A PRIORI LIMITATIONS FOR SOLUTIONS OF MONGE-AMPÈRE EQUATIONS. II*

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In this paper we are concerned with the convergence of solutions of elliptic and analytic Monge-Ampère equations. Theorem 1 gives the principal result of this paper. The example on p. 372 indicates the possibility of certain types of singularities which for linear elliptic equations cannot occur. Theorems 2 and 3 give sufficient conditions for the analyticity of the limit function. These conditions allow applications to certain problems of the differential geometry in the large. Our method consists in introducing a regularizing contact transformation which transforms convex functions into functions with bounded second derivatives and thus makes possible the reduction of Theorem 1 to the principal result of the first part of this paper.

1. **Regularizing contact transformation.** Consider the contact transformation T of an (x, y, z)-space into a (ξ, η, ζ)-space generated by the relation

$$0 = z + \zeta - x\xi - y\eta + \frac{x^2 + y^2}{2}.$$

It leads to the following transformation of a surface $z = z(x, y)$ with continuous first derivatives z_x and z_y

$$\xi = x + z_x(x, y) \qquad\qquad (1)$$
$$\eta = y + z_y(x, y)$$

$$\zeta = -z + x\xi + y\eta - (x^2 + y^2)/2. \qquad\qquad (2)$$

Suppose $z(x, y)$ convex. Then (1) transforms an open rectangle R' of the (x, y)-plane into a domain of the (ξ, η)-plane in a one-to-one way.

For let (ξ_1, η_1) and (ξ_2, η_2) be the images of two distinct points (x_1, y_1) and (x_2, y_2), ϑ the angle between the vector joining (x_1, y_1) to (x_2, y_2) and the positive x-axis so that the distance between these two points is

$$(x_2 - x_1)\cos\vartheta + (y_2 - y_1)\sin\vartheta > 0.$$

As the derivative of z in the direction of the above vector is $z_x \cos\vartheta + z_y \sin\vartheta$, we conclude from the convexity of $z(x, y)$

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$$z_x(x_1, y_1) \cos \vartheta + z_y(x_1, y_1) \sin \vartheta \leqq z_x(x_2, y_2) \cos \vartheta + z_y(x_2, y_2) \sin \vartheta.$$

Hence

$$0 < (x_2 - x_1) \cos \vartheta + (y_2 - y_1) \sin \vartheta \leqq (\xi_2 - \xi_1) \cos \vartheta + (\eta_2 - \eta_1) \sin \vartheta$$

which implies

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 \leqq (\xi_2 - \xi_1)^2 + (\eta_2 - \eta_1)^2. \tag{3}$$

Using (3) for differentiation of (2) we obtain

$$x = \frac{\partial \zeta}{\partial \xi}, \qquad y = \frac{\partial \zeta}{\partial \eta} \tag{4}$$

which together with (3) shows that ζ has continuous derivatives in ξ and η.

If, moreover, z has continuous second derivatives with respect to (x, y), then ζ has continuous second derivatives in ξ and η all of which are numerically $\leqq 1$. For, by (1), the partial derivatives

$$\xi_x = 1 + z_{xx}, \quad \eta_x = \xi_y = z_{xy}, \quad \eta_y = 1 + z_{yy},$$

are continuous in (x, y). Hence the derivatives x_ξ, x_η, y_ξ, y_η of the inverse functions are continuous in (ξ, η) and we have†

$$\begin{aligned} \frac{\partial^2 \zeta}{\partial \xi^2} &= x_\xi = \frac{1 + z_{yy}}{(1 + z_{xx})(1 + z_{yy}) - z_{xy}^2}, \\ \frac{\partial^2 \zeta}{\partial \xi \partial \eta} &= \frac{- z_{xy}}{(1 + z_{xx})(1 + z_{yy}) - z_{xy}^2}, \cdots, \end{aligned} \tag{5}$$

$$\left|\frac{\partial^2 \zeta}{\partial \xi^2}\right|, \quad \left|\frac{\partial^2 \zeta}{\partial \xi \partial \eta}\right|, \quad \left|\frac{\partial^2 \zeta}{\partial \eta^2}\right| \leqq 1, \qquad \frac{\partial^2 \zeta}{\partial \xi^2}\frac{\partial^2 \zeta}{\partial \eta^2} - \left(\frac{\partial^2 \zeta}{\partial \xi \partial \eta}\right)^2 > 0. \tag{6}$$

Now consider a bounded sequence of continuously differentiable convex functions $z_n(x, y)$ defined in R'. Designate by T_n the transformation of a concentric open rectangle R, contained in R', into the (ξ, η)-plane as induced by T for the function $z_n(x, y)$, by T_n^{-1} the inverse of T_n. We proceed to show that there exists a subsequence of n such that

(i) $z_n(x, y)$ converge uniformly in R;

(ii) there exists an open set D in the (ξ, η)-plane contained in all $T_m(R)$ for m sufficiently large;

(iii) the inverse T_m^{-1} converges uniformly in D to a transformation T_∞^{-1};

(iv) $T_\infty^{-1}(D)$ contains a neighborhood of the center of R, and all those

† The denominator in (5) is >0 since the convexity of $z(x, y)$ implies $z_{xx} \geqq 0$, $z_{yy} \geqq 0$, $z_{xx}z_{yy} - z_{xy}^2 \geqq 0$.

points (ξ, η) which by T_∞^{-1} are mapped into the center of R have no limit point on the boundary of D.

The functions $z_n(x, y)$ are uniformly bounded and convex in R', hence equally continuous, so that (i) is satisfied and we have for $m>n$ in R

$$\left| z_m(x, y) - z_n(x, y) \right| < \epsilon_n$$

with $\epsilon_n \to 0$ as $n \to \infty$. If (x_0, y_0) is an arbitrary point of R, a circle γ_n about (x_0, y_0) with radius $\rho_n = 2(\epsilon_n)^{1/2}$ lies for sufficiently large n within R.

Denote by (ξ_n, η_n) the image of (x_0, y_0) in T_n, by ϑ the angle of the vector from (x_0, y_0) to a variable point (x_n', y_n') of the circumference of γ_n and the positive x-axis and by (ξ_m', η_m') the image of (x_n', y_n') in T_m. We have

$$\begin{aligned} z_{nx}(x_0, y_0)\cos\vartheta + z_{ny}(x_0, y_0)\sin\vartheta &\leqq \frac{z_n(x_n', y_n') - z_n(x_0, y_0)}{\rho_n} \\ &\leqq \frac{z_m(x_n', y_n') - z_m(x_0, y_0)}{\rho_n} + \frac{2\epsilon_n}{\rho_n} \\ &\leqq z_{mx}(x_n', y_n')\cos\vartheta + z_{my}(x_n', y_n')\sin\vartheta + \frac{2\epsilon_n}{\rho_n}. \end{aligned}$$

Hence

$$\xi_n \cos\vartheta + \eta_n \sin\vartheta < \xi_m' \cos\vartheta + \eta_m' \sin\vartheta.$$

Thus as ϑ varies from 0 to 2π the projection of the vector $(\xi_m' - \xi_n, \eta_m' - \eta_n)$ on the vector $(\cos\vartheta, \sin\vartheta)$ is positive or the angle between the two vectors is $<\pi/2$, thereby implying that the vector $(\xi_m' - \xi_n, \eta_m' - \eta_n)$ turns by 2π as the point (ξ_m', η_m') describes the image in T_m of the circumference of γ_n. Hence, for all $m>n$, (ξ_n, η_n) lies within $T_m(\gamma_n)$. Denoting by R_n the set of all those points of R whose distance from the boundary of R is $>\rho_n$, we may say that $T_m(R)$ contains $T_n(R_n)$ for $m>n$. Thus (ii) is proved with $D = T_n(R_n)$.

As the transformations T_m^{-1} are bounded and by (3) equicontinuous for all m we may pick out a suitable subsequence satisfying (iii).

Setting $(\xi_m, \eta_m) = T_m(x_0, y_0)$ and $(\xi_n', \eta_n') = T_n(x_n', y_n')$ for (x_n', y_n') on γ_n, we obtain in a way similar to the one used above

$$\xi_m \cos\vartheta + \eta_m \sin\vartheta < \xi_n' \cos\vartheta + \eta_n' \sin\vartheta$$

from which we conclude that for $m>n$ the point (ξ_m, η_m) lies in $T_n(\gamma_n)$. Now take n large enough so that there exists a point (x_0, y_0) of R_n at distance $>\rho_n$ from the boundary of R_n. As the corresponding points (ξ_m, η_m) lie all in $T_n(\gamma_n)$ there exists a converging subsequence tending to a point (ξ_0, η_0). In view of (3), the distance between $T_m^{-1}(\xi_m, \eta_m)$ and $T_m^{-1}(\xi_0, \eta_0)$ tends to zero if the distance between (ξ_m, η_m) and (ξ_0, η_0) tends to zero. Hence for our subsequence $T_m^{-1}(\xi_0, \eta_0) \rightarrow (x_0, y_0)$, or (x_0, y_0) is the image in T_∞^{-1} of a point (ξ_0, η_0) of $T_n(R_n) = D$. Furthermore, as T_m^{-1} maps $T_n(R_n)$ into a domain containing all points (x_0, y_0) of R_n at distance $>\rho_n$ from the boundary of R_n and as the mapping is one-to-one it follows that the boundary of $T_n(R_n)$ is mapped by T_m^{-1} into a set of points whose distance from the boundary of R_n is $\leqq \rho_n$. In view of the uniform convergence we conclude that T_∞^{-1} maps the boundary of $T_n(R_n)$ into a set of points distinct from the center of R. At the same time $T_\infty^{-1}T_n(R_n)$ contains a neighborhood of the center of R, namely, the set of points at distance $>\rho_n$ from the boundary of R_n, which proves (iv).

From (2) we derive by passing to the limit for $T_\infty^{-1}(\xi, \eta) = (x, y)$

$$\lim_{n\to\infty} \zeta_n(\xi, \eta) = \lim_{n\to\infty} (-z_n(x, y) + x\xi + y\eta - (x^2 + y^2)/2);$$

by (4) the derivatives of $\zeta(\xi, \eta)$ converge uniformly, hence evidently

$$\frac{\partial \zeta_n}{\partial \xi} \rightarrow \frac{\partial \lim \zeta_n}{\partial \xi}, \quad \frac{\partial \zeta_n}{\partial \eta} \rightarrow \frac{\partial \lim \zeta_n}{\partial \eta}.$$

If the limit function ζ is analytic in ξ and η, the map T_∞^{-1} is "schlicht" in the neighborhood of a point Q such that $T_\infty^{-1}(Q)$ is a point c near the center of R.

To prove this, let us assume the existence of another point Q' with $T_\infty^{-1}(Q') = c$. Call, for n in the above subsequence, σ_n the segment joining $T_n^{-1}(Q)$ to $T_n^{-1}(Q')$, and k_ρ a circle of small, arbitrary radius $\rho > 0$ about Q and lying in D. The intersections of k_ρ with $T_n(\sigma_n)$ have at least one limit point L and we conclude from the uniform convergence of T_n^{-1} to T_∞^{-1} that $T_\infty^{-1}(L) = c$. Thus it is shown that if there are two points in D, Q and Q', such that $T_\infty^{-1}(Q) = T_\infty^{-1}(Q') = c$, then there exists an infinity of such points. Since the functions $x(\xi, \eta)$ and $y(\xi, \eta)$ determined by the transformation T_∞^{-1} are analytic, there exists a real curve C such that the two equations hold

$$(*) \qquad [x(\xi, \eta), y(\xi, \eta)] = c.$$

All points (ξ, η) such that (*) is satisfied lie within a closed set interior to D because T_∞^{-1} on the boundary of D differs from c as formerly proved. Taking local Puiseux developments it is easy to show that the above analytic curve has at each point a finite number of tangents and that no point of it is a "dead end," i.e., endpoint of one arc and of no other. Furthermore (*) implies that there are at most a finite number of singular points of C in D. Let us choose an arbitrary regular point on C and introduce an orientation on C in the neighborhood of this point. Continue the orientation on the same branch of the curve until we arrive at a singular point. Among the several arcs ending at the singular point we pick one distinct from the one already oriented and continue the orientation on this arc. Proceeding this way we must finally encounter a point twice, as the length of our path is bounded on account of the finiteness of the number of singular points. Hence there exists a closed circuit δ consisting only of arcs of C, in particular one bounding a simply connected domain D'. Now $T_n^{-1}(\delta)$ is the boundary of a simply connected domain $T_n^{-1}(D')$ and $T_n^{-1}(\delta)\rightarrow c$. Hence $T_n^{-1}(D')\rightarrow c$, $T_\infty^{-1}(D')=c$ which is impossible since T_∞^{-1} is analytic and does not map identically into c.

2. We first prove the following theorem.

THEOREM 1. *Let A, B, C, E be analytic functions of five complex arguments u, v, x, p, q in a bounded and closed neighborhood N of a real point u_0, v_0, x_0, p_0, q_0 and suppose A, B, C, E depending on a parameter μ and converging uniformly in N as $\mu\rightarrow\mu^*$. Assume A, B, C, E real for real u, v, x, p, q,*

$$\Delta \equiv 4(AC + E) - B^2 > 0,$$

and $|\Delta|^{-1}$ uniformly bounded in N. Suppose that there exists a rectangle R with u_0, v_0 as center and such that there exists for $\mu\neq\mu^*$ a real and analytic function $x(u, v)$ which is a solution of the Monge-Ampère equation†

$$Ar + Bs + Ct + (rt - s^2) = E; \tag{7}$$

suppose that for this solution (u, v, x, p, q) remains in N, when (u, v) ranges over R. Then there exists a real neighborhood of (u_0, v_0) such that the solutions $x(u, v)$ corresponding to a subsequence of values of μ converge uniformly to a limit function $x^*(u, v)$ as $\mu\rightarrow\mu^*$. There exists an analytic relation between u, v, x^*, and $x^*(u, v)$ has continuous first derivatives. Moreover, $x^*(u, v)$ is analytic in u and v and satisfies the limit relation (7) for $\mu\rightarrow\mu^*$ at each point where a certain not identically vanishing analytic function $G(u, v)$ does not vanish.

Since

$$0 < \Delta = 4(A + t)(C + r) - (B - 2s)^2,$$

we have $A+t$ and $C+r$ for each solution always $\neq 0$ and of the same sign. Hence in the sequence, there are infinitely many solutions for which $A+t$ is of the same sign, and we may even assume $A+t$ positive since in the opposite case we may replace A, B, C, E, and $x(u, v)$ by $-A, -B, -C, +E$, and $-x(u, v)$ without changing hypothesis nor conclusion of the theorem.

Let, for all μ of the subsequence,

$$|A|,\ |C|,\ \left|\frac{B}{2}\right| \leqq a.$$

† Here p, q, r, s, t denote, as usual, the partial derivatives of $x(u, v)$ of the first and second orders, respectively.

We find

$$0 < (r + C)(t + A) - \left(\frac{B}{2} - s\right)^2$$
$$\leqq (r + a)(t + a) - s^2 + \frac{B^2}{4} - B\left(\frac{B}{2} - s\right)$$
$$\leqq (r + a)(t + a) - s^2 + a^2 + 2a\,\frac{r + a + t + a}{2}$$
$$= (r + 2a)(t + 2a) - s^2.$$

Hence $x(u, v)+a(u^2+v^2)$ is convex in R, and satisfies the transformed Monge-Ampère equation with the same discriminant Δ. It is therefore legitimate to *assume henceforth, that for all* μ *of a subsequence* $x(u, v)$ *is convex*. Let us then apply the contact transformation T of the (u, v, x)-space into a (ξ, η, ζ)-space. The Monge-Ampère equation thereby is transformed into another,

$$\tilde{A}\tilde{r} + \tilde{B}\tilde{s} + \tilde{C}\tilde{t} + \tilde{D}(\tilde{r}\tilde{t} - \tilde{s}^2) = \tilde{E}, \tag{8}$$

with

$$\tilde{A} = C - 1,\ \tilde{C} = A - 1,\ \tilde{B} = -B,\ \tilde{D} = 1 - E - A - C,\ \tilde{E} = -1. \tag{9}$$

The discriminant of the new equation equals that of the former:

$$4(\tilde{A}\tilde{C} + \tilde{D}\tilde{E}) - \tilde{B}^2 = 4(AC + E) - B^2 = \Delta.$$

The solution $\zeta(\xi, \eta)$ exists for all μ of a suitable subsequence in a common domain D of the (ξ, η)-plane and converges with its first derivatives to a limit function ζ^* and its derivatives. Let ξ_0, η_0 be an arbitrary, but fixed point in D, π_0 and κ_0 the derivatives of $\zeta^*(\xi, \eta)$ at (ξ_0, η_0). Replace $\zeta(\xi, \eta)$ by $\zeta(\xi, \eta)-\pi_0(\xi-\xi_0)-\kappa_0(\eta-\eta_0)=\bar{\zeta}(\xi, \eta)$. Then

$$\tilde{A}\frac{\partial^2\bar{\zeta}}{\partial\xi^2} + \tilde{B}\frac{\partial^2\bar{\zeta}}{\partial\xi\partial\eta} + \tilde{C}\frac{\partial^2\bar{\zeta}}{\partial\eta^2} + \tilde{D}\left(\frac{\partial^2\bar{\zeta}}{\partial\xi^2}\frac{\partial^2\bar{\zeta}}{\partial\eta^2} - \left(\frac{\partial^2\bar{\zeta}}{\partial\xi\partial\eta}\right)^2\right) = \tilde{E} \tag{10}$$

and

$$u = \frac{\partial\bar{\zeta}}{\partial\xi} + \pi_0, \qquad v = \frac{\partial\bar{\zeta}}{\partial\eta} + \kappa_0,$$
$$x = -\bar{\zeta} - \pi_0(\xi - \xi_0) - \kappa_0(\eta - \eta_0) + \xi\left(\frac{\partial\bar{\zeta}}{\partial\xi} + \pi_0\right) + \eta\left(\frac{\partial\bar{\zeta}}{\partial\eta} + \kappa_0\right)$$
$$- \frac{1}{2}\left[\left(\frac{\partial\bar{\zeta}}{\partial\xi} + \pi_0\right)^2 + \left(\frac{\partial\bar{\zeta}}{\partial\eta} + \kappa_0\right)^2\right],$$

$$p = \xi - \frac{\partial \bar{\zeta}}{\partial \xi} - \pi_0, \qquad q = \eta - \frac{\partial \bar{\zeta}}{\partial \eta} - \kappa_0.$$

For all μ of the subsequence we may assume

$$\left(\frac{\partial \bar{\zeta}(\xi_0, \eta_0)}{\partial \xi}\right)^2 + \left(\frac{\partial \bar{\zeta}(\xi_0, \eta_0)}{\partial \eta}\right)^2 \leqq 1,$$

and our formulas show that the coefficients $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$, $\tilde{E}$ considered as analytic functions of five independent variables $\xi, \eta, \bar{\zeta}, \partial\bar{\zeta}/\partial\xi, \partial\bar{\zeta}/\partial\eta$ (instead of u, v, x, p, q) exist in a complex neighborhood of ξ_0, η_0, $\bar{\zeta} = \zeta^*(\xi_0, \eta_0)$, $\partial\bar{\zeta}/\partial\xi = 0$, $\partial\bar{\zeta}/\partial\eta = 0$ and tend there uniformly to limit functions as $\mu \to \mu^*$.

By (6) we see that for all μ the second derivatives of our solution $\bar{\zeta}(\xi, \eta)$ are absolutely bounded by 1. Hence there exists a neighborhood of the point (ξ_0, η_0) of the (ξ, η)-plane in which, by the theorem on p. 432 of Part I of our paper, all $\bar{\zeta}(\xi, \eta)$ can be developed into power series in $\xi - \xi_0$ and $\eta - \eta_0$ such that there is a common majorant series for all μ which converges. Hence the limit $\zeta^*(\xi, \eta)$ of $\bar{\zeta}(\xi, \eta)$ is analytic in a neighborhood of ξ_0, η_0 and the same holds in the neighborhood of *any* interior point of D. As we have seen in §1 the transformation

$$u = \frac{\partial \zeta^*}{\partial \xi}, \qquad v = \frac{\partial \zeta^*}{\partial \eta}$$

maps the domain D into a set containing a neighborhood of the point (u_0, v_0) and there exists $\lim_{\mu \to \mu^*} x(u, v) = x^*(u, v) = -\zeta^*(\xi, \eta) + u\xi + \eta v - (u^2 + v^2)/2$. Thus it is shown that for a certain neighborhood of (u_0, v_0) the quantities u, v, x^* can be written as analytic functions of two variables ξ, η. Wherever the determinant $\partial(u, v)/\partial(\xi, \eta)$ differs from zero we may introduce u and v instead of ξ and η, and x^* becomes analytic in u and v. The relation $\partial(u, v)/\partial(\xi, \eta) = 0$ cannot be identically satisfied as the mapping $(\xi, \eta) \to (u, v)$ contains a neighborhood of (u_0, v_0). For all points with $\partial(u, v)/\partial(\xi, \eta) \neq 0$ we may differentiate with respect to u, v and find $\partial x^*(u, v)/\partial u = \xi - \partial\zeta^*/\partial\xi$, $\partial x^*/\partial v = \eta - \partial\zeta^*/\partial\eta$. Hence $\partial x^*/\partial u$ and $\partial x^*/\partial v$ are analytic functions of ξ, η and therefore continuous, even at points where $\partial(u, v)/\partial(\xi, \eta) = 0$ (since derivative at limit point equals limit of derivative). The proof of the theorem is now easily completed in view of the fact that the inverse transform of (8) is (7).

The following example shows that there are cases where there exists an analytic relation between u, v, x in a neighborhood of the origin, the function x can be considered as dependent on u and v and having continuous first derivatives there, and $x(u, v)$ is a solution of an analytic and elliptic Monge-

Ampère equation everywhere except on the u-axis where it fails to have second derivatives at all. The function is

$$x(u, v) = \frac{(3u)^{4/3}}{4} + \frac{v^2}{2},$$

the equation is

$$(-1 + p^2)r + rt - s^2 = 1.$$

3. There are wide classes of Monge-Ampère equations important in differential geometry such that the function $\partial(u, v)/\partial(\xi, \eta)$ cannot vanish at any interior point (u_0, v_0). In this case the limit function $x^*(u, v)$ is necessarily analytic.

Let us introduce two new variables α, β and solve the following initial problem for our variables ξ, η to be considered as functions of α and β:

$$(11) \qquad \left(\frac{\partial\eta}{\partial\alpha} + i\frac{\partial\eta}{\partial\beta}\right)(\tilde{B} - 2\tilde{D}\tilde{s} + i\Delta^{1/2}) = \left(\frac{\partial\xi}{\partial\alpha} + i\frac{\partial\xi}{\partial\beta}\right)\cdot 2(\tilde{C} + \tilde{D}\tilde{r}),$$

$$\xi(\alpha, 0) = \alpha, \qquad \eta(\alpha, 0) = 0.$$

By the theorem of Cauchy-Kowalewski we have a solution $\xi(\alpha, \beta)$, $\eta(\alpha, \beta)$, analytic in the neighborhood of the origin. We have at the origin

$$\frac{\partial(\xi, \eta)}{\partial(\alpha, \beta)} = \frac{\partial\eta}{\partial\beta} = -2i(\tilde{C} + \tilde{D}\tilde{r})\frac{\left(1 + i\dfrac{\partial\xi}{\partial\beta}\right)}{(\tilde{B} - 2\tilde{D}\tilde{s} + i\Delta^{1/2})}$$

and hence may assert that there exists a neighborhood of the origin for which $\partial(\xi, \eta)/\partial(\alpha, \beta) \neq 0$ and the mapping $(\alpha, \beta) \to (\xi, \eta)$ is one-to-one. (11) may be written, with $\quad = \partial/\partial\alpha + i\partial/\partial\beta$

$$(12) \qquad (\tilde{B} + i\Delta^{1/2})\nabla\eta - 2\tilde{C}\nabla\xi - 2\tilde{D}\nabla\tilde{\zeta}_\xi = 0.$$

With the aid of $\tilde{\zeta}_\xi = u$, $\xi = u + p^*$, $\eta = v + q^*$ we obtain, in view of (9), for $\partial(u, v)/\partial(\xi, \eta) \neq 0$,

$$[(-B + i\Delta^{1/2})(1 + t^*) - 2(A - 1)s^*]\nabla v + [(-B + i\Delta^{1/2})s^* \\ - 2(A - 1)(r^* + 1) - 2(1 - E - A - C)]\nabla u = 0,$$

and this is seen to be equivalent to

$$(13') \qquad 2(A + t^*)\nabla v + (-B + 2s^* - i\Delta^{1/2})\nabla u = 0$$

or

$$(13) \qquad (-B - i\Delta^{1/2})\nabla u + 2A\nabla v + 2\nabla q^* = 0.$$

From (13′) we have

$$(-B + i\Delta^{1/2})\nabla v + 2C\nabla u + 2\nabla p^* = 0, \tag{14}$$

in view of

$$\Delta = 4(A + t^*)(C + r^*) - (B - 2s^*)^2.$$

Now (ξ, η) is analytic in (α, β), and u, v, p^*, q^*, x^* are analytic in (ξ, η), hence also in (α, β) and the relations (13), (14) hold independently of $\partial(u, v)/\partial(\xi, \eta) \neq 0$. We have moreover

$$\nabla x^* - p^*\nabla u - q^*\nabla v = 0. \tag{15}$$

(13), (14), (15) form the characteristic equations of (7) for the limit function $x^*(u, v)$. The mapping $(\alpha, \beta) \to (u, v)$ is one-to-one as both $(\alpha, \beta) \to (\xi, \eta)$ and $(\xi, \eta) \to (u, v)$ are one-to-one (see p. 368). We have

THEOREM 2. *If, in addition to the hypotheses of Theorem* 1, *we have* $\partial(u, v)/\partial(\alpha, \beta) \neq 0$, *then* $x^*(u, v)$ *is analytic in* (u, v).

The proof follows immediately from Theorem 1 and the relation

$$\frac{\partial(u, v)}{\partial(\alpha, \beta)} = \frac{\partial(u, v)}{\partial(\xi, \eta)} \frac{\partial(\xi, \eta)}{\partial(\alpha, \beta)}$$

which implies $\partial(u, v)/\partial(\alpha, \beta) = 0$ if $\partial(u, v)/\partial(\xi, \eta) = 0$.

We mention the following case which frequently presents itself in differential geometry: Let us operate with $\partial/\partial\alpha - i\partial/\partial\beta$ on (13) and (14) and take imaginary parts. On eliminating the derivatives of x^*, p^*, q^* with respect to (α, β) with the aid of (13), (14), (15), we obtain

$$\frac{\partial^2 u}{\partial\alpha^2} + \frac{\partial^2 u}{\partial\beta^2} = h_1\left[\left(\frac{\partial u}{\partial\alpha}\right)^2 + \left(\frac{\partial u}{\partial\beta}\right)^2\right] + h_2\left[\left(\frac{\partial v}{\partial\alpha}\right)^2 + \left(\frac{\partial v}{\partial\beta}\right)^2\right]$$
$$+ h_3\left(\frac{\partial u}{\partial\alpha}\frac{\partial v}{\partial\alpha} + \frac{\partial u}{\partial\beta}\frac{\partial v}{\partial\beta}\right) + h_4\frac{\partial(u, v)}{\partial(\alpha, \beta)},$$

$$\frac{\partial^2 v}{\partial\alpha^2} + \frac{\partial^2 v}{\partial\beta^2} = h_1'\left[\left(\frac{\partial u}{\partial\alpha}\right)^2 + \left(\frac{\partial u}{\partial\beta}\right)^2\right] + \cdots + h_4'\frac{\partial(u, v)}{\partial(\alpha, \beta)}.$$

Now suppose that the coefficients h_1, h_2, h_3, h_4, h_1', h_2', h_3', h_4' *depend only on* u *and* v. As they are analytic in (u, v), a result of ours† shows that then necessarily $\partial(u, v)/\partial(\alpha, \beta) \neq 0$.

† *On the non-vanishing of the Jacobian in certain one-to-one mappings*, Bulletin of the American Mathematical Society, vol. 42 (1936), pp. 689–692.

Consider the special case where the coefficients of the equation (7) depend only on (u, v). Then the elimination of x^*, p^*, q^* is obviously possible. Furthermore it becomes possible to widen the conditions of Theorem 1, and to state

THEOREM 2′. *Let A, B, C, E be analytic functions of the complex variables u, v for (u, v) in $N[|u-u_0| \leqq \epsilon, |v-v_0| \leqq \epsilon]$ and suppose A, B, C, E depending on μ and converging uniformly as $\mu \to \mu^*$. Assume A, B, C, E real for real u, v,*

$$\Delta \equiv 4(AC + E) - B^2 > 0,$$

and $|\Delta|^{-1}$ uniformly bounded for (u, v) in N. Suppose that there exists a real and analytic function $x(u, v)$ which is a solution of (7) as (u, v) ranges over the real rectangle $R[|u-u_0| < \epsilon, |v-v_0| < \epsilon]$, and that $x(u, v)$ is uniformly bounded as μ varies. Then there exists a subsequence of values μ such that the corresponding solutions $x(u, v)$ converge uniformly in every closed subregion of R to an analytic limit function $x^*(u, v)$, which is a solution of the limit equation (7).

Proceeding precisely as in the proof of Theorem 1, we find a suitable constant α such that $x(u, v) + a(u^2 + v^2)$ is convex for some subsequence of values μ, and, in view of the assumptions, bounded as (u, v) ranges over R. It follows that in every closed concentric rectangle contained in R the first derivatives of $x(u, v)$ are uniformly bounded as μ varies. Thus the assumptions of Theorem 2′ imply those of Theorem 1 and, on the other hand, also those of Theorem 2.

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