# THE GEOMETRY OF WHIRL SERIES*

BY

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## Introduction

In this paper, we shall give some results in addition to those given in a paper, called *The geometry of isogonal and equi-tangential series* by Kasner.†

We begin by considering certain simple operations or transformations on the oriented lineal elements of the plane. A *turn* $T_\alpha$ converts each element into one having the same point and a direction making a fixed angle $\alpha$ with the original direction. By a *slide* $S_k$ the line of the element remains the same and the point moves along the line a fixed distance $k$. These transformations together generate a continuous group of three parameters which is called the group of *whirl* transformations.‡

Applying a turn $T_\alpha$ to the tangential elements of a union produces an *isogonal series* while a slide $S_k$ produces an *equi-tangential series*. When we apply a whirl to the tangential elements of a union we obtain a *whirl series*.

Kasner has proved that (1) *any element transformation which converts every isogonal series into an isogonal series must be the product of a conformal transformation by a turn. He also has proved that* (2) *any element transformation which carries every equi-tangential series into an equi-tangential series must be the product of an equi-long transformation by a magnification by a slide.*

We shall derive certain generalizations of the above results, two of which are the following: *Any element transformation which carries every union into a whirl series must be the product of a contact transformation by a whirl. Any element transformation which converts every whirl series into a whirl series must be the product of a rigid motion by a magnification by a whirl.*

We also find that, *for an arbitrary element transformation, the maximum number of unions, isogonal, equitangential and whirl series which are transformed into whirl series are respectively* $\infty^4$, $\infty^5$, $\infty^5$, $\infty^7$.

## I. The differential equation of all whirl series

For the analytic representation of an element, it will be found convenient to use two systems of coordinates called the cartesian and the hessian co-

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† These Transactions, vol. 42 (1937), pp. 94–106.

‡ Kasner, *The group of turns and slides and the geometry of turbines*, American Journal of Mathematics, vol. 33 (1911), pp. 193–202.

ordinate systems respectively. The cartesian coordinates of an element $E$ are $(x, y, \theta)$ where $(x, y)$ are the cartesian coordinates of the point of $E$ and $\theta = \arctan p$ is the inclination of the line of $E$. The hessian coordinates of an element $E$ are $(u, v, w)$ where $v$ is the length of the perpendicular from the origin to the line of $E$, $u$ is the angle between the perpendicular and the initial line, and $w$ is the distance between the foot of the perpendicular and the point of $E$.

The equations of the slide $S_k$ are

$$U = u, \qquad V = v, \qquad W = w + k.$$

The equations of the turn $T_\alpha$ are

$$U = u + \alpha, \qquad V = v \cos \alpha + w \sin \alpha, \qquad W = - v \sin \alpha + w \cos \alpha.$$

Since any whirl transformation may be given in the form $W = T_\beta S_k T_\alpha$, it follows that the equations of the whirl $W = T_\beta S_k T_\alpha$ are

$$(1) \qquad \begin{cases} U = u + \alpha + \beta, \qquad V = v \cos (\alpha + \beta) + w \sin (\alpha + \beta) + k \sin \beta, \\ W = - v \sin (\alpha + \beta) + w \cos (\alpha + \beta) + k \cos \beta. \end{cases}$$

From (1), it is found that any whirl series is given by the equations

$$(2) \qquad \begin{cases} V = v(U - \alpha - \beta) \cos (\alpha + \beta) + v'(U - \alpha - \beta) \sin (\alpha + \beta) + k \sin \beta, \\ W = - v(U - \alpha - \beta) \sin (\alpha + \beta) + v'(U - \alpha - \beta) \cos (\alpha + \beta) + k \cos \beta, \end{cases}$$

where $(u, v(u), v'(u))$ are the elements of the fundamental union, and $\alpha, k, \beta$ are the constant parameters of the whirl transformation.

THEOREM 1. *In hessian coordinates the necessary and sufficient condition that the series $V = V(U)$, $W = W(U)$ be a whirl series is that the functions $V$ and $W$ satisfy the equation of third order*

$$(3) \qquad \frac{d}{dU} \frac{V'' - W'}{V' + W''} = 0.$$

*In cartesian coordinates the necessary and sufficient condition that the series $X = X(\theta)$, $Y = Y(\theta)$ be a whirl series is that the functions $X$ and $Y$ satisfy the equation*

$$(4) \qquad \frac{d}{d\theta}\left(\frac{- X' + Y''}{X'' + Y'}\right) = 1 + \left(\frac{- X' + Y''}{X'' + Y'}\right)^2.$$

If in hessian coordinates, $V = V(U)$, $W = W(U)$ is a whirl series, then it may be given by the equations (2) which obviously satisfy (3).

Let now the series $V = V(U)$, $W = W(U)$ be such that the functions $V$

and $W$ satisfy (3). By integrating (3) with respect to $U$ and writing the constant of integration as $\tan(\alpha+\beta)$, we obtain the equation

$$(V'' - W')\cos(\alpha+\beta) - (V' + W'')\sin(\alpha+\beta) = 0,$$

which upon a second integration yields

$$V'\cos(\alpha+\beta) - W'\sin(\alpha+\beta) = V\sin(\alpha+\beta) + W\cos(\alpha+\beta) - k\cos\alpha,$$

where $k$ is a constant. From this equation, it is seen that the equations

$$\begin{aligned} u &= U - \alpha - \beta, \\ v &= V\cos(\alpha+\beta) - W\sin(\alpha+\beta) + k\sin\alpha, \\ w &= V\sin(\alpha+\beta) + W\cos(\alpha+\beta) - k\cos\alpha \end{aligned}$$

represent a union. Solving them for $U$, $V$, $W$, and expressing $V$, $W$ in terms of $U$, we are led to (2) and thus conclude that our given series is a whirl series. The proof for cartesian coordinates is similar.

COROLLARY. *In hessian coordinates, the necessary and sufficient condition that the series $U=U(t)$, $V=V(t)$, $W=W(t)$ be a whirl series is that the functions $U$, $V$, and $W$ satisfy the equation*

$$\frac{d}{dt}\left(\frac{U_tV_{tt} - V_tU_{tt} - W_tU_t^2}{V_tU_t^2 + U_tW_{tt} - W_tU_{tt}}\right) = 0. \tag{5}$$

*In cartesian coordinates, the necessary and sufficient condition that the series $X=X(t)$, $Y=Y(t)$, $\theta=\theta(t)$ be a whirl series is that the functions $X$, $Y$, and $\theta$ satisfy the equation*

$$\frac{dM}{dt} = \theta_t(1 + M^2), \tag{6}$$

*where*

$$M = \frac{-X_t\theta_t^2 + \theta_tY_{tt} - Y_t\theta_{tt}}{\theta_tX_{tt} - X_t\theta_{tt} + Y_t\theta_t^2}.$$

## II. UNIONS INTO WHIRL SERIES

THEOREM 2. *Under any element transformation, there exists in general a four-parameter family of unions which are transformed into whirl series. Any element transformation which converts every union into a whirl series must be the product of a contact transformation by a whirl.*

By any element transformation, the elements of a union

$$u, \qquad v = v(u), \qquad w = v'(u),$$

become the elements

$$U = \phi(u, v, v'), \qquad V = \psi(u, v, v'), \qquad W = \chi(u, v, v').$$

If this series is a whirl series, then by (5) we must have

$$\frac{d}{du}\left\{\frac{\frac{d\phi}{du}\frac{d^2\psi}{du^2} - \frac{d\psi}{du}\frac{d^2\phi}{du^2} - \frac{d\chi}{du}\left(\frac{d\phi}{du}\right)^2}{\frac{d\psi}{du}\left(\frac{d\phi}{du}\right)^2 + \frac{d\phi}{du}\frac{d^2\chi}{du^2} - \frac{d\chi}{du}\frac{d^2\phi}{du^2}}\right\} = 0. \tag{7}$$

Upon substituting the values of the derivatives of $\phi$, $\psi$, and $\chi$ into (7), we have

$$\frac{d}{du}\left(\frac{A + Bv'' + Cv''^2 + Dv''^3 + Ev'''}{K + Lv'' + Mv''^2 + Nv''^3 + Pv'''}\right) = 0, \tag{8}$$

where $A, B, C, D, E, K, L, M, N, P$ are functions of $u, v, v'$ only.

Since (8) is a differential equation of the fourth order in $v$, it follows that there is in general a four-parameter family of unions which under the above element transformation are carried into whirl series. All the different cases in the preceding paper have been discussed in a paper by Kasner and De Cicco called *The classification of element transformations by isogonal and equitangential series*, published in the Proceedings of the National Academy of Sciences, vol. 24 (1938), no. 1, pp. 34–38. The different cases of this paper will be considered in a later paper by the author.

Our next problem is to determine all element transformations which convert any union into a whirl series. Then (8) is an identity, and the coefficient of $v''''$ in (8) is zero, whence

$$\frac{A + Bv'' + Cv''^2 + Dv''^3}{K + Lv'' + Mv''^2 + Nv''^3} = \frac{E}{P}.$$

Since this relation is satisfied for every union, the ratios $A/K$, $B/L$, $C/M$, $D/N$, $E/P$ have a common value $\lambda(u, v, v')$ and (8) becomes

$$\frac{d\lambda}{du} = \lambda_u + v'\lambda_v + v''\lambda_{v'} = 0.$$

Hence $\lambda$ is a constant, say $\tan\alpha$, and we obtain

$$\frac{A}{K} = \frac{B}{L} = \frac{C}{M} = \frac{D}{N} = \frac{E}{P} = \tan\alpha. \tag{9}$$

Of the functions $A, B, C, D, E, K, L, M, N, P$, we shall need the explicit expressions for the following ones:

$$
(10)\quad \begin{cases}
A = (\phi_u + v'\phi_v)(\psi_{uu} + 2v'\psi_{uv} + v'^2\psi_{vv}) \\
\qquad - (\psi_u + v'\psi_v)(\phi_{uu} + 2v'\phi_{uv} + v'^2\phi_{vv}) - (\chi_u + v'\chi_v)(\phi_u + v'\phi_v)^2, \\
K = (\psi_u + v'\psi_v)(\phi_u + v'\phi_v)^2 + (\phi_u + v'\phi_v)(\chi_{uu} + 2v'\chi_{uv} + v'^2\chi_{vv}) \\
\qquad - (\chi_u + v'\chi_v)(\phi_{uu} + 2v'\phi_{uv} + v'^2\phi_{vv}), \\
D = \phi_{v'}\psi_{v'v'} - \psi_{v'}\phi_{v'v'} - \chi_{v'}\phi_{v'}^2, \\
N = \psi_{v'}\phi_{v'}^2 + \phi_{v'}\chi_{v'v'} - \chi_{v'}\phi_{v'v'}, \\
E = \psi_{v'}(\phi_u + v'\phi_v) - \phi_{v'}(\psi_u + v'\psi_v), \\
P = \chi_{v'}(\phi_u + v'\phi_v) - \phi_{v'}(\chi_u + v'\chi_v).
\end{cases}
$$

We observe that our element transformation may be expressed as the product of some other transformation

$$U = \Phi(u, v, w), \qquad V = \Psi(u, v, w), \qquad W = \mathrm{X}(u, v, w),$$

by the turn $T_\alpha$, where $\alpha$ is the constant angle of (9). Then we have

$$\phi = \Phi + \alpha, \qquad \psi = \Psi \cos\alpha + \mathrm{X}\sin\alpha, \qquad \chi = -\Psi\sin\alpha + \mathrm{X}\cos\alpha.$$

It is clear that the equation $E/P = \tan\alpha$ may be written as

$$\frac{\dfrac{\psi_{v'}}{\phi_{v'}} - \dfrac{\psi_u + v'\psi_v}{\phi_u + v'\phi_v}}{\dfrac{\chi_{v'}}{\phi_{v'}} - \dfrac{\chi_u + v'\chi_v}{\phi_u + v'\phi_v}} = \tan\alpha.$$

Substituting the equivalent expressions in terms of $\Phi, \Psi, \mathrm{X}$ and simplifying, we obtain the relation

$$(11)\qquad \frac{\Psi_u + v'\Psi_v}{\Phi_u + v'\Phi_v} = \frac{\Psi_{v'}}{\Phi_{v'}}$$

not containing $\alpha$.

It is readily seen that the equation $D/N = \tan\alpha$ may be written as

$$\frac{\dfrac{\partial}{\partial v'}\left(\dfrac{\psi_{v'}}{\phi_{v'}} - \chi\right)}{\dfrac{\partial}{\partial v'}\left(\psi + \dfrac{\chi_{v'}}{\phi_{v'}}\right)} = \tan\alpha.$$

Substituting the equivalent expressions in terms of $\Phi, \Psi, \mathrm{X}$, we obtain eventually the equation

$$\frac{\partial}{\partial v'}\left(\frac{\Psi_{v'}}{\Phi_{v'}} - \mathrm{X}\right) = 0,$$

which upon integration with respect to $v'$ becomes

$$\frac{\Psi_{v'}}{\Phi_{v'}} = X + \mu(u, v), \tag{12}$$

where $\mu$ is a function of $u$ and $v$ only.

The equation $A/K = \tan\alpha$ can be put into the form

$$\frac{\left(\frac{\partial}{\partial u} + v'\frac{\partial}{\partial v}\right)\left(\frac{\psi_u + v'\psi_v}{\phi_u + v'\phi_v} - \chi\right)}{\left(\frac{\partial}{\partial u} + v'\frac{\partial}{\partial v}\right)\left(\psi + \frac{\chi_u + v'\chi_v}{\phi_u + v'\phi_v}\right)} = \tan\alpha.$$

Substituting into this equation the equivalent expressions in terms of $\Phi, \Psi, X$ we obtain, on simplification

$$\left(\frac{\partial}{\partial u} + v'\frac{\partial}{\partial v}\right)\left[\frac{\Psi_u + v'\Psi_v}{\Phi_u + v'\Phi_v} - X\right] = 0. \tag{13}$$

From (11), (12), and (13), we have

$$\left(\frac{\partial}{\partial u} + v'\frac{\partial}{\partial v}\right)\mu(u, v) = \mu_u + v'\mu_v = 0.$$

Therefore $\mu$ is a constant $k$.

We therefore conclude that the functions $\Phi, \Psi, X$ satisfy the equations

$$\frac{\Psi_u + v'\Psi_v}{\Phi_u + v'\Phi_v} = \frac{\Psi_{v'}}{\Phi_{v'}} = X + k. \tag{14}$$

Hence it follows that our given transformation is the product of a contact transformation by a slide by a turn, that is, it is the product of a contact transformation by a whirl. That this condition is sufficient is obvious.

## III. Equi-tangential series into whirl series

THEOREM 3. *Under any element transformation, there exists in general a five-parameter family of equi-tangential series which are transformed into whirl series. Any contact transformation which carries every equi-tangential series into a whirl series must carry every equi-tangential series into an equi-tangential series; that is, it must be the product of an equi-long transformation by a magnification. Any element transformation which converts every equi-tangential series into a whirl series must be the product of an equi-long transformation by a magnification by a whirl.*

By any element transformation the elements of an equi-tangential series $u$, $v=v(u)$, $w=v'(u)+k$, become the elements

$$U = \phi(u, v, w), \qquad V = \psi(u, v, w), \qquad W = \chi(u, v, w).$$

If this series is a whirl series, then by (5) we have

$$\frac{d}{du}\left\{\frac{\dfrac{d\phi}{du}\dfrac{d^2\psi}{du^2} - \dfrac{d\psi}{du}\dfrac{d^2\phi}{du^2} - \dfrac{d\chi}{du}\left(\dfrac{d\phi}{du}\right)^2}{\dfrac{d\psi}{du}\left(\dfrac{d\phi}{du}\right)^2 + \dfrac{d\phi}{du}\dfrac{d^2\chi}{du^2} - \dfrac{d\chi}{du}\dfrac{d^2\phi}{du^2}}\right\} = 0. \tag{15}$$

Upon substituting the values of the derivatives of $\phi$, $\psi$, $\chi$ into (15), we have

$$\frac{d}{du}\left(\frac{A + Bw' + Cw'^2 + Dw'^3 + Ew''}{K + Lw' + Mw'^2 + Nw'^3 + Pw''}\right) = 0, \tag{16}$$

where $A$, $B$, $C$, $D$, $E$, $K$, $L$, $M$, $N$, $P$ are functions of $u$, $v$, $w$, $k$ only.

Since (16) is a differential equation of the fourth order in $v$, the complete solution of (16) contains four constants of integration and the constant $k$ of our equi-tangential series. Therefore, there is in general a five-parameter family of equi-tangential series which under the above element transformation are carried into whirl series.

Our next problem is to determine all element transformations which convert any equi-tangential series into a whirl series. Then (16) is an identity, hence the coefficient of $w'''$ is zero. Thus

$$\frac{A + Bw' + Cw'^2 + Dw'^3}{K + Lw' + Mw'^2 + Nw'^3} = \frac{E}{P}.$$

Since this equation is satisfied for every equi-tangential series, the ratios $A/K$, $B/L$, $C/M$, $D/N$, $E/P$ have a common value $\lambda(u, v, w, k)$, and (16) becomes

$$\frac{d\lambda}{du} = \lambda_u + (w - k)\lambda_v + w'\lambda_w = 0;$$

that is, $\lambda$ is a function of $k$ only. Thus we obtain the result

$$\frac{A}{K} = \frac{B}{L} = \frac{C}{M} = \frac{D}{N} = \frac{E}{P} = \lambda(k), \tag{17}$$

where $\lambda$ is a function of $k$ only.

Now since a union is a special case of an equi-tangential series, it follows by Theorem 2 that the transformation which we are seeking must be the product of a contact transformation by a whirl. It remains therefore to consider the contact transformations which carry any equi-tangential series into a whirl series.

If $\lambda=0$, it follows from (17) that the numerator of the fraction in (16), and hence that of the fraction in (15), vanishes for every equi-tangential series. Thus, the equation

$$\frac{d}{du}\left\{\frac{\dfrac{d\psi}{du}}{\dfrac{d\phi}{du}}-\chi\right\}=0 \tag{18}$$

must be identically satisfied. From (18) it follows that our contact transformation converts every equi-tangential series into an equi-tangential series. Therefore according to Kasner's result our contact transformation must be the product of an equi-long transformation by a magnification.

We shall prove that there is no contact transformation with the desired property for which $\lambda$ is different from zero. In the first place, our contact transformation can not be an extended line transformation. For any such transformation with the desired property must carry every equi-tangential series into an equi-tangential series, and thus $\lambda$ must be zero. Therefore for our contact transformation, we must have

$$\chi=\frac{\psi_u+w\psi_v}{\phi_u+w\phi_v}=\frac{\psi_w}{\phi_w}, \tag{19}$$

where $\phi_w\neq 0$ and $\psi_w\neq 0$.

The equation $D/N=\lambda$ has the form

$$\frac{\phi_w\psi_{ww}-\psi_w\phi_{ww}-\chi_w\phi_w^2}{\psi_w\phi_w^2+\phi_w\chi_{ww}-\chi_w\phi_{ww}}=\lambda.$$

Since $\lambda\neq 0$, and the numerator of the fraction vanishes by (19), the denominator must vanish, and we have

$$\frac{\partial}{\partial w}\left(\psi+\frac{\chi_w}{\phi_w}\right)=0. \tag{20}$$

From (19) and (20), we obtain

$$\begin{aligned}\psi&=\alpha+\beta\sin(\phi+\gamma),\\ \chi&=\beta\cos(\phi+\gamma),\end{aligned} \tag{21}$$

where $\alpha, \beta\neq 0, \gamma$ are functions of $u$ and $v$ only.

The equation $E/P=\lambda$ may be written as

$$\frac{\psi_w[\phi_u + (w-k)\phi_v] - \phi_w[\psi_u + (w-k)\psi_v]}{\chi_w[\phi_u + (w-k)\phi_v] - \phi_w[\chi_u + (w-k)\chi_v]} = \lambda$$

and hence, by (19), may be put in the form

$$\frac{1}{\lambda} = \frac{s}{k} + r, \tag{22}$$

where

$$s = \frac{\chi_w(\phi_u + w\phi_v) - \phi_w(\chi_u + w\chi_v)}{\phi_w\psi_v - \psi_w\phi_v}, \qquad r = \frac{\phi_w\chi_v - \chi_w\phi_v}{\phi_w\psi_v - \psi_w\phi_v}. \tag{23}$$

Since the jacobian of the transformation cannot be zero, it follows by (19) that the common denominator of $r$ and $s$, and also the numerator of $s$, are each different from zero. Hence, since $\lambda$ is a function of $k$ only, $r$ and $s\neq 0$ are finite real constants, independent of $k$.

Substituting the values of $\psi$ and $\chi$ into the second of the equations (23), we obtain

$$(\beta_v - r\beta\gamma_v)\cos(\phi+\gamma) - (\beta\gamma_v + r\beta_v)\sin(\phi+\gamma) - r\alpha_v = 0.$$

Since $\phi_w\neq 0$, it is seen that $\beta_v - r\beta\gamma_v = 0$, $\beta\gamma_v + r\beta_v = 0$, $r\alpha_v = 0$. Thus $\beta\neq 0$ and $\gamma$ are functions of $u$ only.

Substituting the values of $\psi$ and $\chi$ into the first of equations (23) and remembering that $\beta$ and $\gamma$ are functions of $u$ only, we obtain

$$-\beta_u\cos(\phi+\gamma) + \beta\gamma_u\sin(\phi+\gamma) - s\alpha_v = 0.$$

Since $\phi_w\neq 0$, it is seen that

$$\beta_u = 0, \qquad \beta\gamma_u = 0, \qquad s\alpha_v = 0.$$

Thus $\beta$ and $\gamma$ are constants independent of $k$. Then by (21), $\chi$ is a function of $\phi$. Hence there is no contact transformation with the desired property for which $\lambda$ is not zero, and the proof of our theorem is complete.

## IV. Isogonal series into whirl series

Theorem 4. *Under any element transformation, there exists in general a five-parameter family of isogonal series which are transformed into whirl series. Any contact transformation which carries every isogonal series into a whirl series must carry every isogonal series into an isogonal series, that is, it must be a conformal transformation. Any element transformation which converts every* *isogonal series into a whirl series must be the product of a conformal transformation by a whirl.*

By any element transformation, the elements of an isogonal series

$$x, \qquad y = y(x), \qquad \text{arc tan } p = \text{arc tan } y'(x) + \text{arc tan } k$$

become the elements

$$X = \phi(x, y, p), \qquad Y = \psi(x, y, p), \qquad P = \chi(x, y, p).$$

According to (6), this series is a whirl series if and only if

$$(24) \qquad (1 + \chi^2)\frac{df}{dx} = (1 + f^2)\frac{d\chi}{dx},$$

where

$$(25) \qquad f = \frac{-\dfrac{d\phi}{dx}\left(\dfrac{d\theta}{dx}\right)^2 + \dfrac{d\theta}{dx}\dfrac{d^2\psi}{dx^2} - \dfrac{d\psi}{dx}\dfrac{d^2\theta}{dx^2}}{\dfrac{d\theta}{dx}\dfrac{d^2\phi}{dx^2} - \dfrac{d\phi}{dx}\dfrac{d^2\theta}{dx^2} + \dfrac{d\psi}{dx}\left(\dfrac{d\theta}{dx}\right)^2},$$

and $\theta = \text{arc tan } \chi$.

Upon substituting the values of the derivatives of $\phi$, $\psi$, $\chi$ into (25), we have

$$(26) \qquad f = \frac{A + Bp' + Cp'^2 + Dp'^3 + Ep''}{K + Lp' + Mp'^2 + Np'^3 + Rp''},$$

where $A, B, C, D, E, K, L, M, N, R$ are functions of $x, y, p, k$, only.

Since, by (26), (24) is a differential equation of the fourth order in $y$, the complete solution of (24) contains four constants of integration and the constant $k$ of our isogonal series. Thus there is in general a five-parameter family of isogonal series which under the above transformation are carried into whirl series.

Our next problem is to determine all element transformations which convert any isogonal series into a whirl series. Then (24) is an identity, hence the coefficient of $p'''$ is zero, whence

$$(K + Lp' + Mp'^2 + Np'^3)E - (A + Bp' + Cp'^2 + Dp'^3)R = 0.$$

Since this equation is also an identity, the ratios $A/K, B/L, C/M, D/N, E/R$ have a common value $\lambda(x, y, p, k)$. Then obviously $f = \lambda$, and from (24) we obtain

$$(27) \qquad f = \frac{A}{K} = \frac{B}{L} = \frac{C}{M} = \frac{D}{N} = \frac{E}{R} = \frac{\chi + \alpha}{1 - \alpha\chi},$$

where $\alpha$ is a function of $k$ only.

Now since a union is a special case of an isogonal series, it follows by Theorem 2 that the transformation which we are seeking must be the product of a contact transformation by a whirl. It remains therefore to consider the contact transformations which carry any isogonal series into a whirl series.

In the first place, let us suppose that our contact transformation is an extended point transformation. Then it must carry every isogonal series into an isogonal series, and therefore according to Kasner's result, it must be a conformal transformation.

Next let us suppose that our contact transformation carries every point into a line. Then it must be of the form

$$\begin{aligned} \phi &= \frac{g_x + pg_y}{f_x + pf_y}, \\ \psi &= f\phi - g, \\ \chi &= f, \end{aligned} \tag{28}$$

where $f$ and $g$ are functions of $x$ and $y$ only. Then (28) must carry every isogonal series into an equi-tangential series. Thus the elements of the union

$$x, \qquad y = y(x), \qquad y' = y'(x)$$

corresponding to the isogonal series

$$x, \qquad y = y(x), \qquad \text{arc tan } p = \text{arc tan } y'(x) + \text{arc tan } k$$

must be carried into the elements of the union

$$X = \phi(x, y, y'), \qquad Y = \psi(x, y, y'), \qquad P = \chi(x, y, y')$$

corresponding to the equi-tangential series

$$X = \phi(x, y, p), \qquad Y = \psi(x, y, p), \qquad P = \chi(x, y, p).$$

If this series is an equi-tangential series, then

$$\frac{d}{dx}\left([\phi(x, y, p) - \phi(x, y, y')]^2 + [\psi(x, y, p) - \psi(x, y, y')]^2\right)^{1/2} = 0.$$

Upon simplifying this equation by means of (28) and setting the coefficient of $p'$ equal to zero, we obtain (since $f_x g_y - f_y g_x \neq 0$)

$$[(1 + pk)f_x + (p - k)f_y]^2 = (k^2 + 1)(f_x + pf_y)^2. \tag{29}$$

Equation (29) is an identity; hence the coefficient of $k^2$ is zero, whence

$$(pf_x - f_y)^2 = (f_x + pf_y)^2.$$

Since this equation is also an identity, we must have

$$f_x = \pm f_y, \qquad -f_y = \pm f_x,$$

that is, $f$ is a constant. By (28), this is impossible. We have, therefore, proved that there is no contact transformation which carries every point into a line and every isogonal series into a whirl series.

Now we shall prove that there is no contact transformation with the desired property for which $\phi_p \neq 0$, $\psi_p \neq 0$, and $\chi_p \neq 0$. For our contact transformation we must have

$$\chi = \frac{\psi_x + p\psi_y}{\phi_x + p\phi_y} = \frac{\psi_p}{\phi_p}. \tag{30}$$

The equation $E/R = (\chi + \alpha)/(1 - \alpha\chi)$ can be written in the form

$$\frac{\psi_p[(1 + kp)\chi_x + (p - k)\chi_y] - \chi_p[(1 + kp)\psi_x + (p - k)\psi_y]}{\phi_p[(1 + kp)\chi_x + (p - k)\chi_y] - \chi_p[(1 + kp)\phi_x + (p - k)\phi_y]} = \frac{\chi + \alpha}{1 - \alpha\chi},$$

which, when solved for $\alpha$ and combined with (30) gives,

$$\frac{1}{\alpha} = \frac{r}{k} + s, \tag{31}$$

where

$$\begin{aligned} r &= \frac{(1 + \chi^2)[\phi_p(\chi_x + p\chi_y) - \chi_p(\phi_x + p\phi_y)]}{\chi_p[-(p\psi_x - \psi_y) + \chi(p\phi_x - \phi_y)]}, \\ s &= \frac{\phi_p(1 + \chi^2)(p\chi_x - \chi_y) - \chi_p\{(p\phi_x - \phi_y) + \chi(p\psi_x - \psi_y)\}}{\chi_p[-(p\psi_x - \psi_y) + \chi(p\phi_x - \phi_y)]}. \end{aligned} \tag{32}$$

Since the jacobian of the transformation can not be zero, it follows by (30) that the common denominator of $r$ and $s$, and also the numerator of $r$, are each different from zero. Hence, since $\alpha$ is a function of $k$ only, $r \neq 0$ and $s$ are finite real constants and are independent of $k$. From (31), it is then seen that $\alpha \neq 0$.

The equation $D/N = (\chi + \alpha)/(1 - \alpha\chi)$ may be written in the form

$$\frac{-\phi_p\chi_p^2 + 2\chi\psi_p\chi_p^2 + (1 + \chi^2)(\chi_p\psi_{pp} - \psi_p\chi_{pp})}{2\chi\phi_p\chi_p^2 + \psi_p\chi_p^2 + (1 + \chi^2)(\chi_p\phi_{pp} - \phi_p\chi_{pp})} = \frac{\chi + \alpha}{1 - \alpha\chi}.$$

Solving this equation for $\alpha$, we obtain

$$\alpha = \frac{-(1 + 2\chi^2)\phi_p + \chi\psi_p + (1 + \chi^2)\left[\dfrac{\partial}{\partial p}\left(\dfrac{\psi_p}{\chi_p}\right) - \chi\dfrac{\partial}{\partial p}\left(\dfrac{\phi_p}{\chi_p}\right)\right]}{\chi\phi_p + (1 + 2\chi^2)\psi_p + (1 + \chi^2)\left[\dfrac{\partial}{\partial p}\left(\dfrac{\phi_p}{\chi_p}\right) + \chi\dfrac{\partial}{\partial p}\left(\dfrac{\psi_p}{\chi_p}\right)\right]} \tag{33}$$

Since $\alpha \neq 0$ and the numerator of the fraction vanishes by virtue of (30), the denominator must vanish, hence we obtain the relation

$$2\chi\phi_p + \frac{\partial}{\partial p}\left(\frac{\phi_p}{\chi_p}\right) + \chi\frac{\partial}{\partial p}\left(\frac{\psi_p}{\chi_p}\right) = 0. \tag{34}$$

From (30) and (34), we then obtain

$$\begin{aligned} \phi &= \alpha + \frac{\gamma\chi}{(1+\chi^2)^{1/2}}, \\ \psi &= \beta - \frac{\gamma}{(1+\chi^2)^{1/2}}, \end{aligned} \tag{35}$$

where $\alpha$, $\beta$, and $\gamma \neq 0$ are functions of $x$ and $y$ only.

In order that (35) be a contact transformation, we must have

$$\beta_x + p\beta_y = \chi(\alpha_x + p\alpha_y) + (\gamma_x + p\gamma_y)(1+\chi^2)^{1/2}. \tag{36}$$

From (32) and (35), we obtain

$$\begin{aligned} r &= \frac{(\alpha_x + p\alpha_y)(1+\chi^2) + (\gamma_x + p\gamma_y)\chi(1+\chi^2)^{1/2}}{(p\beta_x - \beta_y) - \chi(p\alpha_x - \alpha_y) - (p\gamma_x - \gamma_y)(1+\chi^2)^{1/2}}, \\ s &= \frac{(p\alpha_x - \alpha_y) + \chi(p\beta_x - \beta_y)}{(p\beta_x - \beta_y) - \chi(p\alpha_x - \alpha_y) - (p\gamma_x - \gamma_y)(1+\chi^2)^{1/2}}. \end{aligned} \tag{37}$$

Since $r \neq 0$, it follows from (36) and (37) that

$$\frac{(\alpha_x + p\alpha_y) + \chi(\beta_x + p\beta_y)}{(p\alpha_x - \alpha_y) + \chi(p\beta_x - \beta_y)} = \frac{s}{r} = t,$$

where $t$ is a constant independent of $k$. Solving this equation for $\chi$, we obtain

$$\chi = \frac{(\alpha_x + p\alpha_y) - t(p\alpha_x - \alpha_y)}{-(\beta_x + p\beta_y) + t(p\beta_x - \beta_y)}. \tag{38}$$

It is observed that neither the numerator nor the denominator of the fraction in (38) can be zero. For, if either vanished, both must vanish and it would follow that $\alpha$ and $\beta$ are constants. But, then $\gamma$ would have to be a constant, by (36), and if $\alpha$, $\beta$, $\gamma$ were constants, $\phi$ and $\psi$, as given by (35), would be functions of $\chi$; and this is impossible.

We shall prove that $\gamma$ cannot be a constant. For otherwise by (36) and (38) we would have

$$\frac{(\alpha_x + p\alpha_y) - t(p\alpha_x - \alpha_y)}{-(\beta_x + p\beta_y) + t(p\beta_x - \beta_y)} = \frac{\beta_x + p\beta_y}{\alpha_x + p\alpha_y}.$$

Upon setting the term independent of $p$ and the coefficient of $p^2$ each equal to zero, we obtain

$$\alpha_x(\alpha_x + t\alpha_y) = -\beta_x(\beta_x + t\beta_y),$$
$$\alpha_y(\alpha_y - t\alpha_x) = \beta_y(-\beta_y + t\beta_x).$$

Upon adding these equations, we find that $\alpha$ and $\beta$ are constants. This proves that $\gamma$ cannot be a constant.

By (38) we see that $\chi$ has an expression of the form

$$\chi = \frac{a + bp}{c + dp}, \tag{39}$$

where $a$, $b$, $c$, and $d$ are functions of $x$ and $y$ only. Also we must have

$$ad - bc \neq 0. \tag{40}$$

For otherwise by (35) and (39), $\phi$, $\psi$, $\chi$ would be independent of $p$.

Since $\gamma$ is not a constant, it follows by (36) and (39) that

$$(1 + \chi^2)^{1/2} = \frac{[(a + bp)^2 + (c + dp)^2]^{1/2}}{c + dp}$$

is a rational function of $p$ with coefficients which are functions of $x$ and $y$ only. Hence

$$[(a + bp)^2 + (c + dp)^2]^{1/2}$$

must be a perfect square with respect to the letter $p$. But this can only happen when $ad - bc = 0$. By (40) this is impossible. Therefore we have proved that there is no contact transformation with the desired property for which $\phi_p \neq 0$, $\psi_p \neq 0$, and $\chi_p \neq 0$. This completes the proof of our theorem.

## V. Whirl series into whirl series

Theorem 5. *Under any element transformation, there exists in general a seven-parameter family of whirl series which are transformed into whirl series. Any contact transformation which carries every whirl series into a whirl series must be the product of a rigid motion by a magnification. Any element transformation which converts every whirl series into a whirl series must be the product of a rigid motion by a magnification by a whirl.*

By any element transformation, the elements of a whirl series

$$u, \qquad v = v(u), \qquad w = w(u),$$

become the elements

$$U = \phi(u, v, w), \qquad V = \psi(u, v, w), \qquad W = \chi(u, v, w).$$

If this series is a whirl series, then by (5), we must have

$$\frac{d}{du}\left\{\frac{\dfrac{d\phi}{du}\dfrac{d^2\psi}{du^2}-\dfrac{d\psi}{du}\dfrac{d^2\phi}{du^2}-\dfrac{d\chi}{du}\left(\dfrac{d\phi}{du}\right)^2}{\dfrac{d\psi}{du}\left(\dfrac{d\phi}{du}\right)^2+\dfrac{d\phi}{du}\dfrac{d^2\chi}{du^2}-\dfrac{d\chi}{du}\dfrac{d^2\phi}{du^2}}\right\}=0. \tag{41}$$

Now (41) obviously contains $w'''$. Since $w$ is a combination of the elements $v=v(u)$, $v'=v'(u)$ of the fundamental union, it follows that (41) is a differential equation of the fourth order in $v$. Hence the complete solution of (41) contains four constants of integration and the three parameters of our whirl series. Thus there is in general a seven-parameter family of whirl series which are carried into whirl series. The remainder of the theorem follows from Theorems 3 and 4.

THEOREM 6. *There is no transformation which carries every isogonal series into an equi-tangential series, or every equi-tangential series into an isogonal series.*

This is an immediate consequence of Theorems 3 and 4.

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