BERNSTEIN'S THEOREM AND TRIGONOMETRIC APPROXIMATION*

BY

DUNHAM JACKSON

1. **Introduction.** In a recent paper† the writer has presented a sequence of theorems relating to normalized polynomials and polynomial approximation together with corresponding results for normalized trigonometric sums and trigonometric approximation over an entire period. The present paper is concerned with similar questions in connection with trigonometric sums considered over a part of a period, as well as some additional observations on the polynomial case. The treatment is based largely on Bernstein's theorem on the derivative of a trigonometric sum, and incidentally involves or suggests certain modifications and adaptations of that theorem, both for trigonometric sums and for polynomials.

2. **Standard theorems.** The fundamental theorem of Bernstein for trigonometric sums may be stated as follows.‡

THEOREM B1. *If $T_n(x)$ is a trigonometric sum of the nth order such that $|T_n(x)| \leqq L$ for all real values of x, then $|T_n'(x)| \leqq nL$ for all real values of x.*

From this can be deduced immediately§ the corresponding theorem for polynomials:||

THEOREM B2. *If $P_n(x)$ is a polynomial of the nth degree such that $|P_n(x)| \leqq L$ for $-1 \leqq x \leqq 1$, then $|P_n'(x)| \leqq nL/(1-x^2)^{1/2}$ for $-1 < x < 1$.*

By a linear transformation of the independent variable the theorem can be generalized to an arbitrary interval (a, b), to read thus:

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* Presented to the Society, under two different titles, September 11, 13, 1935; received by the editors, October 15, 1935.

† *Certain problems of closest approximation*, Bulletin of the American Mathematical Society, vol. 39 (1933), pp. 889–906.

‡ S. Bernstein, *Sur l' ordre de la meilleure approximation des fonctions continues par des polynomes de degré donné*, Mémoire couronné, Brussels, 1912, p. 20 (with $2nL$ instead of nL as upper bound for $|T_n'(x)|$); M. Riesz, *Eine trigonometrische Interpolationsformel und einige Ungleichungen für Polynome*, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 23 (1914), pp. 354–368; pp. 357, 360–361; C. de la Vallée Poussin, *Sur le maximum du module de la dérivée d'une expression trigonométrique d'ordre et de module bornés*, Comptes Rendus, vol. 166 (1918), pp. 843–846; also, e.g., the writer's Colloquium, *The Theory of Approximation*, New York, 1930, pp. 80–81.

§ See, e.g., Colloquium, p. 92.

|| S. Bernstein, op. cit., pp. 6–11.

THEOREM B2a. *If* $P_n(x)$ *is a polynomial of the nth degree such that* $|P_n(x)| \leqq L$ *for* $a \leqq x \leqq b$, *then*

$$| P_n'(x) | \leqq \frac{nL}{[(b-x)(x-a)]^{1/2}}$$

for $a < x < b$.

A closely related theorem for polynomials is that of Markoff:*

THEOREM M1. *If* $P_n(x)$ *is a polynomial of the nth degree such that* $|P_n(x)| \leqq L$ *for* $-1 \leqq x \leqq 1$, *then* $|P_n'(x)| \leqq n^2 L$ *throughout the same interval.*

The generalized statement for an arbitrary interval is:

THEOREM M1a. *If* $P_n(x)$ *is a polynomial of the nth degree such that* $|P_n(x)| \leqq L$ *for* $a \leqq x \leqq b$, *then* $|P_n'(x)| \leqq 2n^2L/(b-a)$ *throughout the interval.*

In a somewhat weaker form,† which is nevertheless sufficient for present purposes, namely, with $2n^2L$ as upper bound for $|P_n'(x)|$ instead of n^2L, Theorem M1 can be obtained as an easy corollary of Theorem B1, and Theorem M1a follows with coefficient 4 in place of 2.

Less frequently cited is the following theorem, which is due also to Bernstein:‡

THEOREM B3. *If* $P_{n-1}(x)$ *is a polynomial of degree* $n-1$ *such that*

$$| (1 - x^2)^{1/2} P_{n-1}(x) | \leqq L$$

for $-1 \leqq x \leqq 1$, *then*

$$\left| \frac{d}{dx} [(1 - x^2)^{1/2} P_{n-1}(x)] \right| \leqq \frac{nL}{(1 - x^2)^{1/2}}$$

for $-1 < x < 1$.

For a general interval this becomes

THEOREM B3a. *If* $P_{n-1}(x)$ *is a polynomial of degree* $n-1$ *such that*

$$| (b - x)^{1/2}(x - a)^{1/2} P_{n-1}(x) | \leqq L$$

for $a \leqq x \leqq b$, *then*

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* See, e.g., S. Bernstein, op. cit., pp. 11–13; M. Riesz, loc. cit., pp. 359–360.

† See D. Jackson, *On the convergence of certain trigonometric and polynomial approximations*, these Transactions, vol. 22 (1921), pp. 158–166; pp. 162–163. (In the displayed formula midway between (4) and (5) on p. 163 the absolute value bars inclosing the expression $Q_n(\theta)/\sin\theta$ should be omitted.)

‡ S. Bernstein, op. cit., pp. 17–19; see also D. Jackson, *A generalized problem in weighted approximation*, these Transactions, vol. 26 (1924), pp. 133–154; pp. 140–141.

$$\left| \frac{d}{dx} [(b - x)^{1/2}(x - a)^{1/2}P_{n-1}(x)] \right| \leqq \frac{nL}{[(b - x)(x - a)]^{1/2}}$$

for $a<x<b$.

3. **Variations.** If an upper bound is given for the absolute value of a trigonometric sum over a part of a period, a conclusion is obtained resembling that of Theorem B2a:

THEOREM 1. *If* $T_n(x)$ *is a trigonometric sum of the nth order such that* $|T_n(x)| \leqq L$ *for* $a \leqq x \leqq b$, *where** $0<b-a<2\pi$, *then*

$$| T_n'(x) | \leqq \frac{CnL}{[(b - x)(x - a)]^{1/2}}$$

for $a<x<b$, *where* C *depends only on* a *and* b.

Throughout this paper C will be used as a general notation for a constant, differing in general from one formula to another, and in any particular instance depending on the range assigned to the independent variable, but not depending on anything else. The conclusion of Theorem M1a, for example, has the form $|P_n'(x)| \leqq Cn^2L$. Where the magnitude of the constant is not determined its precise value is immaterial for the purpose in hand.

Theorem 1 has been proved elsewhere.† An alternative form of demonstration will be presented here, to serve as a pattern for other proofs which are to follow.

Let $C_n(x)$ be a cosine sum of the nth order such that $|C_n(x)| \leqq L$ for $a \leqq x \leqq b$, where $0<a<b<\pi$. Then $C_n(x)$ is a polynomial of the nth degree in $z = \cos x$, say $p_n(z)$, and $|p_n(z)| \leqq L$ for $B \leqq z \leqq A$, if $A = \cos a$, $B = \cos b$. According to Theorem B2a, therefore,

$$| p_n'(z) | \leqq \frac{nL}{[(A - z)(z - B)]^{1/2}}$$

in the interior of the interval. By the mean value theorem, inasmuch as $\sin x$ has a positive lower bound for $a \leqq x \leqq b$, $A - z \geqq C(x-a)$, and $z - B \geqq C(b-x)$. Consequently

$$| C_n'(x) | = | \sin x \, p_n'(z) | \leqq \frac{CnL}{[(b - x)(x - a)]^{1/2}}$$

in the interval considered.

Let $S_n(x)$ be a sine sum of the nth order satisfying the condition that

* If $b-a \geqq 2\pi$, Theorem B1 is applicable.

† D. Jackson, these Transactions, vol. 26, loc. cit., pp. 141–145.

$|S_n(x)| \leqq L$ throughout the same interval. It is possible to write $S_n(x)$ as the product of $\sin x$ by a polynomial $q_{n-1}(z)$ of degree $n-1$, and since $\sin x$ has a positive lower bound in the interval $|q_{n-1}(z)| \leqq CL$. Hence

$$| q'_{n-1}(z) | \leqq \frac{C(n-1)L}{[(A-z)(z-B)]^{1/2}} \leqq \frac{CnL}{[(b-x)(x-a)]^{1/2}},$$

with different values of C in the second and third members, and

$$\begin{aligned} | S'_n(x) | &= | \cos x\, q_{n-1}(z) - \sin^2 x\, q'_{n-1}(z) | \\ &\leqq CL + \frac{CnL}{[(b-x)(x-a)]^{1/2}} \leqq \frac{CnL}{[(b-x)(x-a)]^{1/2}}, \end{aligned}$$

again with different values of C.

Now let $T_n(x)$ be an arbitrary trigonometric sum of the nth order such that $|T_n(x)| \leqq L$ for $a \leqq x \leqq b$ *and also* for $-b \leqq x \leqq -a$, still with $0 < a < b < \pi$. Then L is an upper bound likewise for the absolute values of

$$C_n(x) = \tfrac{1}{2}[T_n(x) + T_n(-x)], \qquad S_n(x) = \tfrac{1}{2}[T_n(x) - T_n(-x)],$$

in the interval (a, b). So $|C'_n(x)|$ and $|S'_n(x)|$ have upper bounds of the form obtained in the preceding paragraphs, for $a < x < b$. And as $|C'_n(x)|$ and $|S'_n(x)|$ are even functions,

$$| C'_n(x) | = | C'_n(-x) | \leqq \frac{CnL}{[(b+x)(-x-a)]^{1/2}} = \frac{CnL}{[(a'-x)(x-b')]^{1/2}}$$

for $b' < x < a'$, if $a' = -a$, $b' = -b$, and $S'_n(x)$ is similarly bounded. So $|T'_n(x)| = |C'_n(x) + S'_n(x)|$ has upper bounds of the form

$$\frac{CnL}{[(b-x)(x-a)]^{1/2}}, \qquad \frac{CnL}{[(a'-x)(x-b')]^{1/2}}$$

in the intervals (a, b) and (b', a') respectively.

More generally, let $T_n(x)$ be a trigonometric sum of the nth order having L as an upper bound for its absolute value in the intervals $a_1 \leqq x \leqq b_1$ and $a_2 \leqq x \leqq b_2$, where (a_1, b_1) and (a_2, b_2) are any two intervals of equal length, having no point in common, and both contained within a period:

$$b_1 - a_1 = b_2 - a_2, \qquad a_1 < b_1 < a_2 < b_2 < a_1 + 2\pi.$$

The change of variable $s = x - \frac{1}{2}(b_1 + a_2)$ transforms $T_n(x)$ into a trigonometric sum of the nth order in s with absolute value $\leqq L$ throughout two equal intervals symmetrically situated with respect to the origin. To this sum $\tau_n(s)$ the discussion of the preceding paragraph is applicable. Since $\tau'_n(s) = T'_n(x)$, the conclusion as expressed in terms of the variable x is that $|T_n'(x)|$ has the upper bounds

$$\frac{CnL}{[(b_1 - x)(x - a_1)]^{1/2}}, \qquad \frac{CnL}{[(b_2 - x)(x - a_2)]^{1/2}}$$

respectively in (a_1, b_1) and (a_2, b_2).

Finally, *let* $T_n(x)$ *be a trigonometric sum of the nth order such that* $|T_n(x)| \leqq L$ *for* $a \leqq x \leqq b$, *where* (a, b) *is any interval of length* $<2\pi$. Let the interval (a, b) be covered by four intervals $(a_1, b_1), \cdots, (a_4, b_4)$ of equal length, overlapping two by two but with no point common to any three, extending from a as left-hand extremity of the first interval to b as right-hand extremity of the fourth; specifically, let c_1, c_2, c_3 be the points dividing (a, b) into fourths, in order from left to right, let h be a positive number $<(b-a)/12$, and let

$$a_1 = a, \qquad b_1 = c_1 + 2h; \qquad a_2 = c_1 - h, \qquad b_2 = c_2 + h;$$
$$a_3 = c_2 - h, \qquad b_3 = c_3 + h; \qquad a_4 = c_3 - 2h, \qquad b_4 = b.$$

The preceding paragraph can be applied to the first and third intervals, and again to the second and fourth intervals. If the results are interpreted successively for the interval $a < x \leqq c_1$ (in which $b_1 - x$ has a positive lower bound), for the interval $c_1 \leqq x \leqq c_2$ (in which $(b_2 - x)(x - a_2)$ has a positive lower bound), for the interval $c_2 \leqq x \leqq c_3$ (in which $(b_3 - x)(x - a_3)$ has a positive lower bound), and for the interval $c_3 \leqq x < b$ (in which $x - a_4$ has a positive lower bound), it appears in summary that

$$|T_n'(x)| \leqq \frac{CnL}{[(b - x)(x - a)]^{1/2}}$$

for $a < x < b$, as the assertion of Theorem 1 requires. Since the essential conditions of the problem are unaffected by a change of origin for the independent variable it is clear without re-examination of the details of the proof that the magnitude of C as a function of a and b depends only on the length of the interval, not on the position of its initial point.

A similar method has been used in another passage* to adapt Markoff's theorem to the case of trigonometric sums, with the simplification that since the end points do not have an exceptional status in Markoff's theorem it is not necessary to use overlapping subintervals in dealing with the various parts of the given interval, a simple subdivision of the latter into fourths being all that is required. The conclusion may be repeated here for reference:

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* D. Jackson, *On the application of Markoff's theorem to problems of approximation in the complex domain*, Bulletin of the American Mathematical Society, vol. 37 (1931), pp. 883–890; pp. 887–889.

THEOREM 2. *If $T_n(x)$ is a trigonometric sum of the nth order such that $|T_n(x)| \leqq L$ for $a \leqq x \leqq b$, where $0 < b-a < 2\pi$, then*

$$|T_n'(x)| \leqq Cn^2L$$

throughout the same interval, the magnitude of C depending only on the length of the interval.

A trigonometric analogue of Theorem B3a is this:

THEOREM 3. *If $T_{n-1}(x)$ is a trigonometric sum of order $n-1$ such that*

$$|(b-x)^{1/2}(x-a)^{1/2}T_{n-1}(x)| \leqq L$$

for $a \leqq x \leqq b$, where $0 < b-a < 2\pi$, then

$$\left| \frac{d}{dx} [(b-x)^{1/2}(x-a)^{1/2}T_{n-1}(x)] \right| \leqq \frac{CnL}{[(b-x)(x-a)]^{1/2}}$$

for $a < x < b$.

The proof is obtained by successive stages corresponding to those in the proof of Theorem 1. If $C_{n-1}(x)$ is a cosine sum of order $n-1$ such that

$$|(b-x)^{1/2}(x-a)^{1/2}C_{n-1}(x)| \leqq L$$

for $a \leqq x \leqq b$, where $0 < a < b < \pi$, and if $z = \cos x$, $A = \cos a$, $B = \cos b$, $C_{n-1}(x) = p_{n-1}(z)$, Theorem B3a is applicable to the polynomial $p_{n-1}(z)$ in the interval (B, A), and

$$\left| \frac{d}{dz} [(A-z)^{1/2}(z-B)^{1/2}p_{n-1}(z)] \right| \leqq \frac{nL}{[(A-z)(z-B)]^{1/2}}$$

for $B < z < A$. Hence, as $[(b-x)(x-a)]^{1/2}/[(\cos a - \cos x)(\cos x - \cos b)]^{1/2}$ is bounded and has a bounded derivative in the interval under consideration, it follows that

$$\left| \frac{d}{dx} [(b-x)^{1/2}(x-a)^{1/2}C_{n-1}(x)] \right| \leqq \frac{CnL}{[(b-x)(x-a)]^{1/2}}$$

throughout the same interval. A similar result is obtained if $C_{n-1}(x)$ is replaced by a sine sum $S_{n-1}(x)$, of order $n-1$, the differences between this case and that of the cosine sum being taken care of as at the corresponding stage in the proof of Theorem 1.

Let $T_{n-1}(x)$ be any trigonometric sum of order $n-1$ such that

$$|(b-x)^{1/2}(x-a)^{1/2}T_{n-1}(x)| \leqq L$$

for $a \leqq x \leqq b$, and also

$$|(-a-x)^{1/2}(x+b)^{1/2}T_{n-1}(x)| \leqq L$$

for $-b \leqq x \leqq -a$. The latter condition means that

$$|(b-x)^{1/2}(x-a)^{1/2}T_{n-1}(-x)| \leqq L$$

for $a \leqq x \leqq b$, so that the cosine and sine sums

$$C_{n-1}(x) = \tfrac{1}{2}[T_{n-1}(x) + T_{n-1}(-x)], \quad S_{n-1}(x) = \tfrac{1}{2}[T_{n-1}(x) - T_{n-1}(-x)]$$

satisfy the conditions of the paragraph preceding. Consequently

$$\left| \frac{d}{dx}[(b-x)^{1/2}(x-a)^{1/2}T_{n-1}(x)] \right| \leqq \frac{CnL}{[(b-x)(x-a)]^{1/2}}$$

for $a<x<b$, and at the same time, for $-b<x<-a$,

$$\left| \frac{d}{dx}[(-a-x)^{1/2}(x+b)^{1/2}T_{n-1}(x)] \right| \leqq \frac{CnL}{[(-a-x)(x+b)]^{1/2}}.$$

By a change of origin the two intervals can be replaced by any two intervals of equal length contained within a period, and neither overlapping nor having an end point in common. And the proof of Theorem 3 can be completed in the same manner as that of Theorem 1, by the use of four overlapping subintervals covering the interval to which the theorem refers; if

$$|(b-x)^{1/2}(x-a)^{1/2}T_{n-1}(x)| \leqq L$$

for $a \leqq x \leqq b$, and if the same auxiliary notation is used as in the earlier proof, then certainly

$$|(b_1-x)^{1/2}(x-a_1)^{1/2}T_{n-1}(x)| \leqq L$$

for $a_1 \leqq x \leqq b_1$, and similarly for each of the other subintervals, while the final synthesis of the conclusions pertaining to the various subintervals, though less immediate than in the other case, is complicated only by a multiplicity of simple details, which it would be superfluous to enumerate.

A corollary derivable from Theorems 1 and 3 jointly by the use of overlapping intervals, of which in this case only two are required, is a theorem intermediate between them in form:

THEOREM 4. *If $T_{n-1}(x)$ is a trigonometric sum of order $n-1$ such that*

$$|(x-a)^{1/2}T_{n-1}(x)| \leqq L$$

for $a \leqq x \leqq b$, where $0<b-a<2\pi$, then, for $a<x<b$,

$$\left| \frac{d}{dx}[(x-a)^{1/2}T_{n-1}(x)] \right| \leqq \frac{CnL}{[(b-x)(x-a)]^{1/2}}.$$

Let $c=\frac{1}{2}(a+b)$, $b_1=c+h$, $a_2=c-h$, with $0<h<\frac{1}{2}(b-a)$. Inasmuch as $(b_1-x)^{1/2}\leqq C$ for $a\leqq x\leqq b_1$, application of Theorem 3 to the interval (a, b_1) gives for $a<x\leqq c$ the relations

$$\left| \frac{d}{dx} [(b_1 - x)^{1/2}(x - a)^{1/2}T_{n-1}(x)] \right| \leqq \frac{CnL}{[(b_1 - x)(x - a)]^{1/2}} \leqq \frac{CnL}{(x - a)^{1/2}} \leqq \frac{CnL}{[(b - x)(x - a)]^{1/2}},$$

with different values of C in the successive members of the continued inequality; and hence by further use of the fact that $1/(b_1-x)^{1/2}$ and its derivative are bounded for $a<x\leqq c$ it appears that

$$\left| \frac{d}{dx} [(x - a)^{1/2}T_{n-1}(x)] \right| \leqq \frac{CnL}{[(b - x)(x - a)]^{1/2}} \tag{1}$$

throughout this interval. In (a_1, b), where $|T_{n-1}(x)|\leqq CL$, Theorem 1 is applicable, and for $c\leqq x<b$ the conclusion can be interpreted to give a relation of the form (1). So the assertion of the theorem is valid for the entire interval (a, b).

Naturally the ends of the interval can be interchanged in the statement, the factor $(x-a)^{1/2}$ being replaced by $(b-x)^{1/2}$.

A corresponding theorem for polynomials can be similarly obtained by combination of Theorems B2a and B3a, or deduced from the trigonometric case by means of a cosine substitution:

THEOREM 5. *If $P_{n-1}(x)$ is a polynomial of degree $n-1$ such that*

$$| (x - a)^{1/2}P_{n-1}(x) | \leqq L$$

for $a\leqq x\leqq b$, then

$$\left| \frac{d}{dx} [(x - a)^{1/2}P_{n-1}(x)] \right| \leqq \frac{CnL}{[(b - x)(x - a)]^{1/2}}$$

for $a<x<b$.

The same procedure leads to a proposition relating to the behavior of a trigonometric sum over an entire period.

THEOREM 6. *If $T_{n-1}(x)$ is a trigonometric sum of order $n-1$ such that*

$$| \sin^{1/2} (x/2)T_{n-1}(x) | \leqq L$$

for $0\leqq x\leqq 2\pi$, then

$$\left| \frac{d}{dx} [\sin^{1/2} (x/2)T_{n-1}(x)] \right| \leqq \frac{AnL}{\sin^{1/2} (x/2)}$$

for $0<x<2\pi$, where A is an absolute constant.

It is clear that this is equivalent to a superficially more general statement with $[\sin \frac{1}{2}(x-a)]^{1/2}$ in place of $(\sin \frac{1}{2}x)^{1/2}$, and with the interval from 0 to 2π replaced by that from a to $a+2\pi$. Also it is only for reasons of convenience that $\sin \frac{1}{2}x$ is used rather than an algebraic expression vanishing in the same manner. The same letter A will be used to denote a number of different absolute constants in succession, each in its place determinate once for all.

In consequence of the hypothesis $|x^{1/2}T_{n-1}(x)| \leqq AL$ for $0 \leqq x \leqq 3\pi/2$. (Any other fixed number between π and 2π would serve equally well.) By Theorem 4

$$\left| \frac{d}{dx} [x^{1/2}T_{n-1}(x)] \right| \leqq \frac{AnL}{[x(3\pi/2 - x)]^{1/2}}$$

for $0 < x < 3\pi/2$, the symbol C being replaced by that for an absolute constant because the interval is completely determinate. Here $x^{1/2}$ in the left-hand member can be replaced by $\sin^{1/2}(x/2)$ without changing the form of the upper bound on the right, while

$$\frac{AnL}{[x(3\pi/2 - x)]^{1/2}} \leqq \frac{AnL}{x^{1/2}} \leqq \frac{AnL}{\sin^{1/2}(x/2)}$$

for $0 < x \leqq \pi$. Replacement of x by $2\pi - x$ leads to a similar conclusion for the interval $(\pi, 2\pi)$, and so completes the proof of the theorem.

After successive consideration of $P_n(x)$ and $(1-x^2)^{1/2}P_{n-1}(x)$ it may be pointed out that the function $(1-x^2)P_{n-2}(x)$ (the symbol $P_{n-2}(x)$ being regarded as self-explanatory) is merely a polynomial of the nth degree (specialized, to be sure, by the circumstance that it vanishes at the points $+1$ and -1), and as such comes under Theorem B2, if an upper bound is given for its absolute value in $(-1, 1)$. The fact that the vanishing of the polynomial for $x = \pm 1$ does not affect the order of magnitude of the upper bound obtained for its derivative as x approaches the ends of the interval is illustrated by taking for $(1-x^2)P_{n-2}(x)$ the function $1-\cos n\theta$, with $x = \cos\theta$, for even values of n.

The expression $(b-x)(x-a)T_{n-2}(x)$ of course does not represent a trigonometric sum. It is, however, essentially equivalent, as far as the properties involved in this discussion are concerned, to $\sin \frac{1}{2}(b-x) \sin \frac{1}{2}(x-a)T_{n-2}(x)$, which is a trigonometric sum of order $n-1$.

The hypothesis that $|(1-x^2)^{-1/2}P_n(x)| \leqq L$ in $(-1, 1)$ does lead at once to a different form of conclusion, not with regard to the derivative of the product but with regard to the derivative of $P_n(x)$ itself. Since

$$|P_n(x)| \leqq L(1 - x^2)^{1/2}$$

it follows that $P_n(x)$ vanishes for $x=\pm 1$, and must be of the form $(1-x^2)p_{n-2}(x)$, where $p_{n-2}(x)$ is a polynomial of degree $n-2$. From the relation

$$|(1-x^2)^{1/2}p_{n-2}(x)| \leqq L$$

it appears by Theorem B3 that

$$\left| \frac{d}{dx}[(1-x^2)^{1/2}p_{n-2}(x)] \right| \leqq \frac{(n-1)L}{(1-x^2)^{1/2}} \leqq \frac{nL}{(1-x^2)^{1/2}}.$$

Hence

$$\begin{aligned} |P_n'(x)| &= \left| \frac{d}{dx}\{(1-x^2)^{1/2}[(1-x^2)^{1/2}p_{n-2}(x)]\} \right| \\ &\leqq (1-x^2)^{1/2}\left| \frac{d}{dx}[(1-x^2)^{1/2}p_{n-2}(x)] \right| \\ &\quad + \frac{|x|}{(1-x^2)^{1/2}}|(1-x^2)^{1/2}p_{n-2}(x)| \\ &\leqq nL + \frac{L}{(1-x^2)^{1/2}} \end{aligned}$$

for $-1<x<1$, where in the last member the factor n and the factor $(1-x^2)^{-1/2}$ do not occur in the same term.

On the assumption that $|(1-x^2)^{-1}P_n(x)| \leqq L$ for $-1<x<1$ it is possible by writing

$$P_n(x) = (1-x^2)p_{n-2}(x), \qquad |p_{n-2}(x)| \leqq L,$$

and applying Bernstein's theorem to $p_{n-2}(x)$ to obtain the relation

$$|P_n'(x)| \leqq nL(1-x^2)^{1/2} + 2L.$$

This type of argument will not be carried further here.

4. **Upper bounds for normalized trigonometric sums and polynomials.** The theorems of this section and the next are closely associated in content and arrangement with those of the paper *Certain problems of closest approximation*, previously cited, and henceforth to be referred to simply by the letter A. They are related to the preceding section, not by direct dependence on the results of that section, with the exception of Theorem 2, but by the use of the method of overlapping intervals in making the transition from the case of polynomials to that of trigonometric sums. The conclusions of the

present section have a bearing on the theory set forth in the writer's paper* on *Orthogonal trigonometric sums*. A system of trigonometric sums normalized and orthogonal over an interval (a, b) of length less than 2π can of course be regarded as constituting a system of the same character for an entire period, with weight function vanishing outside (a, b).

An exponent s will be of frequent occurrence. In analogy with the use of the symbol C above, C_s will be used to denote successively a number of different constants, each depending on the interval specified for the independent variable, and on s, but not depending on anything else.

Let $T_n(x)$ be an arbitrary trigonometric sum of the nth order, let s be an arbitrary positive number, and let

$$H_{ns} = \int_a^b | T_n(x) |^s dx,$$

the length of the interval (a, b) being less† than 2π. An upper bound for the absolute values of $T_n(x)$ in (a, b) is to be obtained in terms of H_{ns}. Let μ_n be the maximum of $|T_n(x)|$ for $a \leqq x \leqq b$, taken on for $x = x_0$, so that $|T_n(x_0)| = \mu_n$. By Theorem 2, $|T_n'(x)| \leqq Cn^2\mu_n$ for $a \leqq x \leqq b$. Hence if $|x - x_0| \leqq 1/(2Cn^2)$, with this particular value of C, the point x remaining in (a, b),

$$| T_n(x) - T_n(x_0) | \leqq \tfrac{1}{2}\mu_n,$$

and $|T_n(x)| \geqq \frac{1}{2}\mu_n$. Without impairment of the accuracy of what has been said it can be assumed that $C > 1/(b-a)$, so that $1/(2Cn^2) < \frac{1}{2}(b-a)$ for $n \geqq 1$. Then an interval of length $1/(2Cn^2)$ on one side or the other of x_0 is wholly contained in (a, b), and the integrand in the expression defining H_{ns} is greater than or equal to $(\mu_n/2)^s$ throughout such an interval. Consequently

$$H_{ns} \geqq \frac{1}{2Cn^2}\left(\frac{\mu_n}{2}\right)^s, \qquad \mu_n \leqq C_s(n^2 H_{ns})^{1/s}.$$

This may be summarized as follows:

LEMMA 1. *If $T_n(x)$ is a trigonometric sum of the nth order, if*

$$H_{ns} = \int_a^b | T_n(x) |^s dx, \qquad 0 < b - a < 2\pi, \tag{2}$$

and if μ_n is the maximum of $|T_n(x)|$ for $a \leqq x \leqq b$, then

$$\mu_n \leqq C_s(n^2 H_{ns})^{1/s}.$$

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* Annals of Mathematics, (2), vol. 34 (1933), pp. 799–814.

† For $b-a=2\pi$ see the paper A, p. 891, Lemma 1. The corresponding statement for $b-a>2\pi$ follows as a trivial corollary.

Let $\rho(x)$ be a summable function such that $\rho(x) \geqq v > 0$ for $a \leqq x \leqq b$, where v is constant. Then

$$\int_a^b [T_n(x)]^2 dx \leqq \frac{1}{v} \int_a^b \rho(x)[T_n(x)]^2 dx.$$

An immediate consequence of the lemma for $s=2$ is

THEOREM 7. *If $\rho(x)$ is a summable function having a positive lower bound in (a, b), and if trigonometric sums $T_n(x)$ of the nth order are constructed for successive values of n so that*

$$\int_a^b \rho(x)[T_n(x)]^2 dx \leqq 1, \tag{3}$$

then $|T_n(x)|$ has an upper bound of the order of n for $a \leqq x \leqq b$.

For trigonometric sums normalized over (a, b) with respect to $\rho(x)$ as weight function the integral in (3) is of course equal to 1; the superficially more general statement with the alternative relation of inequality, here and in subsequent theorems, is for the sake of application to the case of sums normalized over a longer interval containing (a, b), but with a weight function not necessarily satisfying the hypothesis throughout the whole of the longer interval.

If $\rho(x)$, without having a positive lower bound, is still non-negative, and is such that $[\rho(x)]^{-r}$ is summable over (a, b) with a positive value of r, let

$$s = 2r/(r+1), \qquad p = 2/s = 1 + (1/r) > 1.$$

The integrand in (2) can be regarded as the product of the factors $[\rho(x)]^{1/p} |T_n(x)|^s$ and $[\rho(x)]^{-1/p}$. By Hölder's inequality, since

$$\{[\rho(x)]^{1/p} |T_n(x)|^s\}^p = \rho(x)[T_n(x)]^2 \text{ and } \{[\rho(x)]^{-1/p}\}^{p/(p-1)} = [\rho(x)]^{-r},$$

$$H_{ns} \leqq \left[\int_a^b \rho(x)[T_n(x)]^2 dx\right]^{1/p} \left[\int_a^b [\rho(x)]^{-r} dx\right]^{(p-1)/p}.$$

If the first integral on the right satisfies (3), it follows that H_{ns} has an upper bound independent of n, and application of the lemma leads to the following conclusion:

THEOREM 8. *If $\rho(x)$ is a non-negative summable function such that $[\rho(x)]^{-r}$ is summable over (a, b), with $r > 0$, and if trigonometric sums $T_n(x)$ of the nth order are constructed for successive values of n so that* (3) *is satisfied, then $|T_n(x)|$ has an upper bound of the order of $n^{(r+1)/r}$ for $a \leqq x \leqq b$.*

Further information with regard to the order of magnitude of $|T_n(x)|$ in the interior of the interval can be obtained by the use of a lemma established in the paper A for polynomials.

Consider first the case of a trigonometric sum $C_n(x)$ of the nth order containing only cosines. Let

$$H_{ns} = \int_a^b | C_n(x) |^s dx,$$

with $0<a<b<\pi$. Let $z=\cos x$, $A=\cos a$, $B=\cos b$. Then $C_n(x)$ is a polynomial $p_n(z)$ of the nth degree in z, and

$$H_{ns} = \int_B^A | p_n(z) |^s \frac{dz}{(1-z^2)^{1/2}},$$

whence

$$\int_B^A | p_n(z) |^s dz \leqq H_{ns}.$$

From this it follows* that

$$| p_n(z) | \leqq \frac{C_s(nH_{ns})^{1/s}}{[(A-z)(z-B)]^{N/2}}$$

for $B<z<A$, where N is the smallest integer $\geqq 1/s$; if $s\geqq 1$, $N=1$. Consequently

$$| C_n(x) | \leqq \frac{C_s(nH_{ns})^{1/s}}{[(b-x)(x-a)]^{N/2}}$$

for $a<x<b$, with a new value of C_s.

Let $S_n(x)$ be a sine sum of the nth order, and let

$$H_{ns} = \int_a^b | S_n(x) |^s dx,$$

with $0<a<b<\pi$ as before. The symbols z, A, B having the same meaning as in the preceding paragraph, $S_n(x)=(1-z^2)^{1/2}q_{n-1}(z)$, where $q_{n-1}(z)$ is a polynomial in z of degree $n-1$, and

$$H_{ns} = \int_B^A (1-z^2)^{(s-1)/2} | q_{n-1}(z) |^s dz, \qquad \int_B^A | q_{n-1}(z) |^s dz \leqq C_s H_{ns}.$$

By the proposition cited for the polynomial $p_n(z)$ above,

$$| q_{n-1}(z) | \leqq \frac{C_s(nH_{ns})^{1/s}}{[(A-z)(z-B)]^{N/2}}, \qquad | S_n(x) | \leqq \frac{C_s(nH_{ns})^{1/s}}{[(b-x)(x-a)]^{N/2}}.$$

* See A, p. 897, Lemma 4.

Let $T_n(x)$ be an arbitrary trigonometric sum of the nth order, let it be supposed again that $0<a<b<\pi$, and let

$$H_{ns} = \int_a^b |T_n(x)|^s dx + \int_{-b}^{-a} |T_n(x)|^s dx$$
$$= \int_a^b \{ |T_n(x)|^s + |T_n(-x)|^s \} dx.$$

Let $C_n(x) = \frac{1}{2}[T_n(x)+T_n(-x)]$, $S_n(x) = \frac{1}{2}[T_n(x)-T_n(-x)]$, so that $C_n(x)$ is a cosine sum and $S_n(x)$ a sine sum. If X and Y are any two real numbers, and if s is any positive exponent, whether greater than 1, equal to 1, or less than 1, $|\frac{1}{2}(X+Y)|^s$ can not exceed the larger of the quantities $|X|^s$, $|Y|^s$, since the maximum of the function $\Phi(X) \equiv |X|^s$ in any finite interval is taken on at one end or the other of the interval, and consequently

$$|\tfrac{1}{2}(X+Y)|^s \leqq |X|^s + |Y|^s$$

in all cases. In the present connection,

$$|C_n(x)|^s \leqq |T_n(x)|^s + |T_n(-x)|^s, \quad |S_n(x)|^s \leqq |T_n(x)|^s + |-T_n(-x)|^s,$$

so that

$$\int_a^b |C_n(x)|^s dx \leqq H_{ns}, \qquad \int_a^b |S_n(x)|^s dx \leqq H_{ns}.$$

Application of the two preceding paragraphs to $C_n(x)$ and $S_n(x)$, and again to $C_n(-x)$ and $S_n(-x)$, gives

$$|T_n(x)| = |C_n(x) + S_n(x)| \leqq \frac{C_s(nH_{ns})^{1/s}}{[(b-x)(x-a)]^{N/2}}$$

for $a<x<b$,

$$|T_n(x)| \leqq \frac{C_s(nH_{ns})^{1/s}}{[(-a-x)(x+b)]^{N/2}}$$

for $-b<x<-a$. These formulas can be adapted by a change of origin to any pair of equal non-contiguous intervals contained within a period.

Let $T_n(x)$ be any trigonometric sum of the nth order, let (a, b) be any interval of length less than 2π, and let

$$H_{ns} = \int_a^b |T_n(x)|^s dx.$$

Let intervals $(a_1, b_1), \cdots, (a_4, b_4)$ be defined as in connection with the proof of Theorem 1. The preceding paragraph is to be applied to the first and third of these intervals, and again to the second and fourth; by combination of the results as interpreted for the four quarters of the interval (a, b) a conclusion is reached which can be stated in the following form:

LEMMA 2. *If $T_n(x)$ is a trigonometric sum of the nth order, if*

$$H_{ns} = \int_a^b | T_n(x) |^s dx, \qquad 0 < b - a < 2\pi,$$

and if N is the smallest integer $\geqq 1/s$, then

$$| T_n(x) | \leqq \frac{C_s(nH_{ns})^{1/s}}{[(b - x)(x - a)]^{N/2}}$$

for $a<x<b$.

An immediate consequence of Lemma 2 for $s=2$ is

THEOREM 9. *If $\rho(x)$ is a summable function having a positive lower bound in (a, b), and if trigonometric sums $T_n(x)$ are constructed so that* (3) *is satisfied, $|T_n(x)|$ has an upper bound of the order of $n^{1/2}$ throughout any closed interval interior to (a, b).*

If $\rho(x)$ does not have a positive lower bound, but does satisfy the hypothesis of Theorem 8, the reasoning with Hölder's inequality in the proof of that theorem, which gave a relation between the integral in (3) and the value of H_{ns} for an appropriate value of s, is applicable again without change; in conjunction with Lemma 2 it gives

THEOREM 10. *If $\rho(x)$ is a non-negative summable function satisfying the hypothesis of Theorem* 8, *and if trigonometric sums $T_n(x)$ are constructed so that* (3) *is satisfied, $|T_n(x)|$ has an upper bound of the order of $n^{(r+1)/(2r)}$ throughout any closed interval interior to (a, b).*

Lemma 2 also leads to a new result for polynomials, intermediate in character between Lemmas 3 and 4 of the paper A.

Let $P_n(x)$ be a polynomial of the nth degree, and s a positive exponent, and let

$$h_{ns} = \int_0^1 (1 - x)^{-1/2} | P_n(x) |^s dx.$$

Then

$$\int_0^1 (1 - x^2)^{-1/2} | P_n(x) |^s dx \leqq h_{ns},$$

since $1+x \geqq 1$ throughout the interval of integration. Let $x=\cos\theta$, so that $P_n(x)=C_n(\theta)$, a cosine sum of the nth order in θ. Then

$$\int_0^{\pi/2} |C_n(\theta)|^s d\theta = \int_0^1 (1-x^2)^{-1/2}|P_n(x)|^s dx \leqq h_{ns},$$

and since $C_n(\theta)$ is an even function of θ

$$\int_{-\pi/2}^{\pi/2} |C_n(\theta)|^s d\theta = 2\int_0^{\pi/2} |C_n(\theta)|^s d\theta \leqq 2h_{ns}.$$

By Lemma 2, therefore,

$$|C_n(\theta)| \leqq \frac{A_s(nh_{ns})^{1/s}}{[(\pi/2)^2-\theta^2]^{N/2}}$$

for $-\pi/2<\theta<\pi/2$, where A_s depends only on s, since the interval is definitely specified. For $0\leqq\theta<\pi/2$ (and so for $0<x\leqq 1$), inasmuch as $(\pi/2)+\theta\geqq\pi/2>1$, while $(\pi/2)-\theta>\cos\theta$,

$$|P_n(x)| = |C_n(\theta)| \leqq \frac{A_s(nh_{ns})^{1/s}}{[(\pi/2)-\theta]^{N/2}} \leqq \frac{A_s(nh_{ns})^{1/s}}{(\cos\theta)^{N/2}} = \frac{A_s(nh_{ns})^{1/s}}{x^{N/2}},$$

with the same A_s throughout. Replacement of x by $(x-a)/(b-a)$, carrying over the interval $(0, 1)$ into an arbitrary interval (a, b), makes it possible to state the conclusion in the following form:

LEMMA 3. *If $P_n(x)$ is a polynomial of the nth degree, if*

$$h_{ns} = \int_a^b (b-x)^{-1/2}|P_n(x)|^s dx,$$

and if N is the smallest integer $\geqq 1/s$, then

$$|P_n(x)| \leqq \frac{C_s(nh_{ns})^{1/s}}{(x-a)^{N/2}}$$

for $a<x\leqq b$.

Similarly, if

$$h_{ns} = \int_a^b (x-a)^{-1/2}|P_n(x)|^s dx,$$

then

$$|P_n(x)| \leqq \frac{C_s(nh_{ns})^{1/s}}{(b-x)^{N/2}}$$

for $a\leqq x<b$.

The second statement follows from the first, without re-examination of the proof in detail, on replacement of x, a, b by $-x$, $-b$, $-a$.

Lemma 3 has the following consequences, one immediate, the other obtained by the use of Hölder's inequality with $[\rho(x)]^{1/p}|P_n(x)|^s$ and $(b-x)^{-1/2}[\rho(x)]^{-1/p}$ as factors under the integral sign, p and s being related to r in the same way as before; each theorem has an alternative statement with the ends of the interval interchanged:

THEOREM 11. *If $\rho(x)$ is a summable function such that $(b-x)^{1/2}\rho(x)$ has a positive lower bound for $a \leqq x < b$, and if polynomials $P_n(x)$ are defined for successive values of n so that*

$$\int_a^b \rho(x)[P_n(x)]^2 dx \leqq 1, \tag{4}$$

then $|P_n(x)|$ has an upper bound of the order of $n^{1/2}$ throughout any interval $a+\delta \leqq x \leqq b$, $\delta > 0$.

THEOREM 12. *If $\rho(x)$ is a non-negative summable function such that $(b-x)^{-(r+1)/2}[\rho(x)]^{-r}$ is summable over (a, b), with $r>0$, and if polynomials $P_n(x)$ are defined satisfying* (4), *then $|P_n(x)|$ has an upper bound of the order of $n^{(r+1)/(2r)}$ throughout any interval $a+\delta \leqq x \leqq b$, $\delta>0$.*

From Lemma 3 it is possible to proceed to a trigonometric analogue of Lemma 3 of the paper A.

Let $C_n(x)$ be a cosine sum of the nth order, and let

$$h_{ns} = \int_a^b (b - x)^{-1/2}|C_n(x)|^s dx, \qquad 0 < a < b < \pi.$$

With the notation used in the proof of Lemma 2,

$$\int_B^A (z - B)^{-1/2}|p_n(z)|^s dz \leqq Ch_{ns},$$

whence by Lemma 3

$$|C_n(x)| = |p_n(z)| \leqq \frac{C_s(nh_{ns})^{1/s}}{(A - z)^{N/2}} \leqq \frac{C_s(nh_{ns})^{1/s}}{(x - a)^{N/2}}$$

in (a, b). A similar argument, with obvious modifications, applies to a sum of sines.

Let $T_n(x)$ be an arbitrary trigonometric sum of the nth order, and with the assumption still that $0<a<b<\pi$ let

$$H'_{ns} = \int_a^b (b^2 - x^2)^{-1/2} | T_n(x) |^s dx + \int_{-b}^{-a} (b^2 - x^2)^{-1/2} | T_n(x) |^s dx$$

$$= \int_a^b (b^2 - x^2)^{-1/2} (| T_n(x) |^s + | T_n(- x) |^s) dx.$$

Then

$$\int_a^b (b - x)^{-1/2} (| T_n(x) |^s + | T_n(- x) |^s) dx \leqq CH'_{ns}.$$

If $S_n(x)$ and $C_n(x)$ are the odd and even parts of $T_n(x)$,

$$\int_a^b (b - x)^{-1/2} | C_n(x) |^s dx \leqq CH'_{ns}, \qquad \int_a^b (b - x)^{-1/2} | S_n(x) |^s dx \leqq CH'_{ns},$$

some of the steps in the proof of Lemma 2 being applicable once more, and

$$| C_n(x) | \leqq \frac{C_s(nH'_{ns})^{1/s}}{(x - a)^{N/2}}, \qquad | S_n(x) | \leqq \frac{C_s(nH'_{ns})^{1/s}}{(x - a)^{N/2}}$$

for $a < x \leqq b$. So $|T_n(x)|$ has an upper bound of the same form in this interval. Replacement of x by $-x$ in the even functions $|C_n(x)|$ and $|S_n(x)|$ gives for each of these functions, and hence for $| T_n(x) |$, an upper bound of the form $C_s(nH'_{ns})^{1/s}/(-x-a)^{N/2}$ in the interval $-b \leqq x < -a$. Consequently

$$| T_n(x) | \leqq \frac{C_s(nH'_{ns})^{1/s}}{(x^2 - a^2)^{N/2}}$$

in both intervals.

By a change of origin, if (a_1, b_1) and (a_2, b_2) are any two equal non-contiguous intervals contained within a period, and if

$$H'_{ns} = \int_{a_1}^{b_1} [(b_2 - x)(x - a_1)]^{-1/2} | T_n(x) |^s dx$$
$$+ \int_{a_2}^{b_2} [(b_2 - x)(x - a_1)]^{-1/2} | T_n(x) |^s dx,$$

the intervals (a_1, b_1) and (a_2, b_2) taking the place of $(-b, -a)$ and (a, b) respectively, then

$$(5) \qquad | T_n(x) | \leqq \frac{C_s(nH'_{ns})^{1/s}}{[(x - a_2)(x - b_1)]^{N/2}}$$

for $a_1 \leqq x < b_1$ and for $a_2 < x \leqq b_2$.

The desired conclusion with regard to an arbitrary trigonometric sum $T_n(x)$ in a single arbitrary interval (a, b) (of length less than 2π) is to be derived now by the use of *three* overlapping intervals covering (a, b). To avoid another definition of symbols the notation used at the corresponding stage in the proofs of Theorem 1 and Lemma 2 may be employed again, the three intervals for the present argument being taken as (a, b_1), (a_2, b_3), and (a_4, b). Let

$$H'_{ns} = \int_a^b [(b-x)(x-a)]^{-1/2} | T_n(x) |^s dx.$$

Then

$$\int_a^{b_1} [(b-x)(x-a)]^{-1/2} | T_n(x) |^s dx$$
$$+ \int_{a_4}^b [(b-x)(x-a)]^{-1/2} | T_n(x) |^s dx \leqq H'_{ns},$$

and by (5)

$$| T_n(x) | \leqq \frac{C_s(nH'_{ns})^{1/s}}{[(x-a_4)(x-b_1)]^{N/2}}$$

in (a, b_1) and in (a_4, b). If x is restricted to the closed intervals (a, c_1) and (c_3, b), in which the last denominator has a positive lower bound, $|T_n(x)| \leqq C_s(nH'_{ns})^{1/s}$. On the other hand,

$$\int_{a_2}^{b_3} | T_n(x) |^s dx \leqq CH'_{ns},$$

so that Lemma 2 gives a relation of the form $|T_n(x)| \leqq C_s(nH_{ns}')^{1/s}$ for the interval $c_1 \leqq x \leqq c_3$, interior to (a_2, b_3). For the whole interval (a, b) a conclusion is thus obtained which can be expressed as

LEMMA 4. *If $T_n(x)$ is a trigonometric sum of the nth order and if*

$$H'_{ns} = \int_a^b [(b-x)(x-a)]^{-1/2} | T_n(x) |^s dx, \qquad 0 < b - a < 2\pi,$$

then $|T_n(x)| \leqq C_s(nH'_{ns})^{1/s}$ for $a \leqq x \leqq b$.

This lemma leads to inferences corresponding to those associated with the earlier lemmas:

THEOREM 13. *If $\rho(x)$ is a summable function such that $[(b-x)(x-a)]^{1/2}\rho(x)$ has a positive lower bound for $a<x<b$, and if trigonometric sums $T_n(x)$ are constructed satisfying* (3), *then $|T_n(x)|$ has an upper bound of the order of $n^{1/2}$ for $a \leqq x \leqq b$.*

THEOREM 14. *If $\rho(x)$ is a non-negative summable function such that*

$$[(b - x)(x - a)]^{-(r+1)/2}[\rho(x)]^{-r}$$

is summable over (a, b), with $r>0$, and if trigonometric sums $T_n(x)$ are constructed satisfying (3), then $|T_n(x)|$ has an upper bound of the order of $n^{(r+1)/(2r)}$ for $a \leqq x \leqq b$.

5. **Convergence theorems.** In this section there will be presented a sequence of applications of the preceding ideas to the theory of convergence of processes of closest approximation. A process of reasoning which has been employed elsewhere,* modified by the use of Theorem 2 in place of the ordinary form of Bernstein's or Markoff's theorem, establishes the following proposition:

LEMMA 5. *If $f(x)$ is a continuous function for $a \leqq x \leqq b$, $T_n(x)$ a trigonometric sum of the nth order, and*

$$G_{ns} = \int_a^b |f(x) - T_n(x)|^s dx,$$

and if there exists a trigonometric sum $t_n(x)$ of the nth order such that

$$|f(x) - t_n(x)| \leqq \epsilon_n$$

throughout the interval, then

$$|f(x) - T_n(x)| \leqq C_s(n^2 G_{ns})^{1/s} + 5\epsilon_n$$

for $a \leqq x \leqq b$.

Now let it be supposed that $f(x)$ is defined and continuous for $\alpha \leqq x \leqq \beta$, where $\alpha \leqq a < b \leqq \beta$ and $\beta - \alpha \leqq 2\pi$. (A more general hypothesis would be admissible for the moment, but unprofitable.) If $\beta = \alpha + 2\pi$ it will be assumed that $f(\beta) = f(\alpha)$, so that $f(x)$ can be thought of as continuous and periodic. Let $\rho(x)$ be a non-negative summable function in (α, β), having a positive constant v as lower bound for $a \leqq x \leqq b$. Let m be a given positive exponent, and let $T_n(x)$ be chosen among all trigonometric sums of the nth order so as to minimize the integral

$$\int_\alpha^\beta \rho(x) |f(x) - T_n(x)|^m dx. \tag{6}$$

The question of the existence of a minimizing sum, as well as of its uniqueness or non-uniqueness, has been treated elsewhere† in a manner adequate

* See, for example, the paper A, p. 899; also cf. the proof of Lemma 1 above.

† See these Transactions, vol. 26, loc. cit., pp. 133–139.

for present purposes; if $\beta-\alpha<2\pi$ the integral can be regarded nevertheless as extended over an entire period, with a weight function $\rho(x)$ vanishing outside (α, β). Let the minimum value of the integral, corresponding to the specific sum $T_n(x)$ under consideration, be γ_n. Then

$$G_{nm} = \int_a^b |f(x) - T_n(x)|^m dx \leqq \frac{1}{v}\int_a^b \rho(x)|f(x) - T_n(x)|^m dx \leqq \frac{\gamma_n}{v},$$

so that by Lemma 5

$$|f(x) - T_n(x)| \leqq C_m(n^2\gamma_n/v)^{1/m} + 5\epsilon_n$$

in (a, b), if ϵ_n has the meaning assigned to it above. Let it be assumed now that $f(x)$ can be approximately represented with an error not exceeding ϵ_n by a trigonometric sum of the nth order for $\alpha\leqq x\leqq\beta$, and not merely in (a, b), if the former interval is more extensive. Then by the minimizing property of $T_n(x)$

$$\gamma_n \leqq R\epsilon_n^m, \qquad R = \int_\alpha^\beta \rho(x)dx.$$

It follows that

$$|f(x) - T_n(x)| \leqq kn^{2/m}\epsilon_n$$

for $a\leqq x\leqq b$, with k independent of x and n. This result will be stated as a theorem.*

Theorem 15. *If $\rho(x)$ is a non-negative summable function over (α, β) having a positive lower bound in (a, b), $\alpha\leqq a<b\leqq\beta$, if trigonometric sums $T_n(x)$ of the nth order are constructed for successive values of n to minimize the integral* (6), *and if trigonometric sums $t_n(x)$ of the nth order exist for each n so that*

$$|f(x) - t_n(x)| \leqq \epsilon_n$$

for $\alpha\leqq x\leqq\beta$, there is a constant k, independent of x and n, such that

$$|f(x) - T_n(x)| \leqq kn^{2/m}\epsilon_n$$

for $a\leqq x\leqq b$.

A sufficient condition for uniform convergence of $T_n(x)$ toward $f(x)$ throughout (a, b) is that $\lim_{n\to\infty} n^{2/m}\epsilon_n=0$. Sums $t_n(x)$ meeting this requirement will exist if the modulus of continuity of $f(x)$, or in case of need, according to the value of m, the modulus of continuity of one of its derivatives, is suitably restricted.† If $m=2$, for example, it is sufficient that $f(x)$ have

* See also these Transactions, vol. 26, loc. cit., pp. 145–150.

† See, for example, Colloquium, Chapter I.

a continuous derivative for $\alpha \leqq x \leqq \beta$. Although the general theorems referred to for the existence of approximating sums are stated for the approximate representation of a periodic function over an entire period they are nevertheless available for use here, since if the interval (α, β) in which $f(x)$ is originally defined is of length less than 2π the definition can in each case be extended through the rest of a period so that the requisite properties are preserved.*

The remaining theorems to be presented below will be stated in a form corresponding to that of Theorem 15, the explicit interpretation of the results in terms of properties of continuity of $f(x)$ being omitted.

If $\rho(x)$ does not have a positive lower bound in (a, b), but is such that $[\rho(x)]^{-r}$ is summable over (a, b), $r > 0$, let

$$s = mr/(r + 1), \qquad p = m/s = 1 + (1/r) > 1.$$

By Hölder's inequality G_{ns}, written in the form

$$\int_a^b [\rho(x)]^{-1/p} \cdot [\rho(x)]^{1/p} \,|\, f(x) - T_n(x) \,|^s \, dx,$$

does not exceed the product of

$$\left[\int_a^b \rho(x) \,|\, f(x) - T_n(x) \,|^m dx \right]^{1/p}$$

by a factor independent of n. The integral last written down does not exceed γ_n, which is the integral of the same integrand from α to β. As $\gamma_n \leqq R\epsilon_n^m$, G_{ns} is not greater than a constant multiple of $\epsilon_n^{m/p} = \epsilon_n^s$, and $|f(x) - T_n(x)|$ has by Lemma 5 an upper bound of the order of $n^{2/s}\epsilon_n$. This gives

THEOREM 16. *If $\rho(x)$ is a non-negative summable function over (α, β) such that $[\rho(x)]^{-r}$ is summable over (a, b), $\alpha \leqq a < b \leqq \beta$, $r > 0$, if trigonometric sums $T_n(x)$ are defined minimizing* (6), *and if ϵ_n has the same meaning as in Theorem* 15, *there is a constant k, independent of x and n, such that*

$$|\, f(x) - T_n(x) \,| \leqq k n^{2(r+1)/(mr)} \epsilon_n$$

for $a \leqq x \leqq b$.

The lemma to be presented next corresponds to Lemma 8 of the paper A, and is to be deduced as a direct consequence of it.

Let $f(x)$ be continuous for $a \leqq x \leqq b$, with the understanding for the present that $0 < a < b < \pi$. Let $C_n(x)$ be a cosine sum of the nth order, and let

* If $f(x)$ has a modulus of continuity $\omega(\delta)$ for $\alpha \leqq x \leqq \beta$ and is defined as linear for $\beta \leqq x \leqq \alpha + 2\pi$, it has a modulus of continuity not exceeding a constant multiple of $\omega(\delta)$ for $\alpha \leqq x \leqq \alpha + 2\pi$; see Colloquium, pp. 52–53. The extension to derivatives offers no difficulty.

$$G_{ns} = \int_a^b |f(x) - C_n(x)|^s dx.$$

Let $z=\cos x$, $A=\cos a$, $B=\cos b$, $f(x)=\phi(z)$. Let $C_n(x)$ as a polynomial in $\cos x$ be denoted by $p_n(z)$. Then

$$G_{ns} = \int_B^A |\phi(z) - p_n(z)|^s(1 - z^2)^{-1/2}dz, \qquad \int_B^A |\phi(z) - p_n(z)|^s dz \leqq G_{ns}.$$

If there is a cosine sum of the nth order differing from $f(x)$ by not more than ϵ_n for $a \leqq x \leqq b$, this is a polynomial of the nth degree in z differing from $\phi(z)$ by not more than ϵ_n for $B \leqq z \leqq A$, and by the lemma* referred to in the preceding paragraph

$$\begin{aligned} |f(x) - C_n(x)| &= |\phi(z) - p_n(z)| \\ &\leqq \frac{C_s[(nG_{ns})^{1/s} + \epsilon_n]}{[(A - z)(z - B)]^{N/2}} \leqq \frac{C_s[(nG_{ns})^{1/s} + \epsilon_n]}{[(b - x)(x - a)]^{N/2}} \end{aligned}$$

for $a<x<b$, N being the smallest integer $\geqq 1/s$.

In carrying through a similar argument with sines in place of cosines some changes are to be noted in details. Let $f(x)$ be as before, let $S_n(x)$ be a sine sum of the nth order, and let G_{ns} now denote the integral

$$G_{ns} = \int_a^b |f(x) - S_n(x)|^s dx, \qquad 0 < a < b < \pi.$$

Let the symbols z, A, B, $\phi(z)$ be interpreted as above. The sum $S_n(x)$ has the form $(1-z^2)^{1/2}q_{n-1}(z)$, where $q_{n-1}(z)$ is a polynomial of degree $n-1$. Let

$$\psi(z) = \frac{\phi(z)}{(1 - z^2)^{1/2}} = \frac{f(x)}{\sin x}.$$

Then

$$\int_B^A |\psi(z) - q_{n-1}(z)|^s dz \leqq C_s G_{ns}.$$

If there is a sine sum $s_n(x)$ of the nth order differing from $f(x)$ by not more than ϵ_n for $a \leqq x \leqq b$ then $s_n(x)/\sin x$ is a polynomial of degree $n-1$ in z differing from $\psi(z)$ by not more than $C\epsilon_n$ for $B \leqq z \leqq A$. Another application of Lemma 8 of the paper A gives for $|f(x)-S_n(x)|$ an upper bound of the form obtained for $|f(x)-C_n(x)|$ at the end of the last paragraph.

Let $T_n(x)$ be an arbitrary trigonometric sum of the nth order, and with another change in the meaning of G_{ns} let

* See A, p. 905.

$$G_{ns} = \int_a^b |f(x) - T_n(x)|^s dx + \int_{-b}^{-a} |f(x) - T_n(x)|^s dx$$

$$= \int_a^b \{ |f(x) - T_n(x)|^s + |f(-x) - T_n(-x)|^s \} dx, \qquad 0 < a < b < \pi.$$

Let

$$u(x) = \tfrac{1}{2}[f(x) + f(-x)], \qquad v(x) = \tfrac{1}{2}[f(x) - f(-x)],$$
$$C_n(x) = \tfrac{1}{2}[T_n(x) + T_n(-x)], \qquad S_n(x) = \tfrac{1}{2}[T_n(x) - T_n(-x)].$$

By an argument used in connection with the proof of Lemma 2,

$$\int_a^b |u(x) - C_n(x)|^s dx \leqq G_{ns}.$$

Let it be supposed that there is a trigonometric sum $t_n(x)$ such that ϵ_n is an upper bound for $|f(x)-t_n(x)|$ throughout both of the intervals (a, b), $(-b, -a)$. Then if $c_n(x)$ is the cosine sum $\frac{1}{2}[t_n(x)+t_n(-x)]$,

$$|u(x) - c_n(x)| \leqq \epsilon_n$$

for $a \leqq x \leqq b$. So the second paragraph preceding gives an upper bound for $|u(x)-C_n(x)|$ in the interval $a<x<b$. Similarly, $|v(x)-S_n(x)|$ has an upper bound of the same form over the same range. As substitution of $-x$ for x leads to corresponding results for the interval $(-b, -a)$, it is found that

$$|f(x) - T_n(x)| \leqq \frac{C_s[(nG_{ns})^{1/s} + \epsilon_n]}{[(-a - x)(x + b)]^{N/2}},$$
$$|f(x) - T_n(x)| \leqq \frac{C_s[(nG_{ns})^{1/s} + \epsilon_n]}{[(b - x)(x - a)]^{N/2}}$$

for $-b<x<-a$ and for $a<x<b$ respectively. These formulas can be adapted immediately to the case of a pair of equal intervals unsymmetrically situated with respect to the origin. The transition to the case of a single arbitrary interval of length less than 2π is then effected by the use of four overlapping subintervals as in the proofs of Theorem 1 and Lemma 2, the conclusion being as follows:

LEMMA 6. *Under the hypotheses of Lemma* 5

$$|f(x) - T_n(x)| \leqq \frac{C_s[(nG_{ns})^{1/s} + \epsilon_n]}{[(b - x)(x - a)]^{N/2}}$$

for $a<x<b$, N being the smallest integer $\geqq 1/s$.

If the reasoning which led from Lemma 5 to Theorem 15 is modified by the use of Lemma 6 in place of Lemma 5 a result is obtained, the most essential content of which can be expressed thus:

THEOREM 17. *Under the hypotheses of Theorem* 15, $|f(x)-T_n(x)|$ *has an upper bound of the order of* $n^{1/m}\epsilon_n$ *throughout any closed interval interior to* (a, b).

This is an appreciable improvement over an earlier result* of similar character, to the extent that the exponent $(1/m)+\eta$ of the earlier work is replaced here by $1/m$.

Lemma 6 in combination with Hölder's inequality (as applied in the proof of Theorem 16) gives

THEOREM 18. *Under the hypotheses of Theorem* 16, $|f(x)-T_n(x)|$ *has an upper bound of the order of* $n^{(r+1)/(mr)}\epsilon_n$ *throughout any closed interval interior to* (a, b).

Lemma 6 also has a bearing on the theory of polynomial approximation, leading to a lemma which is related to it as Lemma 3 is related to Lemma 2.

Let $f(x)$ be a continuous function for $0 \leqq x \leqq 1$, let $P_n(x)$ be a polynomial of the nth degree, and let

$$g_{ns} = \int_0^1 (1 - x)^{-1/2} | f(x) - P_n(x) |^s dx.$$

If $x = \cos\theta$, $f(x) = \psi(\theta)$, $P_n(x) = C_n(\theta)$, application of Lemma 6 in the manner suggested by the proof of Lemma 3 gives in terms of g_{ns} an upper bound for $|\psi(\theta)-C_n(\theta)|$ in the interval $-\pi/2 < \theta < \pi/2$, and so for $|f(x)-P_n(x)|$ in the interval $0 < x \leqq 1$. Interpreted for an arbitrary interval by a linear change of variable, the conclusion may be stated as

LEMMA 7. *If* $f(x)$ *is a continuous function for* $a \leqq x \leqq b$, $P_n(x)$ *a polynomial of the nth degree, and*

$$g_{ns} = \int_a^b (b - x)^{-1/2} | f(x) - P_n(x) |^s dx,$$

and if there exists a polynomial $p_n(x)$ *of the nth degree such that*

$$| f(x) - p_n(x) | \leqq \epsilon_n$$

throughout the interval, then

$$| f(x) - P_n(x) | \leqq \frac{C_s[(ng_{ns})^{1/s} + \epsilon_n]}{(x - a)^{N/2}}$$

for $a < x \leqq b$, *N being the smallest integer* $\geqq 1/s$.

* These Transactions, vol. 26, loc. cit., p. 153. For comparison see also the paper A, p. 906, Theorem 15 and context.

If

$$g_{ns} = \int_a^b (x - a)^{-1/2} |f(x) - P_n(x)|^s dx$$

then

$$|f(x) - P_n(x)| \leqq \frac{C_s[(ng_{ns})^{1/s} + \epsilon_n]}{(b - x)^{N/2}}$$

for $a \leqq x < b$.

The second statement is an immediate corollary of the first.

Theorems on polynomials of closest approximation resulting from Lemma 7 are as follows:

THEOREM 19. *If* $\rho(x)$ *is a non-negative summable function over* (α, β) *such that* $(b-x)^{1/2}\rho(x)$ *has a positive lower bound in* (a, b), $\alpha \leqq a < b \leqq \beta$, *if polynomials* $P_n(x)$ *of the nth degree are constructed for successive values of n to minimize the integral*

$$\int_\alpha^\beta \rho(x) |f(x) - P_n(x)|^m dx, \tag{7}$$

and if polynomials $p_n(x)$ *of the nth degree exist for each n so that*

$$|f(x) - p_n(x)| \leqq \epsilon_n$$

for $\alpha \leqq x \leqq \beta$, *then* $|f(x) - P_n(x)|$ *has an upper bound of the order of* $n^{1/m}\epsilon_n$ *throughout any interval* $a + \delta \leqq x \leqq b$, $\delta > 0$.

THEOREM 20. *If* $\rho(x)$ *is a non-negative summable function over* (α, β) *such that* $(b-x)^{-(r+1)/2}[\rho(x)]^{-r}$ *is summable over* (a, b), $\alpha \leqq a < b \leqq \beta$, $r > 0$, *if polynomials* $P_n(x)$ *are defined minimizing* (7), *and if* ϵ_n *has the same meaning as in Theorem* 19, $|f(x) - P_n(x)|$ *has an upper bound of the order of* $n^{(r+1)/(mr)}\epsilon_n$ *throughout any interval* $a + \delta \leqq x \leqq b$, $\delta > 0$.

Each theorem has an alternative statement with $x - a$ in place of $b - x$ and with $(a, b - \delta)$ in place of $(a + \delta, b)$.

The final lemma of the current sequence is to correspond to Lemma 7 of the paper A. It is derived from Lemma 7 of the present paper, with incidental use of Lemma 6, by adaptation of the procedure which led from Lemmas 2 and 3 to Lemma 4, the last stage of the argument involving consideration of three overlapping intervals. Details of the proof being omitted, the statement is this:

LEMMA 8. *If* $f(x)$ *is continuous for* $a \leqq x \leqq b$, $T_n(x)$ *is a trigonometric sum of the nth order, and*

$$G'_{ns} = \int_a^b [(b - x)(x - a)]^{-1/2} |f(x) - T_n(x)|^s dx,$$

and if there is a trigonometric sum $t_n(x)$ of the nth order such that

$$|f(x) - t_n(x)| \leqq \epsilon_n$$

throughout the interval, then

$$|f(x) - T_n(x)| \leqq C_s[(nG'_{ns})^{1/s} + \epsilon_n]$$

for $a \leqq x \leqq b$.

The following are the corresponding theorems on closest approximation:

THEOREM 21. *If $\rho(x)$ is a non-negative summable function over (α, β) such that $[(b-x)(x-a)]^{1/2}\rho(x)$ has a positive lower bound in (a, b), $\alpha \leqq a < b \leqq \beta$, if trigonometric sums $T_n(x)$ are defined minimizing* (6), *and if ϵ_n has the same meaning as in Theorem* 15, *$|f(x) - T_n(x)|$ has an upper bound of the order of $n^{1/m}\epsilon_n$ for $a \leqq x \leqq b$.*

THEOREM 22. *If $\rho(x)$ is a non-negative summable function over (α, β) such that*

$$[(b - x)(x - a)]^{-(r+1)/2}[\rho(x)]^{-r}$$

is summable over (a, b), $\alpha \leqq a < b \leqq \beta$, $r > 0$, if trigonometric sums $T_n(x)$ are defined minimizing (6), *and if ϵ_n has the same meaning as in Theorem* 15, *$|f(x) - T_n(x)|$ has an upper bound of the order of $n^{(r+1)/(mr)}\epsilon_n$ for $a \leqq x \leqq b$.*

THE UNIVERSITY OF MINNESOTA,
MINNEAPOLIS, MINN.