STEREOGRAPHIC PARAMETERS AND PSEUDO-MINIMAL HYPERSURFACES, II*

BY

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Introduction

In a recent paper† G. Y. Rainich and the writer showed that the Weierstrass formulas for minimal surfaces in terms of stereographic parameters can be generalized naturally to hypersurfaces the sum of whose radii of curvature vanishes. The name pseudo-minimal hypersurfaces was proposed for these, since the name minimal should be reserved for the hypersurfaces, the sum of the curvatures of which is zero everywhere.

As was shown in I, a general hypersurface can be represented in terms of stereographic parameters $x_1, \cdots, x_n$ as follows:

$$X_0 = \frac{1}{\lambda}\left(\phi - x_\alpha \frac{\partial \phi}{\partial x_\alpha}\right) \tag{1}$$

$$X_i = \frac{\partial \phi}{\partial x_i} + \frac{x_i}{\lambda}\left(\phi - x_\alpha \frac{\partial \phi}{\partial x_\alpha}\right) \tag{2}$$

with the abbreviation

$$2\lambda = 1 + x_\alpha x_\alpha. \tag{3}$$

Here $X_0, X_1, \cdots, X_n$ are the coordinates of the embedding E_{n+1}, and ϕ is an arbitrary function of $x_1, \cdots, x_n$ whose geometrical meaning is: distance of the tangent plane from the origin multiplied by λ. If now the hypersurface is to be pseudo-minimal, then, as was shown in I, §7, ϕ has to satisfy the following linear partial differential equation:

$$\lambda \frac{\partial^2 \phi}{\partial x_\alpha \partial x_\alpha} - n x_\alpha \frac{\partial \phi}{\partial x_\alpha} + n\phi = 0. \tag{4}$$

In I an infinity of particular solutions of (4) were given. We shall sometimes refer to ϕ as a pseudo-minimal potential.

The present paper contains extensions of the theory in two respects.

1. The Weierstrass formulas for two-dimensional minimal surfaces established a connection between minimal surfaces and analytic functions. Cor-

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† These Transactions, vol. 39 (1936), pp. 154–182. This paper will be referred to as I.

responding to each potential function $\phi(x_1, \cdots, x_n)$ satisfying (4), there belongs a pseudo-minimal hypersurface; but since the latter is a generalization of a minimal surface, we have to regard the functions ϕ satisfying (4) as a generalization of analytic functions. In §4 of I it was shown that the possibility of rotating the $(n+1)$-dimensional sphere upon whose hypersurface our pseudo-minimal hypersurface is mapped results in the invariance of our formulas under the substitutions of a group Ω, whose infinitesimal operators were obtained. This group is the analogue to the displacement group, and it permits therefore to obtain solutions referred to points other than the origin $x_i=0$. Such solutions will be constructed by applying the group transformation especially to the central symmetric solutions of (4). The new solutions will be found to be symmetric in the coordinates of the singularity and of the point under consideration. Upon development of these solutions into series proceeding according to the particular solutions found in I, §§7 and 8, remarkably symmetrical formulas are obtained; they are, for the type of generalized potential function considered here, the analogue of the geometric series and form therefore the prototype for all other expansions.

2. Two-dimensional one-sided minimal surfaces, i.e., surfaces which like a Möbius strip permit a continuous transition from one side to the other, have been the subject of numerous investigations, especially by Lie, Henneberg, and Schilling* and recently by Douglas† from a quite different point of view. The discussion of the possibility of one-sided pseudo-minimal hypersurfaces will be seen to rest upon the behavior of the potential ϕ under a (negative) transformation of reciprocal radii $x_i' = -x_i/r^2$, which transformation is already contained in our group Ω. The result is obtained that in order for a general hypersurface to be one-sided, the potential must satisfy the functional equation

$$\phi(x_1, \cdots, x_n) = -r^2\phi\left(-\frac{x_1}{r^2}, \cdots, -\frac{x_n}{r^2}\right). \tag{5}$$

An infinity of functions satisfying simultaneously (4) and (5) will be constructed, and, thus, an infinity of one-sided pseudo-minimal hypersurfaces will be obtained in spaces of any number of dimensions.

In order not to interrupt the developments of 1 and 2 with derivations of auxiliary formulas pertaining to hypergeometric functions, a third section containing all such formulas has been added.

The author gladly acknowledges his indebtedness to Professor G. Y. Rainich for frequent advice and numerous suggestions.

* See Darboux, *Théorie Générale des Surfaces*, Part 1, Book III, Chap. VI.

† J. Douglas, these Transactions, vol. 34 (1932), p. 731.

1. Displaced solutions and their expansions. The particular solutions of the differential equation (4) which we have derived in §§7 and 8 of I were all of the form

$$f_l^{(1)}(r)H_l(x_1, \cdots, x_n) \quad \text{and} \quad f_l^{(2)}(r)H_l(x_1, \cdots, x_n), \tag{1.1}$$

where H_l is a homogeneous polynomial solution of the n-dimensional Laplace equation of degree $l>0$, and where, if F means the Gaussian series,

$$f_l^{(1)}(r) = F\left(l - 1, -\frac{n}{2}, l + \frac{n}{2}; -r^2\right), \tag{1.2}$$

$$f_l^{(2)}(r) = r^{2-2l-n}F\left(1 - l - n, -\frac{n}{2}, 2 - l - \frac{n}{2}; -r^2\right). \tag{1.3}$$

The formulas (3.1), (3.2), and (3.8) of §3 define these functions for any value of the argument. For the cases where these series degenerate into Jacobi polynomials see I, §8.

Like their potential-theoretical analogue: H_l and $r^{-2l-n+2}H_l$, the solutions of the set (1.1) have singularities at $r=0$ and $r=\infty$. It is, however, desirable to know "displaced" solutions which have singularities at an arbitrarily given point. This displacing of the singularity is carried out by an application of the group of the differential equation. Comparing again with the potential case we see that since, for it, the group in question is the group of rigid motions, any harmonic function having a singularity at the origin can immediately be "displaced" by replacing the argument x_i by $x_i - x_i^0$. In our case the situation is essentially more complicated because the group Ω of the differential equation (4) as defined by the substitution

$$x_j' = \frac{1}{D}\left\{s_{j\rho}x_\rho + s_{j0}(1 - \lambda)\right\} \tag{1.4}$$

with

$$D = \lambda + s_{00}(1 - \lambda) + s_{0\rho}x_\rho \tag{1.5}$$

differs from the displacement group. It expresses the way in which the stereographic parameters change when the surface as a whole is rotated or, which is the same, when the n-dimensional sphere which defines the parameters is rotated. In order to shift a singularity from the points 0 or ∞ to any others, we must therefore carry out a rotation which affects the component ξ_0 of the unit normal vector. Other rotations, which leave ξ_0 unaffected, will result only in rotations in the $x_1, \cdots, x_n$ hyperplane around the origin. We therefore choose the matrix s_{ab} according to a rotation in the X_0, X_1-plane by an angle η which may be taken as canonical parameter of the group, and put

$$s_{11} = s_{00} = \cos\eta; \qquad s_{10} = -s_{01} = \sin\eta. \tag{1.6}$$

All other $s_{ik}=\delta_{ik}$. Then the transformed solution is according to I, (4.6):

$$\begin{aligned} T\phi &= [\lambda + (1-\lambda)\cos\eta - x_1\sin\eta] \\ &\cdot \phi\left(\frac{(1-\lambda)\sin\eta + x_1\cos\eta}{\lambda + (1-\lambda)\cos\eta - x_1\sin\eta}, \frac{x_2}{\lambda + (1-\lambda)\cos\eta - x_1\sin\eta}, \cdots\right). \end{aligned} \tag{1.7}$$

Since the operator of the infinitesimal transformation

$$Q_1 = (1-\lambda)\frac{\partial}{\partial x_1} + x_1 x_\rho \frac{\partial}{\partial x_\rho} - x_1 \tag{1.8}$$

appears as first coefficient in an expansion of $T\phi$ in powers of η:

$$T\phi = \phi + \eta Q_1\phi + \cdots, \tag{1.9}$$

we can write (1.7) also

$$T\phi = e^{\eta Q_1}\phi. \tag{1.10}$$

Instead of η one may introduce as parameter the radius vector r_I of that point which before the rotation by η was the south pole, which will therefore now contain the singularity. This is done by the substitution

$$\operatorname{tg}\frac{\eta}{2} = r_I, \quad \text{or} \quad e^{i\eta} = \frac{1 + ir_I}{1 - ir_I}. \tag{1.11}$$

It is sufficient to apply the transformation (1.7) to the case of the central symmetric solution; we will get in this way the analogue of the potential solution $|\vec{r}-\vec{r}_I|^{2-n}$. If we write

$$x_1 = r\cos\vartheta, \tag{1.12}$$

where ϑ is the angle between the rays to the two points under consideration, and apply (1.7) to a solution ϕ_0 solely depending upon λ, we obtain

$$\begin{aligned} T\phi_0 &= \frac{1}{\lambda_I}\{\lambda_I\lambda + (1-\lambda_I)(1-\lambda) - r_I r\cos\vartheta\} \\ &\cdot \phi_0\left(\frac{\lambda_I\lambda}{\lambda_I\lambda + (1-\lambda_I)(1-\lambda) - r_I r\cos\vartheta}\right). \end{aligned} \tag{1.13}$$

Here in analogy with (3) the abbreviation

$$2\lambda_I = 1 + r_I^2$$

was used. We see thus that except for a constant factor λ_I^{-1} (which is caused by our using ϕ instead of the generalized Painvin-function; see I, equation (1.5)) $T\phi_0$ is symmetrical in the coordinates of the singular point and of the point under consideration.

It was pointed out in I, §10, that a general solution of the differential equation (4) may be expanded into a series proceeding according to the particular solutions (1.1). We will apply this here to (1.13) and expand $T\phi_0$ which is the simplest solution with a singularity off the origin, into such a series of solutions possessing singularities at zero or infinity. Using the symmetry of the expression for $\lambda_I T\phi_0$ as a function of r_I and of r we may write

$$\lambda_I T\phi_0 = \sum_{l=0}^{\infty} c_l r_I^l f_l(r_I) r^l f_l(r) P_l(\cos\vartheta), \tag{1.14}$$

where the coefficient c_l can only be a function of the summation index l. $f_l(r)$ without upper index can be understood to mean some linear combination of $f_l^{(1)}(r)$ and $f_l^{(2)}(r)$; $P_l(\cos\vartheta)$ means that n-dimensional hyperspherical harmonic, which corresponds to axial symmetry, also called the ultraspherical polynomial. In order to show by an example the determination of c_l and of the special form of the radial function $f_l(r)$ we shall from now on specialize to the case of $n=3$. After having familiarized himself with this case, the reader will have no difficulty in deriving corresponding formulas for hypersurfaces of a higher number of dimensions.

The three-dimensional central symmetric solution was derived in §9 of I and found to be

$$\phi_0 = \frac{1}{r} - 6r + r^3. \tag{1.15}$$

Applying (1.13) to this (considered as a function of λ), the following transformed solution is obtained:

$$T\phi_0 = \frac{1}{2\lambda_I} U, \tag{1.16}$$

$$U = \frac{(1 - 6r_I^2 + r_I^4)(1 - 6r^2 + r^4) - 16r_I r(1 - r_I^2)(1 - r^2)\cos\vartheta - 32r_I^2 r^2 \sin^2\vartheta}{(1 + r_I^2 r^2 - 2r_I r\cos\vartheta)^{1/2}(r_I^2 + r^2 + 2r_I r\cos\vartheta)^{1/2}}.$$

We note here, because of the use we shall make of them later on, the following two relations:

$$U(r_I, r, \vartheta) = U(r, r_I, \vartheta), \tag{1.17}$$

$$U\left(r_I, \frac{1}{r}, \vartheta\right) = \frac{1}{r^2} U(r_I, r, \pi - \vartheta). \tag{1.18}$$

As the denominator of (1.16) indicates, the new solution has two singularities, namely, at

$$(1.19) \qquad r = r_I^{-1}, \qquad \vartheta = 0;$$

and

$$(1.20) \qquad r = r_I, \qquad \vartheta = \pi.$$

As could be anticipated from what was said above, these points are the stereographic projections of the former north pole and its antipodal point, the former south pole.

Geometrically the appearance of a singular point P in the potential ϕ of a hypersurface means that for points on the hypersurface infinitely far from the origin $X_a=0$ of the embedding E_{n+1} the projecting vector passes through P. Now in §9 of I several examples of the surfaces belonging to central symmetric potentials were computed, and it was seen that at points infinitely far from the origin the hypersurfaces possess horizontal tangent planes with normal vectors pointing both vertically up and down. (By vertical and horizontal we mean, of course, directions parallel and perpendicular to the X_0-axis.) The stereographic projections of these normal vectors will be the points 0 and ∞ of the $(x_1, \cdots, x_n)$-hyperplane. If the polar axis is inclined by an angle η, these two points will, of course, become the points (1.19) and (1.20).

The appearance of two singularities complicates the determination of the coefficients and proper linear combinations in (1.14), for three regions of convergence will have to be distinguished:

Region A: $r <$ the smaller of r_I and r_I^{-1}.
Region B: r lies between r_I and r_I^{-1}.
Region C: $r >$ the larger of r_I and r_I^{-1}.

We note here that when interchanging r_I and r the inequality for A becomes that for B, and when replacing r by r^{-1} the inequality for A becomes that for C, while that for B is reproduced.

In region A we now write (1.14) with indetermined coefficients

$$(1.21) \quad U_A(r_I, r, \vartheta) = \sum_{l=0}^{\infty} c_l r^l [f_l^{(1)}(r) + a_l f_l^{(2)}(r)] r_I^l [f_l^{(2)}(r_I) + b_l f_l^{(1)}(r_I)] P_l(\cos\vartheta);$$

$P_l(\cos\vartheta)$ is now the Legendre polynomial.

$U_B(r_I, r, \vartheta)$ is obtained herefrom by exchanging on the right-hand side r and r_I. We shall see how one can determine the coefficients c_l, a_l, and b_l in an elementary way. Firstly we see that the requirement that U_A remain finite for vanishing r causes us to put

(1.22) $$a_l = 0.$$

Secondly we prove that $b_l=0$ in the following way: In the series for U_B which on account of (1.22) is now

(1.23) $$U_B(r_I, r, \vartheta) = \sum_{l=0}^{\infty} c_l r_I{}^l f_l{}^{(1)}(r_I) r^l [f_l{}^{(2)}(r) + b_l f_l{}^{(1)}(r)] P_l(\cos\vartheta)$$

we replace r by r^{-1}. The left side changes according to (1.18), while the right-hand series, due to (3.4) and (3.5) becomes

$$\frac{1}{r^2}\sum_{l=0}^{\infty} c_l r_I{}^l f_l{}^{(1)}(r_I) r^l \{(-1)^l f_l{}^{(2)}(r) + b_l[(-1)^{l+1} f_l{}^{(1)}(r) + \alpha_l f_l{}^{(2)}(r)]\} P_l(\cos\vartheta).$$

Writing for $\pi-\vartheta$ again ϑ we get as a second form of U_B:

$$U_B(r_I, r, \vartheta) = \sum_{l=0}^{\infty} c_l r_I{}^l f_l{}^{(1)}(r_I) r^l \{[1 + (-1)^l \alpha_l b_l] f_l{}^{(2)}(r) - b_l f_l{}^{(1)}(r)\} P_l(\cos\vartheta).$$

Comparing this with (1.23) we see that

(1.24) $$b_l = 0,$$

so that (1.21) assumes the form:

(1.25) $$U_A(r_I, r, \vartheta) = \sum_{l=0}^{\infty} c_l r^l f_l{}^{(1)}(r) r_I{}^l f_l{}^{(2)}(r_I) P_l(\cos\vartheta).$$

Thirdly we determine c_l by giving special values to r and r_I. Let q be a number <1. Put for the moment $r=q/r_I$, and let r_I converge to infinity. The inequalities defining region A remain nevertheless fulfilled. From (1.16) we find

$$\lim_{r_I\to\infty} r_I^{-3} U\left(r_I, \frac{q}{r_I}, \vartheta\right) = (1 + q^2 - 2q\cos\vartheta)^{-1/2}.$$

Then we go similarly to the limit on the right-hand side of (1.25) using auxiliary formula (3.6). Thus, we obtain

(1.26) $$(1 + q^2 - 2q\cos\vartheta)^{-1/2} = \sum_{l=0}^{\infty} c_l (-1)^l q^l P_l(\cos\vartheta).$$

Using the well known definition of the Legendre polynomials we find

$$a_l = (-1)^l.$$

Therefore

(1.27) $$U_A(r_I, r, \vartheta) = \sum_{l=0}^{\infty} (-1)^l r^l f_l{}^{(1)}(r) r_I{}^l f_l{}^{(2)}(r_I) P_l(\cos\vartheta).$$

Exchanging r_I with r carries us into B, as was said above. Therefore,

$$(1.28) \qquad U_B(r_I, r, \vartheta) = \sum_{l=0}^{\infty} (-1)^l r_I{}^l f_l{}^{(1)}(r_I) r^l f_l{}^{(2)}(r) P_l(\cos\vartheta).$$

These expansions are quite as simple as their potential-theoretical analogue. The expansion in region C is obtained by replacing in (1.18) r by r^{-1} and using (3.4). Using (1.10), (1.15), and (1.28) we may now return to the canonical parameter and write:

$$(1.29) \qquad \begin{aligned} &e^{\eta Q_1}\left(\frac{1}{r} - 6r + r^3\right) \\ &\qquad = \cos^2\frac{\eta}{2}\sum_{l=0}^{\infty}(-1)^l\left(\operatorname{tg}\frac{\eta}{2}\right)^l f_l{}^{(1)}\left(\operatorname{tg}\frac{\eta}{2}\right) r^l f_l{}^{(2)}(r) P_l(\cos\vartheta), \end{aligned}$$

where Q_1 is the infinitesimal operator (1.8) of Ω. This relation suggests an entirely different approach which we might have used in arriving at particular solutions of the differential equation (4). By expanding (1.29) on both sides into powers of η we obtain the displaced solution as a linear combination of solutions obtained by an iterated application of Q_1 on ϕ. Each member of this new sequence of particular solutions is, of course, a linear combination of a finite number of solutions (1.1), but this linear relation does not seem to be a simpler one than is already exhibited by (1.29).

An interesting special case of (1.27) is obtained by putting in (1.27) $r_I = 1$, or $\eta = \pi/2$. This results in the spreading of region A and region C to the unit circle, into which region B itself degenerates. Both singularities lie now on the unit circle opposite each other. From (1.16) and (1.27) we have, making use of (3.7) and putting $l = 2j$,

$$(1.30) \qquad \begin{aligned} &\frac{(1+r^2)^2 - 8r^2\cos^2\vartheta}{[(1+r^2)^2 - 4r^2\cos^2\vartheta]^{1/2}} \\ &\qquad = -\frac{1}{4}\sum_{j=0}^{\infty}\binom{2}{j+1}\binom{2}{2j+2}^{-1} r^{2j} f_{2j}^{(1)}(r) P_{2j}(\cos\vartheta). \end{aligned}$$

2. **One-sided pseudo-minimal hypersurfaces.** If a hypersurface is to be one-sided, then we must be able to reach, starting from an initial n-uple of values $x_1, \cdots, x_n$ with normal unit vector $+\xi_a$ by a continuous change in the stereographic parameters a final n-uple of values $\bar{x}_1, \cdots, \bar{x}_n$ with unit normal vector $-\xi_a$, while the values of the coordinates in the embedding space are the same for the two points:*

* From now on we shall write simply $F(x)$ for any function $F(x_1, \cdots, x_n)$ of the n stereographic parameters.

$$X_a(x) = X_a(\bar{x}) \qquad (a = 0, 1, \cdots, n). \tag{2.1}$$

Let us see first, what the relation between $x_1, \cdots, x_n$ and $\bar{x}_1, \cdots, \bar{x}_n$ has to be, so that

$$\xi_a(x) = -\xi_a(\bar{x}).$$

From the zero component $\xi_0 = (1-\lambda)/\lambda$ we conclude

$$\bar{\lambda} = \frac{\lambda}{r^2}, \tag{2.2}$$

and from the other components $\xi_i = x_i/\lambda$ using (2.2)

$$\bar{x}_i = -\frac{x_i}{r^2}. \tag{2.3}$$

This is a transformation of reciprocal radii together with a reversion of sign. A transformation of this sort is contained in the governing group Ω, as we see upon putting the transformation matrix s_{ab} of (1.4) equal to the negative unit matrix. We therefore know that by the transformation (2.3) a pseudo-minimal potential ϕ is transformed into another pseudo-minimal potential ϕ'.

The transformation (2.3), when introduced into equation (2.1) results in a functional equation between $\phi(x)$ and $\phi(\bar{x})$. The formulas giving X_a in terms of ϕ are given in the introduction, equations (1) and (2). We shall first investigate the zero component of (2.1) which is

$$\frac{1}{\bar{\lambda}}\left(\phi(\bar{x}) - \bar{x}_\rho \frac{\partial \phi(\bar{x})}{\partial \bar{x}_\rho}\right) = \frac{1}{\lambda}\left(\phi(x) - x_\rho \frac{\partial \phi(x)}{\partial x_\rho}\right).$$

By means of (2.3) we now replace the barred parameters throughout by the unbarred ones and obtain, using the abbreviation

$$\phi(x) + r^2\phi\left(-\frac{x}{r^2}\right) = g(x), \tag{2.4}$$

the differential equation:

$$g = x_\rho \frac{\partial g}{\partial x_\rho}. \tag{2.5}$$

Now we take the other components of (2.1):

$$\frac{\partial \phi(\bar{x})}{\partial \bar{x}_i} + \frac{\bar{x}_i}{\bar{\lambda}}\left(\phi(\bar{x}) - \bar{x}_\rho \frac{\partial \phi(\bar{x})}{\partial \bar{x}_\rho}\right) = \frac{\partial \phi(x)}{\partial x_i} + \frac{x_i}{\lambda}\left(\phi(x) - x_\rho \frac{\partial \phi(x)}{\partial x_\rho}\right).$$

This can be written using again abbreviation (2.4) after elementary transformations:

$$\frac{\partial g}{\partial x_i} + \frac{x_i}{\lambda}\left(g - x_\rho \frac{\partial g}{\partial x_\rho}\right) = 0$$

or

$$\frac{\partial g}{\partial x_i} = \frac{x_i}{\lambda - 1} g. \tag{2.6}$$

Introducing (2.6) into (2.5) we see that

$$g = 0. \tag{2.7}$$

We see therefore that for a hypersurface to be one-sided the potential ϕ *must fulfill the functional equation*:

$$\phi(x) = - r^2\phi\left(- \frac{x}{r^2}\right). \tag{2.8}$$

Returning to pseudo-minimal hypersurfaces, we will take members of the sequence (1.1) and see if simple linear combinations of them satisfy the above functional equation.

Calling a linear combination of $f_l^{(1)}(r)$ and $f_l^{(2)}(r)$ simply $f_l(r)$, we see that because

$$H_l\left(- \frac{x}{r^2}\right) = (- 1)^l r^{-2l} H_l(x)$$

equation (2.8) can be fulfilled if

$$f_l\left(\frac{1}{r}\right) = (- 1)^{l+1} r^{2(l-1)} f_l(r). \tag{2.9}$$

We shall show that this equation can indeed be satisfied by a proper choice of constants in the $f_l(r)$. As always with discussions concerning properties of the radial functions it is convenient to separate the cases of odd and even dimension number n.

(α) n is an odd integer. Put

$$f_l(r) = f_l^{(1)}(r) + a_l f_l^{(2)}(r),$$

where a_l is constant. Then using (3.4) and (3.5)

$$f_l\left(\frac{1}{r}\right) = (- 1)^{l+1} r^{2(l-1)} \left\{ f_l^{(1)}(r) + \left[\alpha_l(- 1)^{l+1} - a_l\right] f_l^{(2)}(r) \right\}$$

(2.9) will be fulfilled if

$$a_l = \tfrac{1}{2}(-1)^{l+1}\alpha_l$$

or if

$$f_l(r) = f_l^{(1)}(r) + \tfrac{1}{2}(-1)^{l+1}\alpha_l f_l^{(2)}(r). \tag{2.10}$$

(β) n is an even integer. Put, this time,

$$f_l(r) = b_l f_l^{(1)}(r) + f_l^{(2)}(r),$$

where b_l is a constant. Then using (3.9)

$$f_l\left(\frac{1}{r}\right) = b_l r^{2(l-1)} \frac{1}{\beta_l} f_l^{(2)}(r) + \beta_l r^{2(l-1)} f_l^{(1)}(r)$$

(2.9) will be fulfilled if

$$b_l = (-1)^{l+1}\beta_l$$

or if

$$f_l(r) = (-1)^{l+1}\beta_l f_l^{(1)}(r) + f_l^{(2)}(r). \tag{2.11}$$

The formulas (2.10) and (2.11) show how from each pair of particular solutions (1.1) a pseudo-minimal potential may be obtained which will satisfy the functional equation (2.8) and therefore furnish a one-sided hypersurface. More general ϕ's satisfying (2.8) may of course be constructed by combining linearly for different values of l the thus obtained special solutions, or even by infinite series proceeding according to products of type (2.10) or (2.11) multiplied with the proper solid harmonic H_l.

As a special example let us compute the case $n=2$, $l=2$. Formula (2.11) gives

$$f_2(r) = f_2^{(2)}(r) - 3f_2^{(1)}(r),$$

or using (3.1) and (3.8)

$$f_2(r) = \frac{1}{r^4} + \frac{3}{r^2} - 3 - r^2.$$

As harmonic we choose $x_1^2 - x_2^2$, so that

$$\phi = \left(\frac{1}{r^4} + \frac{3}{r^2} - 3 - r^2\right)(x_1^2 - x_2^2).$$

The surface itself becomes, using (1) and (2):

$$X_0 = (x_1^2 - x_2^2)\left(1 + \frac{1}{r^4}\right),$$

$$X_1 = \quad x_1\left(1 - \frac{1}{r^2}\right)\left[1 - \frac{1}{3}(x_1^2 - 3x_2^2)\left(1 + \frac{1}{r^2} + \frac{1}{r^4}\right)\right],$$

$$X_2 = -\, x_2\left(1 - \frac{1}{r^2}\right)\left[1 - \frac{1}{3}(x_2^2 - 3x_1^2)\left(1 + \frac{1}{r^2} + \frac{1}{r^4}\right)\right].$$

This is the famous Henneberg* surface, which was recognized by Lie and by Schilling to be the simplest one-sided minimal surface.

3. **Auxiliary formulas.** Due to the constant appearance of integer and half-integer parameter values it is easier to derive the following formulas from the well known integral representation of the hypergeometric function rather than obtain them by specialization from the general theory.—This integral representation of the radial function $f_l^{(1,2)}$ is

$$\text{const.} \int s^{l-2}(1 - s)^{n/2}(1 + r^2 s)^{n/2} ds.$$

The multiplicative constant will be chosen throughout so that the coefficient of the lowest power of r is unity. The limits or the path of integration have to be selected so that the integrand either resumes its initial value, or vanishes at both limits. As in I the discussion of even and odd dimension numbers is carried out separately. Also we assume l to be $\geqq 2$.

(α) n is an odd integer. The solution I, (8.4) which is regular for $r=0$ is

$$(3.1) \qquad \begin{cases} f_l^{(1)}(r) = C_l^{(1)} \displaystyle\int_0^1 s^{l-2}(1 - s)^{n/2}(1 + r^2 s)^{n/2} ds, \\ C_l^{(1)} = \dfrac{\Gamma(l + n/2)}{\Gamma(l - 1)\Gamma(n/2 + 1)}, \end{cases}$$

and the singular solution is

$$(3.2) \qquad \begin{cases} f_l^{(2)}(r) = C_l^{(2)} \displaystyle\oint s^{l-2}(1 - s)^{n/2}(1 + r^2 s)^{n/2} ds, \\ C_l^{(2)} = \frac{1}{2}(-1)^l \dfrac{\Gamma(l + n)}{\Gamma(l + n/2 - 1)\Gamma(n/2 + 1)}. \end{cases}$$

The path in this integral is a loop surrounding both branch points $s=1$ and $s=-r^{-2}$. Contracting the path into a loop around $s=\infty$ and applying the

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* L. Henneberg, ***Über solche Minimalflächen, die eine vorgeschriebene ebene Kurve zur geodätischen Linie haben***, Zürich Dissertation, 1875.

Cauchy theorem immediately gives the polynomial solutions (I, (8.9)). We wish to derive a relation between $f_l^{(1)}(1/r)$, $f_l^{(1)}(r)$ and $f_l^{(2)}(r)$. Replacing in (3.1) r by r^{-1} and introducing as variable

$$t = -\frac{s}{r^2} \tag{3.3}$$

we obtain

$$f_l^{(1)}\left(\frac{1}{r}\right) = (-1)^{l-1}r^{2(l-1)}C_l^{(1)}\int_0^{-1/r^2} t^{l-2}(1-t)^{n/2}(1+r^2t)^{n/2}dt.$$

Combining this again with (3.1) we can obtain an integral from 1 to $-r^{-2}$; but since these are the branch-points of the integrand, such an integral can be transformed into one taken over a contour surrounding the entire branch cut, which is just the path of (3.2). The result is

$$\left\{\begin{aligned} &f_l^{(1)}\left(\frac{1}{r}\right) + (-1)^l r^{2(l-1)}f_l^{(1)}(r) = \alpha_l r^{2(l-1)}f_l^{(2)}(r), \\ &\alpha_l = \frac{\Gamma(l+n/2)\Gamma(l+n/2-1)}{\Gamma(l-1)\Gamma(l+n)}. \end{aligned}\right. \tag{3.4}$$

The corresponding formula for $f_l^{(2)}(r)$ is obtained by replacing in (3.2) r by r^{-1} and introducing the same t as integration variable:

$$f_l^{(2)}\left(\frac{1}{r}\right) = (-1)^l r^{2(l-1)}f_l^{(2)}(r). \tag{3.5}$$

Since, according to I, (8.6a) and the normalization adopted in this paper,

$$\lim_{r\to 0} r^{2l+n-2}f_l^{(2)}(r) = 1,$$

we have on the basis of (3.5)

$$\lim_{r\to 0} r^n f_l^{(2)}\left(\frac{1}{r}\right) = (-1)^l. \tag{3.6}$$

We furthermore note

$$f_l^{(2)}(1) = 0$$

and

$$f_l^{(2)}(1) = (-1)^{(l+n-1)/2}\binom{n/2}{(l+n-1)/2}\binom{n/2}{l+n-1}^{-1}. \tag{3.7}$$

(β) If n is an even number $=2m$, the s-plane is no longer branched. Therefore, while we may keep the definition (3.1) for $f_l^{(1)}$, the singular solution is now to be defined as:

$$\text{(3.8)} \qquad \begin{cases} f_l^{(2)}(r) = \overline{C}_l^{(2)} \displaystyle\int_0^{-1/r^2} s^{l-2}(1-s)^m(1+r^2s)^m ds \\ \overline{C}_l^{(2)} = (-1)^{l+1} \dfrac{(l+2m-1)!}{(l+m-2)!m!}. \end{cases}$$

The multiplicative constant is as always chosen so that the coefficient of the lowest power of r is unity. Introducing into (3.8) the variable t as defined by (3.3), one obtains easily:

$$\text{(3.9)} \qquad \begin{cases} f_l^{(2)}\left(\dfrac{1}{r}\right) = \beta_l r^{2(l-1)} f_l^{(1)}(r), \\ \beta_l = \dfrac{(l+2m-1)!(l-2)!}{(l+m-2)!(l+m-1)!}. \end{cases}$$

This formula is the counterpart of (3.4) for spaces of even dimension number.

Added in proof, December 14, 1936. It is perhaps not superfluous to note that the partial differential equation (4) is essentially that of a homogeneous harmonic function of degree one in an E_{n+1}. Nevertheless the general theory of hyperspherical harmonics is of little help in obtaining the above formulas, for in it only the $n+1$ homogeneous polynomials which have no singularities on the hypersphere are considered, while for our purpose all solutions are needed.

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