REMARKS ON THE PRECEDING PAPER OF JAMES A. CLARKSON*

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1. In the preceding paper Clarkson has introduced the interesting concept of a Banach space X with a uniformly convex norm and has shown that for such spaces the following theorem holds.

THEOREM. *If the additive function $F(R)$ which is defined for elementary figures R contained in a fixed figure R_0 in Euclidean space of n dimensions and has its values in the Banach space X, is of bounded variation,† then it has a derivative $F'(P)$ for almost all points P in R_0. $F'(P)$ is summable‡ on R_0 and if $F(R)$ is absolutely continuous§ then for every elementary figure R in R_0 we have*

$$F(R) = \int_R F'(P)dP.$$

In this paper it is shown that the theorem holds for all Banach spaces X with a base¶ $\{\phi_i\}$ which satisfies the following postulate:

(A) *If $a_1, a_2, \cdots$ is any sequence of real numbers such that* $\sup_n \left\| \sum_{i=1}^n a_i\phi_i \right\| < \infty$, *then the series $\sum_{i=1}^\infty a_i\phi_i$ converges.*

It is obvious that l_p ($p \geqq 1$) or any Hilbert space satisfies (A).

In §3 it is shown that L_p ($p>1$) does likewise. The method of proof is entirely different from that of Clarkson. In §4 it is shown that if X is any Banach space having the property that every function on a linear interval to X, which satisfies a Lipschitz condition, is differentiable almost everywhere, then also every function of bounded variation from the linear interval to X is differentiable almost everywhere and its derivative is summable.

2. **Proof of the theorem.** It should first be noted that it is no loss of generality to assume besides (A) the property

$$\text{(B)} \qquad \left\| \sum_{i=1}^n a_i\phi_i \right\| \leqq \left\| \sum_{i=1}^{n+1} a_i\phi_i \right\| \quad \textit{for any constants } a_i,\ i = 1, 2, \cdots.$$

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† I.e., $\sum_{i=1}^k \|F(R_i)\|$ is bounded for all finite sets of non-overlapping elementary figures $R_1, \cdots, R_k$ contained in R_0.

‡ Here, as well as elsewhere in this note, the concept of summability is that of Bochner or of Dunford. The two notions are equivalent.

§ I.e., $\lim_{|R|\to 0} F(R) = 0$.

¶ See Banach, *Théorie des Opérations Linéaires*, Warsaw, 1932, p. 110.

This is made evident by the following two considerations. If a Banach space Y is isomorphic* to X then the theorem holds for X if and only if it holds for Y. Now associated with X is a space Y composed of all sequences $y=\{\eta_i\}$ such that $\sum_{i=1}^{\infty}\eta_i\phi_i$ converges. If the metric in Y is defined by $\|y\|=\sup_n\|\sum_{i=1}^{n}\eta_i\phi_i\|$ then† Y is a Banach space and isomorphic to X. By taking $y_k=\{\eta_i^k\}$ where $\eta_i^k=0$ if $k\neq i$, $\eta_i^i=1$, it is easily seen that y_i forms a base for Y which has the properties (A) and (B). Thus it will be assumed in what follows that X has the properties (A) and (B). Now

$$F(R) = \sum_{i=1}^{\infty} a_i(R)\phi_i,$$

where the coefficients $a_i(R)$ are given by means of the limited linear functionals‡ T_i defined on X according to the equation $a_i(R)=T_iF(R)$. Thus it is immediate that the functions $a_i(R)$ are additive, real functions of bounded variation with summable derivatives $a_i'(P)$. The functions $F_n(R)=\sum_{i=1}^{n}a_i(R)\phi_i$ thus have derivatives $F_n'(P)=\sum_{i=1}^{n}a_i'(P)\phi_i$ summable on R_0. Let $V(F_n, R)$ be the total variation of F_n on R, then the positive real function $S_n(R)=V(F_n,R)$ is additive and of bounded variation on R_0 with $S_n(R)\geqq\|F_n(R)\|$. Hence $S_n'(P)\geqq\|F_n'(P)\|$ and thus

$$V(F_n, R) = S_n(R) \geqq \int_R S_n'(P)dP \geqq \int_R \|F_n'(P)\|dP. \tag{1}$$

Now from postulate (B) and (1)

$$\|F_n'(P)\| \leqq \|F_{n+1}'(P)\|, \qquad \int_{R_0}\|F_n'(P)\|dP \leqq V(F_n, R_0) \leqq V(F, R_0). \tag{2}$$

If we let $b(P)=\lim_n\|F_n'(P)\|$, the inequality (2) shows that $b(P)$ is summable and hence finite almost everywhere. Postulate (A) then insures the convergence of the series

$$G(P) = \sum_{n=1}^{\infty} a_n'(P)\phi_n$$

for almost all P in R_0. Since $\|G(P)\|=b(P)$ and $G(P)$ is measurable it is summable on R_0. It will now be shown that $G(P)$ is the derivative of $F(R)$. To do this we need the following lemmas which, in the case of real-valued functions, are well known.

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* Banach, loc. cit., p. 180.

† Banach, loc. cit., p. 111.

‡ Banach, loc. cit., p. 111.

LEMMA 1. *Let $G(R)$ be an additive function of bounded variation defined for elementary figures R contained in a fixed figure R_0 and with values in an arbitrary Banach space. Then $G(R)$ has a derivative equal to zero at almost all points of R_0 if and only if for every $\epsilon>0$ there is an open set E in R_0^0 with measure less than ϵ such that $V(G, E)=V(G, R_0^0)$.*

By the variation $V(G, D)$ of G on an open set D in R_0 is meant the upper bound of all finite sums $\sum_{i=1}^{k}\|G(R_i)\|$ where $R_1, R_2, \cdots, R_k$ are non-overlapping elementary figures in D. Now suppose $|E|<\epsilon$ and $V(G, E)=V(G, R_0^0)$. Define $S(C)=V(G, C)$ where C is either an open set or an elementary figure and let R_n be a sequence of elementary figures such that

$$R_n \subset R_{n+1} \subset E, \qquad S(R_n) \to S(E).$$

Let $R_n' = R_0^0 - R_n$, $E' = R_0^0 - E$, so that

$$S(R_0^0) \geqq S(R_n) + S(R_n') \geqq S(R_n) + \int_{R_n} S'(P)dP.$$

Thus since $R_n' \supset E'$ we have

$$S(R_0^0) \geqq S(R_n) + \int_{E'} S'(P)dP$$

or

$$\int_{E'} S'(P)dP \leqq S(R_0^0) - S(R_n) \to 0.$$

Whence it follows that $S'(P)=0$ almost everywhere on E', and since $|R_0-E'|<\epsilon$ we conclude that $S'(P)=0$ almost everywhere on R_0 which implies $G'(P)=0$ almost everywhere on R_0.

To prove the converse let $\eta>0$ and let $R_1, R_2, \cdots, R_k$ be non-overlapping elementary figures contained in R_0^0 with

$$\left|\sum_{i=1}^{k} R_i\right| > |R_0| - \frac{\epsilon}{2}, \qquad \sum_{i=1}^{k}\|G(R_i)\| \geqq V(G, R_0^0) - \eta.$$

Now define the set E_n as follows: a point P is in E_n if for every cube I containing P with $|I| \leqq 1/n$ it follows that $\|G(I)\|/|I| \leqq \eta$. Thus the set $\lim E_n$ contains all points at which $G'(P)=0$, i.e., almost all points in R_0. Hence $\lim E_n\sum_{i=1}^{k}R_i^0$ contains almost all points of $\sum_{i=1}^{k}R_i^0$. Consequently there is an n_0 and a closed set C contained in $E_{n_0}\sum_{i=1}^{k}R_i^0$ for which $|C|>|R_0|-\epsilon$. Thus $C\subset\sum_{i=1}^{k}R_i^0$ and $\|G(I)\|/|I| \leqq \eta$ for any cube I containing a point of C and having $|I| \leqq 1/n_0$. Let d be the distance from C to the boundary of $\sum_{i=1}^{k}R_i^0$ and $I_1, I_2, \cdots, I_l$ be non-overlapping cubes satisfying the conditions

$$\sum_{i=1}^{l} I_i \supset R_0; \qquad |I_i| \leqq 1/n_0, \qquad \text{diameter } I_i < d, \qquad (i = 1, 2, \cdots, l).$$

Define δ_j $(j=1, 2, \cdots, l)$ to be 0 or 1 according as I_jC is null or non-null and let $\delta_j' = 1-\delta_j$. Now $D=D(\epsilon, \eta)$, the complement of C with respect to R_0^0, is open and since R_iI_j $(j=1, 2, \cdots, l)$ is null or a cube if $\delta_j=1$, it follows that

$$\begin{aligned} V(G, R_0^0) - \eta &\leqq \sum_{i=1}^{k} \|G(R_i)\| \leqq \sum_{i=1}^{k}\sum_{j=1}^{l} \|G(R_iI_j)\| \\ &\leqq \sum_{i=1}^{k}\sum_{j=1}^{l} \delta_j\|G(R_iI_j)\| + \sum_{i=1}^{k}\sum_{j=1}^{l} \delta_j'\|G(R_iI_j)\| \\ &\leqq \sum_{i=1}^{k}\sum_{j=1}^{l} \eta\,|R_iI_j| + V(G, E) \leqq \eta\,|R_0| + V(G, D). \end{aligned}$$

By defining $E=\sum_{n=1}^{\infty} D(\epsilon/2^n, 1/n)$ the conclusion is immediate.

LEMMA 2. *If $F(R)$ is an additive function of bounded variation defined for elementary figures R contained in a fixed figure R_0 with values in a Banach space satisfying postulates* (A) *and* (B) *then there exist additive functions $\alpha(R)$, $\beta(R)$ such that*

$$F(R) = \alpha(R) + \beta(R),$$

$\alpha(R)$ is an indefinite integral and $\beta(R)$ has a derivative equal to zero almost everywhere on R_0.

Let $F(R)=\sum_{i=1} a_i(R)\phi_i$ and define

$$\alpha(R) = \int_R \left(\sum_{i=1}^{\infty} a_i'(P)\phi_i \right) dP, \qquad \beta(R) = F(R) - \alpha(R).$$

If we write $\beta(R)=\sum_{i=1}^{\infty} b_i(R)\phi_i$ and denote $\sum_{i=1}^{n} b_i(R)\phi_i$ by $\beta_n(R)$ then $\beta_n'(P)=0$ almost everywhere on R_0. Take $\epsilon>0$ and let (Lemma 1) E_n be an open set with $|E_n|<\epsilon/2^n$ such that $V(\beta_n, E_n)=V(\beta_n, R_0^0)$ $(n=1, 2, \cdots)$. Let $E=E_1+E_2+E_3+\cdots$. Now the $|E|<\epsilon$ and $V(\beta_n, E)=V(\beta_n, R_0^0)$. Since $V(\beta_n, R_0^0)\to V(\beta, R_0^0)$ by semi-continuity,* we have

$$V(\beta, E) \geqq V(\beta_n, E) \to V(\beta, R_0^0).$$

Hence $V(\beta, E)=V(\beta, R_0^0)$ so that by Lemma 1, $\beta'(P)=0$ almost everywhere on R_0. This completes the proof of Lemma 2.

Returning to the argument of the theorem itself we see immediately that

* It is well known that the relation $\beta_n(R)\to\beta(R)$ for $R\subset R_0$ implies $\liminf_n V(\beta_n, R_0) \geqq V(\beta, R_0)$. It is likewise readily seen that $\liminf_n V(\beta_n, R_0^0) \geqq V(\beta, R_0^0)$.

$F(R)$ has a derivative $F'(P)=G(P)$ almost everywhere on R_0. This follows from the fact that $\alpha(R)$, being an indefinite integral, is differentiable with $\alpha'(P)=G(P)$ almost everywhere on R_0. Now if $F(R)$ is absolutely continuous so are the functions $a_i(R)$ and hence

$$\int_R F'(P)dP = \sum_{i=1}^{\infty}\phi_i T_i \int_R F'(P)dP = \sum_{i=1}^{\infty}\phi_i \int_R T_i F'(P)dP\dagger$$
$$= \sum_{i=1}^{\infty}\phi_i \int_R a_i'(P)dP = \sum_{i=1}^{\infty} a_i(R)\cdot\phi_i = F(R)$$

for every elementary figure R in R_0.

3. **$L_p(p>1)$ has the property (A).** Let $\{\phi_i\}$ be the orthonormal sequence of Haar. Schauder‡ has shown that $\{\phi_i\}$ is a base for L_p $(p\geqq 1)$. The sequence $\{\phi_i\}$ also determines the sequence $\{T_i\}$ of linear functionals on L_p by the formula

$$T_i\psi = \int_0^1 \phi_i(t)\psi(t)dt.$$

If $p>1$ this sequence forms a fundamental set in $\bar{L}_p$ (the space conjugate to L_p) in the sense that every point in $\bar{L}_p$ can be approached by finite linear combinations of the elements of the sequence $\{T_i\}$. Now suppose $a_1, a_2, \cdots$ is an arbitrary sequence of real numbers such that $\|x_n\|$ is bounded, where $x_n=\sum_{i=1}^{n}a_i\phi_i$. We have

$$T_i x_n = a_i, \qquad (i\leqq n), \tag{5}$$

and so§ x_n is a weakly convergent sequence in L_p. Since L_p is weakly complete there is a point $x=\sum_{i=1}^{\infty}T_i x\phi_i$ in L_p such that $Tx_n\rightarrow Tx$ for every T in $\bar{L}_p$. Now from (5) $a_i=\lim_n T_i x_n = T_i x$ and so $x=\sum_{i=1}^{\infty}a_i\phi_i$ which was to be proved.

4. **Differentiability of functions of bounded variation.** It is the purpose of this paragraph to prove the final assertion in the introduction. Let $f(t)$ be of bounded variation on $(0, 1)$ to X, and let E be the set of functional values of the strictly monotone real function

$$\sigma(t) = t + V(f; 0, t), \qquad (0\leqq t\leqq 1).$$

The symbol $V(f; a, b)$ stands for the total variation of f on $a\leqq t\leqq b$. Let $\tau(s)$

† For the interchange of T_i and $\int_R$ see Garrett Birkhoff, these Transactions, vol. 38 (1935), p. 371.

‡ J. Schauder, *Eine Eigenschaft des Haarschen Orthogonalsystem*, Mathematische Zeitschrift, vol. 28 (1928), pp. 317–320.

§ Banach, loc. cit., p. 133, Theorem 1. This theorem needs to be modified so as to apply to weakly convergent sequences rather than sequences weakly convergent to a point.

on E to $(0, 1)$ be the inverse of $\sigma(t)$ and let $g(s)$ be defined on E by the equation $g(s)=f(\tau(s))$. Now for any two points $s<s'$ in E,

$$
\begin{aligned}
\|g(s') - g(s)\| &\leqq V(f; \tau(s), \tau(s')) \\
&\leqq \tau(s') - \tau(s) + V(f; \tau(s), \tau(s')) \\
&\leqq \sigma(\tau(s')) - \sigma(\tau(s)) = s' - s.
\end{aligned} \tag{6}
$$

By first extending the domain of definition of $g(s)$ to $\overline{E}$ (the closure of E) in the natural way and then in a linear fashion on each of the intervals which make up the complement of $\overline{E}$ with respect to the interval $0 \leqq s \leqq 1+V(f; 0, 1)$ it is seen that the extended function satisfies the same Lipschitz condition (6) on the whole of $0 \leqq s \leqq 1+V(f; 0, 1)$. Thus $g(s)$ has a derivative almost everywhere on $(0, 1+V(f; 0, 1))$ and hence almost everywhere on E with respect to E. Now $\tau(s)$ satisfies the Lipschitz condition $|\tau(s')-\tau(s)| \leqq |s'-s|$, and hence if we let E^* be those points of E at which g has a derivative with respect to E we have $m[\tau(E-E^*)]=0$. That is, for almost all t in $(0, 1)$ the point $\sigma(t)$ is in E^*. Thus for almost all t in $(0, 1)$ we have

$$
\begin{aligned}
\lim_{h\to 0} \frac{f(t+h) - f(t)}{h} &= \lim_{h\to 0} \frac{g(\sigma(t+h)) - g(\sigma(t))}{\sigma(t+h) - \sigma(t)} \lim_{h\to 0} \frac{\sigma(t+h) - \sigma(t)}{h} \\
&= g'(\sigma(t)) \cdot \sigma'(t),
\end{aligned}
$$

so that $f(t)$ has a derivative at almost all points of $(0, 1)$, and since this derivative is the product of a bounded measurable function and a real summable function it follows that $f'(t)$ is summable.

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