SOME ARITHMETIC MEANS CONNECTED WITH FOURIER SERIES*

BY

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1. Introduction. It is known that if a series $\sum a_n$ is bounded (C, γ), $\gamma \geq -1$, then it is either summable $(C, \gamma + \delta)$, for every $\delta > 0$, or not summable (A).† Further a necessary and sufficient condition for it to be bounded (C, γ) is that it should be bounded (A) and the sequence na_n bounded $(C, \gamma+1)$.‡ Conditions have been obtained§ under which a Fourier series $\sum A_n \cos nt$ or an allied series $\sum B_n \cos nt$ should be bounded (C) for $t = 0$ and, in the case of the allied series, conditions under which the sequence nB_n should be bounded (C). But the problem of the boundedness (C) of the sequence nA_n has apparently not been considered directly.‖

We suppose that $f(t)$ is integrable L, and periodic with period 2π. We write

$$\phi(t) = \tfrac{1}{2}\{f(x + t) + f(x - t)\},$$

$$\psi(t) = \tfrac{1}{2}\{f(x + t) - f(x - t)\},$$

and suppose that the Fourier series of $\phi(t)$ and $\psi(t)$ are respectively

$$\sum_{n=0}^{\infty} A_n \cos nt \quad \text{and} \quad \sum_{n=1}^{\infty} B_n \sin nt.$$

Then the Fourier series and allied series of $f(t)$ at the point $t = x$ are respectively

$$\sum_{n=0}^{\infty} A_n \quad \text{and} \quad \sum_{n=1}^{\infty} B_n.\, \P$$

We write, for $t > 0$,

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† Proved by Littlewood, **17**, for $\delta = 1$ and $\gamma =$ integer, and completed by Andersen, **1**. See also Hardy and Littlewood, **13a**, Kogbetliantz, **15**, p. 38. Numbers in heavy type refer to the list of references at the end of this paper.

‡ Hardy and Littlewood, **13a**, p. 283. Kogbetliantz, **15**, p. 38.

§ The first systematic results of this type were given by Hardy and Littlewood, **11, 12**.

‖ Sufficient conditions for the existence of the Cesàro limits of nA_n and nB_n have been given by Young, **24, 25**, and the limits of the arithmetic and logarithmic means of nB_n have been considered by other writers. See Zygmund, **27**, where references are given.

¶ In the usual notation $A_0 = \tfrac{1}{2}a_0$, $A_n = a_n \cos nx + b_n \sin nx$ and $B_n = b_n \cos nx - a_n \sin nx$, $n > 0$.

$$\theta(t) = \frac{2}{\pi} \int_t^\infty \frac{\psi(u)}{u} \, du,$$

$$\Phi_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - u)^{\alpha-1} \phi(u) du, \qquad \alpha > 0,$$

$$\Phi_0(t) = \phi(t),$$

$$\Phi_\alpha(t) = \frac{d}{dt} \Phi_{\alpha+1}(t), \qquad -1 < \alpha < 0,$$

$$\phi_\alpha(t) = \Gamma(\alpha + 1)t^{-\alpha}\Phi_\alpha(t), \qquad \alpha > -1,$$

$$\phi_{-1}(t) = \frac{d}{dt}\{t\phi(t)\},$$

and we define $\Psi_\alpha(t)$, $\psi_\alpha(t)$, $\Theta_\alpha(t)$, $\theta_\alpha(t)$ in a similar way. We call $\phi_\alpha(t)$ the mean value of order α of $\phi(t)$.

We also write s_n^α, $\bar{s}_n^\alpha$, τ_n^α, $\bar{\tau}_n^\alpha$ for the Cesàro means of order α of $s_n = A_0 + A_1 + \cdots + A_n$,* $\bar{s}_n = B_1 + B_2 + \cdots + B_n$, $\tau_n = nA_n$ and $\bar{\ }_n = nB_n$ respectively. Finally we write, for $\alpha \geqq 0$,

$$R_\alpha(\omega) = \sum_{n<\omega} (\omega - n)^\alpha A_n, \qquad \overline{R}_\alpha(\omega) = \sum_{n<\omega} (\omega - n)^\alpha B_n,$$

and

$$r_\alpha(\omega) = \omega^{-\alpha} R_\alpha(\omega), \qquad \bar{r}_\alpha(\omega) = \omega^{-\alpha} \overline{R}_\alpha(\omega). \dagger$$

It is known‡ that if $|\phi_\alpha(t)| = O(1)$ $(C, 1)$, i.e.,

$$(1) \qquad\qquad \frac{1}{t} \int_0^t |\phi_\alpha(u)| \, du = O(1)$$

in an interval $(0, \eta)$, then $s_n^\beta = O(1)$ for $\beta > \alpha$, where $\alpha \geqq 0$. The Cesàro summability and boundedness of a Fourier series for negative orders is not a "local" property of the function, since we usually only know that $A_n = o(1)$, but the above result remains true when $-1 \leqq \alpha < 0$ if we take $\eta = \pi$,

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* Thus

$$s_n^\alpha = \frac{1}{A_n^\alpha} \sum_{\nu=0}^n A_{n-\nu}^{\alpha-1} s_\nu,$$

where

$$A_n^\alpha = \frac{\Gamma(\alpha + n + 1)}{\Gamma(\alpha + 1)\Gamma(n + 1)}.$$

† Thus $r_\alpha(\omega)$, $\bar{r}_\alpha(\omega)$ are the Rieszian arithmetic means of order α of the Fourier series and allied series respectively.

‡ See Bosanquet, **6**, where references to previously known special cases are given.

provided $\Phi_{\alpha+1}(t)$ is absolutely continuous and $\Phi_{\alpha+1}(+0)=0$, in the case $\alpha > -1$,[*] and $\phi(t)$ is absolutely continuous except at $t=0$ in the case $\alpha = -1$. Dr. A. C. Offord and I have recently shown that for the class of Fourier series for which $A_n = o(n^\gamma)$, $\gamma > -1$, summability (C, γ) at the point $t=x$ does in fact depend only on the local properties of the function, and there is an analogous result for boundedness (C). It follows from our conditions that, if we restrict ourselves to series for which $A_n = O(n^\beta)$, the above result remains true, with the condition in its localized form, when $\alpha \geqq -1$.[†] Condition (1) may also be replaced by the more general form

$$(2) \qquad\qquad \frac{1}{t} \int_0^t u^{-\alpha} \left| d\Phi_{\alpha+1}(u) \right| = O(1)\text{[‡]}$$

when $\alpha > -1$, or

$$(3) \qquad\qquad \frac{1}{t} \int_0^t \left| d\{u\phi(u)\} \right| = O(1)$$

when $\alpha = -1$.[§] There are also "converse" results,[‖] in which summability or boundedness of the Fourier series is given as an hypothesis. It follows in particular from these results that a necessary and sufficient condition that $s_n^\beta = O(1)$ for some β is that (1) should hold for some α, or, what is the same thing, that $\phi_k(t) = O(1)$ for some k.

The investigations of this paper arise out of certain identities which play an important role in the theory of Cesàro means. If $\alpha > 0$, we have

$$(4) \qquad\qquad \alpha(s_n^{\alpha-1} - s_n^\alpha) = n(s_n^\alpha - s_{n-1}^\alpha) = \tau_n^\alpha,\text{[¶]}$$

and[**]

$$(5) \qquad\qquad \alpha\{\phi_{\alpha-1}(t) - \phi_\alpha(t)\} = t\phi_\alpha'(t) = \chi_\alpha(t),$$

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* In these circumstances $\Phi_{\alpha+1}(t) = \int_0^t \Phi\alpha(u)du$. The last condition is necessary in order to exclude a function like $f(t) = |t-x|^{-\alpha-1}$, $-1 < \alpha < 0$, whose Fourier series diverges for $t=x$. If $\phi(t) = |t|^{-\alpha-1}$, we have $\Phi_{\alpha+1}(t) = \Gamma(-\alpha)$, and hence $\phi_\alpha(t) = 0$ for every $t > 0$.

† See Bosanquet, **8**, Bosanquet and Offord, **9**.

‡ An integral like (2) is to be interpreted in the first instance as $\lim_{\epsilon \to 0} \int_\epsilon^t$, where it is assumed that $\Phi_{\alpha+1}(u)$ is of bounded variation in every interval (ϵ, t). But when condition (2) is satisfied it may be shown that $\Phi_{\alpha+1}(u)$ is of bounded variation in $(0, t)$, and so the integral exists as an ordinary Lebesgue-Stieltjes integral. See Bosanquet, **8**.

§ Condition (3) is Young's well known condition. Young, **26**, stated that it was sufficient for boundedness $(C, -1+\delta)$, for $\delta > 0$, and this was proved by Hardy and Littlewood, **13**, in the case $\eta = \pi$.

‖ See Theorem 6, p. 201, where further references are given.

¶ Kogbetliantz, **15**, and **16**, pp. 23 and 30.

** For (5) and (6) see Bosanquet, **6** and **7**. The analogy between these and (4) is obvious.

where $\chi_\alpha(t)$ is the mean value of order α of $t\phi'(t)$ whenever $t\phi(t)$ is an integral vanishing at $t=0$. We also have

(6) $$\alpha\{\theta_{\alpha-1}(t) - \theta_\alpha(t)\} = t\theta_\alpha'(t) \equiv -\frac{2}{\pi}\psi_\alpha(t).$$

It will be shown here that, if

(7) $$\frac{1}{t}\int_0^t |\phi_\alpha(u) - \phi_{\alpha+1}(u)|\, du = O(1)$$

in an interval $(0, \eta)$, then

(8) $$s_n^\beta - s_n^{\beta+1} = O(1)$$

for $\beta > \alpha$, i.e., $nA_n = O(1)$ $(C, \beta+1)$. This is true with any η, $0 < \eta \leq \pi$, when $\alpha \geq 0$, and with $\eta = \pi$ when $\alpha \geq -1$, provided $\phi_{\alpha+1}$ is absolutely continuous except at $t=0$. For the class of Fourier series for which $A_n = O(n^\beta)$ there is a corresponding "localization" problem, but that belongs to a rather different line of ideas and will not be discussed in this paper. It is clear from (4) and (5) that the results just stated are covered by Theorems 1 and 2. It has been shown elsewhere that (7) is sufficient for the summability (C, β), $\beta > \alpha$, of the Fourier series whenever it is summable (C). This result appears again (Theorem 3) as a corollary of Theorem 1, but for the more general class of Fourier series summable (A). There are "converse" results in which boundedness (C) of the sequence nA_n, or a more general condition, may be taken as hypothesis (Theorem 4), and the problem of the boundedness (C) of nA_n may also be solved in a "necessary and sufficient" form (Theorem 5). Finally this problem and the corresponding one for allied series may be regarded as the starting point of a sequence of more delicate problems of the same nature (Theorems 7 and 8).

2. Before proving the theorems of this section we state as lemmas some results which help to explain the hypotheses.*

LEMMA 1. *If $\alpha > -1$, necessary and sufficient conditions that* (2) *should hold and* $\Phi_{\alpha+1}(+0) = 0$ *are that*

(9) $$\frac{1}{t}\int_0^t u\,|d\phi_{\alpha+1}(u)| = O(1)\dagger$$

and $\phi_k(t) = O(1)$ *for some* $k \geq 0$.

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* These have been given elsewhere, Bosanquet, **8**.
† This integral is to be interpreted in the sense $\lim_{\epsilon \to 0} \int_\epsilon^t$.

LEMMA 2. *Necessary and sufficient conditions that* (3) *should hold are that* (9) *should hold with* $\alpha = -1$ *and* $\phi_k(t) = O(1)$ *for some* $k \geqq 0$.

LEMMA 3. *If* $\alpha \geqq -1$, *and* (9) *holds, then it still holds if* α *is replaced by* $\beta > \alpha$.

There are analogous results with series in place of functions. We also require the following lemma.

LEMMA 4. *If* $\beta > \alpha > -1$, *and* (i) $\Phi_{\alpha+1}(t)$ *is of bounded variation in an interval* $(0, \eta)$, (ii) $\Phi_{\alpha+1}(+0) = 0$, *then* $\Phi_\beta(t)$ *exists for almost all t in* $(0, \eta)$, *and satisfies the relation*

$$(10) \qquad \Phi_\beta(t) = \frac{1}{\Gamma(\beta - \alpha)} \int_0^t (t - u)^{\beta - \alpha - 1} d\Phi_{\alpha+1}(u).$$

THEOREM 1. *If* $\alpha \geqq 0$, *and*

$$(11) \qquad \frac{1}{t} \int_0^t u \, | \, d\phi_\alpha(u) \, | \, = O(1)^*$$

in the interval $(0, \pi)$, *then* $nA_n = O(1)$ $(C, \alpha + \delta)$, *for every* $\delta > 0$.†

There is an analogous theorem with o in place of O, and, more generally, we have the following result.‡

THEOREM 1a. *If* $\alpha \geqq 0$, (11) *holds in the interval* $(0, \pi)$ *and*

$$(12) \qquad \frac{1}{t} \int_0^t | \, ud\phi_\alpha(u) + sdu \, | \, = o(1)$$

as $t \to +0$, *then* $nA_n \to s$ $(C, \alpha + \delta)$, *for every* $\delta > 0$.

When $\alpha \geqq 1$ it is enough to consider the analogous theorem for Rieszian means. We therefore begin by proving the following theorem, which is the same in principle, but rather simpler in detail.

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* It is easily seen, by integration by parts, that (11) is equivalent to $\int_0^t u^\rho | \, d\phi_\alpha(u) | \, = O(t^\rho), \rho > 0$, or

$$\int_t^\pi u^{-\rho} | \, d\phi_\alpha(u) \, | \, = O(t^{-\rho}), \ \rho > 0.$$

† When $\alpha + \delta \geqq 1$ the conclusion is still true if (11) holds in *some* interval $(0, \eta)$.
‡ If we observe that

$$\log | \, \tfrac{1}{2} \operatorname{cosec} \tfrac{1}{2} t \, | \, \sim \sum_1^\infty \frac{\cos nt}{n},$$

we easily see that Theorem 1a may be reduced to the o analogue of Theorem 1 by replacing $\phi(t)$ by $\phi(t) - s \log | \, t^{-1} |$.

Theorem 2. *If* $\alpha \geqq 1$ *and*

$$(13) \qquad \frac{1}{t} \int_0^t u \, | \, \phi_\alpha'(u) \, | \, du = O(1)$$

in an interval $(0, \eta)$, *then*

$$(14) \qquad \sum_{n < \omega} n A_n = O(\omega) \quad (C, \alpha - 1 + \delta)*$$

as $\omega \to \infty$, *for every* $\delta > 0$.

Proof of Theorem 2. We have

$$\beta \omega^{-\beta} \sum_{n < \omega} (\omega - n)^{\beta-1} n A_n = \beta \omega^{-\beta} \sum_{n < \omega} (\omega - n)^{\beta-1} \{ \omega - (\omega - n) \} A_n$$

$$= \beta \{ r_{\beta-1}(\omega) - r_\beta(\omega) \}$$

$$= \omega r_\beta'(\omega),$$

and so we have to show that $\omega r_\beta'(\omega) = O(1)$ as $\omega \to \infty$, for $\beta > \alpha$. There is no loss of generality in supposing that $\beta \leqq \alpha + 1$. Now it has been shown elsewhere† that

$$\tfrac{1}{2} \pi r_\beta(\omega) = \omega \int_0^\infty \phi_\alpha(t) J_\beta^\alpha(\omega t) dt = \int_0^\infty \phi_\alpha\left(\frac{u}{\omega}\right) J_\beta^\alpha(u) du,$$

where

$$(15) \qquad J_\beta^\alpha(x) = \frac{(-1)^{h+1} x^{h+1}}{\Gamma(1 + h - \alpha) \Gamma(\alpha + 1)} \int_1^\infty (t - 1)^{h-\alpha} \gamma_{1+\beta}^{(h+1)}(xt) dt,$$

h being the greatest integer not greater than α, and $\gamma_{1+\beta}(x)$ being given by

$$(16) \qquad \gamma_{1+\beta}(x) = \int_0^1 (1 - u)^\beta \cos xu \, du. \ddagger$$

It has also been shown§ that, for $x > 0$,

$$(17) \qquad | J_\beta^\alpha(x) | \begin{cases} \leqq A \\ \leqq A x^{\alpha-\beta-1}. \end{cases}$$

By differentiation under the integral sign we obtain‖

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* This is equivalent to $n A_n = O(1)$ $(C, \alpha + \delta)$. Cf. Hobson, **14**, pp. 90–98.

† Bosanquet, **6**. The formula is valid for $\beta \geqq \alpha \geqq 0$.

‡ $\gamma_\alpha(x) = \Gamma(\alpha) x^{-\alpha} C_\alpha(x)$, where $C_\alpha(x)$ is Young's generalization of the cosine function. Young, **23**.

§ See Bosanquet, **3** and **6**. Here and elsewhere A denotes some number independent of the variable, or variables, under consideration, and is not necessarily the same at each occurrence.

‖ It will be seen that the resulting integral converges uniformly for $\omega \geqq \epsilon > 0$, when $\beta > \alpha$.

$$\tfrac{1}{2}\pi\omega r'_\beta\,(\omega) = -\frac{1}{\omega}\int_0^\infty u\phi'_\alpha\left(\frac{u}{\omega}\right)J^\beta_\beta\,(u)du = -\omega\int_0^\infty t\phi'_\alpha\,(t)J^\beta_\beta\,(\omega t)dt.$$

Now write, for $\omega > \pi^{-1}$,

$$-\tfrac{1}{2}\pi\omega r'_\beta\,(\omega) = \omega\int_0^\infty = \omega\int_0^{\omega^{-1}} + \omega\int_{\omega^{-1}}^\pi + \omega\int_\pi^\infty = I_1 + I_2 + I_3.$$

Then we have, by (13) and (17),

$$\left|\,I_1\,\right| \leqq A\omega\int_0^{\omega^{-1}}\left|\,t\phi'_\alpha\,(t)\,\right|dt = O(1)$$

and, integrating by parts in the usual manner,

$$\left|\,I_2\,\right| \leqq A\omega^{\alpha-\beta}\int_{\omega^{-1}}^\pi\left|\,t\,\phi'_\alpha\,(t)\,\right|t^{\alpha-\beta-1}dt = O(1)$$

for $\beta > \alpha$. Finally, if we write

$$I_3 = \sum_{\nu=1}^\infty \omega\int_{\nu\pi}^{(\nu+1)\pi} = \sum_{\nu=1}^\infty I_3^{(\nu)},$$

we can show from the periodicity of $\phi(t)$ that $I_3^{(\nu)} = O(\omega^{\alpha-\beta}\nu^{\alpha-\beta-1})$, uniformly in ν, and hence $I_3 = O(\omega^{\alpha-\beta})$, for $\beta > \alpha$. This completes the proof.

Proof of Theorem 1. In what follows we need only consider the case $0 < \alpha < \beta \leqq 1$. The case $\alpha = 0$ follows from Lemma 3, with α in place of $\alpha+1$, but a direct proof proceeds along similar lines.

We first observe that $\Phi_\alpha(t)$ is of bounded variation in $(0, \pi)$, and $\Phi_\alpha(+0) = 0$. For, if $0 < t < \pi$, we have by (11), writing $\Phi^*(t) = \int_0^t u\,|\,d\phi_\alpha(u)\,|$,

$$\int_t^\pi\left|\,d\phi_\alpha(u)\,\right| = \int_t^\pi\frac{1}{u}\,|\,ud\phi_\alpha(u)\,|$$

$$= \left[\frac{1}{u}\,\Phi^*(u)\right]_t^\pi + \int_t^\pi\Phi^*(u)\,\frac{du}{u^2}$$

$$= O(1) + O\left(\log\frac{\pi}{t}\right).$$

It follows that

$$\phi_\alpha(t) = O\left(\log\frac{2\pi}{t}\right),$$

and hence $\Phi_\alpha(+0) = 0$. Hence also, for $0 < t < \pi$,

$$\Gamma(\alpha + 1) \int_t^\tau |d\Phi_\alpha(u)| = \int_t^\tau |d\{u^\alpha \phi_\alpha(u)\}|$$

$$\leq \int_t^\tau u^\alpha |d\phi_\alpha(u)| + \alpha \int_t^\tau u^{\alpha-1} |\phi_\alpha(u)| \, du$$

$$= O(1) + O\left(t^\alpha \log \frac{2\pi}{t}\right)$$

$$= O(1).$$

Now, if $\kappa^\alpha(n, t) + i\bar\kappa^\alpha(n, t)$ denotes the Cesàro mean of order α of the sequence

(18)
$$\frac{1}{\pi} + \frac{2}{\pi} \sum_{\nu=1}^n e^{i\nu t},$$

the Cesàro mean of order β of $(2/\pi) \sin nt$ is $\beta(n+\beta)^{-1} \bar\kappa^{\beta-1}(n, t)$. Since

$$nA_n = \frac{2}{\pi} \int_0^\tau \phi(t) \frac{d}{dt} (\sin nt) dt$$

we now have, by Lemma 4, with 0 in place of β and α in place of $\alpha+1$,

$$\tau_n^\beta = \frac{\beta}{n+\beta} \int_0^\tau \phi(t) \frac{d}{dt} \bar\kappa^{\beta-1}(n, t) dt$$

$$= O\left(\frac{1}{n}\right) \int_0^\tau \frac{d}{dt} \bar\kappa^{\beta-1}(n, t) dt \int_0^t (t - u)^{-\alpha} d\Phi_\alpha(u)$$

$$= O\left(\frac{1}{n}\right) \int_0^\tau d\Phi_\alpha(u) \int_u^\tau (t - u)^{-\alpha} \frac{d}{dt} \bar\kappa^{\beta-1}(n, t) dt$$

$$= O\left(\frac{1}{n}\right) [\Phi_\alpha(u)J(n, u)]_0^\tau + O\left(\frac{1}{n}\right) \int_0^\tau \Phi_\alpha(u) \frac{d}{du} J(n, u) du,$$

where

(19)
$$J(n, u) = \int_u^\tau (t - u)^{-\alpha} \frac{d}{dt} \bar\kappa^{\beta-1}(n, t) dt,$$

provided the last two steps be justified.

We next show that, for $0 < u < \pi$,

(20)
$$|J(n, u)| \begin{cases} \leq An^{1+\alpha} \\ \leq An^{1+\alpha-\beta} u^{-\beta}. \end{cases}$$

Now, for $0 < t < \pi$, $0 < \beta \leq 1$, we have[*]

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[*] Cf. Zygmund, **27**, and Gergen, **10**. Both these papers are concerned with $\kappa^{\beta-1}(n, t)$, but contain enough analysis to show how (21) and (22) are obtained.

(21)
$$\left| \bar{\kappa}^{\beta-1}(n, t) \right| \begin{cases} \leqq An \\ \leqq An^{1-\beta}t^{-\beta} \end{cases}$$

and

(22)
$$\left| \frac{d}{dt} {}^{-\beta-1}(n, t) \right| \begin{cases} \leqq An^2 \\ \leqq An^{2-\beta}t^{-\beta}. \end{cases}$$

Hence writing, for $u+n^{-1}<\pi,$*

$$J(n, u) = \int_u^{u+n^{-1}} + \int_{u+n^{-1}}^{\pi} = J_1 + J_2,$$

we have, by (22),

$$|J_1| \leqq An^{2-\beta} \int_u^{u+n^{-1}} (t - u)^{-\alpha} \min (n^\beta, t^{-\beta})dt$$

$$\leqq An^{2-\beta} \min (n^\beta, u^{-\beta}) \int_u^{u+n^{-1}} (t - u)^{-\alpha}dt$$

$$= O(n^{1+\alpha-\beta}) \min (n^\beta, u^{-\beta}),$$

and, by the second mean-value theorem,

$$J_2 = \left(\frac{1}{n}\right)^{-\alpha} \int_{u+n^{-1}}^{\zeta} \frac{d}{dt} \bar{\kappa}^{\beta-1}(n, t)dt†$$

$$= n^\alpha O(n^{1-\beta}) \min (n^\beta, u^{-\beta})$$

by (21). This establishes (20).

Returning now to the main theme of the proof, we see that the inversion of the repeated integral is justified, the resulting integral being absolutely convergent,‡ and we obtain

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* When $u+n^{-1}\geqq\pi$ the second integral does not occur, and the argument is simpler.
† Where $u+n^{-1}<\zeta<\pi.$
‡ Since, by (22),

$$\int_0^\pi |d\Phi_\alpha(u)| \int_u^\pi (t - u)^{-\alpha} \left| \frac{d}{dt} \bar{\kappa}^{\beta-1}(n, t) \right| dt$$

$$\leqq An^2 \int_0^\pi |d\Phi_\alpha(u)| \int_u^\pi (t - u)^{-\alpha}dt$$

$$\leqq An^2 \int_0^\pi |d\Phi_\alpha(u)|,$$

for $\alpha<1$. We use the analogue for Stieltjes integrals of Fubini's theorem; see Bosanquet, **8**.

$$\tau_n^\beta = O(n^{\alpha-\beta}) + O\left(\frac{1}{n}\right) \int_0^\pi \phi_\alpha(u) u^\alpha \frac{d}{du} J(n, u) du$$

$$= O(n^{\alpha-\beta}) + O\left(\frac{1}{n}\right) \left[\phi_\alpha(u) \int_0^u v^\alpha \frac{d}{dv} J(n, v) dv \right]_0^\pi$$

$$+ O\left(\frac{1}{n}\right) \int_0^\pi d\phi_\alpha(u) \int_0^u v^\alpha \frac{d}{dv} J(n, v) dv.$$

It will be enough now to show that, for $0 < u \leq \pi$,

(23) $$\left| \int_0^u v^\alpha \frac{d}{dv} J(n, v) dv \right| \quad \begin{cases} \leq A n^{1+\alpha} u^\alpha \\ \leq A n^{1+\alpha-\beta} u^{\alpha-\beta}. \end{cases}$$

For, if this be established, it will follow that the integrated term is

$$O\left(\frac{1}{n}\right) \phi_\alpha(\pi) \int_0^\pi v^\alpha \frac{d}{dv} J(n, v) dv = O(n^{\alpha-\beta}),$$

and, writing

$$\frac{1}{n} \int_0^\pi d\phi_\alpha(u) \int_0^u v^\alpha \frac{d}{dv} J(n, v) dv = \frac{1}{n} \int_0^{n^{-1}} + \frac{1}{n} \int_{n^{-1}}^\pi = K_1 + K_2,$$

we shall have, from (11), integrating by parts in the usual way,

$$| K_1 | \leq A n^\alpha \int_0^{n^{-1}} u^{\alpha-1} | u d\phi_\alpha(u) | = O(1)$$

and

$$| K_2 | \leq A n^{\alpha-\beta} \int_{n^{-1}}^\pi u^{\alpha-\beta-1} | u d\phi_\alpha(u) | = O(1)$$

for $\beta > \alpha$. The theorem will therefore be proved.

To establish (23), we first observe that

$$\int_0^u v^\alpha \frac{d}{dv} J(n, v) dv = u^\alpha J(n, u) - \alpha \int_0^u v^{\alpha-1} J(n, v) dv,$$

and the first inequality follows from the first inequality (20). We next observe that, if $\phi(t) \equiv 1$, then $\phi_\alpha(t) = 1$ for all t and $\tau_n^\beta = 0$ for all n. Hence, following through our previous reasoning with this special value of $\phi(t)$, we find that

$$0 = O(n^{\alpha-\beta}) + \frac{\beta}{n+\beta} \int_0^\pi v^\alpha \frac{d}{dv} J(n, v) dv.$$

If we now write

$$\int_0^u v^\alpha \frac{d}{dv} J(n, v) = \int_0^\pi - \int_u^\pi = J' - J'',$$

we have $J' = O(n^{1+\alpha-\beta})$, and finally, observing that

$$\int_u^\pi v^\alpha \frac{d}{dv} J(n, v) dv = [v^\alpha J(n, v)]_u^\pi - \alpha \int_u^\pi v^{\alpha-1} J(n, v) dv,$$

we obtain, from the second inequality (20), $J'' = O(n^{1+\alpha-\beta}) + O(n^{1+\alpha-\beta} u^{-\beta})$. The second inequality (23) now follows, for $0 < u \leq \pi$.

This completes the proof.

We have as a corollary of Theorem 1, after our remarks in the introduction, the following theorem.

THEOREM 3. *If $\alpha \geq 0$ and (11) holds in the interval $(0, \pi)$, then the Fourier series of $f(t)$ is summable $(C, \alpha - 1 + \delta)$ at the point $t = x$, for every $\delta > 0$, if it is summable (A).*

3. The next theorem shows that a condition of the type (11) is also necessary for the boundedness (C) of the sequence.

We confine ourselves to the Rieszian form of the theorem.

THEOREM 4. *If $\alpha \geq 0$ and*

(24)
$$\frac{1}{\omega} \int_0^\omega u \, | \, dr_\alpha(u) \, | \, = O(1)*$$

as $\omega \to \infty$, then $t\phi_\delta'(t) = O(1)$ in the interval $(0, \pi)$, for $\beta > \alpha + 1$.

There is an analogous result with o in place of O.

It follows from (24) that $r_\alpha(\omega) = O(\log \omega)$ as $\omega \to \infty$. If then $\beta > \alpha + 1$, $t > 0$, and h is the greatest integer not greater than α, we have, by an argument used elsewhere,† since $A_n = o(1)$,

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* The condition is satisfied in particular if $\alpha \geq 1$ and $\omega r_\alpha'(\omega) = O(1)$, i.e.,

$$\sum_{n < \omega} n A_n = O(\omega) \ (C, \alpha - 1),$$

or, what is equivalent, $n A_n = O(1)(C, \alpha)$. When $\alpha = 0$ (24) becomes

$$\sum_{n < \omega} n | A_n | = O(\omega).$$

† See Bosanquet, **3** and **4**. The argument was previously used with the hypothesis $R_\alpha(\omega) = o(\omega^\alpha)$. In the present case we have

$$R_\rho(\omega) = o(\omega^{\rho+1}), \qquad 0 \leq \rho \leq \alpha - 1,$$
$$= O(\omega^\alpha \log \omega), \qquad \alpha - 1 < \rho \leq \alpha,$$
$$= O(\omega^\rho \log \omega), \qquad \rho \geq \alpha,$$

as $\omega \to \infty$. Also

$$\gamma_\beta^{(\rho)}(x) = O(x^{-\rho-2}) + O(x^{-\beta})$$

as $x \to \infty$, and $\gamma_\beta^{(\rho)}(x) = O(1)$ for $x > 0$. The inversion of the repeated integral remains valid in the present case.

$$\beta^{-1}\phi_\beta(t) = -\int_0^\infty \gamma_\beta(tu)R_0(u)du$$

$$= \left[\sum_{\rho=1}^{h+1} \frac{(-1)^\rho}{\Gamma(\rho+1)} t^\rho \gamma_\beta^{(\rho)}(tu)R_\rho(u)\right]_0^\infty$$

$$+ \frac{(-1)^{h+2}t^{h+2}}{\Gamma(h+2)} \int_0^\infty \gamma_\beta^{(h+2)}(tu)R_{h+1}(u)du$$

$$= \frac{(-1)^{h+2}t^{h+2}}{\Gamma(1+h-\alpha)\Gamma(\alpha+1)} \int_0^\infty \gamma_\beta^{(h+2)}(tu)du \int_0^u (u-v)^{h-\alpha}R_\alpha(v)dv$$

$$= \frac{(-1)^{h+2}t^{h+2}}{\Gamma(1+h-\alpha)\Gamma(\alpha+1)} \int_0^\infty R_\alpha(v)dv \int_v^\infty (u-v)^{h-\alpha}\gamma_\beta^{(h+2)}(tu)du$$

$$= t\int_0^\infty r_\alpha(v)I_\beta^\alpha(tv)dv = \int_0^\infty r_\alpha'\left(\frac{w}{t}\right)I_\beta^\alpha(w)dw,$$

where

$$I_\beta^\alpha(x) = \frac{(-1)^{h+2}x^{h+1}}{\Gamma(1+h-\alpha)\Gamma(\alpha+1)} \int_1^\infty (y-1)^{h-\alpha}\gamma_\beta^{(h+2)}(xy)dy.$$

It has also been shown* that, for $x>0$, $\beta>\alpha+1\geqq 1$,

(25)
$$|I_\beta^\alpha(x)| \begin{cases} \leqq A \\ \leqq Ax^{\alpha-\beta}. \end{cases}$$

It follows, by differentiation under the integral sign,† that

$$\beta^{-1}t\phi_\beta'(t) = -\frac{1}{t}\int_0^\infty I_\beta^\alpha(w)wdr_\alpha\left(\frac{w}{t}\right) = -t\int_0^\infty I_\beta^\alpha(tv)vdr_\alpha(v).$$

Therefore, writing

$$-\beta^{-1}t\phi_\beta'(t) = t\int_0^{t^{-1}} + t\int_{t^{-1}}^\infty = L_1 + L_2,$$

we have

$$|L_1| \leqq At\int_0^{t^{-1}}|vdv_\alpha(v)| = O(1)$$

and

$$|L_2| \leqq At^{1+\alpha-\beta}\int_{t^{-1}}^\infty v^{\alpha-\beta}|vdv_\alpha(v)| = O(1)$$

for $\beta>\alpha+1$. This proves the theorem.

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* Bosanquet, 3.
† The resulting integral converges uniformly for $t\geqq\epsilon>0$, when $\beta>\alpha+1$.

Combining Theorems 1 and 4 together we now have the following theorem.

THEOREM 5. *A necessary and sufficient condition that* $nA_n = O(1)$ *(C) is that, for some* $\kappa \geq 0$, $\phi_\kappa(t)$ *should be absolutely continuous except at* $t = 0$ *and* $t\phi_\kappa'(t) = O(1)$ *in the interval* $(0, \pi)$.

There is an analogous result with o in place of O.

We also add the following theorem.

THEOREM 6. *If* $\alpha \geq 0$ *and*

$$(26) \qquad\qquad \frac{1}{\omega} \int_0^\omega |r_\alpha(u)| = O(1)$$

as $\omega \to \infty$, *then* $\phi_\beta(t) = O(1)$ *in the interval* $(0, \pi)$, *for* $\beta > \alpha + 1$.

There is an analogous result with o in place of O.*

It follows from (26), by the analogue of Lemma 1, that (24) is satisfied, with $\alpha + 1$ in place of α, and $\sum A_n$ is bounded (C). Hence $t\phi_{\beta+1}'(t) = O(1)$ for $\beta > \alpha + 1$, by Theorem 4, and $\phi(t) = O(1)$ (C).† The result may now be obtained from (5).

4. Finally we can generalize the Rieszian form of Theorem 5 as follows.

THEOREM 7. *If* λ *is a non-negative integer, a necessary and sufficient condition that*

$$\left(\omega \frac{d}{d\omega}\right)^\lambda r_\beta(\omega) = O(1) \ \ as \ \omega \to \infty,$$

for some $\beta \geq \lambda$, *is that*

$$\left(t \frac{d}{dt}\right)^\lambda \phi_\kappa(t) = O(1) \ \ in \ the \ interval \ (0, \pi)$$

for some $\kappa \geq \lambda$, *where* $\phi_\kappa(t)$ *is a* λ*th integral except at* $t = 0$.

Suppose that for some $\alpha \geq \lambda$ we have

$$\left(t \frac{d}{dt}\right)^\lambda \phi_\alpha(t) = O(1) \ \ for \ 0 < t < \pi.$$

Then if $0 \leq \mu \leq \lambda$, we have

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* For the case $\alpha = 0$ see Bosanquet, 4. The hypothesis is satisfied in particular when $\sum A_n$ is summable (C, α). See Hardy and Littlewood, 11, Paley, 18, Verblunsky, 19, Wiener, 21, 22, Bosanquet, 3.

† Hardy and Littlewood, 11.

$$\left(t\frac{d}{dt}\right)^{\lambda-\mu}\phi_\alpha(t) = O\left\{\left(\log\frac{1}{t}\right)^\mu\right\} \text{ as } t \to 0,$$

and it is easily shown by induction that

$$(27) \qquad \tfrac{1}{2}\pi\left(\omega\frac{d}{d\omega}\right)^\lambda r_\beta(\omega) = (-1)^\lambda\omega\int_0^\infty \left(t\frac{d}{dt}\right)^\lambda \phi_\alpha(t)J_\beta^\alpha(\omega t)dt$$

for $\beta > \alpha$. If, on the other hand, for some $\alpha \geq \lambda$ we have

$$\left(\omega\frac{d}{d\omega}\right)^\lambda r_\alpha(\omega) = O(1) \text{ as } \omega \to \infty,$$

then

$$\left(\omega\frac{d}{d\omega}\right)^{\lambda-\mu} r_\alpha(\omega) = O\{(\log\omega)^r\} \text{ as } \omega \to \infty,$$

and we obtain, for $\beta > \alpha+1$,

$$(28) \qquad \beta^{-1}\left(t\frac{d}{dt}\right)^\lambda \phi_\beta(t) = (-1)^\lambda t\int_0^\infty \left(u\frac{d}{du}\right)^\lambda r_\alpha(u)I_\beta^\alpha(tu)du.$$

The result follows from (27) and (28) by arguments analogous to those of Theorems 2 and 4.

The analogue of Theorem 7 for conjugate series is as follows.*

THEOREM 8. *If λ is a non-negative integer, a necessary and sufficient condition that*

$$\left(\omega\frac{d}{d\omega}\right)^\lambda \bar{r}_\beta(\omega) = O(1) \text{ as } \omega \to \infty,$$

for some $\beta \geq \lambda$, is that

$$\left(t\frac{d}{dt}\right)^\lambda \theta_k(t) = O(1) \text{ in the interval } (0, \pi)$$

for some $k \geq \lambda$, where $\theta_k(t)$ is a λth integral except at $t=0$.

It is interesting to note that, by (6), if $k \geq 1$,†

$$t\frac{d}{dt}\theta_k(t) = -\frac{2}{\pi}\psi_k(t)$$

and so, when $\lambda \geq 1$, the condition in Theorem 8 takes the form

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* The previously known cases were $\lambda=0$ or 1. Hardy and Littlewood, **12.**

† When $0 \leq k < 1$ this is true except possibly in a set of measure zero.

$$\left(t\frac{d}{dt}\right)^{\lambda-1}\psi_k(t) = O(1).$$

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