# ON CERTAIN GENERALIZATIONS OF THE CAUCHY-TAYLOR EXPANSION THEORY*

BY

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## I. Introduction

The purpose of this paper is to consider the expression of any function $f(x)$ analytic at zero in an expansion of the form

$$\sum_{n=0}^{\infty} \alpha_n F_n(x) \tag{1}$$

where the functions $F_n(x)$ are given by

$$F_n(x) = x^n + \sum_{s=1}^{\infty} a_s^{(n)} x^{n+s}, \tag{1.1}$$

$F_n(x)$ are analytic in the region $|x| < r$, and $|F_n(x)/x^n|$ are uniformly bounded in the same region.

Pincherle† as early as 1881 showed that such expansions are always possible. We therefore are interested not in the possibility of such expansions but rather in the region of their validity. The facts which have been obtained for this problem from studies of expansions in functions of more general type seem to be of a restricted nature.‡ Okada§ published an important paper on this subject in 1922. Since that time the problem has been studied by numerous other writers.|| The work of Pincherle and Okada has generally been overlooked by later writers. However, the importance of the work of Okada is shown by the fact that it includes as special cases theorems obtained by three of the later writers, namely, Izumi, Widder, and Takahashi.¶ The proofs

* Presented to the Society, April 20, 1935; received by the editors September 23, 1935.

† Memorie della Academia delle Scienze dell' Istituto di Bologna, ser. 4, vol. 3 (1881), pp. 151 ff. See also L. Tonelli, Annali di Matematica, ser. 3, vol. 18, pp. 117 ff.

‡ See, for instance, G. D. Birkhoff, Comptes Rendus, vol. 164 (1917), pp. 942 ff.

§ Tôhoku Mathematical Journal, vol. 22 (1922–23), pp. 325 ff.

|| S. Izumi, Tôhoku Mathematical Journal, vol. 28 (1927), pp. 97 ff. R. F. Graesser, American Journal of Mathematics, vol. 49 (1927), pp. 577–597. S. Narumi, Tôhoku Mathematical Journal, vol. 30 (1928–29), pp. 441 ff. D. V. Widder, these Transactions, vol. 31 (1929), pp. 43–53. S. Takenaka, Proceedings of the Tokyo Physico-Mathematical Society, ser. 3, vol. 13 (1931), pp. 111–117. S. Takahashi, Tôhoku Mathematical Journal, vol. 35 (1932), pp. 242 ff.

¶ A theorem of a different form which is also an easy consequence of Okada's theorem is due to Carmichael. See Bulletin of the American Mathematical Society, vol. 40 (1934), p. 211 (abstract). Theorem VI is a generalization of this theorem due to Carmichael.

offered for these special cases are essentially at least as difficult as the proof of Okada's theorem. Moreover, we give (Theorem III) a slight generalization of his work which includes all the general expansion theorems *on this problem* in the papers cited.

Theorem III, however, is not sufficient to yield the facts obtained by Graesser* concerning expansions in a particular set of confluent hypergeometric functions. Further we show (§V) that these facts cannot be obtained from any theorem, such as III, in which each condition is stated in terms of any set of positive bounds on the coefficients $a_s^{(n)}$. Hence, instead of using bounds on a single coefficient $a_s^{(n)}$, we consider bounds on certain linear combinations of these coefficients, and thereby introduce an auxiliary set of multipliers. We then obtain a general expansion principle (Theorem I) from which by a suitable choice of the multipliers we derive all the known expansion theorems as well as new theorems of the type considered. In particular, by choosing the multipliers from certain binomial sums, we obtain (§V) a generalization of the expansion in hypergeometric functions of Graesser.

The value of Theorem I in actual application rests on a good choice of certain sets of arbitrary constants. Theorems II and III (§II) are simpler and more readily applied. With the exception of the work of Okada and Graesser mentioned above, the known theorems on this subject and generalizations of these theorems are derived in §III. These theorems readily adapt themselves to applications.

In §IV, we obtain a class of sets of integral functions $F_n(x)$ having the property that the expansion of a function $f(x)$ analytic at zero will always converge absolutely and uniformly in any region interior to the circle through a singularity of $f(x)$ nearest the origin. Moreover, this class of functions includes the Bessel coefficients. Expansions in restricted sets of integral functions $F_n(x)$ with leading term $x^n$ have also been treated (see §III) by Julia,† Onofri,‡ and Valiron.§

We apply the results of §III to obtain (§VI) new general theorems on expansions in products of functions of the form $F_n(x)$. These theorems are in turn applied to obtain very considerable generalizations of the known expansions in products of Bessel functions.

## II. General theorems

Consider a set of functions $F_n(x)$ defined as in (1.1) and analytic in the

* Loc. cit., p. 592.

† G. Julia, Acta Mathematica, vol. 54 (1930), pp. 280 ff.

‡ L. Onofri, Annali di Matematica, ser. 4, vol. 13 (1934–35), pp. 209 ff.

§ G. Valiron, Bulletin des Sciences Mathématiques, vol. 69 (1934), pp. 26–28.

region $|x| < r$. As indicated in the introduction, we consider positive numbers $b_\nu^{(\mu)}$ which are bounds for linear combinations of the coefficients $a_\nu^{(\mu)}$, thus

$$b_\nu^{(\mu)} \geqq \left| \sum_{t=0}^{\nu} k_{\mu+\nu}^{(\mu+t)} a_t^{(\mu)} \right|,$$

where $a_0^{(\mu)} = 1$. Here the multipliers $k_{\mu+\nu}^{(\mu)}$ are complex numbers at one's choice subject to the condition that $k_\mu^{(\mu)} = 1$. Let $\delta_{\mu+\nu}^{(\mu)}$ be positive numbers such that $\delta_{\mu+\nu}^{(\mu)} \geqq |k_{\mu+\nu}^{(\mu)}|$. Put

$$M_{\mu+\nu}^{(\mu)} = b_1^{(\mu+\nu)} + \max\,(\delta_{\mu+\nu+1}^{(\mu)}/\delta_{\mu+\nu}^{(\mu)},\ b_{\nu+1}^{(\mu)}/b_\nu^{(\mu)},\ b_\nu^{(\mu+1)}/b_{\nu-1}^{(\mu+1)},\ \cdots,\ b_2^{(\mu+\nu-1)}/b_1^{(\mu+\nu-1)}).$$

Denote by $\overline{M}_{\mu+\nu}^{(\mu)}$ expressions obtained from $M_{\mu+\nu}^{(\mu)}$ by omitting any or all of the outstanding $b$'s or of ratios of $b$'s with a superscript $\mu_1$ such that

$$\sum_{t=0}^{p} k_{\mu_1+p}^{(\mu_1+t)} a_t^{(\mu_1)} = 0 \text{ for all } p > 0.$$

Assuming the notation and conditions stated above we have

THEOREM I. *If there exist numbers $k_{\mu+\nu}^{(\mu)}$, $b_\nu^{(\mu)}$, $\delta_{\mu+\nu}^{(\mu)}$, and a positive number $R$ such that*

$$\left| \frac{F_n(x)}{x^n} \right| \leqq Q \text{ for } |x| \leqq R < r, \tag{2.1}$$

*where $Q$ is independent of $n$, and such that*

$$\sum_{\mu,\nu=0}^{\infty} |\gamma_\mu|\, \overline{M}_{\mu+\nu-1}^{(\mu)} \overline{M}_{\mu+\nu-2}^{(\mu)} \cdots \overline{M}_{\mu}^{(\mu)} \delta_\mu^{(\mu)} R^{\mu+\nu} \tag{2.2}$$

*converges,* then the function $f(x) \equiv \sum_{\mu=0}^{\infty} \gamma_\mu x^\mu$ has an absolutely and uniformly convergent expansion of the form* (1) *valid for $|x| \leqq R$. Further, there is only one such expansion converging uniformly for $|x| \leqq R'$, $R' > 0$.*

The proof will be given for the particular case $\overline{M}_{\mu+\nu}^{(\mu)} = M_{\mu+\nu}^{(\mu)}$ but the necessary changes to obtain the more general case are obvious.

Let $c_{n+\nu}^{(n)}$, $n = 0, 1, 2, \cdots$, be defined by the equations†

$$\begin{aligned} &c_n^{(n)} = 1, \\ &c_{n+\nu}^{(n)} + c_{n+\nu-1}^{(n)} a_1^{(n+\nu-1)} + c_{n+\nu-2}^{(n)} a_2^{(n+\nu-2)} + \cdots + c_n^{(n)} a_\nu^{(n)} = 0, \qquad \nu > 0. \end{aligned} \tag{2.3}$$

* We shall see later that if (2.1) is satisfied for a value $R_1 > 0$ and $f(x)$ is analytic for $|x| \leqq R_1$, it is possible to choose the quantities $k_{\mu+\nu}^{(\mu)}$, $\delta_{\mu+\nu}^{(\mu)}$, $b_\nu^{(\mu)}$ so that (2.2) converges for some positive $R \leqq R_1$. In applications the problem is to choose these quantities so as to obtain a value of $R$ as large as possible.

† These equations are obtained by equating coefficients of $x^\nu$ in the formal relation

$$x^n = \sum_{p=0}^{\infty} c_{n+p}^{(n)} F_{n+p}(x).$$

Then

$$(2.4)\qquad c_{n+\nu}^{(n)} = (-1)^{\nu}\begin{vmatrix} 1 & 1 & 0 & 0 & \cdots 0 \\ 0 & a_1^{(n)} & 1 & 0 & \cdots 0 \\ 0 & a_2^{(n)} & a_1^{(n+1)} & 1 & \cdots 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots \cdot \\ 0 & a_{\nu-1}^{(n)} & a_{\nu-2}^{(n+1)} & a_{\nu-3}^{(n+2)} & \cdots 1 \\ 0 & a_{\nu}^{(n)} & a_{\nu-1}^{(n+1)} & a_{\nu-2}^{(n+2)} & \cdots a_1^{(n+\nu-1)} \end{vmatrix}.$$

We next modify the determinant in (2.4) by multiplying the $r$th row, $r=1, 2, \cdots, p-1$, by $k_{n+p-1}^{(n+r-1)}$ and adding to the $p$th row. Hence

$$\left| c_{n+\nu}^{(n)} \right| \leqq \left\{ \begin{matrix} \delta_n^{(n)} & 1 & 0 & 0 & \cdots 0 \\ \delta_{n+1}^{(n)} & b_1^{(n)} & 1 & 0 & \cdots 0 \\ \delta_{n+2}^{(n)} & b_2^{(n)} & b_1^{(n+1)} & 1 & \cdots 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots \cdot \\ \delta_{n+\nu-1}^{(n)} & b_{\nu-1}^{(n)} & b_{\nu-2}^{(n+1)} & b_{\nu-3}^{(n+2)} & \cdots 1 \\ \delta_{n+\nu}^{(n)} & b_{\nu}^{(n)} & b_{\nu-1}^{(n+1)} & b_{\nu-2}^{(n+2)} & \cdots b_1^{(n+\nu-1)} \end{matrix} \right\} \equiv A_{n+\nu}^{(n)},$$

where the expression in braces is to be expanded by the same law as a determinant except that all signs are taken positive.

Hence expanding $A_{n+\nu}^{(n)}$ by the last column

$$\left| c_{n+\nu}^{(n)} \right| \leqq b_1^{(n+\nu-1)} A_{n+\nu-1}^{(n)} + \left\{ \begin{matrix} \delta_n^{(n)} & 1 & 0 & \cdots 0 \\ \delta_{n+1}^{(n)} & b_1^{(n)} & 1 & \cdots 0 \\ \delta_{n+2}^{(n)} & b_2^{(n)} & b_1^{(n+1)} & \cdots 0 \\ \cdot & \cdot & \cdot & \cdots \cdot \\ \delta_{n+\nu-2}^{(n)} & b_{\nu-2}^{(n)} & b_{\nu-3}^{(n+1)} & \cdots 1 \\ \delta_{n+\nu}^{(n)} & b_{\nu}^{(n)} & b_{\nu-1}^{(n+1)} & \cdots b_2^{(n+\nu-2)} \end{matrix} \right\},$$

$$\begin{aligned} &\leqq b_1^{(n+\nu-1)} A_{n+\nu-1}^{(n)} + A_{n+\nu-1}^{(n)} \\ &\qquad \cdot \max\left(\delta_{n+\nu}^{(n)}/\delta_{n+\nu-1}^{(n)},\ b_{\nu}^{(n)}/b_{\nu-1}^{(n)},\ b_{\nu-1}^{(n+1)}/b_{\nu-2}^{(n+1)}, \cdots, b_2^{(n+\nu-2)}/b_1^{(n+\nu-2)}\right), \\ &\leqq M_{n+\nu-1}^{(n)} A_{n+\nu-1}^{(n)}, \\ &\leqq M_{n+\nu-1}^{(n)} M_{n+\nu-2}^{(n)} \cdots M_n^{(n)} \delta_n^{(n)}. \end{aligned}$$

Thus the double series $\sum_{\mu,\,\nu=0}^{\infty} \gamma_\mu c_{\mu+\nu}^{(\mu)} F_{\mu+\nu}(x)$ is dominated for $|x| \leqq R$ by the series (2.2) multiplied by $Q$. Hence the series

$$\sum_{\mu=0}^{\infty}\left(\sum_{\nu=0}^{\mu}\gamma_{\mu-\nu}c_{\mu}^{(\mu-\nu)}\right)F_{\mu}(x)$$

converges absolutely and uniformly for $|x| \leqq R$ to an analytic function $g(x)$. By a theorem due to Weierstrass the coefficients of powers of $x$ in the expansion of $g(x)$ may be obtained by expanding $F_\mu(x)$ and collecting like powers of $x$. But it is readily seen from the relations (2.3) that the coefficient of $x^n$ is $\gamma_n$. Hence $g(x)=f(x)$.

THEOREM II. *Let the functions $F_n(x)$ be defined as in* (1.1) *and analytic in the region $|x|<r$. Let $b_\nu^{(\mu)}$ be any set of positive numbers such that $b_\nu^{(\mu)} \geqq |a_\nu^{(\mu)}+k_{\mu+\nu}a_{\nu-1}^{(\mu)}|$, $\nu=1, 2, \cdots ; \mu=0, 1, \cdots$, where $a_0^{(\mu)}=1$ and $k_n$ is a set of complex numbers such that* $\limsup_{n\to\infty}|k_n|^{1/n}\leqq 1$. *If $R<\rho\leqq r$ and*

$$0 < R < \liminf_{n\to\infty} M_n^{-1}, \tag{2.5}$$

$$M_n \equiv b_1^{(n)} + \max\left\{b_{n+1}^{(0)}/b_n^{(0)},\ b_n^{(1)}/b_{n-1}^{(1)}, \cdots, b_2^{(n-1)}/b_1^{(n-1)}\right\} \qquad (n = 0, 1, 2, \cdots),$$

*and if*

$$\left|\frac{F_n(x)}{x^n}\right| \leqq Q \text{ for } |x| = R, \tag{2.6}$$

*where $Q$ is independent of $n$, then any function $f(x)$ analytic for $|x|<\rho$ has an absolutely and uniformly convergent expansion of the form* (1) *valid for $|x|\leqq R$. Further, this is the only such expansion converging uniformly for $|x|\leqq R'$, $R'>0$.*

In Theorem I take $k_{\mu+1}^{(\mu)}=k_{\mu+1}$, $k_{\mu+p}^{(\mu)}=0$, $p=2, 3, \cdots$. For every positive $\epsilon$ there corresponds a $K_\epsilon$ greater than $\epsilon$ and such that $|k_{\mu+1}^{(\mu)}| \leqq K_\epsilon(1+\epsilon)^\mu$. Now let $\delta_{\mu+\nu}^{(\mu)}=K_\epsilon\epsilon^{\nu-1}(1+\epsilon)^\mu$. Then

$$M_{\mu+\nu}^{(\mu)} = b_1^{(\mu+\nu)} + \max\left(\epsilon,\ b_{\nu+1}^{(\mu)}/b_\nu^{(\mu)},\ b_\nu^{(\mu+1)}/b_{\nu-1}^{(\mu+1)}, \cdots, b_2^{(\mu+\nu-1)}/b_1^{(\mu+\nu-1)}\right)$$

and

$$M_n^{(\mu)} \leqq b_1^{(n)} + \max\left(\epsilon,\ b_{n+1}^{(0)}/b_n^{(0)},\ b_n^{(1)}/b_{n-1}^{(1)}, \cdots, b_2^{(n-1)}/b_1^{(n-1)}\right) \qquad (n = \mu, \mu+1, \cdots).$$

From (2.5) there exists a positive number $\lambda_1$, such that $M_nR\leqq\lambda_1<1$ for all $n\geqq n_0$ where $n_0$ is a suitably chosen number. But $M_n^{(\mu)}\leqq M_n+\epsilon$ so that $M_n^{(\mu)}R\leqq\lambda_1+\epsilon R<\lambda<1$ for $n=\mu+\nu\geqq n_0$ and for $\epsilon$ sufficiently small. Now $\sum_{\nu=0}^{\infty}M_{\mu+\nu-1}^{(\mu)}M_{\mu+\nu-2}^{(\mu)}\cdots M_\mu^{(\mu)}R^\nu$ converges for all $\mu$ since the ratio of the

($\nu+1$ts) to the $\nu$th term is $M_{\mu+\nu}^{(\mu)}R$ which is less than $\lambda$ for $\nu>n_0$. Moreover, if $\mu\geqq n_0$ this last series is dominated by the series $\sum_{\nu=0}^{\infty}\lambda^\nu=(1-\lambda)^{-1}$. Hence (2.2) converges if the series

$$(1-\lambda)^{-1}\sum_{\mu=n_0}^{\infty}|\gamma_\mu|\,\delta_\mu^{(\mu)}R^\mu$$

converges. This in turn is dominated by

$$(1-\lambda)^{-1}\sum_{\mu=n_0}^{\infty}|\gamma_\mu|\,(1+\epsilon)^\mu R^\mu. \tag{2.7}$$

But $f(x)$ is analytic at $x=R$ and $\limsup_{\mu\to\infty}|\gamma_\mu|^{1/\mu}R<1$. Therefore (2.7) converges if $\epsilon$ is chosen sufficiently small and hence the series (2.2) converges. Theorem II is then proved.

*If we specialize Theorem* II *by taking* $k_n\equiv 0$, *then* $b_s^{(n)}$ *is a bound for the single coefficient* $a_s^{(n)}$. *The resulting theorem we shall denote by Theorem* III.*

Theorem III is a generalization of the theorem of Okada referred to in the introduction. In the latter theorem condition (2.5) is replaced by

$$0<R<\liminf_{n\to\infty}\left[b_1^{(n)}+\max\left(1/\rho,\ b_{n+1}^{(0)}/b_n^{(0)},\ b_n^{(1)}/b_{n-1}^{(1)},\ \cdots,\ b_2^{(n-1)}/b_1^{(n-1)}\right)\right]^{-1}$$

and condition (2.6) by $1+|a_1^{(n)}|R+|a_2^{(n)}|R^2+\cdots\leqq Q$.

We now point out instances where Theorem II yields results which cannot be established by Theorem III. Consider the set of functions $F_n(x)=x^n(1+a^{(n)}x)(1-x^2)^{-1}$ where $a_{2s}^{(n)}=1$ and $a_{2s-1}^{(n)}=a^{(n)}$ for all $s$, and $\lim_{n\to\infty}a^{(n)}=1$. Then for a function $f(x)$ analytic for $|x|<1$, using $k_n=-1$ for all $n$, Theorem II establishes an expansion for $|x|<1$. However, Theorem III would yield at most $|x|<\frac{1}{2}$.

Further Theorem II may be applied with ease to the class of sets of functions which have every other coefficient zero. Application of Theorem II to obtain Neumann's† expansion in Bessel coefficients is immediate if we choose $k_n=(n)^{-1/2}$, or if we choose $k_n=\epsilon$, where $\epsilon$ is arbitrarily small.

Theorem III possesses more than a formal advantage over the theorem of Okada. In case $\limsup_{n\to\infty}(M_n-b_1^{(n)})=\infty$, neither theorem yields any results. Hence we assume $M_n-b_1^{(n)}$ bounded for all $n$. Now consider any

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* In (2.5) of Theorem II and Theorem III any or all of the outstanding $b$'s or of ratios of $b$'s with a superscript $\mu_1$ such that $a_\nu^{(\mu_1)}+k_{\mu_1+\nu}a_{\nu-1}^{(\mu_1)}=0$ for all $\nu$ may be omitted. If $a_\nu^{(\mu)}+k_{\mu+\nu}a_{\nu-1}^{(\mu)}=0$ for all $\nu$ and $\mu$ then the only restriction on $R$ is $R<\rho\leqq r$.

† *Theorie der Bessel'schen Functionen*, Leipzig, 1867, pp. 33–35. Also compare with Okada, loc. cit., p. 332.

set of functions $F_n(x)$ such that $\limsup_{n\to\infty}|a_1^{(n)}|\neq 0$ and an $R$ such that $1+|a_1^{(n)}|R+|a_2^{(n)}|R^2+\cdots\leqq Q$. Then there exists a positive number $\rho<R$ and also less than $1/(M_n-b_1^{(n)})$ for all $n$, and it is easily shown that Theorem III will establish a larger region of convergence than the Okada theorem for the expansion of any analytic function $f(x)$ whose nearest singularity is on the circle $|x|=\rho$. In particular, the expansion of $f(x)$, analytic for $\rho<\frac{1}{2}$, in the set of functions $F_n(x)=x^n(1+a^{(n)}x)(1-x^2)^{-1}$ as defined above is established for $|x|<\frac{1}{2}$ by Theorem III and for $|x|<\frac{1}{3}$ by Okada's theorem.

## III. Forms of Theorem III using the Cauchy bounds

We shall state here two sets of parallel theorems since in the development of a theory for expansions in products of analytic functions in §VI, it is convenient to have the hypotheses stated in terms of a bound on the functions $|F_n(x)/x^n|$ rather than on the coefficients $a_s^{(n)}$. It is also convenient in order to emphasize this to use the notation $g_n(x)=x^n(1+h_n(x))$ instead of $F_n(x)$. The functions $h_n(x)$ shall be assumed to satisfy the following conditions which we denote by

Conditions (3.1). *$h_n(x)$ vanishes at $x=0$, and $h_n(x)$ is analytic for $|x|<r$.*

Corresponding to any set of functions $h_n(x)$ and to a given positive $N$ less than $r$, there exists an $M_{N,n}$ such that $|h_n(x)|\leqq M_{N,n}$ for $|x|\leqq N$.

We shall hereafter indicate by $K_N$ $[\overline{K}_N]$ the expression $\limsup_{n\to\infty}M_{N,n}$ $[\limsup_{n\to\infty}\overline{M}_{N,n}]$ and by $N$ a positive number less than $r$.

The set of functions $g_n(x)$ satisfying conditions (3.1) belongs to one of two classes:

*to $A_r$ if $K_N$ is finite for every $N$;*

*to $B_r$ if $K_N$ is infinite for some $N$.*

*Moreover, the set of functions $F_n(x)$ shall be said to belong to the class $\overline{A}_r$ if $|a_s^{(n)}|\leqq\overline{M}_{N,n}N^{-s}$ and* $\limsup_{n\to\infty}\overline{M}_{N,n}\equiv\overline{K}_N$ *is finite for every $N$.*

Theorem IV.* *If the set of functions $g_n(x)$ belongs to the class $A_r$, then any function $f(x)$ analytic for $|x|<\rho$ has a unique uniformly convergent expansion*

$$(3.2)\quad f(x)=\sum_0^\infty \alpha_n g_n(x)\ \ \textit{for}\ \ |x|\leqq R<G\equiv\min\left(\rho,\ \max_N N(1+K_N)^{-1}\right)$$

*and the expansion converges absolutely for $|x|<G$.*

Theorem IV′. *If the functions $F_n(x)$ belong to the class $\overline{A}_r$, then any function $f(x)$ analytic for $|x|<\rho$ has a unique uniformly convergent expansion of the form* (1) *for*

* Theorem IV is essentially equivalent to a theorem due to Takenaka (loc. cit.). He states that $K_N$ is positive but the case $K_N=0$ need not be excluded.

$$|x| \leqq R < \bar{G} \equiv \min \left[ \rho, \max_N N(1 + \bar{K}_N)^{-1} \right]$$

*and absolutely convergent for* $|x| < \bar{G}$.*

The proof of Theorem IV′ follows at once from Theorem III. It is evident that $F_n(x)$ is analytic for $|x| < r$ since

$$\limsup_{s\to\infty} [\bar{M}_{N,n} N^{-s}]^{1/s} = N^{-1}.$$

The evaluation of (2.5) using $k_n = 0$ gives $0 < R < N(1+\bar{K}_N)^{-1}$. For such a range of values of $R$ (2.6) would be satisfied since $\bar{M}_{N,n}$ is bounded with respect to $n$. Now Theorem IV follows at once from Theorem IV′. For if we write $h_n(x) = a_1^{(n)}x + a_2^{(n)}x^2 + a_3^{(n)}x^3 + \cdots$, then from the Cauchy inequalities $|a_s^{(n)}| \leqq M_{N,n} N^{-s}$.

It is clear that if in either case $M_{N,n} \leqq P_n$, then the expansion converges for $|x| < \min [\rho, r(1 + \limsup_{n\to\infty} P_n)^{-1}]$.

If in Theorems IV and IV′, $K_N = 0$ for every $N < \rho \leqq r$ then the expansion of $f(x)$ is valid for $|x| < \rho$. The importance of this property justifies the explicit statement of the theorems for this subclass of sets of functions. First, however, we shall state without proof a lemma which will be of value here and also in our later development of expansions in products.

Lemma 1. *The totality of sets of functions of the class* $A_\rho$ *where* $K_N = 0$ *for every* $N < \rho \leqq r$ *is identical with the totality of sets of functions of the class* $\bar{A}_\rho$ *where* $\bar{K}_N = 0$ *for every* $N < \rho$.

Theorem V† [V′]. *If the set of functions* $g_n(x)$ $[F_n(x)]$ *belongs to the class* $A_\rho$ $[\bar{A}_\rho]$ *and* $K_N = 0$ $[\bar{K}_N = 0]$ *for every* $N < \rho \leqq r$ *then any function* $f(x)$ *analytic for* $|x| < \rho$ *has an absolutely and uniformly convergent expansion of the form* (3.2) [(1)] *valid for any region* $|x| \leqq R < \rho$. *Further, there is only one such expansion converging uniformly to* $f(x)$ *in a region* $|x| \leqq R'$, $R' > 0$.

From Lemma I, Theorems V and V′ are equivalent.

To obtain the expansion theorem of Valiron‡ from Theorem V′, put $F_n(x) = n!\phi_n(x)$, where $\phi_n(x)$ is an integral function of the form

* Graesser's expansion theorem (loc. cit.) is obtained from Theorem IV′ by replacing $K_N$ by $M_N$ where $M_N$ is the upper bound of $M_{N,n}$ with respect to $n$. Every case coming under Theorem IV′ can also be treated under Graesser's theorem but IV′ will in general give a larger region of validity for the expansion. The expansion theorems of Pincherle and Graesser are readily shown to be equivalent. See also the work of Izumi and Narumi referred to previously.

† This theorem was proved by Takahashi (1932) and the special case where $M_{N,n}$ goes to zero as $1/n$ by Widder (1929).

‡ Loc. cit. This particular expansion was previously established by Okada (loc. cit., p. 331) for the more general case where $\phi_n(x)$ is analytic for $|x| < r$. See also E. T. Whittaker, *A Course of Modern Analysis*, 1902, p. 110.

$$\phi_n(x) = \frac{x^n}{n!} + a_1\frac{x^{n+1}}{(n+1)!} + \cdots + a_s\frac{x^{n+s}}{(n+s)!} + \cdots.$$

Then

$$| a_s^{(n)} | = \left| \frac{a_s}{(n+1)(n+2)\cdots(n+s)} \right| \leqq A_N N^{-s}(n+1)^{-1},$$

where $N$ is arbitrary.

As an application of Theorem V let

$$g_0(x) = 1, \qquad g_n(x) = x^n f_n[\phi^{p_n}(x)] \qquad (n = 1, 2, \cdots),$$

where $f_n(u)$ is analytic for $|u| < R_1^{p_n}$, $\phi(x)$ is analytic for $|x| < R_2$, $\phi(x)$ is not a constant, $\phi(0)=0$, $f_n(0)=1$, and the sequence of positive integral numbers $p_n \to \infty$. Using the notation of Theorem V, write $g_n(x) = x^n(1+G_n(u)) = x^n(1+h_n(x))$, where $u=\phi^{p_n}(x)$. Let $M_n(\sigma^{p_n})$ denote the maximum of $|G_n(u)|$ on the circle $|u| = \sigma^{p_n}$, $\sigma'$ the upper bound of values of $\sigma$ such that $\sigma \leqq R_1$ and $M_n(\sigma^{p_n})$ is bounded, $\delta'$ the upper bound of values of $\delta$ such that $\delta \leqq R_2$ and maximum of $|\phi(x)| \leqq \sigma'$ for $|x| = \delta$. It is easy to prove that* $\lim_{n\to\infty} M_n(\sigma^{p_n}) = 0$ for $\sigma < \sigma'$. Hence in any region $|x| \leqq \delta < \delta'$, the $\lim_{n\to\infty} |h_n(x)| = 0$. Further $g_n(x)$ is analytic in the region $|x| \leqq \delta$. Hence any function $f(x)$ analytic for $|x| < \rho$ has an absolutely and uniformly convergent expansion of the form

$$f(x) = \sum_{n=0}^{\infty} c_n x^n f_n[\phi^{p_n}(x)] \text{ for } |x| \leqq R < \min(\delta', \rho).$$

The special case where $f_n(u)$ and $\phi(x)$ are integral was obtained by Onofri.† In some cases he obtains a region of validity greater than this but the expression which determines this region involves the coefficients $c_n$.

The above results obtained by Onofri are a generalization of the work of Julia. Julia‡ takes $\phi(x) = x$, $p_n = n$, $f_0(u) = 1$, and $f_n(u) = f_1(u)$ $(n = 1, 2, \cdots)$.

Theorem VI.§ *Let $a_k^{(n)}(x)$ be a finite or infinite set of functions each analytic for $|x| < \rho$ such that $|a_k^{(n)}(x)| \leqq M_{N,k}$ for $|x| \leqq N$ and every $N < \rho$. Let $\lambda_k$ be a non-decreasing sequence of positive constants with the limit infinity. If*

$$M_{N,1}\frac{N}{\lambda_1} + M_{N,2}\frac{N^2}{\lambda_1\lambda_2} + M_{N,3}\frac{N^3}{\lambda_1\lambda_2\lambda_3} + \cdots$$

* Onofri, loc. cit., p. 211.

† Loc. cit.

‡ Loc. cit.

§ Theorem VI reduces to a theorem due to Carmichael (loc. cit.), when $a_k^{(n)}(x) = a_k(x)$. While Theorem VI is proved as a special case of Theorem V, conversely, Theorem V is a special case of Theorem VI obtained by putting $a_1^{(n)}(x)x/\lambda_{n+1} = h_n(x)$ and $a_r^{(n)}(x) = 0$ for $r = 2, 3, \cdots$.

*converges to a sum $M_N$ for every $N<\rho$, then any function $f(x)$ analytic for $|x|<\rho$ has an expansion of the form*

$$f(x) = \sum_0^\infty c_n x^n \left\{1 + a_1^{(n)}(x)\frac{x}{\lambda_{n+1}} + a_2^{(n)}(x)\frac{x^2}{\lambda_{n+1}\lambda_{n+2}} + \cdots\right\}.$$

*This expansion converges uniformly and absolutely in any region $|x|\leqq R<\rho$ and is valid for $|x|<\rho$. Further, there is only one such expansion converging uniformly to $f(x)$ in a region $|x|\leqq R'$, $R'>0$.*

Now

$$g_n(x) = x^n\left(1 + a_1^{(n)}(x)\frac{x}{\lambda_{n+1}} + a_2^{(n)}(x)\frac{x^2}{\lambda_{n+1}\lambda_{n+2}} + \cdots\right)$$

and under the hypotheses given $|h_n(x)|\leqq\lambda_1 M_N/\lambda_{n+1}$ which $\to 0$ as $n\to\infty$. Hence the results follow from Theorem V.

Theorems III and IV establish an expansion theory for any set of functions belonging to a class $A_\rho$ for some $\rho\neq 0$. If the set of functions $g_n(x)$ belongs to the class $B_\rho$ for every $\rho\neq 0$, then neither theorem yields any results. Further if $K_N=0$ for every $N<\rho$, then the region of convergence asserted by each theorem is the same, namely, $|x|<\rho$. In case $K_N\neq 0$ the question as to whether by selecting $b_\nu^{(\mu)}$ different from the Cauchy bounds it were possible to establish the validity of the expansion for a larger region by application of Theorem III is left unanswered.

## IV. Expansion of an arbitrary analytic function in a class of integral functions

Theorem VII. *Let $\lambda_n$ be a sequence of positive numbers such that* $\lim_{n\to\infty}\lambda_n=\infty$, *and let $M$ be a positive number independent of $n$ and $s$. If the functions $F_n(x)$ are such that*

$$|a_s^{(n)}| < \frac{M^s}{\lambda_{n+1}\lambda_{n+2}\cdots\lambda_{n+s}} \qquad (n=0,1,2,\cdots;\ s=1,2,3,\cdots),$$

*then any function $f(x)$ analytic for $|x|<\rho$ has a unique expansion of the form* (1) *valid for $|x|<\rho$ and this expansion converges absolutely and uniformly for values of $x$ such that $|x|\leqq R<\rho$.*

We shall make use of Theorem V′ to establish Theorem VII.*

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* Theorem VII may be established more easily by direct application of Theorem III. However, we desire to make use in a later discussion of the fact that it follows from Theorem V′ and hence from Theorem V.

The functions $F_n(x)$ are integral functions so it is sufficient to show that corresponding to any $N$, there exists an $M'_N$ such that

$$\frac{M^s}{\lambda_{n+1}\lambda_{n+2}\cdots\lambda_{n+s}} < \frac{M'_N}{N^s\lambda_{n+1}} \qquad (n = 0, 1, 2, \cdots; s = 1, 2, 3, \cdots).$$

Now $MN/\lambda_{n+s}\to 0$ as $n+s\to\infty$, therefore there exists a $q$ such that $MN/\lambda_{n+s}<1$ for $n+s>q$. But there are only a finite number of values of $\lambda_{n+s}$ for which $n+s\leqq q$ and hence if we pick $M'_N$ larger than 1 and so large that $(MN)^s/(\lambda_{n+2}\lambda_{n+3}\cdots\lambda_{n+s})<M'_N$, for $n+s\leqq q$, this $M'_N$ will satisfy the desired condition and Theorem VII is established.

To obtain the Neumann* expansion of $f(x)$ in Bessel coefficients we write $F_n(x)=2^n n!J_n(x)$ and obtain the expansion in the set of functions $F_n(x)$ and hence in $J_n(x)$. For $s$ odd $|a_s^{(n)}|=0$ and for $s$ even

$$\begin{aligned} |a_s^{(n)}| &= [2^s(s/2)!(n+1)(n+2)\cdots(n+s/2)]^{-1} \\ &< [(2n+2)(2n+4)\cdots(2n+s)]^{-1} \\ &\leqq (n+1)^{-1/2}(n+2)^{-1/2}\cdots(n+s)^{-1/2}. \end{aligned}$$

In this case $M=1$, and $\lambda_{n+s}=(n+s)^{1/2}$.

We may extend Theorem VII to obtain expansions of the form

$$x^\sigma f(x) = \sum_{n=0}^{\infty}\alpha_n F_{\sigma+n}(x).$$

THEOREM VIII. *Let $\lambda_n$ be a sequence of positive numbers such that* $\lim_{n\to\infty}\lambda_n=\infty$ *and let $M_\sigma$ be a positive number independent of $n$ and $s$. Also let*

$$F_{\sigma+n}(x) = x^{\sigma+n} + \sum_{s=1}^{\infty} a_s^{(n)}\,x^{\sigma+n+s},$$

*where $a_s^{(n)}$ will in general depend on $\sigma$, and $|a_s^{(n)}|\leqq M_\sigma^s/(\lambda_{n+1}\lambda_{n+2}\cdots\lambda_{n+s})$. If $f(x)$ is any function analytic for $|x|<\rho$, then $x^\sigma f(x)$ has an absolutely and uniformly convergent expansion of the form*

$$x^\sigma f(x) = \sum_{n=0}^{\infty}\alpha_n F_{\sigma+n}(x)$$

*valid for $|x|\leqq R<\rho$.*

It is sufficient to observe that the function $x^{-\sigma}F_{\sigma+n}(x)$ is of the form $F_n(x)$ as defined in Theorem VII.

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* Loc. cit., pp. 33–35.

We now apply this theorem to obtain the Gegenbauer† expansion in Bessel functions. Write

$$F_{\sigma+n}(x) = 2^{\sigma+n}\Gamma(\sigma+n+1)J_{\sigma+n}(x) = \sum_{n=0}^{\infty}(-1)^r \frac{\Gamma(\sigma+n+1)x^{\sigma+n+2r}}{2^{2r}r!\Gamma(\sigma+n+r+1)}.$$

Then for $s$ odd $|a_s^{(n)}|=0$, and for $s$ even

$$|a_s^{(n)}| = [2^s(s/2)!|\sigma+n+1||\sigma+n+2| \cdots |\sigma+n+s/2|]^{-1}.$$

Hence if $\sigma$ is any point of the complex plane not a negative integer, $|\sigma+n+s|(n+s)^{-1}>Q_\sigma>0$ and therefore

$$|a_s^{(n)}| < M_\sigma^s(n+1)^{-1/2}(n+2)^{-1/2}\cdots(n+s)^{-1/2}$$

and the Gegenbauer expansion is established.

## V. Generalization of expansions in confluent hypergeometric functions

In the preceding section, Theorem VII exhibited a class of sets of functions defined by a condition

$$|a_s^{(n)}| < \phi(n, s) \tag{5.1}$$

such that any set $F_n(x)$ has the following property (which we call Property A): *The expansion of any function $f(x)$ analytic at zero in the functions $F_n(x)$ converges for $|x|<\rho$ where the singularity of $f(x)$ nearest the origin is on the circle $|x|=\rho$.*

We shall now obtain some necessary conditions on $\phi(n, s)$ in order for (5.1) to define a non-null class of sets of functions each set of which possesses Property A. In the first place $\phi(n, s)$ must be positive for all $n$ and $s$ else no functions $F_n(x)$ are determined. Also for every $s$,

$$\Phi(s) \equiv \liminf_{n\to\infty} \phi(n, s) = 0. \tag{5.2}$$

For if not, let $\nu$ be the first value of $s$ for which $\Phi(s)>0$ and let $B$ be a positive number less than 1 and less than $\phi(n, \nu)$ for all $n$. If $F_p(x)=x^p-B^\nu x^{p+\nu}$, then $\sum_{n=0}^{\infty}B^{n\nu}F_{n\nu}(x)$ converges to 1 if $|x|<1/B$ and diverges if $|x|>1/B$. Hence this set $F_n(x)$ does not possess Property A. But $|a_\nu^{(n)}|=B^\nu<\phi(n, \nu)$ and $|a_s^{(n)}|=0<\phi(n, s)$ for $s\neq\nu$. Hence no such $\phi(n, s)$ could exist without the condition (5.2).

Now consider any theorem such that for every particular set of functions $F_n(x)$, with coefficients $a_s^{(n)*}$, possessing Property A by virtue of the theorem; there corresponds a positive function $\phi(n, s)\geqq|a_s^{(n)*}|$ such that every set of

† Wiener Sitzungsberichte, (2), vol. 74 (1876), pp. 125–127.

functions $F_n(x)$ for which $|a_s^{(n)}| < \phi(n, s)$ also possesses Property A. We designate such a theorem as of Type O.

Every expansion theorem in which each condition is expressed entirely in terms of an arbitrary set of positive bounds on $a_s^{(n)}$ would be of Type O. Theorem III and all its corollaries are of Type O.†

If there exists an $s$ such that $\lim\inf_{n\to\infty} a_s^{(n)} \neq 0$, then it is not possible to prove by any theorem of Type O that a particular set of functions $F_n(x)$ possesses Property A. For otherwise there would be determined a class of sets of functions defined by a relation (5.1) such that (5.2) is not satisfied which has just been shown to be impossible. In particular, it is impossible to prove by any such theorem the known fact that Graesser's set of confluent hypergeometric functions‡ possesses Property A.

We next exhibit a general class of sets of functions including this exceptional set of Graesser which possesses Property A.

THEOREM IX. *Let the functions $F_n(x)$ satisfy the conditions*:

$$(5.3)\qquad a_s^{(n)} = \frac{\psi(n+1)\psi(n+2)\cdots\psi(n+s)}{(2cn+d+c)(2cn+d+2c)\cdots(2cn+d+sc)}\,\frac{1}{s!},$$

*where $c$ and $d$ are any non-negative real§ constants not both zero and $\psi(n)$ is any function of $n$ such that $|\psi(n)|/(cn+d) < L$, $(n=1, 2, \cdots)$. Then any function $f(x)$ analytic for $|x| < \rho$ has an absolutely and uniformly convergent expansion of the form* (1) *valid for $|x| \leqq R < \rho$.*

In Theorem I, take

$$k_{\mu+\nu}^{(\mu)} = \frac{(-1)^\nu\psi(\mu+1)\psi(\mu+2)\cdots\psi(\mu+\nu)}{(2c\mu+d+c\nu)(2c\mu+d+c\nu+c)\cdots(2c\mu+d+2c\nu-c)}\,\frac{1}{\nu!}$$
$$(\nu = 1, 2, \cdots).$$

Then|| $\sum_{t=0}^{\nu} k_{\mu+\nu}^{(\mu+t)} a_t^{(\mu)} = 0$. Let $\delta_{\mu+\nu}^{(\mu)} = |k_{\mu+\nu}^{(\mu)}|$ for $\nu = 0, 1, 2, \cdots$, so that

† The first condition of Theorem III is already expressed by means of any positive bounds $b_s^{(n)}$. This first condition being satisfied for every $R$, the theorem will yield Property A if and only if $|P_n(x)| < Q_R$ for $|x| = R$ and for every $R$. But this could happen if and only if $\sum_{s=0}^{\infty} |a_s^{(n)}| R^s < Q_R'$ for every $R$. In turn this can happen if and only if $\sum_{s=0}^{\infty} [|a_s^{(n)}| + (s!)^{-1}] R^s < Q_R''$ for every $R$. Given any set of functions with coefficients $a_s^{(n)*}$ and bounds $b_s^{(n)*}$ which yield Property A by this theorem, if we take $\phi(n, s)$ as the smaller of $b_s^{(n)*}$ and $|a_s^{(n)*}| + (s!)^{-1}$ then the conditions for a theorem of Type O are seen to be satisfied.

‡ These functions are obtained from the functions of Theorem IX by taking $\psi(n) = n - k - \frac{1}{2}$, $c = 1$, $d = 0$, where $k$ is any complex number. In this case $\lim_{n\to\infty} a_1^{(n)} = \frac{1}{2}$.

§ Evidently this restriction may be lightened.

|| This result follows from the equality

$$\sum_{t=0}^{\nu} (-1)^{\nu-t}(t+1)(t+2)\cdots(t+k)\binom{\nu}{t} = 0 \qquad (k = 1, 2, \cdots, \nu-1).$$

$$(5.4)\quad \overline{M}^{(\mu)}_{\mu+\nu} = \frac{\delta^{(\mu)}_{\mu+\nu+1}}{\delta^{(\mu)}_{\mu+\nu}} = \left| \frac{\psi(\mu+\nu+1)(2c\mu+d+c\nu)}{(\nu+1)(2c\mu+d+2c\nu)(2c\mu+d+2c\nu+c)} \right| < \frac{L}{\nu+1},$$

for $\nu=1, 2, \cdots$, and $\overline{M}_\mu{}^{(\mu)} = |\psi(\mu+1)/(2c\mu+d+c)| < L$. Hence (2.2) is dominated by the series

$$\sum_{\mu,\nu} |\gamma_\mu| \frac{L^\nu R^{\mu+\nu}}{\nu!} = e^{LR} \sum_{\mu=1}^{\infty} |\gamma_\mu| R^\mu.$$

But the last series converges since $f(x)$ is analytic for $|x| \leqq R$. Also for $|x| \leqq R$, $|F_\mu(x)/x^\mu| < e^{LR}$. Hence Theorem IX follows.

Evidently these functions $F_n(x)$ are exceptional in the above sense, that is, $\liminf_{n\to\infty} a_1{}^{(n)} \neq 0$, provided $|\psi(n)|/(cn+d) > L'$ for an infinite number of values of $n$.

A simple corollary of Theorem IX would be obtained by taking $c=0$, $d=1$.

## VI. Expansions in products of analytic functions

Throughout the present section, we shall make the following assumptions:

A point $\sigma$ belongs to a region or set of points $S$ of the complex plane.

$g_\sigma(x)$ and $g_\sigma'(x)$ are defined for each point $\sigma$ of $S$ and are of the form $g_\sigma(x) = x^\sigma(1+h_\sigma(x))$, $g_\sigma'(x) = x^\sigma(1+h_\sigma'(x))$, where $h_\sigma$ and $h_\sigma'$ are analytic for $|x| < r$ and vanish at the origin.

$N$ shall denote a positive number less than $r$.

$f(x)$ is any function analytic for $|x| < \rho$.

$\kappa$ is any complex number such that $\kappa+n = \sigma_n + \sigma_n'$ for $n = 0, 1, 2, \cdots$, where $\sigma_n$ and $\sigma_n'$ are points of $S$.

$F_\sigma(x)$ and $F_\sigma'(x)$ are defined for each $\sigma$ of $S$ and are of the form

$$F_\sigma(x) = x^\sigma + \sum_{s=1}^{\infty} a_s{}^{(\sigma)} x^{\sigma+s}, \qquad F_\sigma'(x) = x^\sigma + \sum_{s=1}^{\infty} \alpha_s{}^{(\sigma)} x^{\sigma+s}.$$

Theorem X. *If the upper bounds $M_{N,\sigma}$ and $M'_{N,\sigma}$ for $|h_\sigma(x)|$ and $|h_\sigma'(x)|$ respectively in the region $|x| \leqq N$ are such that* $\limsup_{n\to\infty} M_{N,\sigma_n}$ *and* $\limsup_{n\to\infty} M'_{N,\sigma_n}$ *are finite for every $N$, then $x^\kappa f(x)$ has an absolutely and uniformly convergent expansion of the form*

$$(6.1)\qquad x^\kappa f(x) = \sum_{n=0}^{\infty} c_n g_{\sigma_n}(x) g'_{\sigma_n'}(x) \quad \textit{for } |x| \leqq R,$$

*where $R < \rho$ and also less than*

$$N\left\{1 + \limsup_{n\to\infty}\ M_{N,\sigma_n} + M'_{N,\sigma_n'} + M_{N,\sigma_n} M'_{N,\sigma_n'}\right\}^{-1}.$$

If we write $x^{-\kappa}g_{\sigma_n}g'_{\sigma'_n}=x^n(1+H_{\kappa+n}(x))$, then the functions $H_{\kappa+n}(x)$ satisfy the conditions in Theorem IV.

THEOREM XI. *If $M_{N,\sigma_n}$ and $M'_{N,\sigma'_n}$ defined as in Theorem X satisfy the further condition that* $\limsup_{n\to\infty} M_{N,\sigma_n}$ *and* $\limsup_{n\to\infty} M'_{N,\sigma'}$ *are zero for every $N<r=\rho$, then the expansion* (6.1) *is valid for $|x|<\rho$ and the series converges absolutely and uniformly for $|x|\leqq R<\rho$.*

In view of Theorem XI and Lemma 1 we have

THEOREM XII. *Let $F_\sigma(x)$ and $F'_\sigma(x)$ satisfy the conditions*

$$|a_s^{(\sigma)}| \leqq M_{N,\sigma}N^{-s}, \qquad |\alpha_s^{(\sigma)}| \leqq M'_{N,\sigma}N^{-s}, \qquad (s = 1, 2, 3, \cdots),$$

*where $M_{N,\sigma}$ and $M'_{N,\sigma}$ are positive constants. If* $\limsup_{n\to\infty} M_{N,\sigma_n}=0$ and $\limsup_{n\to\infty} M'_{N,\sigma'_n}=0$ *for every $N<r=\rho$, then $x^\kappa f(x)$ has an absolutely and uniformly convergent expansion of the form*

$$x^\kappa f(x) = \sum_{n=0}^{\infty} c_n F_{\sigma_n}(x)F'_{\sigma'_n}(x) \quad for \quad |x| \leqq R < \rho. \tag{6.2}$$

THEOREM XIII. *Let $\sigma$ be restricted to an unbounded region or set of points $S$ of the complex plane such that $|s(\sigma+s)^{-1}|$ $(s=1, 2, \cdots)$ is bounded with respect to both $s$ and $\sigma$ for $\sigma$ in $S$. If $\lim_{n\to\infty}|\sigma_n| = \infty$, $\lim_{n\to\infty}|\sigma'_n| = \infty$, and $F_\sigma(x)$, $F'_\sigma(x)$ are such that $|a_s^{(\sigma)}|$ and $|\alpha_s^{(\sigma)}|$ are bounded by $M^s/(|\sigma+1|^m|\sigma+2|^m \cdots |\sigma+s|^m)$ for all $\sigma$ in $S$, $M$ being a positive constant independent of $\sigma$ and $s$, and $m$ a fixed positive number; then $x^\kappa f(x)$ has a uniformly and absolutely convergent expansion of the form* (6.2) *for $|x|\leqq R<\rho$.*

We shall make the proof of this theorem depend on Theorem XII. It is sufficient to show under the named conditions that we can determine an $\overline{M}$ corresponding to any $N$ arbitrarily large such that

$$\frac{M^s}{|\sigma_n+1|^m|\sigma_n+2|^m\cdots|\sigma_n+s|^m} \leqq \frac{\overline{M}N^{-s}}{|\sigma_n+1|^m}.$$

Now for $\sigma$ in $S$, $s/|\sigma+s|<K$ and hence

$$\frac{(MN)^s}{|\sigma_n+2|^m|\sigma_n+3|^m\cdots|\sigma_n+s^m} < \frac{(MN)^s}{K^{m(s-1)}2^m\cdot 3^m\cdots s^m}.$$

But $MNK^{-m}/s^m\to 0$ as $s\to\infty$. Therefore

$$\frac{(MN)^s}{|\sigma_n+2|^m|\sigma_n+3|^m\cdots|\sigma_n+s|^m} \leqq \overline{M}.$$

A region $S$ suitable for application of Theorem XIII would be obtained by excluding the negative integers $-1, -2, \cdots, -j$ by circles of radius $l$ and by excluding any sector containing the negative axis of reals in its interior and radiating from the point $-(j+\frac{1}{2})$. Here $j$ is a positive integer and $l$ is any positive number.

To apply Theorem XIII to the Bessel functions, write

$$F_\sigma(x) = F_\sigma'(x) = \Gamma(\sigma + 1)2^\sigma J_\sigma(x)$$
$$= x^\sigma + \sum_{r=1}^{\infty} (-1)^r \frac{x^{\sigma+2r}}{r!2^{2r}(\sigma + 1)(\sigma + 2) \cdots (\sigma + r)}.$$

Then for $s$ odd $|a_s^{(\sigma)}| = 0$, and for $s$ even

$$|a_s^{(\sigma)}| = \left((s/2)!2^s|\sigma + 1||\sigma + 2| \cdots |\sigma + s/2|\right)^{-1}$$
$$\leqq M^s(|\sigma + 1||\sigma + 2| \cdots |\sigma + s|)^{-1/2},$$

where $M$ is a properly chosen positive number and $\sigma$ is restricted to any region $S$ for which $s/|\sigma+s|$ is uniformly bounded.

The Neumann-Gegenbauer* expansion in products of Bessel functions,

$$x^{\mu+\nu}f(x) = \sum_{n=0}^{\infty} c_n J_{\mu+n/2}(x)J_{\nu+n/2}(x),$$

is obtained from Theorem XIII if we put $F_\sigma(x) = F_\sigma'(x) = \Gamma(\sigma+1)2^\sigma J_\sigma(x)$, $\mu+\nu = \kappa$, $\sigma_n = \mu+n/2$, $\sigma_n' = \nu+n/2$, and restrict $\mu$ and $\nu$ so that $\mu+n/2$ and $\nu+n/2$ are not negative integers. Then $\sigma_n$, $\sigma_n'$ are in $S$ and $|\sigma_n|$, $|\sigma_n'| \to \infty$ as $n \to \infty$.

If we take $\kappa = 0$, $\sigma_n = n/2$, $\sigma_n' = n/2$ for $n$ even and $\sigma_n = (n-1)/2$, $\sigma_n' = (n+1)/2$ for $n$ odd, we obtain the expansion of any function $f(x)$ analytic about zero in the form

$$f(x) = c_0J_0^2(x) + c_1J_0(x)J_1(x) + c_2J_1^2(x) + c_3J_1(x)J_2(x) + \cdots.$$

From this expansion we have at once the Neumann† expansion of an even analytic function and for an odd analytic function: namely, $f(x) = \sum_{n=0}^{\infty} c_n J_n^2(x)$ and $f(x) = \sum_{n=0}^{\infty} c_n J_n(x)J_{n+1}(x)$.

The expansion due to Nielsen‡ of the form

$$x^\nu f(x) = \sum_{n=0}^{\infty} a_n x^{(\nu+n)/2} J_{(\nu+n)/2}(x)$$

* See Watson, *Theory of Bessel Functions*, p. 525.

† Berichte der K. Sächsische Gesellschaft der Wissenschaften, vol. 21 (1869), pp. 221–256.

‡ Nyt Tidsskrift for Matematik, vol. 9 (1898), pp. 77–79.

is an immediate consequence of Theorem XIII. Evidently much more general expansions of the form

$$x^{\kappa}f(x) = \sum_{n=0}^{\infty} \alpha_n J_{\sigma_n}(x) x^{\sigma_n'}$$

also follow from Theorem XIII.

Obviously the theorems of this section may be extended immediately to expansions in series each term of which is made up of the product of $k$ functions. We shall state the theorem corresponding to Theorem XIII for expansions in products of $k$ Bessel functions.

THEOREM XIV. *Let $\sigma$ be restricted to an unbounded region $S$ of the complex plane such that $s/|\sigma+s|$ $(s=1, 2, \cdots)$ is uniformly bounded; and let $\kappa$ be a complex number such that $\kappa+n=\sigma_n^{(1)}+\sigma_n^{(2)}+\cdots+\sigma_n^{(k)}$, where $\sigma_n^{(1)}, \sigma_n^{(2)}, \cdots, \sigma_n^{(k)}$ are in $S$ and $\lim_{n\to\infty}|\sigma_n^{(r)}|=\infty$ $(r=1, 2, \cdots, k)$. Then if $f(x)$ is any function analytic for $|x|<\rho$, $x^{\kappa}f(x)$ has the absolutely and uniformly convergent expansion*

$$x^{\kappa}f(x) = \sum_{n=0}^{\infty} c_n J_{\sigma_n^{(1)}}(x) J_{\sigma_n^{(2)}}(x) \cdots J_{\sigma_n^{(k)}}(x) \quad \textit{for } |x| \leqq R < \rho.$$

A special case of this theorem which is a generalization of the Neumann-Gegenbauer expansion is

$$x^{\mu}f(x) = \sum_{n=0}^{\infty} c_n J_{\mu_1+\theta_1 n}(x) J_{\mu_2+\theta_2 n}(x) \cdots J_{\mu_k+\theta_k n}(x),$$

where $\mu_r$ and $\theta_r$ $(r=1, 2, \cdots, k)$ are any complex numbers subject to the conditions that $\mu_r+\theta_r n$ is not a negative integer, $\theta_r$ does not vanish, $\mu=\mu_1+\mu_2+\cdots+\mu_k$, and $\theta_1+\theta_2+\cdots+\theta_k=1$. Here $f(x)$ is analytic for $|x|<\rho$ and the expansion converges absolutely and uniformly for $|x|\leqq R<\rho$.

It is not necessary that each term of an expansion be the product of exactly $k$ functions. Under appropriate hypotheses the theorems of this section may be generalized in such a way that we obtain expansions where any term may be the product of one, two, up to $k$ functions so long as $k$ is bounded for all terms.

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