DIFFERENTIABLE FUNCTIONS DEFINED IN ARBITRARY SUBSETS OF EUCLIDEAN SPACE*

BY

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1. **Introduction.** In a former paper† we studied the differentiability of a function defined in closed subsets of Euclidean n-space E. We consider here the differentiability "about" an arbitrary point of a function defined in an arbitrary subset of E. We show in Theorem 1 that any function defined in a subset A of E which is differentiable about a subset B of E may be extended over E so that it remains differentiable about B. This theorem is a generalization of AE Lemma 2. We show further that any function of class C^m about a set B is of class C^{m-1} about an open set B' containing B. In the second part of the paper we consider some elementary properties of differentiable functions, such as: the sum or product of two such functions is such a function.‡ We end with the theorem that differentiability is a local property.§

2. **Definitions and elementary properties.** We use a one-dimensional notation as in AE. Thus $f_k(x) = f_{k_1 \cdots k_n}(x_1, \cdots, x_n)$, $x^l = x_1^{l_1} \cdots x_n^{l_n}$, $l! = l_1! \cdots l_n!$, $D_k f(x) = \partial^{k_1 + \cdots + k_n} f(x)/\partial x_1^{k_1} \cdots \partial x_n^{k_n}$, etc.; we set $\sigma_k = k_1 + \cdots + k_n$, r_{xy} = distance from x to y. We always set $f(x) = f_0(x)$. Suppose the functions $f_k(x)$ for $\sigma_k \leqq m$ are defined in the subset A of Euclidean n-space E. Define $R_k(x'; x)$ for x, x' in A by

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† *Analytic extensions of differentiable functions defined in closed sets*, these Transactions, vol. 36 (1934), pp. 63–89. We refer to this paper as AE. See also *Functions differentiable on the boundaries of regions*, Annals of Mathematics, vol. 35 (1934), pp. 482–485, and *Differentiable functions defined in closed sets*, I, these Transactions, vol. 36 (1934), pp. 369–387, which we refer to as F and D respectively.

P. Franklin in Theorem 1 of a paper *Derivatives of higher order as single limits*, Bulletin of the American Mathematical Society, vol. 41 (1935), pp. 573–582, has given a necessary and sufficient condition for the existence of a continuous mth derivative. We remark that this theorem is exactly the special case of Theorem I of D obtained by letting $f(x)$ be defined in an interval. It is also a special case of Theorem 2 of the author's *Derivatives, difference quotients, and Taylor's formula*, Bulletin of the American Mathematical Society, vol. 40 (1934), pp. 89–94 (see also Errata, p. 894). For his assumption is easily seen to imply the needed uniformity condition; it also implies at once that $f(x)$ is continuous, so that no considerations of measurability are necessary. His Theorem 2 should be compared with Theorems II and III of D.

‡ If the set is closed, these theorems may be proved by first extending the functions throughout E.

§ For the case of one variable this follows from D, Theorem I.

$$(1) \qquad f_k(x') = \sum_{\sigma_l \leqq m-\sigma_k} \frac{f_{k+l}(x)}{l!}(x'-x)^l + R_k(x';x).$$

Let x^0 be an arbitrary point of E. If for each k ($\sigma_k \leqq m$) and every $\epsilon>0$ there is a $\delta>0$ such that

$$(2) \qquad |R_k(x';x)| \leqq r_{xx'}^{m-\sigma_k}\epsilon \text{ if } x,\ x' \text{ in } A,\ r_{xx^0}<\delta,\ r_{x'x^0}<\delta,$$

we shall say that $f(x)$ is *of class C^m in A about x^0 in terms of the $f_k(x)$*, or, $f(x)$ is $(C^m, A, x^0, f_k(x))$. If this is true for each x^0 in B, we say $f(x)$ is $(C^m, A, B, f_k(x))$, and replace "about x^0" by "about B." We say $f(x)$ (defined in A) is *of class C^m in A about B*, or, $f(x)$ is (C^m, A, B), if there exist functions $f_k(x)$ ($\sigma_k \leqq m$) defined in A such that $f(x)$ is $(C^m, A, B, f_k(x))$. If $B=A$ in the last two definitions, we leave out the words "about B"; this is in agreement with the previous definitions. We say $f(x)$ is $(C^\infty, A, B, f_k(x))$ if $f(x)$ is $(C^m, A, B, f_k(x))$ for each m. Any function defined in A is (C^{-1}, A, E).

Remark. We might define in an obvious manner such relations as (C^m, A, x^0), (C^∞, A, B). To study them would require a study of the different possible definitions of the $f_k(x)$ if $f(x)$ is (C^m, A, B). The $f_k(x)$ are not in general determined by $f(x)$. Thus if $A=B$ is the x_1-axis, only the $f_k(x)$ with $k_2=\cdots=k_n=0$ are determined by $f(x)$. It is not obvious for what point sets A the $f_k(x)$ are all determined by $f(x)$.

If $f(x)$ is $(C^m, A, B, f_k(x))$ ($m\geqq 0$), then the $f_k(x)$ are continuous at each point of B;* that is, the $f_k(x)$ may be defined in $B-B\cdot A$ so that this will be true. To show this, take x^0 in B, set $\epsilon=1$, and choose δ so that (2) holds for any k ($\sigma_k\leqq m$). Take x in A within δ of x^0 (if there is such a point); then (1) and (2) show that $f_k(x')$ is bounded for x' in A within δ of x^0 ($\sigma_k \leqq m$). Now let $\{x^i\}$ be any sequence of points of A, $x^i \to x^0$; (1) and (2) show that $\{f_k(x^i)\}$ is a regular sequence.

If A is open and $f(x)$ is $(C^m, A, A, f_k(x))$, then $D_k f(x)$ exists and equals $f_k(x)$ in A ($\sigma_k \leqq m$). (See AE.) If x^0 is an isolated point of A or is at a positive distance from A, then $f(x)$ is $(C^m, A, x^0, f_k(x))$ for any $f_k(x)$. If $f(x)$ is $(C^m, A, B, f_k(x))$ [or (C^m, A, B)], and A' is in A, B' is in B, then $f(x)$ is $(C^m, A', B', f_k(x))$ [or (C^m, A', B')]. Also $f(x)$ is (C^0, A, B) if and only if it is continuous at each point of B. If $f(x)$ is $(C^m, A, B, f_k(x))$, then it is $(C^{m'}, A, B, f_k(x))$ for all $m'<m$; a stronger theorem is proved in Theorem 2. If $f(x)$ is $(C^m, A, B, f_k(x))$, then $f_k(x)$ is $(C^{m-\sigma_k}, A, B, f_l(x))$.

3. **Extension theorems.** We prove here a theorem which gives the maximum range of differentiability of a function, and a theorem about the still larger range of differentiability of a function to an order one less.

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* Or better, "continuous in A about B."

THEOREM 1. *If $f(x)$ is $(C^m, A, B, f_k(x))$* (m finite or infinite), then the $f_k(x)$ may be extended throughout E so that $f(x)$ is $(C^m, E, B, f_k(x))$.*†

We note, conversely, that if $f(x)$ is not $(C^m, A, x^0, f_k(x))$, then no extension of $f(x)$ will be so. We remark also that $f(x)$ may be made analytic in $E-\bar{A}$ ($\bar{A}=A$ plus limit points).

To prove the theorem, we first extend the $f_k(x)$ through $\bar{A}-A$ as follows: Take any x^0 in $\bar{A}-A$. Let $f_k(x^0)$ be the upper limit of $f_k(x^i)$ for sequences $\{x^i\}$, $x^i\to x^0$, x^i in A, if this is finite; otherwise, set $f_k(x^0)=0$. Next we extend the $f_k(x)$ throughout $E-\bar{A}$ by the method of AE Lemma 2. We shall assume in the proof that m is finite. If $m=\infty$, we prove $C^{m'}$ for every integer m'. The only alteration needed in the proof is that AE §12 should be used; but this makes no essential change.

As $E-\bar{A}$ is open, $f(x)$ is $(C^m, E, E-\bar{A}, f_k(x))$; we must show that $f(x)$ is $(C^m, E, B\cdot\bar{A}, f_k(x))$. Take a fixed point x^0 in $B\cdot\bar{A}$. Let us say (k, ϵ, A_1, A_2) holds if there is a $\delta>0$ such that (2) holds whenever x is in A_1, x' is in A_2, and $r_{xx^0}<\delta$, $r_{x'x^0}<\delta$. We must prove (k, ϵ, E, E) for each k ($\sigma_k\leqq m$) and each $\epsilon>0$.

First we prove $(k, \epsilon, \bar{A}, \bar{A})$. Set $\epsilon'=\epsilon/[2(m+1)^n]$, and let δ be the smallest of the δ's given by (l, ϵ', A, A) for $\sigma_l\leqq m$. Let U be the spherical neighborhood of x^0 of radius δ; then $f_l(x)$ is bounded in $U\cdot A$ ($\sigma_l\leqq m$). Given x, x' in $U\cdot\bar{A}$, choose sequences $\{x^i\}$, $\{x'^i\}$ of points of $U\cdot A$, with $x^i\to x$, $x'^i\to x'$. Suppose first $\sigma_k=m$. Then we may take these sequences so that $f_k(x^i)\to f_k(x)$, $f_k(x'^i)\to f_k(x')$, and the desired inequality for $R_k(x'; x)$ follows from that for $R_k(x'^i; x^i)$. Suppose now that $\sigma_k<m$. Relations (1) and (2) with k, x', x replaced by l, x^i, x^j show that for any such $\{x^i\}$, $\{f_l(x^i)\}$ is a regular sequence ($\sigma_l<m$); hence $f_l(x^i)\to f_l(x)$, and similarly $f_l(x'^i)\to f_l(x')$ ($\sigma_l<m$). Relation (1) now shows that for i large enough, $\Delta=R_k(x'; x)-R_k(x'^i; x^i)$ differs as little as we please from

$$-\sum_{\sigma_l=m-\sigma_k}\frac{f_{k+l}(x)-f_{k+l}(x^i)}{l!}(x'-x)^l.$$

As $|f_j(x)-f_j(x^i)|\leqq\epsilon'$ ($\sigma_j=m$) and $|(x'-x)^l|\leqq r_{xx'}^{\sigma_l}$, $|\Delta|\leqq(m+1)^n\epsilon' r_{xx'}^{m-\sigma_k}$ for i large enough; the inequality again follows.

Next we prove $(k, \epsilon, \bar{A}, E-\bar{A})$. Set $\epsilon'=\epsilon/[2\cdot 4^m(m+1)^n]$, and define η in terms of ϵ' and then δ as in AE §11, using $(k, \eta, \bar{A}, \bar{A})$. Take x in $\bar{A}$ and x' in $E-\bar{A}$, each within $\delta/4$ of x^0. By AE (6.3) and the equation following (11.6),

* Or merely locally (C^m, A, B); see Theorem 6.

† If $A=B$ is closed, then B may be replaced by E; the present proof then gives a proof of AE Lemma 2 which makes no use of AE Lemma 1.

$$R_k(x';x) = D_k f(x') - \psi_k(x';x)$$
$$= \sum_l \frac{R_{k+l}(x^*;x)}{l!}(x'-x^*)^l + \sum_{s=1}^{t}\sum_l \binom{k}{l} D_l\phi_{\lambda_s}(x')\zeta_{\lambda_s;k-l}(x'),$$

where x^* is a point of $\bar{A}$ distant $\delta_*/4$ from x', $\delta_*/4$ being the distance from x' to $\bar{A}$. As $r_{x^*x} \leqq 2r_{xx'}$, $r_{x'x^*} \leqq 2r_{xx'}$, and $\delta_* \leqq 4r_{xx'}$, we find with the help of AE (11.8)

$$|R_k(x';x)| \leqq (m+1)^n (2r_{xx'})^{m-\sigma_k}\eta + (4r_{xx'})^{m-\sigma_k}\epsilon'/2 < r_{xx'}^{m-\sigma_k}\epsilon.$$

Next we prove $(k, \epsilon, E-\bar{A}, \bar{A})$. As is easily seen from AE (6.3) or by F (6) with x^{i-1}, x^i replaced by x, x',

$$R_k(x';x) = \sum_l \frac{R_{k+l}(x;x')}{l!}(x'-x)^l.$$

Set $\epsilon' = \epsilon/(m+1)^n$, and take the smallest δ given by $(k+l, \epsilon', \bar{A}, E-\bar{A})$ for $\sigma_l \leqq m-\sigma_k$. The required inequality now follows at once.

Finally we must show $(k, \epsilon, E-\bar{A}, E-\bar{A})$. Set $\epsilon' = \epsilon/[2n(m+1)^n]$, and take δ smaller than the $\delta/4$ given by AE §11 with ϵ replaced by ϵ' and smaller than the δ's given by $(k+l, \epsilon', \bar{A}, E-\bar{A})$ and $(k+l, \epsilon', E-\bar{A}, \bar{A})$ for $\sigma_l \leqq m-\sigma_k$. Now take x and x' in $E-\bar{A}$ within δ of x^0; we must consider two cases. Case I: The line segment $S = xx'$ lies wholly in $E-\bar{A}$. By AE (11.2), $|f_l(y) - f_l(x')| < 2\epsilon'$ for y on S $(\sigma_l \leqq m)$; the desired inequality now follows from F, Lemma 3. Case II: There is a point x^* of $\bar{A}$ on S. From AE (6.3), or F (6) with x^{i-1}, x^i replaced by x, x^*, we find

$$R_k(x';x) = R_k(x';x^*) + \sum_l \frac{R_{k+l}(x^*;x)}{l!}(x'-x^*)^l,$$

and the inequality again follows.

THEOREM 2. *If $f(x)$ is $(C^m, A, B, f_k(x))$ (m finite), then there is an open set B' containing B such that $f(x)$ is $(C^{m-1}, A, B', f_k(x))$.*

For each x in B, let $\delta(x)$ be the largest of the numbers δ for which (2) holds for all k $(\sigma_k \leqq m)$ with ϵ replaced by 1. Let $U(x)$ be the set of all points x' within $\delta(x)$ of x; then B' is the sum of all $U(x)$. The set B' is open. To prove $(C^{m-1}, A, B', f_k(x))$, take any x^0 in B' and any $\epsilon > 0$. For some x^* in B, $r_{x^*x^0} < \delta(x^*)$. There is an M such that $|f_k(y)| < M$ for y in $A \cdot U(x^*)$ $(\sigma_k \leqq m)$.† Let δ be the smaller of $\delta(x^*) - r_{x^*x^0}$ and $\epsilon/[2(m+1)^n M + 2]$. Now take any x and x' in A within δ of x^0. We are interested in the remainders

† For the proof, see the paragraph following the remark.

$$R_k'(x';x) = \sum_{\sigma_l = m-\sigma_k} \frac{f_{k+l}(x)}{l!}(x'-x)^l + R_k(x';x)$$

with $\sigma_k < m$. As $r_{xx'} < 2\delta$,

$$\left| R_k'(x';x) \right| \leqq (m+1)^n M r_{xx'}^{m-\sigma_k} + r_{xx'}^{m-\sigma_k} < r_{xx'}^{m-1-\sigma_k}\epsilon.$$

Corollary. *If $f(x)$ is of class C^m in any given point set about B, then it may be extended through an open set B' containing B so that it is of class C^{m-1} in B' and of class C^m in B' about B.*

4. **Composite functions, etc.** We prove here three theorems.

Theorem 3. *If f and g are of class C^m in A about B, then so are $f+g$ and $f-g$, with*

$$(f \pm g)_k = f_k \pm g_k. \tag{3}$$

This is obvious.

Theorem 4. *If f and g are of class C^m in A about B, then so is fg, and f/g if $g \neq 0$. The derivatives are given by the ordinary formulas. Thus*

$$(fg)_k = \sum_l \binom{k}{l} f_l g_{k-l}. \tag{4}$$

We might prove this theorem directly, but it follows from Theorem 5: fg and f/g are functions (of two variables) of class C^∞ of the functions f and g. (The condition B in A is obtained by using Theorem 1.)

Theorem 5. *Let A and B be subsets of n-space E_n, and let A' and B' be subsets of ν-space E_ν. Let $f^i(x)$ be $(C^m, A, B, f_k^i(x))$ $(i=1, \cdots, \nu)$, and let $g(y)$ be $(C^m, A', B', g_k(y))$ (m finite or infinite). Suppose B is in A, x in A implies*

$$y = (y_1, \cdots, y_\nu) = (f^1(x), \cdots, f^\nu(x)) = f(x)$$

in A', and x in B implies $f(x)$ in B'. Then the function

$$h(x) = g(f^1(x), \cdots, f^\nu(x)) = g(f(x))$$

is $(C^m, A, B, h_k(x))$; the $h_k(x)$ are given by the ordinary formulas (9) *for derivatives.*

As a consequence of this theorem, the definition of being of class C^m is independent of the coordinate system chosen. If the condition x in A [or B] does not imply $f(x)$ in A' [or B'], we may apply the theorem to any subset A_1 [or B_1] of A [or B] for which it does. We shall suppose m is finite; if $m = \infty$, we merely apply the reasoning below for each positive integer.

Suppose first $u^1(x), \cdots, u^\nu(x)$ are functions of class C^m in an open set Γ of E_n, suppose $v(y)$ is of class C^m in an open set Γ' of E_ν, and suppose x in Γ implies $u(x)$ in Γ'. Letting R'^i, S' denote remainders for u^i, v, Taylor's formula gives

$$u_k^i(x') = D_k u^i(x') = \sum_{\sigma_l \leqq m-\sigma_k} \frac{u_{k+l}^i(x)}{l!}(x'-x)^l + R_k'^i(x';x), \tag{5}$$

$$v_k(y') = D_k v(y') = \sum_{\sigma_l' \leqq m-\sigma_k'} \frac{v_{k+l}(y)}{l!}(y'-y)^l + S_k'(y';y), \tag{6}$$

certain inequalities on the $R_k'^i$ and S_k' being satisfied. We have set $\sigma_k' = k_1 + \cdots + k_\nu$. Set $w(x) = v(u(x))$; then (5) and (6) with $k=0$ give

$$w(x') = \sum_t \frac{v_t(u(x))}{t!}\left\{\sum_{\sigma_j \geqq 1} \frac{u_j(x)}{j!}(x'-x)^j + R'(x';x)\right\}^t + S'(u(x');u(x)), \tag{7}$$

where $S' = S_0'$. Also, by Taylor's formula,

$$w_k(x') = \sum_l \frac{w_{k+l}(x)}{l!}(x'-x)^l + T_k'(x';x). \tag{8}$$

Subtract (8) with $k=0$ from (7); then as R'^i, S', and T' all approach 0 to the mth order as $x' \to x$,† we may equate coefficients of $(x'-x)^k$ for $\sigma_k \leqq m$.‡ Thus we find polynomials

$$P_k(u_p^i, v_q) \qquad (\sigma_p \leqq \sigma_k, \sigma_q' \leqq \sigma_k; \sigma_k \leqq m)$$

such that, for any x in Γ,

$$w_k(x) = P_k(u_p^i(x), v_q(u(x))). \tag{9}$$

Using (8) gives for $w_k(x')$

$$w_k(x') = \sum_l \frac{P_{k+l}(u_p^i(x), v_q(u(x)))}{l!}(x'-x)^l + T_k'(x';x). \tag{10}$$

We may also evaluate it by replacing x by x' in (9) and using (5) and (6). (In (6) we replace y' by $u(x')$ and use (5) again.) Each variable in the resulting polynomial P_k consists of a polynomial in quantities R', S', and other quantities; if we multiply out and collect all terms with an R' or an S' as a factor, we obtain

† This is clear for S' if $m=0$; if $m>0$, then

$$S'/r_{xx'}^m = [S'/|u(x')-u(x)|^m]\cdot[|u(x')-u(x)|/r_{xx'}]^m,$$

where $|y'-y| = r_{yy'}$, and the last factor is bounded in $U \cdot A$.

‡ This is easily proved in succession for $\sigma_k = 0, 1, \cdots$ on letting $x' \to x$.

$$w_k(x') = P_k\Big[\sum_s \frac{u^i_{p+s}(x)}{s!}(x'-x)^s,\qquad (11)$$
$$\sum_t \frac{v_{q+t}(u(x))}{t!}\Big\{\sum_{\sigma_j\geqq 1}\frac{u_j(x)}{j!}(x'-x)^j\Big\}^t\Big] + Q_k,$$

where Q_k is a polynomial containing an R' or an S' as a factor in each term. It must be understood that $\sum u^i_{p+s}(x)(x'-x)^s/s!$ appears as the variable in the position of u^i_p, etc., in $P_k(u^i_p, v_q)$.

We now prove: *If* u^i_k $(\sigma_k\leqq m;\ i=1,\cdots,\nu)$, v_k $(\sigma'_k\leqq m)$ *are any numbers, then*

$$P^*_k(x; u^i_p, v_q) = P_k\Big[\sum_s \frac{u^i_{p+s}}{s!}x^s, \sum_t \frac{v_{q+t}}{t!}\Big\{\sum_{\sigma_j\geqq 1}\frac{u_j}{j!}x^j\Big\}^t\Big] \qquad (12)$$
$$-\sum_l \frac{P_{k+l}(u^i_p, v_q)}{l!}x^l,$$

considered as a polynomial in x, contains no terms of degree $\leqq m-\sigma_k$. To prove this, define the polynomials

$$u^i(x) = \sum_{\sigma_l\leqq m}\frac{u^i_l}{l!}x^l, \qquad v(y) = \sum_{\sigma'_l\leqq m}\frac{v_l}{l!}(y-u_0)^l; \qquad (13)$$

then $u^i_k(0) = D_k u^i(0) = u^i_k$, $v_k(u_0) = D_k v(u_0) = v_k$. Set $w(x) = v(u(x))$. Replacing x', x by x, 0 in (10) and (11) and putting in (12) gives, as $Q_k = 0$ in this case,

$$P^*_k(x; u^i_p, v_q) = T'_k(x; 0). \qquad (14)$$

As $T'_k\to 0$ to the $(m-\sigma_k)$th order as $x\to 0$, P^*_k cannot contain any terms of degree $\leqq m-\sigma_k$.

We return now to the functions $f^i(x)$, $g(y)$, $h(x)$. Set $h_k(x) = P_k(f^i_p(x), g_q(f(x)))$. The formulas (10) and (11) hold equally well for the f^i, g, h. Hence using (10), (11), and (12), we find for the remainder for $h_k(x)$

$$T_k(x'; x) = P^*_k(x'-x; f^i_p(x), g_q(f(x))) + Q_k. \qquad (15)$$

To show that $h(x)$ is $(C^m, A, B, h_k(x))$, take any x^0 in B, and set $y^0 = f(x^0)$. As $f(x)$ is continuous in A about B, for each neighborhood V of y^0 there is a neighborhood $U(V)$ of x^0 such that x in $U(V)\cdot A$ implies $f(x)$ in $V\cdot A'$. As y^0 is in B', we may take V so that the $g_k(y)$ are bounded in $V\cdot A'$. We may take U in $U(V)$ so small that the $f_k(x)$ are bounded in $U\cdot A$. Because of the property of P^*_k, we may obviously take δ small enough so that P^*_k satisfies an inequality of the nature of (2). Moreover each term in Q_k contains an $R_p(x'; x)$ or an $S_q(u(x'); u(x))$ with $\sigma_p\leqq\sigma_k$ or $\sigma'_q\leqq\sigma_k$; as each such remainder satisfies an inequality (2) (see a recent footnote) and all other quantities entering into Q_k are bounded, we may take δ small enough so that Q_k also satisfies an inequality (2). Hence the same is true of T_k, and the theorem is proved.

5. **Differentiability a local property.** Our object is to prove

THEOREM 6. *Let $f(x)$ be locally (C^m, A, B) (m finite or infinite). For each point x^0 of B there is a neighborhood U of x^0 and functions $f_k^{(x^0)}(x)$ defined in $U\cdot A$ such that $f(x)$ is $(C^m, U\cdot A, U\cdot B, f_k^{(x^0)}(x))$.† Then $f(x)$ is (C^m, A, B). If the $f_k^{(x^0)}(x)$ for $\sigma_k \leqq p$ are independent (at any x for which they are defined) of x^0, then these functions may be included among the $f_k(x)$ $(\sigma_k\leqq m)$.*

We may take each neighborhood U as an open n-cube, so small that the $f_k^{(x^0)}(x)$ are bounded in U. A finite or denumerable number of them, $C_1, C_2, \cdots$, cover B; we may take them so that any one touches at most a finite number of the others, and so that any boundary point of any C_i is interior to some C_j.‡ By hypothesis, to each i there correspond functions $f_k^i(x)$, $\sigma_k\leqq m$, such that $f(x)$ is $(C^m, C_i\cdot A, C_i\cdot B, f_k^i(x))$. In each C_i we define the function $\pi_i(x)$ as it was defined in I_i in AE §9; set

$$\phi_i(x) = \pi_i(x)/\sum_j \pi_j(x) \tag{16}$$

in $C_1+C_2+\cdots$. Set $g^i(x)=\phi_i(x)f(x)$ in $C_i\cdot A$. By Theorem 4, $g^i(x)$ is $(C^m, C_i\cdot A, C_i\cdot B)$, and

$$g_k^i(x) = \sum_l \binom{k}{l} D_l\phi_i(x)f_{k-l}^i(x). \tag{17}$$

As the $f_k^i(x)$ are bounded in $C_i\cdot A$ and the $D_l\phi_i(x)\to 0$ to infinite order as x approaches the boundary of C_i (see AE §9), the latter statement is true also of the $g_k^i(x)$. Hence, evidently, if we set $g_k^i(x)=0$ in $A-C_i\cdot A$, $g^i(x)$ is $(C^m, A, B, g_k^i(x))$. Set

$$f_k(x) = g_k^1(x) + g_k^2(x) + \cdots, \tag{18}$$

which in any $C_i\cdot A$ is a finite sum; this reduces to $f(x)$ for $k=0$. Theorem 3 shows at once that $f(x)$ is $(C^m, A, B, f_k(x))$. (Given x^0 in B, to apply Theorem

† Note that "$f(x)$ is locally $(C^\infty, \cdots)$" is not the same statement as "$f(x)$ is locally $(C^m, \cdots)$ for each m."

‡ Let $C^1, C^2, \cdots$ be a denumerable set of the cubes which cover B. Express each C^i as the sum of a denumerable number of cubes $C_j{}^i$ with the following properties: Each $C_j{}^i$ is, with its boundary, interior to C^i; the diameter of $C_j{}^i$, $\delta(C_j{}^i)$, is $<1/i$; $\delta(C_j{}^i)\to 0$ as $j\to\infty$; the cubes $C_j{}^i$ approach the boundary of C^i as $j\to\infty$. Now drop out all cubes $C_j{}^i$ which are interior to larger cubes $C_l{}^k$; the remaining cubes $C_1, C_2, \cdots$ still cover B. To each cube $C_j{}^i$ corresponds a number $\eta>0$ such that any point set of diameter $<\eta$ having points in common with $C_j{}^i$ lies interior to some $C_k{}^i$; using this fact, it is easily seen that any C_i has points in common with but a finite number of the C_j.

3, we choose δ so small that the points within δ of x^0 lie in but a finite number of the C_i.)

To prove the second statement, let $f_l'(x)$ denote the common value of $f_l^i(x)$ for $\sigma_l \leqq p$. Differentiating $\sum \phi_i = 1$ gives

$$\sum_i D_l\phi_i(x) = \begin{cases} 1 \text{ if } l = 0, \\ 0 \text{ if } \sigma_l > 0. \end{cases} \tag{19}$$

Define the $f_k(x)$ as before. Take any k with $\sigma_k \leqq p$; then (17) and (18) give

$$f_k(x) = \sum_l \binom{k}{l} f_{k-l}'(x) \sum_i D_l\phi_i(x) = f_k'(x)$$

in $C_1 + C_2 + \cdots$. It does not matter how $f_k(x)$ is defined outside this set.

The second statement in the theorem does not hold for an arbitrary set of $f_k(x)$, at least using the above method. To see this, take $n = m = 2$, $A = B$ = the interval $(-1, 1)$ of the x_1-axis, $C_1 = C_2$ = the square with corners $(\pm 1, \pm 1)$; set $f = 0$,

$$f_{10}^1 = f_{20}^1 = f_{11}^1 = f_{02}^1 = 0, \qquad f_{01}^1 = 1,$$

and $f_{ij}^2 = -f_{ij}^1$ on A. Also set

$$\phi_1(x, y) = \tfrac{1}{2} + \tfrac{3}{4}x - \tfrac{1}{4}x^3, \qquad \phi_2(x, y) = \tfrac{1}{2} - \tfrac{3}{4}x + \tfrac{1}{4}x^3.$$

(Though ϕ_1 and ϕ_2 are not the functions defined above, they have the necessary properties.) We find on A

$$g_{11}^1(x, y) = g_{11}^2(x, y) = \tfrac{3}{4} - \tfrac{3}{4}x^2, \qquad f_{11}(x, y) = \tfrac{3}{2} - \tfrac{3}{2}x^2 \not\equiv 0.$$

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