ON THE THEOREM OF JORDAN-HÖLDER*

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In a group we have the well known theorem on *principal series*, that any two principal maximal series have the same length and the quotient groups in the two series are isomorphic in some order. In a paper entitled *Über die von drei Moduln erzeugte Dualgruppe*, Dedekind† analyzed the axiomatic foundation of this theorem, particularly the fact that the length of two maximal principal series is the same. He showed that this theorem can be considered as a theorem on *structures* (Dualgruppen), i.e., systems with two operations called union and cross-cut. In order that the theorem shall hold in such a structure it is necessary that it satisfy a further condition which I have called the *Dedekind axiom*. Similar considerations have been made by G. Birkhoff.‡ In a recent paper§ I have shown that in Dedekind structures one can prove a general theorem corresponding to the theorem of Schreier-Zassenhaus|| for principal series in groups. This theorem contains the analogue of the theorem of Jordan-Hölder for Dedekind structures and yields also the fact that the quotients are isomorphic in some order.

All these investigations apply only to Dedekind structures, and give the analogues to the theorems on principal series, i.e., series of sub-groups where each group is a normal sub-group of the full group. They do *not* apply to *composition series* where one only supposes that each term is normal under the preceding. In this paper we shall investigate the possibility of deriving a theory applicable to arbitrary structures and giving an analogue to the theorem of Jordan-Hölder for composition series. The first step is to examine the validity of the analogue to the second theorem of isomorphism (Theorem 1). Next we have to introduce some notion of normality and normal elements. It turns out that two suitable types of normality, α- and β-normality may be

* Presented to the Society, September 5, 1936; received by the editors May 6, 1936.

† Mathematische Annalen, vol. 53 (1900), pp. 371–403; Werke, vol. 2, pp. 236–271.

‡ Proceedings of the Cambridge Philosophical Society, vol. 29 (1933), pp. 441–464; vol. 30 (1934), pp. 115–122, p. 200.

§ Oystein Ore, *On the foundation of abstract algebra*, I, Annals of Mathematics, vol. 36 (1935), pp. 406–437. For further elucidation on the concepts used in the following I shall have to refer to this paper.

|| O. Schreier, *Über den Jordan-Hölderschen Satz*, Abhandlungen aus dem Mathematischen Seminar, Hamburg, vol. 6 (1928), pp. 300–302.

H. Zassenhaus, *Zum Satz von Jordan-Hölder-Schreier*, Abhandlungen aus dem Mathematischen Seminar, Hamburg, vol. 10 (1934), pp. 106–108.

defined, each corresponding to some particular property of the decomposition theorem. For normal sub-groups both properties are always satisfied. The main theorem is then Theorem 7, which gives the analogue of the Schreier-Zassenhaus theorem for composition series. In the last part I discuss the difference between the theorems in structures and the corresponding theorems for groups.

1. **Structures and quotients.** We shall in the following consider an arbitrary *structure* Σ, i.e., an algebraic system consisting of elements $A, B, \cdots$ with an inclusion relation $A > B$ holding for certain pairs of elements. Furthermore, we suppose that to any two elements A and B there exists a *union* $[A, B]$ which is a minimal element of Σ containing A and B and a *cross-cut* (A, B) which is a maximal element contained in A and B. For these symbols we have the ordinary axioms

$$(A, B) = (B, A), \qquad [A, B] = [B, A],$$
$$(A, A) = A, \qquad [A, A] = A,$$
$$(A, (B, C)) = ((A, B), C), \qquad [A, [B, C]] = [[A, B], C],$$
$$[A, (A, B)] = A, \qquad (A, [A, B]) = A.$$

We shall say Σ has a *unit element* E_0 and an *all-element* O_0 if these elements satisfy the relations

$$[A, O_0] = O_0, \qquad (A, E_0) = E_0$$

for all A in Σ.

The structures Σ and Σ' shall be said to be *structure isomorphic* if there exists a one-to-one correspondence $A \rightleftarrows A'$ between the elements of the two structures such that if

$$A \to A', \qquad B \to B'$$

then

$$(A, B) \to (A', B'), \qquad [A, B] \to [A', B'].$$

To any two elements $A > B$ it is convenient to associate a symbol, the *quotient* $\mathfrak{A} = A/B$. These quotients may themselves be made into a structure by defining that for

$$\mathfrak{A}_1 = A_1/B_1, \qquad \mathfrak{A}_2 = A_2/B_2$$

we shall have

$$[\mathfrak{A}_1, \mathfrak{A}_2] = [A_1, A_2]/[B_1, B_2], \qquad (\mathfrak{A}_1, \mathfrak{A}_2) = (A_1, A_2)/(B_1, B_2).$$

We shall usually apply these operations only in the case where $\mathfrak{A}_1$ and $\mathfrak{A}_2$ have the same denominator or the same numerator. Furthermore if

$$\mathfrak{A} = A/B, \qquad \mathfrak{L} = B/C$$

we define the *product*

$$\mathfrak{A} \times \mathfrak{L} = A/B \times B/C = A/C.$$

The existence of a *chain*

$$A_1 \geqq A_2 \geqq \cdots \geqq A_{n+1}$$

may then also be expressed by saying that the quotient

$$\mathfrak{A} = A_1/A_{n+1}$$

has a factorization

$$\mathfrak{A} = \mathfrak{A}_1 \times \mathfrak{A}_2 \times \cdots \times \mathfrak{A}_n, \qquad \mathfrak{A}_i = A_i/A_{i+1}.$$

Furthermore we associate with each quotient $\mathfrak{A} = A/B$ a *quotient structure* consisting of all elements S in Σ satisfying the condition

$$A \geqq S \geqq B.$$

We have formerly studied in detail the so-called *Dedekind structures*, i.e., structures satisfying the

DEDEKIND AXIOM. *For any three elements A, B, C in Σ such that $C > A$ we have*

$$(C, [A, B]) = [A, (C, B)].$$

In the following we shall not suppose that the Dedekind axiom is satisfied. In Dedekind structures important considerations were based upon the notion of *transformation*. A large part of this theory may also be developed in structures not satisfying the Dedekind axiom. We define for any two quotients with the same denominator

$$\mathfrak{A} = A/B, \qquad \mathfrak{T} = C/B \tag{1}$$

the (right-hand) *transform of* $\mathfrak{A}$ *by* $\mathfrak{T}$ to be the quotient

$$\mathfrak{A}' = \mathfrak{T}\mathfrak{A}\mathfrak{T}^{-1} = [\mathfrak{A}, \mathfrak{T}] \times \mathfrak{T}^{-1} = [A, C]/C. \tag{2}$$

We mention without proof that most of the theorems established for transformations in Dedekind structures will also hold in general structures.

As before we shall call (2) an *extension* of $\mathfrak{A}$ by $\mathfrak{T}$ if $\mathfrak{A}$ and $\mathfrak{T}$ in (1) are relatively prime, i.e., if in (1) we have $(A, C) = B$. Conversely, we shall call $\mathfrak{A}$ in (2) a *contraction* of $\mathfrak{A}'$. A series of extensions and contractions shall be called a *similarity transformation* of $\mathfrak{A}$, and $\mathfrak{A}$ and $\mathfrak{B}$ are said to be *similar* when one can be obtained from the other by a similarity transformation.

2. **The second law of isomorphism.** A fundamental result for Dedekind structures was the result that similar quotients were associated with isomorphic quotient structures. The proof for this fact was based upon the fundamental theorem that the two quotient structures

$$\mathfrak{A} = [A, B]/A, \qquad \mathfrak{B} = B/(A, B) \tag{3}$$

were isomorphic. This is the analogue of the so-called second law of isomorphism for groups and ideals.

Let us now determine some condition for the two quotients (3) to be isomorphic even when the Dedekind axiom is not satisfied in Σ. We denote by $\overline{A}$ and $\overline{B}$ arbitrary elements such that

$$[A, B] \geqq \overline{A} \geqq A, \qquad B \geqq \overline{B} \geqq (A, B). \tag{4}$$

One can then easily obtain a correspondence between the two quotients $\mathfrak{A}$ and $\mathfrak{B}$ by putting

$$\overline{A} \rightarrow (\overline{A}, B), \qquad \overline{B} \rightarrow [A, \overline{B}]. \tag{5}$$

We shall call (5) the *regular correspondence* between $\mathfrak{A}$ and $\mathfrak{B}$. We can then prove:

THEOREM 1. *The necessary and sufficient condition for the regular correspondence to establish a structure isomorphism between the quotients* (3) *is that for every* $\overline{A}$ *and* $\overline{B}$ *defined by* (4) *we have*

$$\overline{A} = [A, (B, \overline{A})], \qquad \overline{B} = (B, [A, \overline{B}]). \tag{6}$$

The conditions (6) are obviously necessary and sufficient in order that the regular correspondence be a one-to-one correspondence, one correspondence (5) being the inverse of the other. To prove that it also establishes an isomorphism let

$$\overline{B}_1 \rightarrow \overline{A}_1 = [A, \overline{B}_1], \qquad \overline{B}_2 \rightarrow \overline{A}_2 = [A, \overline{B}_2].$$

Then obviously

$$[\overline{B}_1, \overline{B}_2] \rightarrow [A, [\overline{B}_1, \overline{B}_2]] = [\overline{A}_1, \overline{A}_2].$$

In the same way

$$(\overline{B}_1, \overline{B}_2) \rightarrow [A, (\overline{B}_1, \overline{B}_2)] = [A, (\overline{A}_1, \overline{A}_2, B)] = (\overline{A}_1, \overline{A}_2).$$

Let us observe that this proof also simplifies the demonstration of Theorem 1 in the case of Dedekind structures.

3. **Types of normality.** In order to obtain for arbitrary structures an analogue to the theorem of Jordan-Hölder it is necessary to introduce some notion of *normality* and *normal element.* It is of interest that this may be done in several ways and that each such condition of normality has some particular meaning for the theorem of Jordan-Hölder. We shall begin by defining:

α. An element A_0 contained in M shall be said to be α-normal in M if it satisfies the condition:

For any $B \geqq C$ contained in M we have

$$(B, [A_0, C]) = [C, (B, A_0)]. \tag{7}$$

This condition for α-normality may be formulated in various equivalent ways obtainable from (7) by a suitable choice of B and C. We mention only:

α'. For every B and C contained in M we have

$$\begin{aligned} ([B, C], [A_0, C]) &= [C, (A_0, [B, C])], \\ (B, [A_0, (B, C)]) &= [(B, C), (A_0, B)], \\ ([B, C], [A_0, (B, C)]) &= [(B, C), (A_0, [B, C])]. \end{aligned} \tag{8}$$

Furthermore:

α''. If $B \geqq C$ are contained in M and

$$[A_0, C] = [A_0, B], \qquad (A_0, C) = (A_0, B),$$

then we can conclude $B = C$.

We may also define α-normality of A_0 with respect to elements not containing A_0 in the following manner:

An element A_0 is said to be α-normal with respect to B if A_0 is α-normal in $[B, A_0]$.

Obviously if A_0 is α-normal with respect to B it is α-normal with respect to any element contained in B. We can now prove

THEOREM 2. *If A_0 is α-normal with respect to B, i.e., A_0 is α-normal in $[A_0, B]$, then (A_0, B) is α-normal in B.*

Let B_1 and B_2 be any elements such that

$$B \geqq B_1 \geqq B_2.$$

We find then

$$(B_1, [B_2, (B, A_0)]) = (B_1, [B_2, A_0]) = [B_2, (B_1, A_0)] = [B_2, (B_1, (B, A_0))]$$

and our theorem is proved.

A second type of normality may be introduced as follows:

β. An element A_0 contained in M shall be said to be β-normal in M when it satisfies the condition:

For any B and C contained in M such that $B \geqq A_0$ we shall have

$$(B, [A_0, C]) = [A_0, (B, C)]. \tag{9}$$

This condition for β-normality may again be stated in other equivalent forms, for instance:

β'. For any B and C contained in M we have

$$([B, A_0], [C, A_0]) = [A_0, (B, [A_0, C])] = [A_0, (C, [A_0, B])].$$

Also here we can define normality of A_0 with respect to an arbitrary element.

THEOREM 3. *If A_0 and A_1 are both β-normal in M then $[A_0, A_1]$ has the same property.*

If, namely, $B \geqq [A_0, A_1]$ then

$$(B, [A_0, A_1, C]) = [A_0, (B, [A_1, C])] = [A_0, A_1, (B, C)].$$

4. **Semi-normality.** We shall now join the two notions of α-normality and β-normality and define:

Semi-normality. An element A_0 is semi-normal in M if it is both α-normal and β-normal in M.

We say further that A_0 is semi-normal with respect to B if it is semi-normal in $[B, A_0]$.

THEOREM 4. *The necessary and sufficient condition for A_0 to be semi-normal in M is that for any B and C in M we have*

$$[(A_0, [B, C]), (C, [B, A_0])] = ([A_0, B], [A_0, C], [B, C]). \tag{10}$$

We obtain namely by supposing $B > A_0$

$$(B, [A_0, C]) = [A_0, (B, C)]$$

and by supposing $B > C$

$$(B, [A_0, C]) = [C, (B, A_0)],$$

and conversely these relations suffice to derive (10).

The principal theorem on semi-normality is:

THEOREM 5. *When A_0 is semi-normal with respect to B; i.e., A_0 is semi-normal in $[B, A_0]$, then (B, A_0) is semi-normal in B and the two quotients*

$$\mathfrak{A} = [A_0, B]/A_0, \qquad \mathfrak{K} = B/(A_0, B), \tag{11}$$

are structure isomorphic.

It follows from Theorem 2 that (A_0, B) is α-normal in B. The β-normality implies that for any B_1 and B_2 in B such that $B_1 \geqq (A_0, B)$ we shall have

$$(B_1, [(A_0, B), B_2]) = [(A_0, B), (B_1, B_2)]. \tag{12}$$

To prove this relation we observe that on account of the α-normality of A_0 we have

$$B_1 = (B, [A_0, B_1])$$

and hence

$$[(A_0, B), (B_1, B_2)] = [(A_0, B), (B_2, [A_0, B_1])] = (B, [A_0, (B_2, [A_0, B_1])]).$$

The β-normality shows that this last expression is identical with

$$(B, [A_0, B_1], [A_0, B_2]) = (B_1, [A_0, B_2]) = (B_1, [(A_0, B), B_2])$$

and (12) is proved.

The structure isomorphism between the two quotients (11) follows from Theorem 1. For any A_1 such that

$$[A_0, B] \geqq A_1 \geqq A_0$$

we have, namely,

$$[A_0, (A_1, B)] = (A_1, [A_0, B]) = A_1,$$

and similarly for any B_1 such that

$$B \geqq B_1 \geqq (A_0, B)$$

we have

$$(B, [A_0, B_1]) = [B_1, (B_1, A_0)] = B_1.$$

5. **The theorem of Schreier-Zassenhaus.** We shall now consider the possibility of extending the theorem of Jordan-Hölder, or rather its generalization by Schreier-Zassenhaus, to an arbitrary structure Σ. Let us suppose that we have two descending chains between two elements A and B,

$$\begin{aligned} &A > B_1 > \cdots > B_{r-1} > B, \\ &A > C_1 > \cdots > C_{s-1} > B, \end{aligned} \tag{13}$$

or in the terminology of §1, that the quotient $\mathfrak{A} = A/B$ has two product representations

$$\mathfrak{A} = \mathfrak{B}_1 \times \mathfrak{B}_2 \times \cdots \times \mathfrak{B}_r = \mathfrak{C}_1 \times \mathfrak{C}_2 \times \cdots \times \mathfrak{C}_s \tag{14}$$

where

$$\mathfrak{B}_i = B_{i-1}/B_i, \qquad \mathfrak{C}_j = C_{j-1}/C_j.$$

We can now prove

THEOREM 6. *Let there exist two sequences* (13) *between A and B where each term is α-normal under the preceding. In the corresponding factorizations* (14) *of the quotient* $\mathfrak{A} = A/B$ *it is then possible to decompose the factors further*

$$\mathfrak{B}_i = \mathfrak{B}_{i,1} \times \mathfrak{B}_{i,2} \times \cdots \times \mathfrak{B}_{i,s}, \qquad \mathfrak{C}_j = \mathfrak{C}_{j,1} \times \mathfrak{C}_{j,2} \times \cdots \times \mathfrak{C}_{j,r} \tag{15}$$

such that the two factors $\mathfrak{B}_{i,j}$ *and* $\mathfrak{C}_{j,i}$ *correspond in the manner that they may both be obtained by extension from the same quotient* $\mathfrak{R}_{i,j}$.

We put

$$\begin{aligned} \mathfrak{B}_{i,j} &= [B_i, (B_{i-1}, C_{j-1})]/[B_i, (B_{i-1}, C_j)], \\ \mathfrak{C}_{j,i} &= [C_i, (C_{i-1}, B_{j-1})]/[C_i, (C_{i-1}, B_j)] \end{aligned} \tag{16}$$

and write

$$L = [B_i, (B_{i-1}, C_j)], \qquad M = (B_{i-1}, C_{j-1}).$$

We then find

$$[L, M] = [B_i, (B_{i-1}, C_{j-1})], \qquad \mathfrak{B}_{i,j} = [L, M]/L$$

and on account of the α-normality of B_i in B_{i-1} we have

$$(L, M) = (B_{i-1}, C_{j-1}, [B_i, (B_{i-1}, C_j)]) = [(B_{i-1}, C_j), (B_i, C_{j-1})].$$

This shows that $\mathfrak{B}_{i,j}$ may be obtained by extension from

$$\mathfrak{R}_{i,j} = M/(L, M) = (B_{i-1}, C_{j-1})/[(B_{i-1}, C_j), (B_i, C_{j-1})],$$

and since similar considerations show that $\mathfrak{C}_{j,i}$ may be obtained by extension from $\mathfrak{R}_{i,j}$, our theorem is proved.

We also mention without proof that, when α-normality is assumed in the sequences (13), repeated applications of the decompositions (16) yield no new factorizations.

Under the assumption of semi-normality we can prove the more exact theorem:

THEOREM 7. *Let there exist two chains* (13) *between A and B such that each term is semi-normal under the preceding. The corresponding factorizations* (14) *of* $\mathfrak{A} = A/B$ *may then be factored further into* (15) *such that the new factors* $\mathfrak{B}_{i,j}$ *and* $\mathfrak{C}_{j,i}$ *have isomorphic quotient structures.*

Since we have

$$\mathfrak{B}_{i,j} = [L, M]/L, \qquad \mathfrak{R}_{i,j} = M/(M, L),$$

it follows from Theorem 5 that $\mathfrak{B}_{i,j}$ and $\mathfrak{R}_{i,j}$ are structure isomorphic. Since the same holds for $\mathfrak{R}_{i,j}$ and $\mathfrak{C}_{j,i}$ our theorem is proved. We can obtain the explicit correspondence between $\mathfrak{B}_{i,j}$ and $\mathfrak{C}_{j,i}$ in the following way: Let $\overline{B}$ and $\overline{C}$ be arbitrary elements such that

$$[B_i, (B_{i-1}, C_{j-1})] \geqq \overline{B} \geqq [B_i, (B_{i-1}, C_j)],$$
$$[C_j, (C_{j-1}, B_{i-1})] \geqq \overline{C} \geqq [C_j, (C_{j-1}, B_i)].$$

We obtain the correspondence by putting

$$\overline{B} \rightarrow [C_j, (C_{j-1}, \overline{B})], \qquad \overline{C} \rightarrow [B_i, (B_{i-1}, \overline{C})].$$

6. **Comparison with normal sub-groups.** In the preceding we have tried to derive results similar to those known in the case of normal sub-groups. The main theorem, 7, is almost identical with the theorem of Schreier-Zassenhaus for groups.

Let us now take the opposite view of the matter and consider the difference of our results from the group theorems. To any two semi-normal chains (13) in a structure we have constructed new chains by intercalation such that the quotients in the two new chains are structure isomorphic in pairs. Due to our weak condition of semi-normality we can not, however, prove that in the new decompositions each term is semi-normal under the preceding.

In the case of groups this deficiency is easily remedied. To show that the group

$$D_{i,j-1} = [B_i, (B_{i-1}, C_{j-1})]$$

contains the normal sub-group

$$D_{i,j} = [B_i, (B_{i-1}, C_j)]$$

when B_i is normal in B_{i-1} and C_j normal in C_{j-1}, it is only necessary to show that B_i and (B_{i-1}, C_j) are transformed into themselves by transformation with elements in B_i and (B_{i-1}, C_j). This follows directly from the definition of normality in groups.

On account of this difference Theorem 7 may not be specialized into the analogue of the theorem of Jordan-Hölder by supposing that in the chains (13) each term is maximal semi-normal under the preceding. Let us try to determine however some conditions under which the theorem of Jordan-Hölder is valid for structures.

In order to do this, let us consider the structure formed by *all* sub-groups of a given group $\mathfrak{G}$. We then have:

I. The set of normal sub-groups of a given sub-group M forms a Dedekind structure.

II. For a given sub-group A those sub-groups M in which A is normal form a structure.

III. Let A be normal in $[A, B]$. The structure isomorphism

$$[A, B]/A \cong B/(A, B)$$

is also a one-to-one correspondence between the normal sub-groups of $[A, B]$ containing A and the normal sub-groups of B containing (A, B).

None of these properties is ordinarily satisfied for semi-normal elements in structures. For instance, the set of semi-normal elements in M usually does not even form a structure. With regard to III one can prove that every semi-normal element in $[A, B]/A$ corresponds to a semi-normal element in $B/(A, B)$ but not conversely.

For the proof of an analogue to the theorem of Jordan-Hölder one needs III and a part of I, namely, that the union of two semi-normal elements in M is again semi-normal. If one then has two series (13) in which each term is maximal semi-normal in the preceding, the ordinary inductional proof carries through without difficulty. It is an interesting fact that one does not need all the properties of the normal sub-groups and I shall use this fact to prove in a following paper a new theorem of Jordan-Hölder which is valid also for certain classes of non-normal sub-groups.

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