$$

where $d(T)$ is the determinant of T, and where the elements of A and T belong to a field D.

Under T, A becomes

$$\bar{A} = T' \cdot A \cdot T, \tag{2.5}$$

where T' is the transpose of T.

If $\bar{A}_{r_1 \cdots r_s}$ is $\bar{A}$ with the $r_1, \cdots, r_s$ rows and columns deleted, the r_i's being all distinct and less than or equal to m, then

$$\bar{A}_{r_1 \cdots r_s} = T'_{r_1 \cdots r_s} \cdot A_{r_1 \cdots r_s} \cdot T_{r_1 \cdots r_s}, \tag{2.6}$$

where $T_{r_1 \cdots r_s}$ and $A_{r_1 \cdots r_s}$ are T and A, respectively, with the $r_1, \cdots, r_s$ rows and columns deleted. Thus $A_{r_1 \cdots r_s}$ is an *invariant matrix* of A under T in the sense that $\bar{A}_{r_1 \cdots r_s}$ can be found either (i) by transforming and then deleting the $r_1, \cdots, r_s$ rows and columns, or (ii) by deleting the $r_1, \cdots, r_s$ rows and columns of A and T and then transforming.

Let the *ranks* and *signatures* of $A_{r_1 \cdots r_s}$ be denoted by $\rho_{r_1 \cdots r_s}$, $\sigma_{r_1 \cdots r_s}$, respectively; the $\sigma_{r_1 \cdots r_s}$ being meaningless if $A_{r_1 \cdots r_s}$ cannot be reduced to a diagonal matrix, or if the field is not ordered.

As is well known,*

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* See list of references, under (4).

THEOREM 2.1. *The $\rho, \rho_1, \cdots, \rho_{r_1 \cdots r_s}$ are integer invariants* of $A, A_1, \cdots, A_{r_1 \cdots r_s}$, respectively, and hence of A and F. If the field D is ordered, the $\sigma_{r_1 \cdots r_s}$ are integer invariants.*

By taking determinants of (2.6), it follows that

$$d(\overline{A}_{r_1 \cdots r_s}) = [d(T_{r_1 \cdots r_s})]^2 \cdot d(A_{r_1 \cdots r_s}), \tag{2.7}$$

whence

THEOREM 2.2. *The $d(A), d(A_1), \cdots, d(A_{r_1 \cdots r_s})$ are relative invariants of A and F under T.*

Likewise, it is easy to show

THEOREM 2.3. *If rows $r_1 \cdots r_t$ and columns $s_1 \cdots s_t$ be deleted from A and yield $A_{r_1 \cdots r_t}^{s_1 \cdots s_t}$; and if $T_{s_1 \cdots s_t} \equiv T_{s_1 \cdots s_t}^{s_1 \cdots s_t}$ be T with the $s_1 \cdots s_t$ rows and columns deleted, and $T'_{r_1 \cdots r_t}$ be T' with the $r_1 \cdots r_t$ rows and columns deleted, where all the s_i and r_i are less than or equal to m, then*

$$\overline{A}_{r_1 \cdots r_t}^{s_1 \cdots s_t} = T'_{r_1 \cdots r_t} \cdot A_{r_1 \cdots r_t}^{s_1 \cdots s_t} \cdot T_{s_1 \cdots s_t} \tag{2.8}$$

is an invariant matrix; and $d(A_{r_1 \cdots r_t}^{s_1 \cdots s_t})$ is a relative invariant of A and F under T.

Since $d(T_{r_1 \cdots r_t}) = d(T_{s_1 \cdots s_t}) = d(T)$,

THEOREM 2.4. *If R_1 and R_2 be any two of the above relative invariants, then*

$$I_{1,2} \equiv R_1/R_2 \tag{2.9}$$

is an absolute invariant of A and F under T.

(In case $I_{1,2}$ is indeterminate, recourse may be had to a limiting process to define and determine $I_{1,2}$.)

Thus, with each form F, of matrix A, there is associated a set of matric, integer, relative and absolute invariants.

If the transformation $T_{1, \cdots, q-1, q+1, \cdots, m}$ is insufficient to reduce $A_{1, \cdots, q-1, q+1, \cdots, m}$ to a diagonal form, then $A_{1, \cdots, q-1, q+1, \cdots, m}$ is *parabolic* and A is *q-parabolic.*

As in the paper cited under (3) in list of references,

THEOREM 2.5. *A necessary and sufficient condition that A be q-parabolic is that $\rho_{1 \cdots m} - \rho_{1, \cdots, q-1, q+1, \cdots, m} = 2$.*

Two matrices A and B in D are said to be *m-affine congruent* if and only if there exists a non-singular matrix T of type (2.4) in D such that

$$A = T' \cdot B \cdot T.$$

* Integer invariants are known as arithmetic invariants in paper (3) in list of references. The term *integer invariant* was adopted at the suggestion of Professor Arthur B. Coble.

3. **Reduction to canonical forms.** It is known* that there exists a transformation T which reduces F to a form for which $A_{1\cdots m-1}$ becomes (see paper (3) in list of references for proof)

$$\left(\begin{array}{c|cccccc} a_{mm} & 0 & 0 & \cdots & 0 & 0 \\ \hline 0 & \delta_{m+1} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \delta_{m+2} & \cdots & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \vdots & \cdot & \cdot & \cdots & \delta_{n-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \delta_n \end{array}\right), \text{ if } \nu \equiv \rho_{1\cdots m-1} - \rho_{1\cdots m} \neq 2, \tag{3.1}$$

and

$$\left(\begin{array}{c|cccccc} 0 & 0 & 0 & \cdots & 0 & 1 \\ \hline 0 & \delta_{m+1} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \delta_{m+2} & \cdots & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \vdots & \cdot & \cdot & \cdot & \delta_{n-1} & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{array}\right), \text{ if } \nu = 2. \tag{3.2}$$

If the field is real, each positive δ can be reduced to 1, and each negative δ to -1. The number of positive δ's in $A_{1\cdots m}$ is $(\rho_{1\cdots m}+\sigma_{1\cdots m})/2$ and the number of negative δ's is $(\rho_{1\cdots m}-\sigma_{1\cdots m})/2$, the remaining δ's being zero. If the field is algebraically closed, each non-zero δ can be reduced to 1. No further reduction of $A_{1\cdots m-1}$ is possible. The *parameter* a_{mm} is an absolute invariant (by Theorem 2.4).

Thus,

THEOREM 3.1. *Every symmetric matrix $A_{1\cdots m-1}$ of A with elements in a field D not of characteristic 2 is m-affine congruent in D with a diagonal matrix* (3.1) *if $\nu\neq 2$, and with a parabolic matrix* (3.2) *if $\nu=2$, the number of non-zero δ's in $A_{1\cdots m}$ being equal to the rank $\rho_{1\cdots m}$ of $A_{1\cdots m}$.*

THEOREM 3.2. *Every symmetric matrix $A_{1\cdots m-1}$ of A with elements in a real field R is m-affine congruent in R with a diagonal matrix* (3.1) *if $\nu\neq 2$, and with a parabolic matrix* (3.2) *if $\nu=2$; the number of positive δ's being $(\rho_{1\cdots m}+\sigma_{1\cdots m})/2$ and the number of negative δ's being $(\rho_{1\cdots m}-\sigma_{1\cdots m})/2$.*

* See (4) in list of references.

THEOREM 3.3. *A necessary and sufficient condition for the m-affine congruence of two matrices* $A^{(1)}_{1\cdots m-1}$ *and* $A^{(2)}_{1\cdots m-1}$ *of the symmetric matrices* $A^{(1)}$ *and* $A^{(2)}$, *whose elements belong to any algebraically closed field, is that these matrices have the same ranks* $\rho^{(1)}_{1\cdots m-1}$, $\rho^{(1)}_{1\cdots m}$, *and* $\rho^{(2)}_{1\cdots m-1}$, $\rho^{(2)}_{1\cdots m}$, *respectively, with* $\nu^{(1)}=\nu^{(2)}=2$, *and if* $\nu^{(1)}, \nu^{(2)}\neq 2$, *that in addition to the above, their parameters* $a^{(1)}_{mm}$ *and* $a^{(2)}_{mm}$ *be identical.*

THEOREM 3.4. *Two matrices* $A^{(1)}_{1\cdots m-1}$ *and* $A^{(2)}_{1\cdots m-1}$ *of the symmetric matrices* $A^{(1)}$ *and* $A^{(2)}$, *whose elements belong to a real field R, are m-affine congruent in R if and only if these matrices have the same ranks and signatures,* $\rho^{(1)}_{1\cdots m-1}$, $\rho^{(1)}_{1\cdots m}$, $\sigma^{(1)}_{1\cdots m-1}$, $\sigma^{(1)}_{1\cdots m}$ *and* $\rho^{(2)}_{1\cdots m-1}$, $\rho^{(2)}_{1\cdots m}$, $\sigma^{(2)}_{1\cdots m-1}$, $\sigma^{(2)}_{1\cdots m}$, *respectively, with* $\nu^{(1)}=\nu^{(2)}=2$; *and if* $\nu^{(1)}, \nu^{(2)}\neq 2$, *that in addition to the above, their parameters* $a^{(1)}_{mm}$ *and* $a^{(2)}_{mm}$ *be identical.*

THEOREM 3.5. *The quadratic form* $F_{1\cdots m-1}$ *of* F *can be reduced by a nonsingular m-affine transformation to the form*

$$(3.3)\qquad a_{mm}x_m^2+\sum_{j=m+1}^{n}\delta_j x_j^2, \qquad \text{if } \nu\neq 2,$$

and to the form

$$(3.4)\qquad 2x_1x_n+\sum_{j=m+1}^{n}\delta_j x_j^2, \qquad \text{if } \nu=2.$$

THEOREM 3.6. *A necessary and sufficient condition for the m-affine equivalence of two quadratic forms* $F^{(1)}_{1\cdots m-1}$, $F^{(2)}_{1\cdots m-1}$ *of forms* $F^{(1)}$ *and* $F^{(2)}$, *with elements in a field D, is that their matrices* $A^{(1)}_{1\cdots m-1}$ *and* $A^{(2)}_{1\cdots m-1}$ *be m-affine congruent.*

The classification of quadratics $F_{1\cdots m-1}$ can now be made as in paper (3)* in terms of the parameter a_{mm}, and the ranks (and signatures) of $A_{1\cdots m-1}$.

The above theorems hold, in a like manner, for $A_{1,\cdots,q-1,q+1,\cdots,m}$.

As an aid to the reduction, in case D is real, it is agreed that the δ's will be so ordered that all the positive δ's are followed by all the negative δ's and then by the zero δ's. No loss of generality will result.

Case $\rho_{1\cdots m}=r-m-1$. Suppose that $\delta_r=\cdots=\delta_n=0$ for $r\geqq m+1$, that is, that $A_{1\cdots m}$ is of rank $(r-m-1)$. Transformation T with $b_{jj}=1$, $j=1,\cdots,n$; $a_{sr}+b_{sr}\cdot\delta_s=0$, $r=1,\cdots,m-1$, $s=m+1,\cdots,n$; $b_{sr}=0$ for $s=1,\cdots,m$ and $r=1,\cdots,n$; $b_{sr}=0$ for $s=m+1,\cdots,n$, $r=m,\cdots,n$, $r\neq s$; reduces A to the forms

* See list of references at end of paper.

$$
(3.5)\qquad \left(\begin{array}{cccc|ccc|ccc}
a_{11} & \cdots & a_{1,m-1} & a_{1m} & 0 & \cdots & 0 & a_{1r} & \cdots & a_{1n} \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
\vdots & & \vdots & a_{m-1,m} & 0 & \cdots & 0 & a_{m-1,r} & \cdots & a_{m-1,n} \\
a_{m1} & \cdots & a_{m,m-1} & a_{mm} & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\hline
0 & \cdots & 0 & 0 & \delta_{m+1} & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
0 & \cdots & 0 & 0 & 0 & \cdots & \delta_{r-1} & 0 & \cdots & 0 \\
\hline
a_{r1} & \cdots & a_{r,m-1} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
a_{n1} & \cdots & a_{n,m-1} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0
\end{array}\right), \quad \text{if } \nu \neq 2,
$$

and

$$
(3.6)\qquad \left(\begin{array}{cccc|ccc|ccc}
a_{11} & \cdots & a_{1,m-1} & a_{1m} & 0 & \cdots & 0 & a_{1r} & \cdots & a_{1n} \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
\vdots & & \vdots & a_{m-1,m} & 0 & \cdots & 0 & a_{m-1,r} & & a_{m-1,n} \\
a_{m1} & \cdots & a_{m,m-1} & 0 & 0 & \cdots & 0 & 0 & \cdots 0 & 1 \\
\hline
0 & \cdots & 0 & 0 & \delta_{m+1} & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
0 & \cdots & 0 & 0 & 0 & \cdots & \delta_{r-1} & 0 & \cdots & 0 \\
\hline
a_{r1} & \cdots & a_{r,m-1} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
\vdots & & \vdots & 0 & \vdots & & \vdots & \vdots & & \vdots \\
a_{n1} & \cdots & a_{n,m-1} & 1 & 0 & \cdots & 0 & 0 & \cdots & 0
\end{array}\right), \quad \text{if } \nu = 2.
$$

The case where $m=2$ will now be considered in detail. (This case is of interest in the 4-pole equivalence problem.)

Case $m=2$; $\nu=\rho_1-\rho_{12}\neq 2$. If $a_{1n}\neq 0$, transformation T with

$$
a_{1n}\cdot x_n = \left(-\frac{a_{11}}{2}\bar{x}_1 - a_{12}\bar{x}_2 - a_{1r}\bar{x}_r - a_{1,r+1}\cdot\bar{x}_{r+1} - \cdots - a_{1,n-1}\bar{x}_{n-1} + \bar{x}_n\right),
$$

$$
x_j = \bar{x}_j \qquad (j = 1, \cdots, n-1),
$$

reduces (3.5), $m=2$, to the form

$$
\left(\begin{array}{cc|ccc|cccc}
0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1\\
0 & a_{22} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0\\
\hline
0 & 0 & \delta_3 & \cdots & 0 & 0 & \cdots & & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & & & \vdots\\
0 & 0 & 0 & \cdots & \delta_{r-1} & 0 & \cdots & & 0\\
\hline
0 & 0 & 0 & \cdots & 0 & 0 & \cdots & & 0\\
\vdots & \vdots & \vdots & & \vdots & \vdots & & & \vdots\\
0 & 0 & & & & & & & \\
1 & 0 & 0 & \cdots & 0 & 0 & \cdots & & 0
\end{array}\right)
\equiv
\left(\begin{array}{cc|c|cccc}
0 & 0 & & 0 & \cdots & 0 & 1\\
0 & a_{22} & 0 & 0 & \cdots & 0 & 0\\
\hline
 & & & & & & \\
\multicolumn{2}{c|}{0} & \delta & & & 0 & \\
 & & & & & & \\
\hline
0 & 0 & & & & & \\
\vdots & \vdots & 0 & & & 0 & \\
0 & 0 & & & & & \\
1 & 0 & & & & &
\end{array}\right). \tag{3.7}
$$

It is easy to show that no further reduction is possible which preserves this form.

If $a_{1n}=a_{1,n-1}= \cdots =a_{1,n-k-1}=0$, $a_{1,n-k}\neq 0$, $(n-k)\geqq r$, (3.5) with $m=2$ may be reduced to a form similar to (3.7), whence by a simple transformation to form (3.7).

If $a_{1n}= \cdots =a_{1r}=0$, then A becomes

$$
\left(\begin{array}{cc|c|c}
a_{11} & a_{12} & & \\
 & & 0 & 0\\
a_{21} & a_{22} & & \\
\hline
\multicolumn{2}{c|}{0} & \delta & 0\\
\hline
\multicolumn{2}{c|}{0} & 0 & 0
\end{array}\right). \tag{3.8}
$$

No further reduction is possible.

Case $m=2$, $\nu=2$. If $a_{1,n-1}\neq 0$, the transformation

$$
\begin{aligned}
a_{1,n-1}x_{n-1} &= (-\ a_{11}/2)\ \bar{x}_1 - a_{12}\bar{x}_2 - a_{1r}\bar{x}_r - a_{1,r+1}\bar{x}_{r+1}\\
&\qquad - \cdots - a_{1,n-2}\bar{x}_{n-2} + \bar{x}_{n-1} - a_{1n}\bar{x}_n),\\
x_j &= \bar{x}_j \qquad\qquad (j = 1, \cdots , n-2, n),
\end{aligned}
$$

reduces (3.6), with $m=2$, to the form

$$
\left(\begin{array}{cc|c|ccccc}
 & & & 0 & \cdots & 0 & 1 & 0\\
\multicolumn{2}{c|}{0} & 0 & 0 & \cdots & 0 & 0 & 1\\
\hline
\multicolumn{2}{c|}{0} & \delta & & & 0 & & \\
\hline
0 & 0 & & & & & & \\
\vdots & \vdots & & & & & & \\
0 & 0 & 0 & & & 0 & & \\
1 & 0 & & & & & & \\
0 & 1 & & & & & &
\end{array}\right), \tag{3.9}
$$

and no further reduction is possible.

If $a_{1,n-1}= \cdots =a_{1,k-1}=0$, $a_{1k}\neq 0$, $k\geqq r$, (3.6) may be reduced to a form similar to (3.9), whence to form (3.9).

If $a_{1,n-1}= \cdots =a_{1r}=0$, then (3.6), with $m=2$, becomes

$$\left(\begin{array}{cc|c|cc}
a_{11} & a_{12} & 0 & 0\cdots 0 & a_{1n}\\
a_{21} & 0 & & 0\cdots 0 & 1\\ \hline
\multicolumn{2}{c|}{0} & \delta & \multicolumn{2}{c}{0}\\ \hline
0 & 0 & & & \\
\vdots & \vdots & 0 & \multicolumn{2}{c}{0}\\
0 & 0 & & & \\
a_{n1} & 1 & & &
\end{array}\right). \tag{3.10}$$

The transformation $x_n=-a_{21}\bar{x}_1+\bar{x}_n$ reduces (3.10) to the form

$$\left(\begin{array}{cc|c|cc}
a_{11} & 0 & 0 & 0\cdots 0 & a_{1n}\\
0 & 0 & & 0\cdots 0 & 1\\ \hline
\multicolumn{2}{c|}{0} & \delta & \multicolumn{2}{c}{0}\\ \hline
0 & 0 & & & \\
\vdots & \vdots & 0 & \multicolumn{2}{c}{0}\\
0 & 0 & & & \\
a_{n1} & 1 & & &
\end{array}\right), \quad \text{if } a_{1n}\neq 0, \tag{3.11}$$

and to the form

$$\left(\begin{array}{cc|c|cc}
a_{11} & 0 & 0 & 0\cdots 0 & 0\\
0 & 0 & & 0\cdots 0 & 1\\ \hline
\multicolumn{2}{c|}{0} & \delta & \multicolumn{2}{c}{0}\\ \hline
0 & 0 & & & \\
\vdots & \vdots & 0 & \multicolumn{2}{c}{0}\\
0 & 0 & & & \\
0 & 1 & & &
\end{array}\right), \quad \text{if } a_{1n} = 0. \tag{3.12}$$

No further reduction is possible.

Each of the forms (3.7), (3.8), (3.9), (3.11), (3.12) may be subdivided, in case of the ordered (and real) field, according to the signature of A_{12} (which is of rank $(r-3)$). For each value of $r=3, 4, \cdots, (n+1)$, there corresponds a set of forms (3.7), $\cdots$, (3.12). By Theorem 2.4, the parameters a_{11}, a_{12}, a_{22},

a_{1n} are absolute invariants.* It can be shown that the various forms thus obtained cannot be reduced any further with preservation of the invariance of their forms. The forms thus obtained are called *canonical forms*. The various parameters $a_{11}, \cdots, a_{1n}$ may or may not vanish, the conditions for this being indicated in the following table. The separation into forms can be made according to the ranks (and signatures) of A as indicated.

CLASSIFICATION OF MATRIX A (AND FORM F) FOR CASE $m=2$

TABLE I

$\rho_{12}=r-3,\ r=3,\ 4, \cdots, n+1$

$\rho_1-\rho_{12}$	$\rho_2-\rho_{12}$	ρ	ρ_1	ρ_2	ρ_1^2	a_{11}	a_{12}	a_{22}	a_{1n}	Form
$\neq 2$	2		$r-2$					$\neq 0$		3.7
$\neq 2$	2		$r-3$					$=0$		3.7
$\neq 2$	$\neq 2$		$r-2$	$r-2$	$r-2$	$\neq 0$	$\neq 0$	$\neq 0$		3.8
$\neq 2$	$\neq 2$		$r-3$	$r-2$	$r-2$	$\neq 0$	$\neq 0$	$=0$		3.8
$\neq 2$	$\neq 2$		$r-3$	$r-2$	$r-3$	$\neq 0$	$=0$	$=0$		3.8
$\neq 2$	$\neq 2$		$r-2$	$r-2$	$r-3$	$\neq 0$	$=0$	$\neq 0$		3.8
$\neq 2$	$\neq 2$		$r-2$	$r-3$	$r-2$	$=0$	$\neq 0$	$\neq 0$		3.8
$\neq 2$	$\neq 2$		$r-3$	$r-3$	$r-2$	$=0$	$\neq 0$	$=0$		3.8
$\neq 2$	$\neq 2$		$r-3$	$r-3$	$r-3$	$=0$	$=0$	$=0$		3.8
$\neq 2$	$\neq 2$		$r-2$	$r-3$	$r-3$	$=0$	$=0$	$\neq 0$		3.8
2	2	$r+1$								3.9
2	2	r				$\neq 0$			$(\neq 0)$	3.11
2	2	$r-1$				$=0$			$(\neq 0)$	3.11
2	$\neq 2$	r		$(r-2)$		$\neq 0$				3.12
2	$\neq 2$	$r-1$		$(r-3)$		$=0$				3.12

Each form is subdivided according to signature of A_{12}, if the field is real.

Case $m=m$. The reduction of A to canonical form for the m-affine case is done in a manner similar to that used above in the case where $m=2$. Reduce A to the forms for which $A_{1\cdots m-2}$ assume the canonical forms exhibited above for $m=2$. Continue the reduction by T, as was done above, until no further reduction of $A_{1\cdots m-3}$ can be made. Next, repeat the operations on A, by means of T, with respect to $A_{1\cdots m-4}$, etc., until no further reduction is possible. In this manner certain canonical forms $C_1, C_2, \cdots$ for matrix A are found, with corresponding forms $g_1, g_2, \cdots$ for the form F. The following theorems are evident:

* Suppose $r=n+1$, and F belongs to form (3.8). Then

$$I \equiv d(A_2)/d(A_{12}) = (a_{11}\delta_3 \cdots \delta_{r-1}\delta_r \cdots \delta_n)/(\delta_3 \cdots \delta_{r-1}\delta_r \cdots \delta_n).$$

Note that $\lim_{\delta\to 0} I = a_{11}$. If $d(A_2)/d(A_{12})$ is indeterminate, as is the case when $r<n+1, \cdots$, define I to be the $\lim_{\delta\to 0}[d(A_2)/d(A_{12})]$. Similarly, the other parameters $a_{11}, \cdots, a_{1n}$ may be handled. (See Theorem 2.4.)

THEOREM 3.7. *The matrix A of F can be reduced by a non-singular m-affine transformation to one of the forms $C_1, C_2, \cdots$, according to the ranks (and signatures in real field) of the invariant matrices of A, with corresponding canonical forms $g_1, g_2, \cdots$ for F, as indicated in Table* I. *In case $m=2$ these forms are*

$$f_1 = a_{22}x_2^2 + 2x_1x_n + \sum_{j=3}^{r-1} \delta_j x_j^2,$$

$$f_2 = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2 + \sum_{j=3}^{r-1} \delta_j x_j^2,$$

$$f_3 = 2x_1x_{n-1} + 2x_2x_n + \sum_{j=3}^{r-1} \delta_j x_j^2,$$

$$f_4 = a_{11}x_1^2 + 2a_{1n}x_1x_n + 2x_2x_n + \sum_{j=3}^{r-1} \delta_j x_j^2,$$

$$f_5 = a_{11}x_1^2 + 2x_2x_n + \sum_{j=3}^{r-1} \delta_j x_j^2.$$

THEOREM 3.8. *A necessary and sufficient condition for the m-affine congruence of two matrices $A^{(1)}$ and $A^{(2)}$ whose elements belong to the real field is that their invariant matrices have the same ranks and signatures, $\rho^{(1)}, \rho_1^{(1)}, \cdots, \sigma_{12}^{(1)}, \cdots$ and $\rho^{(2)}, \rho_1^{(2)}, \cdots, \sigma_{12}^{(2)}, \cdots$, respectively; and that their parameters (in case they appear in the canonical form dictated by the ranks named above) $a_{11}^{(1)}, a_{12}^{(1)}, a_{22}^{(1)}, a_{1n}^{(1)}, \cdots$ and $a_{11}^{(2)}, a_{12}^{(2)}, a_{22}^{(2)}, a_{1n}^{(2)}, \cdots$ be identical. If the elements of $A^{(1)}$ and $A^{(2)}$ belong to an algebraically closed field, the above holds without the signatures.*

THEOREM 3.9. *A necessary and sufficient condition for the m-affine equivalence of quadratic forms $F^{(1)}$ and $F^{(2)}$, with elements in a field, is that their matrices $A^{(1)}$ and $A^{(2)}$ be m-affine congruent in that field.*

Theorems similar to Theorems 3.7, 3.8, 3.9 hold for any invariant matrix of A; e.g., Theorems 3.1 to 3.6.

4. **Application to the locus $F=0$.** In a manner similar to that given in paper (3),* a classification of locus $F=0$ can be made, the numerical value of the signature being used instead of the signature in case the field is real. With the various interpretations that can be placed upon the transformation T, a geometric study of the quadrics can be made.

5. **Relation to the theory of linear networks.** Consider a linear network of a finite number n of meshes. Let R_{st}, L_{st}, D_{st}, $s \neq t$ (real numbers) be the mutual circuit parameters (the resistance, inductance and *elastance*, respec-

* See list of references at end of paper.

tively), between mesh s and mesh t; and R_{ss}, L_{ss}, D_{ss} the total circuit parameters of mesh s. If $I_1, \cdots, I_m$ be the (complex numbers) currents through the m-terminal pairs of $2m$-poles and $E_1, \cdots, E_m$ (complex numbers) be the corresponding electromotive forces, subject to the restriction that the currents through the terminals be linearly independent, and if I_j be the current in jth mesh, then the Kirchoff equations of the network may be written

$$A(I) = (E), \tag{5.1}$$

where $(I) = (I_1, \cdots, I_n)$ and $(E) = (E_1, \cdots E_m, 0, \cdots, 0)$ are column matrices and $A = (a_{st})$ is the *network matrix*, with

$$a_{st} = R_{st} + L_{st}\lambda + D_{st}\lambda^{-1}, \tag{5.2}$$

and $\lambda = i\omega$, $i^2 = -1$, the "imaginary frequency parameter."

The total power loss, instantaneous magnetic energy and electrostatic energy for the complete network are given by the (symmetric) quadratic forms

$$\mathcal{R} = \sum_{j,k}^{n} R_{jk}\cdot I_j\cdot I_k, \quad \mathcal{L} = \tfrac{1}{2}\sum_{j,k}^{n} L_{jk}\cdot I_j\cdot I_k, \quad \mathcal{D} = \tfrac{1}{2}\sum_{j,k}^{n} D_{jk}\cdot Q_j\cdot Q_k, \tag{5.3}$$

where Q_j is the instantaneous charge in mesh j, I_j is the corresponding current, and $L_{jk} = L_{kj}$, $D_{jk} = D_{kj}$, $R_{jk} = R_{kj}$.

The pencil of forms

$$\mathcal{A} = \mathcal{R} + 2\mathcal{L}\lambda + 2\mathcal{D}\lambda^{-1} \tag{5.4}$$

has the (energy) matrix

$$A = R + L\lambda + D\lambda^{-1}. \tag{5.5}$$

Thus the energy matrix is identical with the network matrix A of system (5.1).

If $d(A) \neq 0$, that is, A is of rank n, (5.1) may be solved for the currents

$$(I) = A^{-1}(E). \tag{5.6}$$

Let $(I)_m$ and $(E)_m$ denote (I) and (E), respectively, with all but the first m rows and columns deleted. If $Y \equiv (Y_{st})$ be A^{-1} with all but the first m rows and columns deleted, then

$$(I)_m = Y(E)_m. \tag{5.7}$$

Cauer has called Y a *characteristic coefficient matrix* of the network A. Two $2m$-pole linear networks are *equivalent* if, for all frequencies $(\omega = -\lambda i)$, they have equal characteristic coefficient matrices $Y(\lambda)$ (or $Z(\lambda)$); i.e., for all ω, they have equal electrical characteristics.

To each $2m$-pole linear network of matrix A, there corresponds a set of equivalent networks (1)* whose matrices (3)* may be obtained one from the other by a non-singular linear transformation (5)* of matrix T. If the driving-point currents (and charges) across the terminal pairs in meshes $1, \cdots, m$ be left invariant, T is m-affine.

It is evident that the methods and results given in the earlier parts of this paper are available for use in the study of linear networks (3).*

By Theorem 2.4, the elements of Y

$$(5.8) \qquad Y_{st} = d(A_t^s)/d(A) \qquad (s, t = 1, \cdots, m)$$

are absolute invariants of A (and F) under non-singular m-affine linear transformations of matrix T. Y_{st}, $s \neq t$, is the short circuit transfer admittance between terminal pairs s and t and Y_{ss} is the short circuit driving point admittance at terminal pairs s. In fact, Y is an *absolutely invariant matrix* of A (and F) under T. The ranks of A, A_t^s, Y are integer invariants. The rank of A^{-1} is the number of linearly independent mesh currents; the rank of Y, the number of linearly independent driving-point currents.

In view of the assumption made upon the independence of currents $I_1, \cdots, I_m$, the rank of Y must equal m. Whence (5.7) gives

$$(5.9) \qquad (E)_m = Z(I)_m$$

where

$$(5.10) \qquad Z \equiv (z_{st}) = Y^{-1} \qquad (s, t = 1, \cdots, m).$$

Z is also known as a *characteristic coefficient matrix* of the network.

Equation (5.9) may also be obtained from (5.1) by eliminating the inner currents $I_{m+1}, \cdots, I_n$, provided the rank of $A_{1 \cdots m}$ is $(n-m)$, in which case $I_{m+1}, \cdots, I_n$ are linear functions of $I_1, \cdots, I_m$. The number of linearly independent inner mesh currents is equal to the rank of $A_{1 \cdots m}$.

Evidently, each z_{st} is an absolute invariant of A, whence Z is an absolutely invariant matrix of A. In fact, z_{st}, $s \neq t$, is the open circuit transfer impedance between terminal pairs s and t; and z_{ss} the open circuit driving point impedance at terminal pairs s. The rank of Z is equal to the number of linearly independent e.m.f.'s imposed across the terminal pairs $1, \cdots, m$.

The various invariant matrices and their several invariants may be given physical interpretations. For example, A_k^k $(k \leq m)$ is the matrix of the network derived from a network of matrix A by removing the imposed e.m.f. in mesh k and leaving the circuit open across terminal pairs k; i.e., the network matrix corresponding to the original network with mesh k removed.

* Numbers refer to list of references at end of paper.

A_k^k $(k>m)$ is the network matrix derived from A upon removing mesh k from the original network.

If the terminal pairs in mesh k are shorted, $k \leqq m$, the network becomes a $2(m-1)$-pole network and the corresponding mathematical theory is that of $(m-1)$-affine n-space, provided I_k is no longer held invariant. If mesh $k(k>m)$ be opened and an e.m.f. inserted, this increases the number of pole pairs by one and the mathematical theory becomes that of $(m+1)$-affine n-space.

If in addition to the invariance of the currents through terminal pairs $1, \cdots, m$, it is required to preserve the invariance of the current in an inner mesh $k(k>m)$, a restriction on T is imposed which dictates the theory used for $(m+1)$-affine n-space.

The various theorems given in the earlier part of this paper may be interpreted physically. For example, Theorem 3.6 may be interpreted as a theorem on the equivalence of two networks with one type of circuit parameter. The various canonical forms may be interpreted in terms of *canonical network forms* and the parameters appearing therein interpreted in terms of circuit parameters. The detailed treatment of the 2-affine case given above should be of particular interest because of the importance of 4-pole networks.

Two networks may be "equivalent" and yet one or both of them may not be physically realizable. Necessary and sufficient conditions for the physical realizability of a network corresponding to forms (5.3), in the case of networks containing but two types of circuit parameters, have been discussed by Cauer. The forms (5.3) used in existing theory have been positive definite because the networks considered have been passive. Other restrictions such as $2|R_{11}| \geqq \sum_{j=1}^{n} |R_{1j}|$ are necessitated by the nature of the physical problem. Should some future development occur that would make the study of non-passive circuits desirable, the generalizations of this paper are applicable.

It is evident that two $2m$-pole networks with a different number of meshes, of numbers p and q respectively, may be equivalent. The theory corresponding to this situation is really that of one quadric in p-space embedded in a q-space. The theorems of this paper include this possibility.

In conclusion, it should be noted that this paper deals essentially with the mathematical structure underlying the theory relating to electrical networks.

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