LINEAR TRANSFORMATIONS IN $\mathfrak{L}_p$, $p>1$†

BY

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INTRODUCTION

In this paper, we study linear and closed linear transformations in $\mathfrak{L}_p$. $\mathfrak{L}_p$ is the space of measurable complex-valued functions defined in the interval (0, 1) for which

$$\int_0^1 |f|^p dx$$

exists. We shall use certain results which may be found in Banach's treatise§ and which are there shown for the space of real-valued functions but which can easily be extended to the space of complex-valued functions. We follow the notation of (B) in general; otherwise an explicit definition is given.

We shall study transformations by means of their "graphs," i.e., the set of pairs $\{f, Tf\}$ in the product space $\mathfrak{L}_p \times \mathfrak{L}_p$. The graph is used in (B) at one place, but Banach confines his attention to limited transformations, while the set of linear and even closed linear transformations is known to be a wider class.‖ The graph has also been used by J. von Neumann¶ to obtain certain results for $\mathfrak{L}_2$ which we generalize to $\mathfrak{L}_p$.

The graph permits us to study linear transformations by studying linear manifolds, and in §1 we obtain the relationship between the operations of taking the orthogonal complement of a linear manifold ($\mathfrak{M}^\perp$; cf. Definition 1.1), of forming the intersection with a space of higher index, and of closure in a space of lower index. In §2, we obtain the analogous results for transformations. In §3, we apply the above results to the study of the closure in $\mathfrak{L}_p$, $1<p<2$, of transformations in $\mathfrak{L}_2$. In particular, projections are discussed in terms of the closure of their ranges. We also generalize the notion of an Hermitian transformation on the basis of the familiar inclusion $H^* \supseteq H$.††

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§ *Théorie des Opérations Linéaires*, Warsaw, 1932. This reference will be denoted by (B) in what follows.

‖ For a consideration of linear and closed linear transformations in $\mathfrak{L}_2$, see Stone, *Linear Transformations in Hilbert Space*, American Mathematical Society Colloquium Publications, vol. 15, 1932. We shall refer to this work as (S).

¶ Annals of Mathematics, (2), vol. 33 (1932), pp. 294–310. We shall refer to this memoir as (N).

†† We prefer to use the perpendicular instead of the adjoint, but one class is obtained from the other by multiplication by i.

In §4, an example is given to illustrate the application of the above to the study of a particular transformation. A second example shows that even for the bounded closure of a self-adjoint transformation in $\mathfrak{L}_p$, the spectral theory does not hold in general.

A word as to the extension to $\mathfrak{L}_p$ of the results of (B) for real functions spaces. The only essential addition, it will be found, is the application of (B), Theorem 2, chapter IV, p. 55. It may be stated as follows. *Let $\mathfrak{M}$ be a closed linear manifold (cf. Definition 1.1) in $\mathfrak{L}_p$. Let $f(y)$ be a limited linear (complex-valued) functional defined on $\mathfrak{M}$. Let C be such that $|f(y)| \leqq C\|y\|_p$, $y \epsilon \mathfrak{M}$. There exists a linear functional F defined on $\mathfrak{L}_p$, such that $F(y) = f(y)$, $y \epsilon \mathfrak{M}$, $|F(x)| \leqq C\|x\|_p$, for all $x \epsilon \mathfrak{L}_p$.*

We sketch the proof. $\mathfrak{L}_p$ may be put in correspondence with a Banach space $\mathfrak{L}_p^2$, of pairs of real functions $\{f_1, f_2\}$ with the metric $(\int_0^1(|f_1|^2 + |f_2|^2)^{p/2}dx)^{1/p}$, i.e., if $y \epsilon \mathfrak{L}_p$, then $y \sim \{f_1, f_2\}$ if $y = f_1 + if_2$. A linear manifold $\mathfrak{M}$ in $\mathfrak{L}_p$, however, corresponds to a linear vector subspace $\mathfrak{M}^2$ of $\mathfrak{L}_p^2$, which is invariant under the operation $U\{f_1, f_2\} = \{-f_2, f_1\}$, the operation which corresponds to multiplication by i in $\mathfrak{L}_p$. Now, corresponding to $f(y)$, we have a complex-valued linear functional on $\mathfrak{M}^2$, $f(\{f_1, f_2\}) = \Re f(\{f_1, f_2\}) + i\Im f(\{f_1, f_2\})$. Since $f(U(\{f_1, f_2\})) = if(\{f_1, f_2\}) = -\Im f(\{f_1, f_2\}) + i\Re f(\{f_1, f_2\}) = \Re f(\{-f_2, f_1\}) + i\Im f(\{-f_2, f_1\})$, $\Im f(\{f_1, f_2\}) = -\Re f(\{-f_2, f_1\})$ or $f(\{f_1, f_2\}) = \Re f(\{f_1, f_2\}) - i\Re f(\{-f_2, f_1\})$.

We now extend by the above mentioned theorem of (B) the linear functional $\Re f(\{f_1, f_2\})$ to the whole space $\mathfrak{L}_p^2$. Calling the extension $\Re(\{f_1, f_2\})$, we let $F(\{f_1, f_2\}) = \Re(\{f_1, f_2\}) - i\Re(\{-f_2, f_1\})$ and by elementary considerations one can show that F corresponds to a complex-valued linear functional $F(z)$ on $\mathfrak{L}_p$.

Now since multiplication in $\mathfrak{L}_p$, by a number of the form $e^{i\theta}$, corresponds to a unitary transformation in $\mathfrak{L}_p^2$, one sees that the bound of $\Re(\{f_1, f_2\})$ in $\mathfrak{L}_p^2$ is exactly the bound of $F(z)$ in $\mathfrak{L}_p$. For if $F(z) = ce^{i\theta}$, then for $z' = e^{-i\theta}z = f_1' + if_2'$, $F(z') = \Re(\{f_1', f_2'\}) = |F(z)|$, while $\|\{f_1', f_2'\}\| = \|z'\|_p = \|z\|_p$. This implies that the bound of the linear functional $\Re(\{f_1, f_2\})$ on $\mathfrak{L}_p^2$ is greater than or equal to the bound of $F(z)$ on $\mathfrak{L}_p$, and of course this means the equality.

The above reasoning may be applied to $f(z)$ to show that its bound is the same as that of $\Re f(\{f_1, f_2\})$ and since in the extension process the bound is not altered, we have that the bound of F is the same as that of f.

1. Linear manifolds in $\mathfrak{L}_p$

Definition 1.1. *A set of functions, $\mathfrak{M}$, such that $\mathfrak{M} \subseteq \mathfrak{L}_r$, and such that if f and $g \epsilon \mathfrak{M}$, then $af + bg \epsilon \mathfrak{M}$ where a and b are any two complex constants, is called a linear manifold in $\mathfrak{L}_r$. If $\mathfrak{M}$ is also closed, then $\mathfrak{M}$ is called a closed linear* *manifold in* $\mathfrak{L}_r$. *If* $\mathfrak{M}$ *is a linear manifold* $\mathfrak{L}_r$, *then the closure of* $\mathfrak{M}$ *in* $\mathfrak{L}_r$ *is a closed linear manifold in* $\mathfrak{L}_r$, *and is denoted by* $[\mathfrak{M}]^r$. *If* $\mathfrak{M}$ *is a linear manifold in* $\mathfrak{L}_r$, *and if* $t<r$, *then* $\mathfrak{M}$ *is also a linear manifold in* $\mathfrak{L}_t$; *also if* $t>r$, *we denote by* $\mathfrak{M}\cdot\mathfrak{L}_t$ *the linear manifold of all those functions which are in both* $\mathfrak{M}$ *and* $\mathfrak{L}_t$. *If* $1/p+1/p'=1$ *and if* $\mathfrak{M}$ *is a linear manifold in* $\mathfrak{L}_p$, *then the set of functions* f *in* $\mathfrak{L}_{p'}$, *such that* $\int_0^1 g\bar{f}dx=0$ *for every* $g\epsilon\mathfrak{M}$, *will be denoted by* $\mathfrak{M}^\perp$.

It follows from the Hölder inequality that $\mathfrak{M}^\perp$ is a closed linear manifold. In what follows, $\mathfrak{M}$ will denote a linear manifold.

THEOREM 1.1. *If* $\mathfrak{M}\subseteq\mathfrak{L}_s$ *is a closed linear manifold, then* $\mathfrak{M}\cdot\mathfrak{L}_r$, $r>s$, *is closed in* $\mathfrak{L}_r$. *If* $\mathfrak{M}$ *is a linear manifold in* $\mathfrak{L}_s$ *and* $t<s$, *then* $[\mathfrak{M}]^t\cdot\mathfrak{L}_s\supseteq[\mathfrak{M}]^s$.

This follows easily from the fact that if $r>s$, and if $f_n\rightarrow f$ in $\mathfrak{L}_r$, then $f_n\rightarrow f$ in $\mathfrak{L}_s$, which is shown by using the Hölder inequality.

THEOREM 1.2. *If* $t\leqq r\leqq q$ *and* $\mathfrak{M}\subseteq\mathfrak{L}_q$, *then* $[[\mathfrak{M}]^r]^t=[\mathfrak{M}]^t$. *Also if* $[\mathfrak{M}]^t\cdot\mathfrak{L}_q=\mathfrak{M}$, *then* $[\mathfrak{M}]^r\cdot\mathfrak{L}_q=\mathfrak{M}$.

Since $[\mathfrak{M}]^r\supseteq\mathfrak{M}$, we have $[[\mathfrak{M}]^r]^t\supseteq[\mathfrak{M}]^t$. But we also have $[\mathfrak{M}]^r\subseteq[\mathfrak{M}]^t$, from the fact mentioned in the proof of the previous theorem. Hence $[[\mathfrak{M}]^r]^t\subseteq[\mathfrak{M}]^t$. Hence $[[\mathfrak{M}]^r]^t=[\mathfrak{M}]^t$. Also when $[\mathfrak{M}]^t\cdot\mathfrak{L}_q=\mathfrak{M}$, by Theorem 1.1, we have

$$\mathfrak{M} = [\mathfrak{M}]^t\cdot\mathfrak{L}_q = [[\mathfrak{M}]^r]^t\cdot\mathfrak{L}_q\supseteq[\mathfrak{M}]^r\cdot\mathfrak{L}_q\supseteq[\mathfrak{M}]^q\supseteq\mathfrak{M}.$$

Hence $[\mathfrak{M}]^r\cdot\mathfrak{L}_q=\mathfrak{M}$.

THEOREM 1.3. *If* $\mathfrak{M}$ *is a linear manifold in* $\mathfrak{L}_p$, *then* $\mathfrak{M}^\perp$ *is a closed linear manifold and* $(\mathfrak{M}^\perp)^\perp=[\mathfrak{M}]^p$. *If* $\mathfrak{M}\supset\mathfrak{M}'$, *and* $\mathfrak{M}'$ *is closed in* $\mathfrak{L}_p$, *then* $\mathfrak{M}^\perp\subset\mathfrak{M}'^\perp$. *If* $\mathfrak{M}\supseteq\mathfrak{M}'$, *then* $\mathfrak{M}^\perp\subseteq\mathfrak{M}'^\perp$.

We have indicated the proof that $\mathfrak{M}^\perp$ is a closed linear manifold. It follows readily from the definition that $(\mathfrak{M}^\perp)^\perp\supseteq[\mathfrak{M}]^p$. Now suppose there is a function $f\epsilon\mathfrak{L}_p$, $f\neq 0$, which is in $(\mathfrak{M}^\perp)^\perp$, but not in $[\mathfrak{M}]^p$. By (B), chapter IV, p. 57, lemma, and chapter IV, p. 64, there exists a function $g(x)\epsilon\mathfrak{L}_{p'}$, such that

$$\int_0^1 g\bar{f}dx = 1; \qquad \int_0^1 g\bar{h}dx = 0 \text{ for all } h\epsilon[\mathfrak{M}]^p.$$

Hence $g\epsilon\mathfrak{M}^\perp$, but we also see that f cannot be in $(\mathfrak{M}^\perp)^\perp$, contrary to hypothesis, hence $(\mathfrak{M}^\perp)^\perp=[\mathfrak{M}]^p$.

Now it is easy to see that $\mathfrak{M}^\perp\subseteq\mathfrak{M}'^\perp$. But if $\mathfrak{M}\supset\mathfrak{M}'$ and $\mathfrak{M}^\perp=\mathfrak{M}'^\perp$, then $\mathfrak{M}\subseteq[\mathfrak{M}]^p=(\mathfrak{M}^\perp)^\perp=(\mathfrak{M}'^\perp)^\perp=[\mathfrak{M}']^p=\mathfrak{M}'$, contrary to the assumption that $\mathfrak{M}\supset\mathfrak{M}'$.

THEOREM 1.4. *If* $p\geqq r$, $p'\leqq r'$, $1/p+1/p'=1$, $1/r+1/r'=1$, *and* $\mathfrak{M}$ *is a linear manifold in* $\mathfrak{L}_p$, *then* $([\mathfrak{M}]^r)^\perp=\mathfrak{M}^\perp\cdot\mathfrak{L}_{r'}$.

$\mathfrak{M}^{\perp}\cdot\mathfrak{L}_{r'}$ is orthogonal to $\mathfrak{M}$ considered in $\mathfrak{L}_r$ and hence to $[\mathfrak{M}]^r$. Thus $([\mathfrak{M}]^r)^{\perp}\supseteq\mathfrak{M}^{\perp}\cdot\mathfrak{L}_{r'}$. But $([\mathfrak{M}]^r)^{\perp}$, which is in $\mathfrak{L}_{r'}$ hence in $\mathfrak{L}_{p'}$, is orthogonal to $\mathfrak{M}$, and hence $([\mathfrak{M}]^r)^{\perp}\subseteq\mathfrak{M}^{\perp}$ and $([\mathfrak{M}^r])^{\perp}\subseteq\mathfrak{M}^{\perp}\cdot\mathfrak{L}_{r'}$. The combination of the two inclusions gives the desired result.

THEOREM 1.5. *If $p\leqq r$, $p'\geqq r'$, $1/p+1/p'=1$, $1/r+1/r'=1$, and $\mathfrak{M}$ is a closed linear manifold in $\mathfrak{L}_p$, then* $(\mathfrak{M}\cdot\mathfrak{L}_r)^{\perp}=[\mathfrak{M}^{\perp}]^{r'}$.

Now $(\mathfrak{M}\cdot\mathfrak{L}_r)^{\perp}\supseteq\mathfrak{M}^{\perp}$ since if $f\epsilon\mathfrak{M}^{\perp}$, $f\epsilon\mathfrak{L}_{p'}\subseteq\mathfrak{L}_{r'}$, and is orthogonal to $\mathfrak{M}\cdot\mathfrak{L}_r$ for it is orthogonal to $\mathfrak{M}$. Since $(\mathfrak{M}\cdot\mathfrak{L}_r)^{\perp}$ is closed, we must also have $(\mathfrak{M}\cdot\mathfrak{L}_r)^{\perp}\supseteq[\mathfrak{M}^{\perp}]^{r'}$. But $([\mathfrak{M}^{\perp}]^{r'})^{\perp}$ is orthogonal to $[\mathfrak{M}^{\perp}]^{r'}$, hence to $\mathfrak{M}^{\perp}$, and hence must be in $(\mathfrak{M}^{\perp})^{\perp}=\mathfrak{M}$ and of course in $\mathfrak{L}_r$. Hence $\mathfrak{M}\cdot\mathfrak{L}_r\supseteq([\mathfrak{M}^{\perp}]^{r'})^{\perp}$. Theorem 1.3 now implies that $(\mathfrak{M}\cdot\mathfrak{L}_r)^{\perp}\subseteq[\mathfrak{M}^{\perp}]^{r'}$. This and our previous result imply the theorem.

THEOREM 1.6. *If the set $\mathfrak{S}$ is everywhere dense in the linear manifold $\mathfrak{M}$ in $\mathfrak{L}_r$, and $s<r$, then $\mathfrak{S}$ is everywhere dense in* $[\mathfrak{M}]^s$.

$\mathfrak{S}$ is everywhere dense in $\mathfrak{M}$ considered in $\mathfrak{L}_s$, since if a sequence $\{f_\nu\}$ is such that $f_\nu\to f$ in $\mathfrak{L}_r$, then $f_\nu\to f$ in $\mathfrak{L}_s$. $\mathfrak{M}$ is everywhere dense in $[\mathfrak{M}]^s$, hence by a well known argument $\mathfrak{S}$ is everywhere dense in $[\mathfrak{M}]^s$.

DEFINITION 1.2. *Let $\mathfrak{M}$ be a closed linear manifold in $\mathfrak{L}_p$. Let $p_1<p$. Now if for all r such that $p_1\leqq r\leqq p$, $[\mathfrak{M}]^{p_1}\cdot\mathfrak{L}_r=[\mathfrak{M}]^r$, then $\mathfrak{M}$ is said to be of simple lace between p and p_1.*

DEFINITION 1.3. *Let $\mathfrak{M}$ be a closed linear manifold in $\mathfrak{L}_p$. Let $p_1<p$ and $p_1\leqq r\leqq p$, $p_1\leqq u_i\leqq r\leqq v_i\leqq p$, $i=1, 2, 3, \cdots$. We define $[\mathfrak{M}]^r=\mathfrak{M}_1^{(r)}$; $\mathfrak{M}_2^{(r)}=[\mathfrak{M}]^{p_1}\cdot\mathfrak{L}_r$, $\mathfrak{M}_1^{(r)}(u_1)=[\mathfrak{M}]^{u_1}\cdot\mathfrak{L}_r$, $\mathfrak{M}_2^{(r)}(v_1)=[[\mathfrak{M}]^{p_1}\cdot\mathfrak{L}_{v_1}]^r$, $\mathfrak{M}_1^{(r)}(u_1, v_2)=[\mathfrak{M}_1^{(r)}(u_1)\cdot\mathfrak{L}_{v_2}]^r$, $\mathfrak{M}_2^{(r)}(v_1, u_2)=[\mathfrak{M}_2^{(r)}(v_1)]^{u_2}\cdot\mathfrak{L}_r$, and for $k=1, 2, 3, \cdots$,*

$$
\begin{aligned}
\mathfrak{M}_1^{(r)}(u_1, v_2, \cdots, v_{2k}, u_{2k+1}) &= [\mathfrak{M}_1^{(r)}(u_1, \cdots, v_{2k})]^{u_{2k+1}}\cdot\mathfrak{L}_r,\\
\mathfrak{M}_1^{(r)}(u_1, v_2, \cdots, u_{2k+1}, v_{2k+2}) &= [\mathfrak{M}_1^{(r)}(u_1, \cdots, u_{2k+1})\cdot\mathfrak{L}_{v_{2k+2}}]^r,\\
\mathfrak{M}_2^{(r)}(v_1, \cdots, u_{2k}, v_{2k+1}) &= [\mathfrak{M}_2^{(r)}(v_1, \cdots, u_{2k})\cdot\mathfrak{L}_{v_{2k+1}}]^r,\\
\mathfrak{M}_2^{(r)}(v_1, \cdots, v_{2k+1}, u_{2k+2}) &= [\mathfrak{M}_2^{(r)}(v_1, \cdots, v_{2k+1})]^{u_{2k+2}}\cdot\mathfrak{L}_r.\dagger
\end{aligned}
$$

† These are the only ways in which distinct manifolds can arise by a finite number of applications of these processes. For if we repeat the same process after the initial steps, we get the same manifold as we had applied once in a suitable way, for instance, $[\mathfrak{M}_1^{(r)}(u_1)]^{u_2}\cdot\mathfrak{L}_r=\mathfrak{M}_1^{(r)}(\min(u_1, u_2))$. For suppose $u_2\leqq u_1$. Then obviously $[\mathfrak{M}]^{u_2}\cdot\mathfrak{L}_r=[[\mathfrak{M}]^{u_1}]^{u_2}\cdot\mathfrak{L}_r\supseteq[[\mathfrak{M}]^{u_1}\cdot\mathfrak{L}_r]^{u_2}\cdot\mathfrak{L}_r$. But $[\mathfrak{M}]^{u_1}\cdot\mathfrak{L}_r\supseteq[\mathfrak{M}]^r$ and $[[\mathfrak{M}]^{u_1}\cdot\mathfrak{L}_r]^{u_2}\supseteq[[\mathfrak{M}]^r]^{u_2}=[\mathfrak{M}]^{u_2}$ or $[[\mathfrak{M}]^{u_1}\cdot\mathfrak{L}_r]^{u_2}\cdot\mathfrak{L}_r\supseteq[\mathfrak{M}]^{u_2}\cdot\mathfrak{L}_r$, which with our previous inclusion gives the result. But if on the other hand $u_1\leqq u_2$, we have

$$[[\mathfrak{M}]^{u_1}\cdot\mathfrak{L}_r]^{u_2}\cdot\mathfrak{L}_r\supseteq([\mathfrak{M}]^{u_1}\cdot\mathfrak{L}_r)\cdot\mathfrak{L}_r=[\mathfrak{M}]^{u_1}\cdot\mathfrak{L}_r.$$

But $[\mathfrak{M}]^{u_1}\cdot\mathfrak{L}_{u_2}\supseteq[\mathfrak{M}]^{u_1}\cdot\mathfrak{L}_r$. Hence since $[\mathfrak{M}]^{u_1}\cdot\mathfrak{L}_{u_2}$ is closed, $[\mathfrak{M}]^{u_1}\cdot\mathfrak{L}_{u_2}\supseteq[[\mathfrak{M}]^{u_1}\cdot\mathfrak{L}_r]^{u_2}$ or $[\mathfrak{M}]^{u_1}\cdot\mathfrak{L}_r=([\mathfrak{M}]^{u_1}\cdot\mathfrak{L}_{u_2})\cdot\mathfrak{L}_r\supseteq([[\mathfrak{M}]^{u_1}\cdot\mathfrak{L}_r]^{u_2})\cdot\mathfrak{L}_r$, which with our previous inclusion proves the result.

Now if for some value of r such that $p_1 \leqq r \leqq p$, there are n distinct linear manifolds among the $\mathfrak{M}_1^{(r)}$, $\mathfrak{M}_2^{(r)}$, $\mathfrak{M}_1^{(r)}(u_1)$, etc., defined above, and for no value of r between p_1 and p are there more than n, $\mathfrak{M}$ is said to have an n-valued lace between p and p_1.

DEFINITION 1.4. *If $\mathfrak{M}$ is a closed linear manifold in $\mathfrak{L}_p$, and $p < p_1$, $p \leqq r \leqq p_1$, $p \leqq u_i \leqq r \leqq v_i \leqq p_1$, we define $\mathfrak{M} \cdot \mathfrak{L}_r = \mathfrak{M}_2^{(r)}$, $[\mathfrak{M} \cdot \mathfrak{L}_{p_1}]^r = \mathfrak{M}_1^{(r)}$, $\mathfrak{M}_2^{(r)}(v_1) = [\mathfrak{M} \cdot \mathfrak{L}_{v_1}]^r$, $\mathfrak{M}_1^{(r)}(u_1) = [\mathfrak{M}_1^{(r)}]^{u_1} \cdot \mathfrak{L}_r$ and $\mathfrak{M}_2^{(r)}(v_1, u_2)$, $\mathfrak{M}_1^{(r)}(u_1, v_2)$, etc., by the inductive relations given in Definition* 1.3.

If, for some value of r such that $p \leqq r \leqq p_1$, there are n distinct manifolds among the $\mathfrak{M}_1^{(r)}$, $\mathfrak{M}_2^{(r)}$, etc., thus defined, and for no such value of r are there more than n, $\mathfrak{M}$ is said to have an n-valued lace between p and p_1.

THEOREM 1.7. *Let $\mathfrak{M}$ be a closed linear manifold in $\mathfrak{L}_p$. Let $p_1 < p$, $p_1 \leqq r \leqq p$; then if $1/p + 1/p' = 1/r + 1/r' = 1/p_1 + 1/p_1' = 1$, we have $p_1' \geqq r' \geqq p'$. Let $1/u_i + 1/u_i' = 1/v_i + 1/v_i' = 1$; $p_1 \leqq u_i \leqq r \leqq v_i \leqq p$; then $p_1' \geqq u_i' \geqq r' \geqq v_i' \geqq p'$. Then*

$$\mathfrak{M}_2^{(r)} \supseteq \mathfrak{M}_j^{(r)}(\quad, \cdots, \quad) \supseteq \mathfrak{M}_1^{(r)};$$

$$(\mathfrak{M}_1^{(r)})^\perp = (\mathfrak{M}^\perp)_2^{(r')}; \qquad (\mathfrak{M}_2^{(r)})^\perp = (\mathfrak{M}^\perp)_1^{(r')};$$

$$(\mathfrak{M}_1^{(r)}(u_1, \cdots, v_{2k}))^\perp = (\mathfrak{M}^\perp)_2^{(r')}(u_1', \cdots, v_{2k}') \qquad (k = 1, 2, \cdots);$$

$$(\mathfrak{M}_1^{(r)}(u_1, \cdots, u_{2k+1}))^\perp = (\mathfrak{M}^\perp)_2^{(r')}(u_1', \cdots, u_{2k+1}') \qquad (k = 0, 1, \cdots);$$

$$(\mathfrak{M}_2^{(r)}(v_1, \cdots, u_{2k}))^\perp = (\mathfrak{M}^\perp)_1^{(r')}(v_1', \cdots, u_{2k}') \qquad (k = 1, 2, \cdots);$$

$$(\mathfrak{M}_2^{(r)}(v_1, \cdots, v_{2k+1}))^\perp = (\mathfrak{M}^\perp)_1^{(r')}(v_1', \cdots, v_{2k+1}') \qquad (k = 0, 1, \cdots).$$

We first show that $\mathfrak{M}_2^{(r)} \supseteq \mathfrak{M}_i^{(r)}(\quad, \cdots, \quad)$. For $\mathfrak{M}_2^{(r)} \supseteq \mathfrak{M}_2^{(r)}(v_1)$ and $\mathfrak{M}_2^{(r)} \supseteq \mathfrak{M}_1^{(r)}(u_1)$, since $[\mathfrak{M}]^{p_1} \cdot \mathfrak{L}_r \supseteq [\mathfrak{M}]^{p_1} \cdot \mathfrak{L}_{v_1}$, and since $[\mathfrak{M}]^{p_1} \cdot \mathfrak{L}_r$ is closed, $[\mathfrak{M}]^{p_1} \cdot \mathfrak{L}_r \supseteq [[\mathfrak{M}]^{p_1} \cdot \mathfrak{L}_{v_1}]^r = \mathfrak{M}_2^{(r)}(v_1)$. Also $[\mathfrak{M}]^{p_1} \cdot \mathfrak{L}_{u_1} = [[\mathfrak{M}]^{u_1}]^{p_1} \cdot \mathfrak{L}_{u_1} \supseteq [\mathfrak{M}]^{u_1}$, hence $[\mathfrak{M}]^{p_1} \cdot \mathfrak{L}_r = [\mathfrak{M}]^{p_1} \cdot \mathfrak{L}_{u_1} \cdot \mathfrak{L}_r \supseteq [\mathfrak{M}]^{u_1} \cdot \mathfrak{L}_r \;\; \mathfrak{M}_1^{(r)}(u_1)$. Now if $\mathfrak{N}$ is such that $[\mathfrak{M}]^{p_1} \cdot \mathfrak{L}_r \supseteq \mathfrak{N}$, then $\mathfrak{N} \subseteq [\mathfrak{M}]^{p_1} \cdot \mathfrak{L}_{u_i}$, and since $[\mathfrak{M}]^{p_1} \cdot \mathfrak{L}_{u_i}$ is closed, $[\mathfrak{M}]^{p_1} \cdot \mathfrak{L}_{u_i} \supseteq [\mathfrak{N}]^{u_i}$, hence $[\mathfrak{N}]^{u_i} \cdot \mathfrak{L}_r \subseteq [\mathfrak{M}]^{p_1} \cdot \mathfrak{L}_{u_i} \cdot \mathfrak{L}_r = [\mathfrak{M}]^{p_1} \cdot \mathfrak{L}_r$, also $[\mathfrak{M}]^{p_1} \cdot \mathfrak{L}_r \supseteq [[\mathfrak{M}]^{p_1} \cdot \mathfrak{L}_r \cdot \mathfrak{L}_{v_i}]^r \supseteq [\mathfrak{N} \cdot \mathfrak{L}_{v_i}]^r$. An inductive proof will now give the desired result. A similar type of proof will show that $\mathfrak{M}_1^{(r)} \subseteq \mathfrak{M}_i^{(r)}(\quad, \cdots, \quad)$.

It is easy to give an inductive proof of the statements concerning orthogonal complements based on Theorems 1.4 and 1.5.

The proof of the following theorem is similar to that of the above.

Now we notice that $\mathfrak{M}_i^{(r)}(\quad, \cdots, v_s)$ is the closure in $\mathfrak{L}_r$ of some manifold in $\mathfrak{L}_v$, $v \geqq r$, and hence the above proof goes through with $\mathfrak{M}_i^{(r)}(\quad, \cdots, v_s)$ substituted for $\mathfrak{M}$.

THEOREM. 1.8. *Let $\mathfrak{M}$ be a closed linear manifold in $\mathfrak{L}_p$. Let $p \leqq r \leqq p_1$, $1/p+1/p'=1/r+1/r'=1/p_1+1/p_1'=1$, $p' \geqq r' \geqq p_1'$, and let $1/u_i+1/u_i'=1/v_i'+1/v_i=1$, $p \leqq u_i \leqq r \leqq v_i \leqq p_1$, $p' \geqq u_i' \geqq r' \geqq v_i' \geqq p_1'$. Then*

$$\mathfrak{M}_2^{(r)} \supset \mathfrak{M}_i^{(r)}(\ \ ,\cdots,\ \ ) \supseteq \mathfrak{M}_1^{(r)};$$

$$(\mathfrak{M}_1^{(r)})^{\perp} = (\mathfrak{M}^{\perp})_2^{(r')}; \qquad (\mathfrak{M}_2^{(r)})^{\perp} = (\mathfrak{M}^{\perp})_1^{r'};$$

$$(\mathfrak{M}_1^{(r)}(u_1,\cdots,v_{2k}))^{\perp} = (\mathfrak{M}^{\perp})_2^{(r')}(u_1',\cdots,v_{2k}');$$

$$(\mathfrak{M}_1^{(r)}(u_1,\cdots,u_{2k+1}))^{\perp} = (\mathfrak{M}^{\perp})_2^{(r')}(u_1',\cdots,u_{2k+1}');$$

$$(\mathfrak{M}_2^{(r)}(v_1,\cdots,u_{2k}))^{\perp} = (\mathfrak{M}^{\perp})_1^{(r')}(v_1',\cdots,u_{2k}');$$

$$(\mathfrak{M}_1^{(r)}(v_1,\cdots,v_{2k+1}))^{\perp} = (\mathfrak{M}^{\perp})_1^{(r')}(v_1',\cdots,v_{2k+1}').$$

2. LINEAR TRANSFORMATION IN $\mathfrak{L}_p$

The set of pairs $\{f_1, f_2\}$, f_1 and $f_2 \in \mathfrak{L}_p$ with the norm $(\|f_1\|^p+\|f_2\|^p)^{1/p}$ constitutes a Banach space $\mathfrak{L}_p \times \mathfrak{L}_p$. The set of linear functionals on $\mathfrak{L}_p \times \mathfrak{L}_p$ is simply isomorphic with $\mathfrak{L}_{p'} \times \mathfrak{L}_{p'}$, $1/p+1/p'=1$ (cf. (B), p. 64, pp. 181–183). Since $\mathfrak{L}_p \times \mathfrak{L}_p$ is simply isomorphic with $\mathfrak{L}_p$, the discussion of linear manifolds in $\mathfrak{L}_p$ applies to $\mathfrak{L}_p \times \mathfrak{L}_p$ also.

DEFINITION 2.1. *A set $\mathfrak{T} \subseteq \mathfrak{L}_p \times \mathfrak{L}_p$ is said to constitute a transformation T in $\mathfrak{L}_p$, if no two distinct pairs of $\mathfrak{T}$ have the same first elements. If $\{f_1, f_2\} \in \mathfrak{T}$, then we also write $Tf_1 = f_2$. The set $\mathfrak{D}$ of first elements of the pairs of $\mathfrak{T}$ is called the domain of T, the set $\mathfrak{R}$ of second elements of $\mathfrak{T}$ is called the range of T.*

DEFINITION 2.2. *A transformation T in $\mathfrak{L}_p$ is said to be linear if $\mathfrak{T}$ is a linear manifold.*

DEFINITION 2.3. *A transformation T in $\mathfrak{L}_p$ is said to be closed if $\mathfrak{T}$ is closed.*

The definitions of T_1T_2, T_1+T_2, aT_1, where a is a complex number, are standard, and will be assumed as will such facts as that if T_1 and T_2 are linear, T_1T_2, T_1+T_2 are linear. Our procedure would be just that of (S), chapter 2.

In what follows we shall restrict our discussion to linear transformations and we shall use the symbols $[T]^r$, $T \cdot \mathfrak{L}_r$ applied to transformations to mean the transformations corresponding to $[\mathfrak{T}]^r$, $\mathfrak{T} \cdot \mathfrak{L}_r \times \mathfrak{L}_r$, when these last sets constitute transformations. $T_1 \subseteq T_2$ is to mean of course $\mathfrak{T}_1 \subseteq \mathfrak{T}_2$.

A somewhat different procedure is used to define $T^{\perp}$.

DEFINITION 2.4. *If the pairs of $\mathfrak{T}^{\perp}$ with their order inverted constitute a transformation in $\mathfrak{L}_{p'}$, it will be denoted by $T^{\perp}$. The adjoint of T, T^*, is defined as $-T^{\perp}$ when the latter exists.*†

† This definition coincides, when T is limited, with the adjoint defined in (B), p. 99.

THEOREM 2.1. *If T is a closed linear transformation in $\mathfrak{L}_s$, then $\mathfrak{T}\cdot\mathfrak{L}_r\times\mathfrak{L}_r$, $r>s$, constitutes a closed linear transformation in $\mathfrak{L}_r$.*

This follows from Theorem 1.1 and Definition 2.1.

THEOREM 2.2. *If T is a linear transformation in $\mathfrak{L}_s$ and if for some $t<s$, $[T]^t$ exists, then for every r such that $t\leqq r\leqq s$, $[T]^r$ exists and $[T]^r\cdot\mathfrak{L}_s\supseteq T$. If $[T]^t\cdot\mathfrak{L}_s=T$ then $[T]^r\cdot\mathfrak{L}_s=T$.*

By Theorem 1.2, $[\mathfrak{T}]^t\cdot\mathfrak{L}_r\times\mathfrak{L}_r=[[\mathfrak{T}]^r]^t\cdot\mathfrak{L}_r\times\mathfrak{L}_r\supseteq[\mathfrak{T}]^r$. By Theorem 2.1, $[\mathfrak{T}]^t\cdot\mathfrak{L}_r\times\mathfrak{L}_r$ constitutes a transformation. Hence by Definition 2.1 $[\mathfrak{T}^r]$ must also, since it is included in $[\mathfrak{T}]^t\cdot\mathfrak{L}_r\times\mathfrak{L}_r$. The remaining statements of the theorem follow immediately from Theorem 1.2.

THEOREM 2.3. *If T is a linear transformation in $\mathfrak{L}_s$, with domain everywhere dense in $\mathfrak{L}_s$, and if for some $t<s$, $[T]^t$ exists, then for every r such that $t\leqq r\leqq s$, $[T]^r$ exists and has domain everywhere dense in $\mathfrak{L}_r$, and $[T]^t\cdot\mathfrak{L}_r\supseteq[T]^r$.*

This follows from Theorems 2.2 and 1.6, since by Theorems 1.2 and 1.1, $[\mathfrak{T}]^t\cdot\mathfrak{L}_r\times\mathfrak{L}_r=[[\mathfrak{T}]^r]^t\cdot\mathfrak{L}_r\times\mathfrak{L}_r\supseteq[\mathfrak{T}]^r$.

THEOREM 2.4. *If T is a closed linear transformation in $\mathfrak{L}_s$, and for some $t>s$, $T\cdot\mathfrak{L}_t$ has a domain everywhere dense in $\mathfrak{L}_t$, then for any r such that $t\geqq r\geqq s$, $T\cdot\mathfrak{L}_r$ is closed and linear and has domain everywhere dense in $\mathfrak{L}_r$.*

This follows from Theorem 2.1 and the fact that the domain of $T\cdot\mathfrak{L}_r$ includes that of $T\cdot\mathfrak{L}_t$ and Theorem 1.6.

THEOREM 2.5. *If T is a linear transformation in $\mathfrak{L}_p$, $T^\perp$ (and T^*) exists if and only if $\mathfrak{D}$ is everywhere dense in $\mathfrak{L}_p$. If $T^\perp$ exists it is a closed linear transformation.*

Since $\mathfrak{T}^\perp$ consists of all pairs $\{f^*,f\}$ of $\mathfrak{L}_{p'}\times\mathfrak{L}_{p'}$, such that

$$\int_0^1 f^*\bar{g}dx+\int_0^1 f\overline{Tg}dx=0$$

for all $g\epsilon\mathfrak{D}$, one can readily obtain a proof of this Theorem analogous to the proof of (S), Theorem 2.6 and Theorem 2.7.

THEOREM 2.6. *If T is a linear transformation in $\mathfrak{L}_p$, and $\mathfrak{D}$ is everywhere dense, $[T]^p$ exists if and only if $T^\perp$ has domain everywhere dense in $\mathfrak{L}_{p'}$. $[T]^p=(T^\perp)^\perp$ if $[T]^p$ exists.*†

If $[\mathfrak{T}]^p$ constitutes a transformation, since by Theorem 1.3, $[\mathfrak{T}]^p=(\mathfrak{T}^\perp)^\perp$, $(\mathfrak{T}^\perp)^\perp$ constitutes a transformation, hence $T^\perp$ has domain everywhere dense in $\mathfrak{L}_{p'}$, by Theorem 2.5. If $T^\perp$ has domain everywhere dense in $\mathfrak{L}_{p'}$, then $[\mathfrak{T}]^p=(\mathfrak{T}^\perp)^\perp$ constitutes a transformation by Theorem 2.5.

† This is of course a simple generalization of Theorem 2 of (N).

DEFINITION 2.5. *Let p, r, p_1, u_i, v_i be as in Definition* 1.3. *Then if T is a closed linear transformation in $\mathfrak{L}_p$, we define $T_i^{(r)}(\ ,\cdots,\ )$ as the transformation corresponding to $\mathfrak{T}_i^{(r)}(\ ,\cdots,\ )$, when the latter manifold constitutes a transformation.*

DEFINITION 2.6. *Let p, r, p_1, u_i, v_i be as in Definition* 1.4. *Then if T is a closed linear transformation in $\mathfrak{L}_p$, we define $T_i^{(r)}(\ ,\cdots,\ )$ as the transformation corresponding to $\mathfrak{T}_i^{(r)}(\ ,\cdots,\ )$, when the latter manifold constitutes a transformation.*

THEOREM 2.7. *Let $p, r, p_1, p', r', p_1', u_i, u_i', v_i, v_i'$ be as in Theorem* 1.7. *Let T be a linear transformation with domain everywhere dense in $\mathfrak{L}_p$, and such that $[T]^{p_1}$ exists. Then*

(a) *$T_i^{(r)}(\ ,\cdots,\ )$ exists, is closed and linear, and has domain everywhere dense in $\mathfrak{L}_r$;*

(b) *$(T_i^{(r)}(\ ,\cdots,\ ))^\perp$ exists, is closed and linear, and has domain everywhere dense in $\mathfrak{L}_{r'}$;*

(c) $T_2^{(r)} \supseteq T_i^{(r)}(\ ,\cdots,\ ) \supseteq T_1^{(r)}$;

(d) $(T_1^{(r)})^\perp = (T^\perp)_2^{(r')}$; $(T_2^{(r)})^\perp = (T^\perp)_1^{(r')}$;

$$(T_1^{(r)}(u_1,\cdots,v_{2k}))^\perp = (T^\perp)_2^{(r')}(u_1',\cdots,v_{2k}');$$

$$(T_1^{(r)}(u_1,\cdots,u_{2k+1}))^\perp = (T^\perp)_2^{(r')}(u_1',\cdots,u_{2k+1}');$$

$$(T_2^{(r)}(v_1,\cdots,u_{2k}))^\perp = (T^\perp)_1^{(r')}(v_1',\cdots,u_{2k}');$$

$$(T_2^{(r)}(v_1,\cdots,v_{2k+1}))^\perp = (T^\perp)_1^{(r')}(v_1',\cdots,v_{2k+1}').$$

That $T_1^{(r)}$ and $T_2^{(r)}$ exist, follows from Theorems 2.3, 2.2, and 2.1. Theorem 2.3 also implies that $T_1^{(r)}$ has domain everywhere dense in $\mathfrak{L}_r$, and that $T_2^{(r)} \supseteq T_1^{(r)}$. Hence $T_2^{(r)}$ has domain everywhere dense. From Theorem 1.7, we see that $\mathfrak{T}_2^{(r)} \supseteq \mathfrak{T}_i^{(r)}(\ ,\cdots,\ ) \supseteq \mathfrak{T}_1^{(r)}$. Hence $T_i^{(r)}(\ ,\cdots,\ )$ exists, is closed and linear by Definitions 2.2 and 2.3, and has domain everywhere dense, i.e., (a) holds. Theorems 2.5 and 2.6 and (a) imply (b); (c) and (d) follow from Definition 2.4, Theorem 1.7 and (b).

THEOREM 2.8. *Let $p, r, p_1, p', r', p_1', u_i, u_i', v_i, v_i'$ be as in Theorem* 1.7. *Let T be a linear transformation in $\mathfrak{L}_p$, such that $T^\perp$ exists and $T^\perp\cdot\mathfrak{L}_{p_1'}$ has domain everywhere dense in $\mathfrak{L}_{p_1'}$. Then the statements* (a), (b), (c), *and* (d) *of Theorem* 2.7 *hold.*

Since $T^\perp$ exists, T has domain everywhere dense in $\mathfrak{L}_p$ (Theorem 2.5). Since $[\mathfrak{T}]^{p_1\perp} = \mathfrak{T}^\perp\cdot\mathfrak{L}_{p_1}\times\mathfrak{L}_{p_1'}$ and $T^\perp\cdot\mathfrak{L}_{p_1'}$ has domain everywhere dense in $\mathfrak{L}_{p_1'}$, and by Theorem 1.6, T, considered in $\mathfrak{L}_{p_1}$, has domain everywhere dense in $\mathfrak{L}_{p_1}$, Theorem 2.6 implies that $[T]^{p_1}$ exists. Hence we have satisfied the hypothesis of Theorem 2.7 and may infer its conclusion.

The proof of the following two theorems is similar to that of Theorems 2.7 and 2.8.

THEOREM 2.9. *Let p, r, p_1, p', r', p_1', u_i, u_i', v_i, v_i' be as in Theorem* 1.8. *Let T be a closed linear transformation in $\mathfrak{L}_p$, such that $T\cdot\mathfrak{L}_{p_1}$ has domain everywhere dense in $\mathfrak{L}_{p_1}$. Then*

(a) *$T_i^{(r)}(\ ,\cdots,\ )$ exists and is a closed linear transformation with domain everywhere dense in $\mathfrak{L}_r$;*

(b) *$(T_i^{(r)}(\ ,\cdots,\ ))^\perp$ exists and has domain everywhere dense in $\mathfrak{L}_{r'}$;*

(c) *$T_2^{(r)}\supseteq T_i^{(r)}(\ ,\cdots,\ )\supseteq T_1^{(r)}$;*

(d) *$(T_1^{(r)})^\perp=(T^\perp)_2^{(r')}$; $(T_2^{(r)})^\perp=(T^\perp)_1^{(r')}$;*

$$(T_1^{(r)}(u_1,\cdots,v_{2k}))^\perp=(T^\perp)_2^{(r')}(u_1',\cdots,v_{2k}');$$

$$(T_1^{(r)}(u_1,\cdots,u_{2k+1}))^\perp=(T^\perp)_2^{(r')}(u_1',\cdots,u_{2k+1}');$$

$$(T_2^{(r)}(v_1,\cdots,u_{2k}))^\perp=(T^\perp)_1^{(r')}(v_1',\cdots,u_{2k}');$$

$$(T_2^{(r)}(v_1,\cdots,v_{2k+1}))^\perp=(T^\perp)_1^{(r')}(v_1',\cdots,v_{2k}').$$

THEOREM 2.10. *Let p, r, p_1, p', r', p_1', u_i, u_i', v_i, v_i' be as in Theorem* 1.8. *Let T be a closed linear transformation in $\mathfrak{L}_p$, such that $T^\perp$ exists and has domain everywhere dense in $\mathfrak{L}_{p'}$, and such that $[T^\perp]^{p_1'}$ exists. Then statements* (a), (b), (c), *and* (d) *of Theorem* 2.9 *hold.*

3. SKEW-SYMMETRIC TRANSFORMATIONS IN $\mathfrak{L}_p$

DEFINITION 3.1. *Let $1<p\leqq 2$. A closed linear transformation H with domain everywhere dense in $\mathfrak{L}_p$ is said to be p-skew-symmetric if $H^\perp\subseteq H$. It is said to be p-auto-perpendicular if $[H^\perp]^p=H$.*

THEOREM 3.1. *If H is p-skew-symmetric in $\mathfrak{L}_p$, and $p\leqq p_1\leqq 2$, then $H\cdot\mathfrak{L}_{p_1}$ is p_1-skew-symmetric and $(H\cdot\mathfrak{L}_{p_1})^\perp=[H^\perp]^{p_1'}$.*

Since $H^\perp\subseteq H$, $H^\perp\subseteq H\cdot\mathfrak{L}_{p'}\subseteq H\cdot\mathfrak{L}_{p_1}$. By Theorem 2.6, $H^\perp$ has domain everywhere dense in $\mathfrak{L}_{p'}$. Hence $H\cdot\mathfrak{L}_{p'}$ has domain everywhere dense in $\mathfrak{L}_{p'}$. From Theorem 2.9, we obtain that $(H\cdot\mathfrak{L}_{p_1})^\perp=(H_2^{(p_1)})^\perp=(H^\perp)_1^{(p_1')}=[H^\perp]^{p_1'}$. From $H^\perp\subseteq H\cdot\mathfrak{L}_{p_1}$, and the fact that $H\cdot\mathfrak{L}_{p_1}$ is closed, we see that $(H\cdot\mathfrak{L}_{p_1})^\perp\subseteq H\cdot\mathfrak{L}_{p_1}$.

THEOREM 3.2. *If H is p-skew-symmetric in $\mathfrak{L}_p$, $[iH^\perp]^2$ is an Hermitian transformation in $\mathfrak{L}_2$, and $([iH^\perp]^2)^*=iH\cdot\mathfrak{L}_2$. $[iH^\perp]^2$ is self-adjoint, if and only if $H\cdot\mathfrak{L}_2=[H^\perp]^2$.*

$([iH^\perp]^2)^*=-([iH^\perp]^2)^\perp=i([H^\perp]^2)^\perp=iH\cdot\mathfrak{L}_2$, by Theorem 3.1 and Theorem 2.6. By Theorem 3.1, $iH\cdot\mathfrak{L}_2\supseteq[iH^\perp]^2$. Since $H^\perp$ has domain everywhere dense in $\mathfrak{L}_{p'}$, $[H^\perp]^2$ has domain everywhere dense in $\mathfrak{L}_2$, by Theorem 2.9, as above. Hence $[iH^\perp]^2$ is Hermitian.

THEOREM 3.3. *When H is p-auto-perpendicular, then $H\cdot\mathfrak{L}_{p'}=H^\perp$.*

By Theorem 2.9, $(H\cdot\mathfrak{L}_{p'})^\perp=[H^\perp]^p=H$. Hence by Theorem 2.6, $H\cdot\mathfrak{L}_{p'}=H^\perp$.

THE CLOSURE IN $\mathfrak{L}_p$ OF TRANSFORMATIONS IN $\mathfrak{L}_2$

THEOREM 3.4. *Let T be a closed linear transformation in $\mathfrak{L}_2$, with domain everywhere dense in $\mathfrak{L}_2$. Then if $1<p\leqq 2$, and $p\leqq r\leqq 2$, then $[T]^r$ exists for all such r's, if and only if $T^\perp\cdot\mathfrak{L}_{p'}$ has domain everywhere dense in $\mathfrak{L}_{p'}$.*

From Theorems 2.8 and 2.5, we see that if $T^\perp\cdot\mathfrak{L}_{p'}$ has domain everywhere dense, then $T_1^{(r)}=[T]^r$ exists. If, on the other hand, $[T]^p$ exists, by Theorem 2.7, we see that $T^\perp\cdot\mathfrak{L}_{p'}$ has domain everywhere dense in $\mathfrak{L}_{p'}$.

THEOREM 3.5. *Let H be 2-auto-perpendicular. Let $p'\geqq 2$, $1/p+1/p'=1$, $p\leqq r\leqq 2$. If $H\cdot\mathfrak{L}_{p'}$ has domain everywhere dense in $\mathfrak{L}_{p'}$, and $[H\cdot\mathfrak{L}_{p'}]^2=H$, then $[H]^r$ exists and is r-auto-perpendicular.*

$[H]^r$ exists by Theorem 3.4 and we shall show that it is r-auto-perpendicular. Now $H=[H\cdot\mathfrak{L}_{p'}]^2\subseteq[H\cdot\mathfrak{L}_{r'}]^2\subseteq H$. Hence $H=[H\cdot\mathfrak{L}_{r'}]^2$ and $[H\cdot\mathfrak{L}_{r'}]^r=[H]^r$. Since H is 2-auto-perpendicular, $H^\perp=H$. Now by Theorem 2.7, $[H]^{r\perp}=(H_1^{(r)})^\perp=(H^\perp)_2^{(r')}=H_2^{(r')}=H\cdot\mathfrak{L}_{r'}$. Since $[H\cdot\mathfrak{L}_{r'}]^r=[H]^r$, we have that $[H]^r$ is r-auto-perpendicular.

It should be pointed out in connection with Theorem 3.5, that if the conditions of the theorem are satisfied we cannot conclude that H is of simple lace between p and 2, since we are not able to infer that $[H]^r=[H]^p\cdot\mathfrak{L}_r$.

THE CLOSURE OF PROJECTIONS IN $\mathfrak{L}_p$

THEOREM 3.6. *Let E be a projection in $\mathfrak{L}_2$, $E^\perp=-E$. Let $\mathfrak{M}$ be the range of E in $\mathfrak{L}_2$. Then $[E]^p$ exists if and only if $[\mathfrak{M}]^p\cdot[\mathfrak{M}^\perp]^p=\{0\}$.*

$[E]^p$ exists if and only if $-E\cdot\mathfrak{L}_{p'}$ has domain everywhere dense in $\mathfrak{L}_{p'}$, by Theorem 3.4. The domain of $E\cdot\mathfrak{L}_{p'}$ is the set in $\mathfrak{L}_{p'}$ of all elements in the form f_1+f_2, $f_1\epsilon\mathfrak{M}\cdot\mathfrak{L}_{p'}$, $f_2\epsilon\mathfrak{M}^\perp\cdot\mathfrak{L}_{p'}$.

Now suppose $[\mathfrak{M}]^p\cdot[\mathfrak{M}^\perp]^p\ni g\neq 0$. Since $[\mathfrak{M}]^{p\perp}=\mathfrak{M}^\perp\cdot\mathfrak{L}_{p'}$, we have $\int_0^1 g\bar{f}_2dx=0$, for all $f_2\epsilon\mathfrak{M}^\perp\cdot\mathfrak{L}_{p'}$, since $g\epsilon[\mathfrak{M}]^p$. Similarly $\int_0^1 g\bar{f}_1dx=0$, for all $f_1\epsilon\mathfrak{M}\cdot\mathfrak{L}_{p'}$. Hence $\int_0^1 g(\bar{f}_1+\bar{f}_2)dx=0$, hence the domain of $E\cdot\mathfrak{L}_{p'}$ is not everywhere dense. Now conversely, if the set $\{f_1+f_2\}$ does not determine $\mathfrak{L}_{p'}$, then there is a $g\neq 0$, such that $\int_0^1 g\bar{f}_1dx=0$ and $\int_0^1 g\bar{f}_2dx=0$ for all $f_1\epsilon\mathfrak{M}\cdot\mathfrak{L}_{p'}$ and $f_2\epsilon\mathfrak{M}^\perp\cdot\mathfrak{L}_{p'}$, and hence $g\epsilon(\mathfrak{M}\cdot\mathfrak{L}_{p'})^\perp\cdot(\mathfrak{M}^\perp\cdot\mathfrak{L}_{p'})^\perp=[\mathfrak{M}^\perp]^p\cdot[\mathfrak{M}]^p$.

THEOREM 3.7. *Let E be a projection in $\mathfrak{L}_2$, such that $[E]^p$ exists. Then $[E]^p$ is a limited transformation if and only if for every closed linear manifold $\mathfrak{M}'$ in $\mathfrak{L}_{p'}$, such that $\mathfrak{M}'\subset\mathfrak{M}\cdot\mathfrak{L}_{p'}$, $\mathfrak{M}'^\perp([\mathfrak{M}]^p)\ni g\neq 0$.*

Let the condition be satisfied. Let f be any element in $\mathfrak{L}_p$. Let $\{cf\}^\perp = \mathfrak{N}$; $\mathfrak{N}\cdot\mathfrak{M}\cdot\mathfrak{L}_{p'} = \mathfrak{M}\cdot\mathfrak{L}_{p'}$ if and only if $f\epsilon(\mathfrak{M}\cdot\mathfrak{L}_{p'})^\perp$. Now if $f\epsilon(\mathfrak{M}\cdot\mathfrak{L}_{p'})^\perp = [\mathfrak{M}^\perp]^p$, then $[E]^p f$ exists and is zero. Suppose however that f is not in $(\mathfrak{M}\cdot\mathfrak{L}_{p'})^\perp$. Let $\mathfrak{M}' = \mathfrak{N}\cdot\mathfrak{M}\cdot\mathfrak{L}_{p'}$, and since $\mathfrak{M}'\subset\mathfrak{M}\cdot\mathfrak{L}_{p'}$, by hypothesis there is a $g\epsilon\mathfrak{M}'^\perp\cdot[\mathfrak{M}]^p$ and $g\neq 0$. Now there is an x in $\mathfrak{M}\cdot\mathfrak{L}_{p'}$, such that $\mathfrak{M}\cdot\mathfrak{L}_{p'} = \{ax\}\cdot+\cdot\mathfrak{M}'$. For let $x\epsilon\mathfrak{M}\cdot\mathfrak{L}_{p'}$ be such that $\int_0^1 f\bar{x}dt\neq 0$. Then if y is any element of $\mathfrak{M}\cdot\mathfrak{L}_{p'}$, $y-(\int_0^1 f\bar{y}dt/\int_0^1 f\bar{x}dt)x\epsilon\mathfrak{N}\cdot\mathfrak{M}\cdot\mathfrak{L}_{p'} = \mathfrak{M}'$. Now $\int_0^1 g\bar{x}dt\neq 0$, otherwise g would be in $(\mathfrak{M}\cdot\mathfrak{L}_{p'})^\perp$, since $g\epsilon\mathfrak{M}'^\perp$, but we know that $g\epsilon[\mathfrak{M}]^p$ and since $[E]^p$ exists and $g\neq 0$, this is impossible, by Theorem 3.6. Let $h=f-(\int_0^1 f\bar{x}dt/\int_0^1 g\bar{x}dt)g$; then $\int_0^1 h\bar{x}dt=0$ and $\int_0^1 h\bar{z}dt=0$ for all z in $\mathfrak{M}'$, hence $\int_0^1 h\bar{y}dt=0$ for all $y\epsilon\mathfrak{M}\cdot\mathfrak{L}_{p'}$. Hence $h\epsilon(\mathfrak{M}\cdot\mathfrak{L}_{p'})^\perp = [\mathfrak{M}^\perp]^p$ and h is in the domain of $[E]^p$. So also is g, since $g\epsilon[\mathfrak{M}]^p$. Hence $f=h+cg$ also is in the domain of $[E]^p$. Hence the domain of $[E]^p$ is $\mathfrak{L}_p$ and since $[E]^p$ is closed, it is limited. (Cf. (B), chapter III, p. 41, Theorem 7.)

Now conversely, let $[E]^p$ be limited and let $\mathfrak{M}'$ be any closed linear manifold such that $\mathfrak{M}'\subset\mathfrak{M}\cdot\mathfrak{L}_{p'}$. Let $x\epsilon\mathfrak{M}\cdot\mathfrak{L}_{p'}$ but x not in $\mathfrak{M}'$. There is an f in $\mathfrak{L}_p$, such that $\int_0^1 f\bar{x}dt\neq 0$ and $\int_0^1 f\bar{z}dt=0$ for all z in $\mathfrak{M}'$. (Cf. proof of Theorem 1.3.) Since $[E]^p$ is closed, limited, and has domain everywhere dense, the domain of $[E]^p$ is $\mathfrak{L}_p$. Let $[E]^p f=g, f=g+h$. Now g is an element of $\mathfrak{M}'^\perp$, since $f\epsilon\mathfrak{M}'^\perp$, and† $h\epsilon[\mathfrak{M}^\perp]^p = [\mathfrak{M}\cdot\mathfrak{L}_{p'}]^\perp\subseteq\mathfrak{M}'^\perp$. We also have $\int_0^1 g\bar{x}dt\neq 0$, since $\int_0^1 h\bar{x}dt=0$ and $\int_0^1 f\bar{x}dt\neq 0$. Hence $g\neq 0$. Since $[E]^p f=g$, $g\epsilon[\mathfrak{M}]^p$. Hence $\mathfrak{M}'^\perp\cdot[\mathfrak{M}]^p\ni g\neq 0$.

4. Examples

Example 1. We give an example of a skew-symmetric transformation having a double-valued lace between 3/2 and 3.

Definition 4.1. *Let T_r' be the transformation in $\mathfrak{L}_r$, such that $\{g, g_1\}\epsilon\mathfrak{L}_r\times\mathfrak{L}_r$ is in $\mathfrak{T}_r'$ if $g=(k(y))^{-1/2}(c+\int_{1/2}^y g_1 k^{-1/2}dx)$, where $k(y)=y^{2/3}(1-y)^{2/3}\cdot(y^{2/3}+(1-y)^{2/3})^{-1}$ and $\lim_{y\to 0}(k(y))^{1/2}g=0$, $\lim_{y\to 1}(k(y))^{1/2}g=0$. Let T_r'' be the transformation in $\mathfrak{L}_r$, such that $\{g, g_1\}\epsilon\mathfrak{L}_r\times\mathfrak{L}_r$ is in $\mathfrak{T}_r''$, if $g=(k(y))^{-1/2}(c+\int_{1/2}^y g_1 k^{-1/2}dx)$.*

4.1. *$T_r'\subset T_r''$. Both T_r' and T_r'' have domains everywhere dense in $\mathfrak{L}_r$.*

T_r' is obviously included in T_r'' and if we show that T_r' has domain everywhere dense, then we have proved the statement. Now note that if $g\epsilon\mathfrak{D}_r'$, then

$$T_r' g = (k)^{1/2}\frac{d}{dy}(k)^{1/2}g,$$

also that $k(y)$ is bounded. Let $\mathfrak{S}$ denote the set of absolutely continuous functions g with bounded right and left derivatives, such that for some ϵ such that $0<\epsilon\leqq\frac{1}{2}$, $g(y)=(g(\epsilon)/\epsilon)y$ for $0\leqq y\leqq\epsilon$, and $g(y)=(g(1-\epsilon)/\epsilon)(1-y)$, for

† Since $(I-[E]^p)=[(I-E)]^p$ and $(I-[E]^p)f=h$.

$1-\epsilon \leqq y \leqq 1$. Now $\mathfrak{S} \subseteq \mathfrak{D}_r'$, for every r, since one can readily show that both g and $T_r'g$ are bounded. Since it is readily shown that $\mathfrak{S}$ is dense in $\mathfrak{L}_r$, so are $\mathfrak{D}_r'$ and $\mathfrak{D}_r''$.

4.2. *If* $1 < r \leqq 3/2$, *the ranges of* T_r' *and* T_r'' *are everywhere dense. If* $3/2 < r < 3$, *the range of* T_r' *is the closed linear manifold of all* $g_1 \epsilon \mathfrak{L}_r$ *such that*

$$\int_0^1 (k(x))^{-1/2}\bar{g}_1(x)dx = 0, \tag{*}$$

and that of T_r'' *is* $\mathfrak{L}_r$. *For* $r \geqq 3$, *the range of* T_r' *and that of* T_r'' *is the closed linear manifold such that* (*) *is satisfied.*

For $\mathfrak{L}_r$, we recall that the range of T_r' consists of all functions $g_1 \epsilon \mathfrak{L}_r$, such that

$$g = k^{-1/2}\left(c + \int_{1/2}^{y} k^{-1/2}g_1 dx\right) \text{ is in } \mathfrak{L}_r,$$

and such that

$$\lim_{y\to 0} k^{1/2}g = \lim_{y \to 1} k^{1/2}g = 0$$

for some value of c, and the range of T_r'', all $g_1 \epsilon \mathfrak{L}_r$ such that $g \epsilon \mathfrak{L}_r$, for some value of c.

First suppose that $1 < r \leqq 3/2$. Those $g \epsilon \mathfrak{L}_r$ for which

$$\int_0^1 gk^{-1/2}dx = K(g)$$

exists are everywhere dense, since they include $\mathfrak{L}_2$. The same is true for the set $\mathfrak{N}$ of such g's for which $K(g)=0$. For given an $\epsilon > 0$ and an f in $\mathfrak{L}_r$, there exists a g such that $K(g)$ exists and such that $\|f-g\| < \epsilon/2$. Since $K(g)$ is not a limited functional on $\mathfrak{L}_r$, there exists an h such that $K(h)$ exists, $\|h\|_r = 1$ and $|K(h)| > 2|K(g)|/\epsilon$, and $g' = g - (K(g)/K(h))h$ is such that $K(g')=0$ and $\|g-g'\|_r = \|(K(g)/K(h))h\|_r < \epsilon/2$. Hence

$$\|f - g'\|_r \leqq \|f - g\|_r + \|g - g'\|_r < \epsilon.$$

But it is easy to see that the range of T_r' is exactly $\mathfrak{N}$. Thus the range of T_r' is everywhere dense in $\mathfrak{L}_r$, and since $\mathfrak{R}_r'' \geqq \mathfrak{R}_r'$, so is $\mathfrak{R}_r''$.

Since for $3/2 < r < 3$, also $3/2 < r' < 3$ and $(k(y))^{-1/2} \epsilon \mathfrak{L}_{r'}$, and $K(g)$ exists for all $g \epsilon \mathfrak{L}_r$, the proofs of the given statements are extremely simple and we omit them.

Since $(k(y))^{-1/2}$ is not in $\mathfrak{L}_r$, $r \geqq 3$, if for g_1 there is a c such that

$$k^{-1/2}\left(c + \int_{1/2}^{y} g_1 k^{-1/2} dx\right) \epsilon \mathfrak{L}_r,$$

then c is uniquely determined and equals $\int_0^{1/2} g_1 k^{-1/2} dx$.

Thus we must consider those g_1's for which

$$g = k^{-1/2}\left(\int_0^{y} g_1 k^{-1/2} dx\right) \epsilon \mathfrak{L}_r.$$

We can easily verify that g_1 must be such that $\int g k^{-1/2} dx = 0$.

This is also sufficient for g_1 to be in $\mathfrak{R}_r'$. For if $g_1 \epsilon \mathfrak{L}_r$ and such that $K(g_1) = 0$, then

$$\int_0^{x} g_1 k^{-1/2} dy = o(x^{1/r'-1/3}); \qquad \int_0^{1-x} (g_1/k^{-1/2}) dy = o(x^{1/r'-1/3});$$

and

$$k^{-1/2}\left(\int_0^{x} g_1 k^{-1/2} dy\right) = o(x^{1/r'-2/3}); \quad k^{-1/2}\left(\int_0^{1-x} g_1 k^{-1/2} dy\right) = o(x^{1/r'-2/3});$$

and since $1/r' + 1/r = 1$, $1/r' = 1 - 1/r \geqq 2/3$, g is bounded and hence in $\mathfrak{L}_r$, and

$$\lim_{y \to 0} k^{1/2} g = \lim_{y \to 1} k^{1/2} g = 0.$$

We have also shown that for $3 \leqq r$, $T_r' = T_r''$.

We shall use certain results, which may easily be proved by elementary methods. We collect them here.

4.3. (a) *If* $r > 3/2$, $h(x) \epsilon \mathfrak{L}_r$, *then*

$$\int_0^{\eta} k^{-1/2}(x) h(x) dx = o(\eta^{1/r'-1/3}), \qquad \int_{1-\eta}^{1} k^{-1/2}(x) h(x) dx = o(\eta^{1/r'-1/3}).$$

If furthermore $\int_0^1 k^{-1/2}(x) h(x) dx = 0$, *then* $\int_0^{1-\eta} k^{-1/2}(x) h(x) dx = o(\eta^{1/r'-1/3})$, $\int_\eta^1 k^{-1/2}(x) h(x) dx = o(\eta^{1/r'-1/3})$.

(b) *If* $r \leqq 3/2$, $0 < a < 1$, $h(x) \epsilon \mathfrak{L}_r$, *then*

$$\int_\eta^{a} k^{-1/2}(x) h(x) dx = o(\eta^{1/r'-1/3}), \qquad \int_a^{1-\eta} k^{-1/2}(x) h(x) dx = o(\eta^{1/r'-1/3}).$$

4.4. *For* $1 < r \leqq 3/2$, $T_r'^{\perp} = T_{r'}''$, $T_r''^{\perp} = T_{r'}'$, $1/r + 1/r' = 1$. *For* $3/2 < r < 3$, $T_r'^{\perp} = T_{r'}''$, $T_r''^{\perp} = T_{r'}'$. *For* $3 \leqq r$, $T_r'^{\perp} = T_r''^{\perp} = T_{r'}''$

Let us first consider the case $3 \leqq r$. Then $T_r'^{\perp}$ consists of all pairs $\{f, f^*\}$ in $\mathfrak{L}_{r'} \times \mathfrak{L}_{r'}$, such that

$$\int_0^1 f\bar{g} dx + \int_0^1 f^* \bar{g} dx = 0 \tag{1}$$

when $\int_0^1 g_1 k^{-1/2}dx=0$ and $g=k^{-1/2}\int_0^x g_1 k^{-1/2}dx$ by 4.2. Now

$$\int_0^1 f^*\bar{g}dx = \int_0^1 \left(k^{-1/2}\int_0^x \bar{g}_1 k^{-1/2}dy\right) f^*dx$$

$$= \lim_{\eta\to 0}\int_\eta^{1-\eta} f^*\left(k^{-1/2}\int_0^x \bar{g}_1 k^{-1/2}dy\right)dx.$$

Integrating by parts the expression on the right and using 4.3 (a) and (b) yields

$$\int_0^1 f^*\bar{g}dx = -\lim_{\eta\to 0}\int_\eta^{1-\eta}\bar{g}_1\left(k^{-1/2}\int_\eta^y f^*k^{-1/2}dx\right)dy.$$

Thus (1) implies

$$\int_0^1 f\bar{g}_1 dx = \lim_{\eta\to 0}\int_\eta^{1-\eta}\bar{g}_1\left(k^{-1/2}\int_\eta^y f^*k^{-1/2}dx\right)dy.$$

Let η be fixed and let $\mathfrak{M}'$ be the set of all g_1's in $\mathfrak{R}_r'$ such that $g_1=0$ for $0\leqq x\leqq\eta$, $1-\eta\leqq x\leqq 1$. Then

$$\int_\eta^{1-\eta} f\bar{g}_1 dx = \int_\eta^{1-\eta}\bar{g}_1\left(k^{-1/2}\int_\eta^y f^*k^{-1/2}dx\right)dy.$$

This, by 4.2 and the definition of $\mathfrak{M}'$, implies that there is a constant α_η depending only on η, such that after an inconsequential change in the definition of f if necessary

$$f = \alpha_\eta k^{-1/2} + k^{-1/2}\int_\eta^x f^*k^{-1/2}dy$$

for $\eta<x<1-\eta$. Hence for this range of x,

$$k^{1/2}\frac{d}{dy}k^{1/2}f = f^*$$

or

$$f = k^{-1/2}\left(c+\int_{1/2}^x f^*k^{-1/2}dy\right).$$

This last result is independent of η. Thus we have shown that $T_r'^{\perp}\subseteq T_r''$. But if $\{f,f^*\}\epsilon\mathfrak{T}_r''$ then by integration by parts and using 4.3 (a) and (b), we can show that (1) holds for all $\{g,g_1\}\epsilon\mathfrak{T}_r'$. Thus $T_r''\subseteq T_r'^{\perp}$ and from our previous inclusion we infer that $T_r'^{\perp}=T_r''$.

One can easily simplify the above argument to prove the desired results for $3/2<r<3$. This we omit.

For $r \leqq 3/2$, we have that $T_r'^{\perp}$ consists of all pairs $\{f, f^*\}$ in $\mathfrak{L}_{r'} \times \mathfrak{L}_{r'}$, such that (1) holds for all $\{g, g_1\} \epsilon \mathfrak{T}_r'$. But

$$\begin{aligned}
\int_0^1 f^* \bar{g}\, dx &= \int_0^1 f^* k^{-1/2}\left(c + \int_{1/2}^x \bar{g}_1 k^{-1/2} dy\right) dx \\
&= \lim_{\eta \to 0}\left(\left(c' + \int_{1/2}^{1-\eta} f^* k^{-1/2} dy\right)\left(c + \int_{1/2}^{1-\eta} \bar{g}_1 k^{-1/2} dy\right)\right. \\
&\qquad \left. - \left(c' + \int_{1/2}^{\eta} f^* k^{-1/2} dy\right)\left(c + \int_{1/2}^{\eta} \bar{g}_1 k^{-1/2} dy\right)\right) \\
&\qquad - \lim_{\eta \to 0} \int_\eta^{1-\eta} k^{-1/2}\left(c' + \int_{1/2}^x f^* k^{-1/2} dy\right) \bar{g}_1 dx \\
&= - \lim_{\eta \to 0} \int_\eta^{1-\eta} k^{-1/2}\left(c' + \int_{1/2}^x f^* k^{-1/2} dy\right) \bar{g}_1 dx,
\end{aligned}$$

since

$$\lim_{\eta \to 0}\left(c + \int_{1/2}^{1-\eta} \bar{g}_1 k^{-1/2} dy\right) = \lim_{\eta \to 0} \int_0^{1-\eta} \bar{g}_1 k^{-1/2} dy = 0,$$

$$\lim_{\eta \to 0}\left(c + \int_{1/2}^{\eta} \bar{g}_1 k^{-1/2} dy\right) = \lim_{\eta \to 0} \int_0^{\eta} \bar{g}_1 k^{-1/2} dy = 0,$$

and the integral $\int_0 f^* k^{-1/2} dy$ exists. But

$$\begin{aligned}
&\lim_{\eta \to 0} \int_\eta^{1-\eta} k^{-1/2}\left(c' + \int_{1/2}^x f^* k^{-1/2} dy\right) \bar{g}_1 dx \\
&\qquad = \lim_{\eta \to 0} \int_\eta^{1-\eta} k^{-1/2}\left(c'' + \int_0^x f^* k^{-1/2} dy\right) \bar{g}_1 dx \\
&\qquad = \lim_{\eta \to 0} \int_\eta^{1-\eta} k^{-1/2}\left(\int_0^x f^* k^{-1/2} dy\right) \bar{g}_1 dx,
\end{aligned}$$

since

$$\lim_{\eta \to 0} c'' \int_\eta^{1-\eta} \bar{g}_1 k^{-1/2} dx = c'' \int_0^1 \bar{g}_1 k^{-1/2} dx = 0.$$

Hence by (1),

$$\int_0^1 f \bar{g}_1 dx = \lim_{\eta \to 0} \int_\eta^{1-\eta}\left(k^{-1/2} \int_0^x f^* k^{-1/2} dy\right) \bar{g}_1 dx.$$

For a fixed $\eta > 0$, if we restrict ourselves to such g_1 in $\mathfrak{R}_1'$ for which $g_1 = 0$ for $0 \leqq x \leqq \eta$ and $1 - \eta \leqq x \leqq 1$, we see that

$$f = ck^{-1/2} + k^{-1/2}\int_0^x f^* k^{-1/2} dy.$$

One can easily show that c is independent of η. But since f is in $\mathfrak{L}_{r'}$, $r' \geqq 3$, considering f in the neighborhood of the origin leads us to infer that $c=0$, and

$$f = k^{-1/2}\int_0^x f^* k^{-1/2} dy.$$

But from the proof of 4.2, we see that f is in $\mathfrak{L}_{r'}$, only if $\int_0^1 f^* k^{-1/2} dy = 0$. Hence $\{f, f^*\} \epsilon \mathfrak{T}_{r'}$. Thus we have shown that $T_r'^{\perp} \subseteq T_{r'}'$. But a familiar argument will easily show that $T_r'^{\perp} \supseteq T_{r'}'$. Hence $T_r'^{\perp} = T_{r'}'$. We also note that this shows that $T_{r'}'$ is closed.

Now we have also shown before that $T_{r'}'^{\perp} = T_r''$. Since $T_{r'}'$ is closed and has domain everywhere dense, this implies that $T_{r'}' = T_{r'}'^{\perp\perp} = T_r''^{\perp}$. This concludes the proof of the statement.

Our theorems on the relationships of closure and the perpendicular of transformations now permit us to show the general closure relationships involved.

4.5. *If* $3/2 < t \leqq r$, *then* $[T_r']^t = T_t'$. *If* $t \leqq r$ *and* $t \leqq 3/2$, $[T_r']^t = T_t''$. *If* $t \leqq r < 3$, $[T_r'']^t = T_t''$. *If* $t < r$, $T_t' \cdot \mathfrak{L}_r = T_r'$, $T_t'' \cdot \mathfrak{L}_r = T_r''$.

The statements concerning intersections are immediate consequences of the definitions. The rest follows from Theorems 2.7 and 2.6 and 4.4, and the intersection relationships.

4.5 implies that T_3' has a double lace between 3 and 3/2. It also tells us that $T_{3/2}$ is (3/2)-auto-perpendicular, while $T_{3/2}'' \cdot \mathfrak{L}_2 = T_2''$ is not 2-auto-perpendicular.

Example 2. We give an example of an operator, which is self-adjoint in $\mathfrak{L}_2$, of simple lace between 2 and $p \geqq 1\frac{1}{4}$ ($p<2$), but for which two projections in its corresponding resolution of the identity do not have transformations as closures in $p < 3/2$.

Let $\phi_1(x) = x^{-2/3}$, $1 \geqq x \geqq 1/8$; $\phi_1(x) = -x^{-2/3}$, $1/27 \leqq x < 1/8$, $\phi_1(x)=0$, $0 \leqq x < 1/27$.

Let $\phi_n(x) = x^{-2/3}$, $(2^{n-1}+1)^{-3} \leqq x \leqq 1$; $\phi_n(x) = -x^{-2/3}$, $(2^n+1)^{-3} \leqq x < (2^{n-1}+1)^{-3}$; $\phi_n(x)=0$, $0 \leqq x < (2^n+1)^{-3}$.

One can verify by direct calculation that if $n < m$,

$$\int_0^1 \phi_n\phi_m dx = -\int_{(2^n+1)^{-3}}^{(2^{n-1}+1)^{-3}} x^{-4/3} dx + \int_{(2^{n-1}+1)^{-3}}^1 x^{-4/3} dx = 0.$$

Hence if $n \neq m$, $\int_0^1 \phi_n\phi_m dx = 0$. Also that if $p < 1\frac{1}{2}$, then

$$\left\| \phi_n - \frac{1}{x^{2/3}} \right\|_p = \left(\frac{1}{1 - \frac{2}{3}p} \left(\frac{2^p}{(2^{n-1} + 1)^{-2p+3}} - \frac{2^p - 1}{(2^n + 1)^{-2p+3}} \right) \right)^{1/p}.$$

Hence for such a p,

$$\lim_{n\to\infty} \|\phi_n(x) - x^{-2/3}\|_p = 0.$$

Now let

$$b_n = \max_{5/4 \leqq p \leqq 5} \|\phi_n\|_p$$

and let H_p be the operator in $\mathfrak{L}_p$, $1\frac{1}{4} \leqq p \leqq 5$, which is constituted by the set of pairs, $\{f, f_1\}$, in which f_1 is related to f in such a manner that

$$f_1 = \sum_1^\infty a_{2n}\phi_{2n}, \text{ if } a_{2n} = b_{2n}^{-2} 2^{-n} \int_0^1 f\phi_{2n} dx.$$

We shall show that for such p's the domain of H_p is $\mathfrak{L}_p$. We have for any f in $\mathfrak{L}_p$,

$$\left\| \sum_{j=1}^n a_{2j}\phi_{2j} - \sum_{k=1}^n a_{2k}\phi_{2k} \right\|_p = \left\| \sum_{k=m}^n a_{2k}\phi_{2k} \right\|_p \leqq \sum_{k=m}^n |a_{2k}| \, \|\phi_{2k}\|_p.$$

But

$$|a_{2k}| \, \|\phi_{2k}\|_p \leqq 2^{-k} b_{2k}^{-1} \left| \int_0^1 f\phi_{2k} dx \right| \leqq 2^{-k} \|f\|_p.$$

Hence

$$\left\| \sum_{j=1}^n a_{2j}\phi_{2j} - \sum_{k=1}^m a_{2k}\phi_{2k} \right\|_p \leqq \sum_{k=m}^n \frac{1}{2^k} \|f\|_p \leqq \frac{1}{2^{m-1}} \|f\|_p.$$

Hence f_1 exists for every f and H_p has domain $\mathfrak{L}_p$. A similar proof will show that H_p has a bound less than or equal to 1.

If $1/p + 1/p' = 1$, then

$$\int_0^1 H_p f \bar{g} dx = \int_0^1 \left(\sum_{n=1}^\infty \frac{\int_0^1 f\phi_{2n} dx}{2^n b_{2n}} \phi_n(x) \right) \bar{g} dx = \sum_{n=1}^\infty \frac{\int_0^1 f\phi_{2n} dx \cdot \int_0^1 \phi_{2n}\bar{g} dx}{2^n b_{2n}}$$
$$= \int_0^1 f \overline{H_{p'} g} dx.$$

Hence $H_p{}^\perp \supseteq -H_{p'}$, but since $H_{p'}$ has domain $\mathfrak{L}_{p'}$, $H_p{}^\perp = -H_{p'}$ and $H_p^* = H_{p'}$. In particular this holds for $p = 2$; thus H_2 is self-adjoint. Now if $p > p_1$, $p' < p_1'$, $5 \geqq p > p_1 \geqq 1\frac{1}{4}$, then since $H_p{}^\perp \cdot \mathfrak{L}_{p_1'} = -H_{p'} \cdot \mathfrak{L}_{p_1'} = -H_{p'} = H_{p_1}{}^\perp$, we have $[H_p]^{p_1} = H_{p_1}$. We also have that $H_{p_1} \cdot \mathfrak{L}_p = H_p$. Thus the lace of H_p is simple in the range stated.

Now consider the case $p = 2$. For the resolution of the identity associated with H_2, we have that if $\mathfrak{M}$ is the closed linear manifold determined by the set $\{\phi_{2n}\}$, then $E(0-) = 0$, $E(0)$ is the projection on $\mathfrak{M}^\perp$, and $E(1) - E(0)$ is the projection on $\mathfrak{M}$. But since $[\mathfrak{M}]^p \cdot [\mathfrak{M}^\perp]^p \supseteq \{x^{-2/3}\}$, $p < 3/2$, we see by Theorem 3.6 that while $H_p = [H_2]^p$ is bounded for $1\frac{1}{4} < p < 2$, there are projections in the resolution of the identity associated with H_2 which do not have transformations for closures in $\mathfrak{L}_p$, if $p < 3/2$.

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