GENERALIZED INTEGRALS AND DIFFERENTIAL EQUATIONS*

BY

HANS LEWY

Introduction. The idea of the following considerations can best be explained in the simplest case of an integral

$$\int a(x, f(x))db(x, f(x))$$

in which a and b are continuously differentiable functions of two variables, $f(x)$ a continuously differentiable function. The routine estimate of $\int adb$ gives bounds depending either on $\int|df(x)|$ or on the maximum of $|df/dx|$ in the interval of integration. It is, however, possible, as shown in Theorem 1, to give a bound that is entirely independent of the derivative of $f(x)$, and, consequently, to define, by a limiting process, $\int adb$, even in the case where $f(x)$ not only has no derivative, but is no longer continuous, provided $f(x)$ belongs to Baire's first class. The same observation holds for a great number of functionals of $f(x)$ whose construction depends on the derivative of $f(x)$, but for which bounds can be found nevertheless without reference to df/dx. In this paper we are concerned mainly with ordinary differential equations (Theorems 2–3′) and systems of hyperbolic equations in two independent variables (Theorem 6 and corollaries) whose treatment is based on a detailed study of the double integral (33).

1. **Simple integrals.** The theory of the Lebesgue-Stieltjes integral contains the following statement: If in a closed interval J the function $\beta_0(x)$ is monotone and the sequence of continuous functions $\alpha_\mu(x)$ is uniformly bounded and tends to a limit function $\alpha_0(x)$, then the Stieltjes integral $\int\alpha_\mu(x)d\beta_0(x)$ tends to the Lebesgue-Stieltjes integral $\int\alpha_0(x)d\beta_0(x)$ as $\mu\to 0$. If, furthermore, a sequence of functions $\beta_\mu(x)$ of bounded variation tends to $\beta_0(x)$ as $\mu\to\infty$ so that the total variation of the difference $\beta_\mu(x)-\beta_0(x)$ tends to zero, then

$$\limsup\left|\int \alpha_\mu(x)d\beta_\mu(x) - \int \alpha_0(x)d\beta_0(x)\right|$$
$$\leqq \limsup \left\{\max \mid \alpha_\mu(x) \mid \cdot \int \mid d(\beta_\mu - \beta_0) \mid\right\} \to 0,$$

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which relation leads to the following lemma:

LEMMA 1. *If in the interval $X_0 \leqq x \leqq X_1$ the continuous functions $\alpha_\mu(x)$, $\mu = 1, 2, \cdots$, are uniformly bounded and tend to $\alpha_0(x)$ as $\mu \to \infty$, and if the functions $\beta_\mu(x)$ of bounded variation tend to a function $\beta_0(x)$ of bounded variation while the total variation of $\beta_\mu(x) - \beta_0(x)$ tends to zero as $\mu \to \infty$, then*

$$\lim_{\mu\to\infty} \int_{X_0}^{x} \alpha_\mu(x) d\beta_\mu(x) \text{ exists and } = \int_{X_0}^{x} \alpha_0(x) d\beta_0(x),$$

where the integral on the right is the Lebesgue-Stieltjes integral.

We proceed to introduce a new notion of integral which is essentially different from the Lebesgue-Stieltjes integral and is based upon the following theorem:

THEOREM 1. *In the interval J $(X_0 \leqq x \leqq X_1)$ there are given a function $f(x)$ satisfying the inequality $|f(x) - f(X_0)| < F$, a sequence $\{f_\mu(x)\}$ of continuously differentiable functions with $|f_\mu(x) - f(X_0)| < F$ such that $f_\mu(x) \to f(x)$ as $\mu \to \infty$, and n continuous functions $g_1(x), \cdots, g_n(x)$ with bounded total variations $T[g_1], \cdots, T[g_n]$. Denote by $f^0, g_1^0, \cdots, g_n^0$ the values of $f(X_0), g_1(X_0), \cdots, g_n(X_0)$ and by $\gamma_1, \cdots, \gamma_n$ upper bounds of $|g_1(x) - g_1^0|, \cdots, |g_n(x) - g_n^0|$ in J. Let $\{g_{1\mu}(x)\}, \cdots, \{g_{n\mu}(x)\}$ be n sequences of continuous functions of bounded variation, tending to $g_1(x), \cdots, g_n(x)$, respectively, as $\mu \to \infty$ while the total variations of the differences $T[g_{1\mu} - g_1], \cdots, T[g_{n\mu} - g_n]$ tend to zero and $|g_{i\mu}(x) - g_i^0| < \gamma_i$ for all x in J, $i = 1, 2, \cdots, n$ and $\mu = 1, 2, \cdots$. Suppose that in the $(n+2)$-dimensional domain*

$$D: \quad \begin{aligned} &|z - f^0| \leqq F, \; X_0 \leqq x \leqq X_1, \\ &|y_1 - g_1^0| \leqq \gamma_1, \cdots, |y_n - g_n^0| \leqq \gamma_n \end{aligned}$$

the functions $a(z, x, y_1, \cdots, y_n)$ and $b(z, x, y_1, \cdots, y_n)$ are continuously differentiable. Then the Stieltjes integral

$$(1) \quad S_\mu(x) = \int_{X_0}^{x} a(f_\mu(x), x, g_{1\mu}(x), \cdots, g_{n\mu}(x)) db(f_\mu(x), x, g_{1\mu}(x), \cdots, g_{n\mu}(x))$$

tends to a limit $L(x)$ as $\mu \to \infty$.

Remarks. It is clear that any function $f(x)$ with $|f(x) - f^0| < F$ which is a limit of continuous functions may be considered as a limit of continuously differentiable functions $f_\mu(x)$ with $|f_\mu(x) - f^0| < F$. Moreover, the limit $L(x)$ is independent of the approximating sequences $f_\mu(x)$ and $g_{i\mu}(x)$. For the statement of Theorem 1 implies the existence of a limit no matter which sequences are used and hence any two sequences may be considered as subsequences of a third one containing both. Consequently we may state the following as a definition:

DEFINITION.

$$L(x) = \int_{X_0}^{x} a(f(x), x, g_1(x), \cdots, g_n(x))db(f(x), x, g_1(x), \cdots, g_n(x)).$$

In order to prove Theorem 1, we first assume that $b(z, x, y_1, \cdots, y_n)$ has continuous derivatives of second order of the mixed type. We determine a function $A(z, x, y_1, \cdots, y_n)$ in D as the solution of the differential equation

$$\frac{\partial A}{\partial z} = a\frac{\partial b}{\partial z}$$

with the initial condition $A(f^0, x, y_1, \cdots, y_n)=0$. We obtain

$$\begin{aligned} A(z, x, y_1, \cdots, y_n) \\ = \int_{f^0}^{z} a(z', x, y_1, \cdots, y_n)b_z(z', x, y_1, \cdots, y_n)dz', \end{aligned}$$

where we may differentiate with respect to $x, y_1, \cdots, y_n$ under the integral sign. Thus we find

$$\begin{aligned} adb &= ab_z dz + ab_x dx + \sum_{i=1}^{n} ab_{y_i}dy_i \\ &= dA + (ab_x - A_x)dx + \sum (ab_{y_i} - A_{y_i})dy_i, \end{aligned}$$

$$\begin{aligned} A_x(z, x, y_1, \cdots, y_n) &= \int_{f^0}^{z} (ab_z)_x dz' \\ &= \int_{f^0}^{z} [(ab_x)_z + (a_x b_z - a_z b_x)]dz' \\ &= a(z, x, y_1, \cdots)b_x(z, x, y_1, \cdots) \\ &\quad - a(f^0, x, y_1, \cdots)b_x(f^0, x, y_1, \cdots) + \int_{f^0}^{z} (a_x b_z - a_z b_x)dz' \end{aligned}$$

and

$$\begin{aligned} A_{y_i}(z, x, y_1, \cdots, y_n) &= a(z, x, y_1, \cdots)b_{y_i}(z, x, y_1, \cdots) \\ &\quad - a(f^0, x, y_1, \cdots)b_{y_i}(f^0, x, y_1, \cdots) \\ &\quad + \int_{f^0}^{z} (a_{y_i}b_z - a_z b_{y_i})dz'. \end{aligned}$$

Hence

$$
\begin{aligned}
S_\mu(x) &\equiv \int_{X_0}^{x} a(f_\mu(x'), x', g_{i\mu}(x'))\,db(f_\mu(x'), x', g_{i\mu}(x')) \\
&= A(f_\mu(x), x, g_{i\mu}(x)) + \int_{X_0}^{x} a(f^0, x', g_{i\mu}(x'))b_x(f^0, x', g_{i\mu}(x'))dx \\
&\quad + \sum_{i=1}^{n} \int_{X_0}^{x} a(f^0, x', g_{1\mu}(x'), \cdots)b_{y_i}(f^0, x', g_{1\mu}(x'), \cdots)dg_{i\mu}(x') \qquad (2)\\
&\quad + \int_{X_0}^{x} dx' \int_{f^0}^{f_\mu(x')} [a_z(z', x', g_{1\mu}(x'), \cdots)b_x(z', x', g_{1\mu}(x'), \cdots) \\
&\qquad\qquad - a_x b_z(\cdots)]dz' \\
&\quad + \sum_i \int_{X_0}^{x} dg_{i\mu}(x') \int_{f^0}^{f_\mu(x')} [a_z(z', x', g_{1\mu}(x'), \cdots)b_{y_i} - a_{y_i}b_z]dz'.
\end{aligned}
$$

This formula, derived under the assumption that b has continuous second derivatives of mixed type, still holds under the conditions of Theorem 1. For any b which is continuously differentiable in D may be uniformly approximated by a polynomial such that its first derivatives uniformly approximate those of b. Introducing the approximations instead of b into (2) and passing to the limit we obtain again the formula (2) as both sides of (2) involve only first derivatives of b and the passage to the limit under the integral signs is legitimate in view of the uniform convergence of the derivatives of the polynomials to those of b.

In the right-hand member of (2) we can effect the passage $\mu\to\infty$ by simply cancelling all reference to μ. This may be seen as follows. An integral $\int_{f^0}^{z} a_z(z', x, y_1, \cdots, y_n)b_{y_i}dz'$, for instance, is continuous in $z, x, y_1, \cdots, y_n$. Thus $\int_{f^0}^{f_\mu(x)} a(z', x, g_{1\mu}(x), \cdots)b_{y_i}dz'$ is a continuous function of x, bounded as $\mu\to\infty$, and converging as $\mu\to\infty$. Now the convergence of

$$\int_{X_0}^{x} dg_{i\mu}(x') \int_{f^0}^{f_\mu(x)} a_z(z', x', g_{1\mu}(x'), \cdots)b_{y_i}dz'$$

follows from Lemma 1.

Thus Theorem 1 is proved.

From (2) we have the following estimate:

$$
\begin{aligned}
|L(x)| \leqq MN\left\{ |f(x) - f^0| + \sum_{i=1}^{n} T_{X_0}^{x}[g_i] + |x - X_0| \right\} \\
+ 2N^2F\left(\sum_{i}^{n} T_{X_0}^{x}[g_i] + |x - X_0| \right),
\end{aligned}
$$

where M is an upper bound for $|a|$ and $|b|$, N an upper bound for the moduli of the first derivatives of a and b in D.

Remark. We have, for instance, for every admissible $f(x)$

$$L(x) \equiv \int_0^x f(x')df(x') = \tfrac{1}{2}(f^2(x) - f^2(0)),$$

which leads to $L(1)=\frac{1}{2}$ for $f(x)=0$ if $0 \leqq x<1, f(1)=1$. The Lebesgue-Stieltjes integral, however, would be 1.

2. **Ordinary differential equations.** We may now prove the following theorem:

THEOREM 2. *Let the functions $f(x), g_1(x), \cdots, g_n(x)$ be continuously differentiable in $0 \leqq x \leqq X$ and $f(0)=f^0$, $g_1(0)= \cdots =g_n(0)=0$. Denote by F an upper bound of $|f(x)|$ and by $G_1, G_2, \cdots, G_n$ the total variations of $g_1(x)$, $g_2(x), \cdots, g_n(x)$ in $[0, X]$. Suppose, for $\epsilon>0$, that in the domain*

$$D_{3,\epsilon}: \quad |z| \leqq F, \ |y_i| \leqq G_i, \ |u| < \epsilon + 3FM_0 + \sum_{i=1}^n (e^{2FN} - e^{FN} + M_i e^{FN})G_i$$

the functions $a_0, a_1, \cdots, a_n$ are continuous functions of z, y_i, u satisfying the inequalities $|a_k| \leqq M_k$, $k=0, 1, \cdots, n$, and that a_0 has continuous derivatives of first order, bounded in absolute value by N. Then the solution $u(x)$ of the equation

$$\text{(E)} \qquad \begin{aligned} du(x) &= a_0(f(x), g_1(x), \cdots, g_n(x), u(x))df(x) \\ &\quad + \sum_{i=1}^n a_i(f(x), g_1(x), \cdots, g_n(x), u(x))dg_i(x) \end{aligned}$$

with $u(0)=0$ can be extended over the whole interval $0 \leqq x \leqq X$. It satisfies, moreover, the inequality

$$|u(x)| \leqq 2FM_0 + \sum_{i=1}^n (e^{2FN} - e^{FN} + M_i e^{FN})G_i.$$

Let us solve the auxiliary partial equation for $A(z, y_1, \cdots, y_n, u)$

$$\text{(3)} \qquad \frac{\partial A}{\partial z} + \frac{\partial A}{\partial u}\cdot a_0 = a_0, \qquad 1 - \frac{\partial A}{\partial u} \neq 0$$

under the initial condition

$$\text{(4)} \qquad A(z, y_i, u) = 0 \text{ for } z = 0.$$

The characteristic equations of (3) are

$$\text{(5)} \qquad dz:du:dA = 1:a_0:a_0.$$

Consider the family C of curves satisfying the differential equation $du/dz=a_0$ and passing through any point P of the domain

$$D_{2,\epsilon}: \quad |z| \leqq F, \ |y_i| \leqq G_i, \ |u| < \epsilon + 2FM_0 + \sum_{i=1}^{n} (e^{2FN} - e^{FN} + M_i e^{FN}) G_i.$$

Since a_0 has continuous derivatives of first order throughout $D_{2,\epsilon}$ and is bounded by M_0, there exists one and only one curve through an arbitrary point P, and the corresponding value $\bar{u}$ of u for $z=0$ lies within the range

$$|\bar{u}| < \epsilon + 3FM_0 + \sum_{i=1}^{n} (e^{2FN} - e^{FN} + M_i e^{FN}) G_i.$$

Conversely, an arbitrary curve of C is uniquely determined by the quantities $\bar{u}, y_1, \cdots, y_n$, and a point of the curve is determined by giving in addition the corresponding value of z. On writing $u=u(z, y_i, \bar{u})$, we find bounds for the derivatives of u with respect to the arguments $\bar{u}, y_1, \cdots, y_n, z$. We have, in fact,

$$\frac{d}{dz}\frac{\partial u}{\partial \bar{u}} = \frac{\partial a_0}{\partial u}\frac{\partial u}{\partial \bar{u}} \quad \text{with} \quad \frac{\partial u}{\partial \bar{u}} = 1 \quad \text{for} \quad z = 0,$$

hence

$$e^{-N|z|} \leqq \frac{\partial u}{\partial \bar{u}} \leqq e^{N|z|}.$$

Similarly

$$\frac{d}{dz}\frac{\partial u}{\partial y_i} = \frac{\partial a_0}{\partial y_i} + \frac{\partial a_0}{\partial u}\frac{\partial u}{\partial y_i} \quad \text{with} \quad \frac{\partial u}{\partial y_i} = 0 \quad \text{for} \quad z = 0,$$

hence

$$\left|\frac{\partial u}{\partial y_i}\right| \leqq e^{N|z|} - 1.$$

On introducing the new variables $z, y_1, \cdots, y_n, u$ instead of $z, y_1, \cdots, y_n, \bar{u}$ we find throughout $D_{2,\epsilon}$

$$\begin{gathered} \frac{\partial(z, y_i, u)}{\partial(z, y_i, \bar{u})} = \frac{\partial u}{\partial \bar{u}}, \quad \frac{\partial(z, y_i, \bar{u})}{\partial(z, y_i, u)} = \frac{\partial \bar{u}}{\partial u} = \left(\frac{\partial u}{\partial \bar{u}}\right)^{-1}, \quad e^{-|z|N} \leqq \frac{\partial \bar{u}}{\partial u} \leqq e^{|z|N}, \\ \frac{\partial \bar{u}}{\partial y_i} = -\frac{\partial u}{\partial y_i}\Big/\frac{\partial u}{\partial \bar{u}}, \quad \left|\frac{\partial \bar{u}}{\partial y_i}\right| \leqq e^{|z|N}(e^{|z|N} - 1). \end{gathered} \tag{6}$$

Since $\bar{u}$ is constant along any curve of C, we have

$$\frac{\partial \bar{u}}{\partial z} + a_0 \frac{\partial \bar{u}}{\partial u} = 0, \qquad \left| \frac{\partial \bar{u}}{\partial z} \right| \leqq M_0 e^{N|z|}.$$

Now put, throughout $D_{2,\epsilon}$,

$$A(z, y_i, u) = u - \bar{u}. \tag{7}$$

Evidently A satisfies (3) and (4). Furthermore we have

$$\begin{aligned} &|A| \leqq |z| M_0, \qquad \left| \frac{\partial A}{\partial y_i} \right| \leqq e^{|z|N}(e^{|z|N} - 1), \\ &\left| \frac{\partial A}{\partial z} \right| \leqq M_0 e^{N|z|}, \qquad e^{-|z|N} \leqq \left| 1 - \frac{\partial A}{\partial u} \right| \leqq e^{|z|N}. \end{aligned} \tag{8}$$

Returning to the ordinary differential equation (E), we remark that the conditions of our theorem allow us to write (E) in the form

$$\frac{du}{dx} = \phi(x, u)$$

with $\phi(x, u)$ continuous in the rectangle R determined by

$$0 \leqq x \leqq X, \quad |u| \leqq \frac{\epsilon}{2} + 2FM_0 + H, \quad H = \sum_{i=1}^{n} (e^{2FN} - e^{FN} + M_i e^{FN}) G_i.$$

The fundamental existence theorem for differential equations shows that a solution through $[x=0, u(0)=0]$ always may be continued until it reaches the boundary of R. Hence a solution which cannot be continued across a certain point x_1 with $0 \leqq x_1 < X$ may be assumed to exist for $0 \leqq x \leqq x_1$ and to satisfy the condition

$$|u(x_1)| = \frac{\epsilon}{2} + 2FM_0 + H.$$

Thus our theorem is proved as soon as we show the following property of $u(x)$. If, in the interval $0 \leqq x \leqq x_1$, the solution $u(x)$ of (E) satisfies the inequality

$$|u(x)| \leqq \frac{\epsilon}{2} + 2FM_0 + H,$$

it satisfies the stronger inequality

$$|u(x)| \leqq 2FM_0 + H.$$

Indeed, by (E) we have, since $z = f(x)$, $y_i = g_i(x)$, $u = u(x)$ stay in $D_{2,\epsilon}$,

$$(9)\qquad \begin{aligned} d(u - A) &= (a_0 - A_z)df(x) + \sum_{i=1}^{n} (a_i - A_{y_i})dg_i(x) - A_u du \\ &= (a_0 - A_z - A_u a_0)df(x) + \sum_{i=1}^{n} (a_i(1 - A_u) - A_{y_i})dg_i(x), \end{aligned}$$

with $z=f(x)$, $y_i=g_i(x)$ and $u(0)=0$. Hence we conclude from (3) and (8)

$$(10)\qquad | u(x) - A(f(x), g_i(x), u(x)) | \leqq FM_0 + H,$$

and, by (8),

$$(11)\qquad | u(x) | \leqq 2FM_0 + H.$$

Remarks. The function $u(z, y_i, \bar{u})$ is continuous. This may be expressed by the statement: If the quantities z, y_i, $u-A(z, y_i, u)$ tend to limiting values which belong to the domain $|z| \leqq F$, $|y_i| \leqq G_i$, $|u-A(z, y_i, u)| \leqq H+FM_0$, then u itself tends to a limiting value which in absolute value does not exceed $2FM_0+H$.

Any function satisfying a Lipschitz condition of exponent 1 in z, y_i, u satisfies also a Lipschitz condition of exponent 1 in the variables z, y_i, $u-A$. This follows from (8), for we have

$$(12)\qquad | u_1 - A(z, y_i, u_1) - u_2 + A(z, y_i, u_2) | \geqq | u_2 - u_1 | e^{-FN}.$$

THEOREM 3. *Assume that in the interval* $0 \leqq x \leqq X$

(i) *the functions* $f_\mu(x)$, $\mu=1, 2, \cdots$, *are continuously differentiable*, $|f_\mu(x)| < F$, *and* $f_\mu(x)$ *converges to a function* $f(x)$ *as* $\mu \to \infty$;

(ii) *the functions* $g_{1\mu}(x), \cdots, g_{n\mu}(x)$, $\mu=1, 2, \cdots$, *are continuously differentiable*, $g_{i\mu}(x) \to g_i(x)$ *as* $\mu \to \infty$, $i=1, 2, \cdots, n$, *where* $g_i(x)$ *is continuous, and the total variations of the differences* $T[g_{i\mu}-g_i]$ *tend to zero as* $\mu \to \infty$; *furthermore* $g_{i\mu}(0)=0$ *and* $T[g_{i\mu}] \leqq G_i$ *for all* i *and* μ;

(iii) *the functions* $a_0, a_1, \cdots, a_n$ *are defined in*

$$D_{3,\epsilon}:\qquad \begin{aligned} &| z | \leqq F, \quad | y_i | \leqq G_i, \quad | u | < \epsilon + 3FM_0 + H, \\ &H = \sum_{i=1}^{n} (e^{2FN} - e^{FN} + M_i e^{FN})G_i, \qquad (\epsilon > 0), \end{aligned}$$

and we have in $D_{3,\epsilon}$

$$(13)\qquad | a_0 | \leqq M_0, \ | a_1 | \leqq M_1, \cdots, | a_n | \leqq M_n.$$

Furthermore, $a_0, a_1, \cdots, a_n$ *have continuous derivatives of first order in* $D_{3,\epsilon}$, *and those of* a_0 *are bounded in absolute value by* N *and satisfy a Lipschitz condition of exponent* 1.

Then the solution of

$$(\mathrm{E}_\mu) \quad du_\mu(x) = a_0(f_\mu(x), g_{i\mu}(x), u_\mu(x))df_\mu(x) + \sum_{i=1}^{n} a_i dg_{i\mu}(x), \quad u_\mu(0) \equiv 0,$$

exists for $0 \leqq x \leqq X$, *satisfies*

$$(14) \qquad | u_\mu(x) | \leqq 2FM_0 + H,$$

and tends to a limit function $u(x)$ *as* $\mu \to \infty$. $u(x)$ *is said to be the solution of* (E) *for the initial condition* $u(0) = 0$.

From Theorem 2 we conclude the existence of $u_\mu(x)$ and the inequality (14). The classical statement about uniqueness of the solution of the initial problem may, incidentally, be used to establish the uniqueness of $u_\mu(x)$. On putting

$$B_\mu(x) = u_\mu(x) - A(f_\mu(x), g_{i\mu}(x), u_\mu(x)),$$

we conclude from (9) and (8)

$$(15) \qquad | B_\mu(x_1) - B_\mu(x_2) | \leqq \sum_{i=1}^{n} (M_i e^{FN} + e^{2FN} - e^{FN}) \int_{x_1}^{x_2} | dg_{i\mu}(x) |.$$

Now a theorem by Adams and Clarkson† shows that the total variation between any two points x_1 and x_2, of $g_{i\mu}(x)$ tends uniformly to that of $g_i(x)$ on account of the continuity of $g_i(x)$, and of the assumption (iii) that $T[g_{i\mu} - g_i] \to 0$, $g_{i\mu} \to g_i$. Thus (15) establishes equicontinuity for $B_\mu(x)$, while (14) gives boundedness. Hence, by Ascoli's theorem, we may select a subsequence $B_{\mu'}(x)$ tending uniformly to a function $B^*(x)$. From the remark on page 444 we conclude that also the corresponding subsequence of $u_\mu(x)$, say $u_{\mu'}(x)$, converges to a function $u^*(x)$. $B^*(x)$ satisfies the following integral equation

$$(16) \qquad B^*(x) = \int_0^x \sum_{i=1}^{n} (a_i(1 - A_u) - A_{y_i}) dg_i(x) - A(f^0, 0, 0, \cdots)$$

in which the expressions in a_i and A are to be considered as functions of z, y_i, u with $z = f(x)$, $y_i = g_i(x)$, $u - A = B^*(x)$. This follows from Lemma 1 and (9).

Any two subsequences of $B^*(x)$ converge to the same limit. In the opposite case we would have two functions $B^*(x)$ and $B^{**}(x)$, both satisfying (16). In view of (iii) the coefficients $a_0, a_1, \cdots, a_n$ admit of continuous derivatives with respect to z, y_i, u, whence also with respect to $z, y_i, u - A$, and thus satisfy a Lipschitz condition of exponent 1 in these variables, in the closed domain $|z| \leqq F$, $|y_i| \leqq G_i$, $|u - A| \leqq FM_0 + H$. On account of (3) and (4), the

† C. R. Adams and J. A. Clarkson, *On convergence in variation*, Bulletin of the American Mathematical Society, vol. 40 (1934), pp. 413–417.

derivatives $A_{y'}$ and A_u satisfy a Lipschitz condition with respect to z, y_i, u, whence with respect to z, y_i, $u-A$. Therefore we conclude from (16)

$$(17)\qquad |B^*(x) - B^{**}(x)| \leqq K\sum_i \int_0^x |B^*(x) - B^{**}(x)|\,|dg_i(x)|,$$

where K is a certain constant. On iterating (17) m times we easily find

$$(18)\quad \max_{0\leqq x\leqq X} |B^*(x) - B^{**}(x)| \leqq K^m \max_{0\leqq x\leqq X} |B^*(x) - B^{**}(x)| \left(\sum_{i=1}^n G_i\right)^m \Big/ m!$$

which gives $B^*(x) - B^{**}(x) = 0$ as $M\to\infty$.

Now the uniqueness of $B^*(x)$ implies, by the remark on page 444, the uniqueness of $u^*(x)$, which in turn justifies defining $u(x) = u^*(x)$ as the solution of (E) for the initial condition $u(0) = 0$.

Remark. The assumptions of Theorem 3 state bounds for the functions $a_0, a_1, \cdots, a_n(z, y_1, \cdots, y_n, u)$ holding in a domain that depends on these same bounds. One may ask for a formulation of the theorem such that a statement results for *any* functions $a_0, a_1, \cdots, a_n$, defined in an *arbitrary* neighborhood of the origin. Therefore we observe that there always exists a sub-neighborhood of the form $D_{3,\epsilon}$, provided the constants $F, G_1, \cdots, G_n$, ϵ can be decreased sufficiently. Since the functions $g_{1\mu}(x), \cdots, g_{n\mu}(x)$ are continuous, their total variations are also continuous and may be shown† to converge *uniformly* to those of $g_1(x), \cdots, g_n(x)$. Thus, omitting at most a finite number of values of μ and taking the upper end of the x-interval sufficiently small, makes it possible to choose $G_1, G_2, \cdots, G_n$ arbitrarily small. Whence we conclude the following theorem:

THEOREM 3′. *Suppose that, in a neighborhood of the origin of a* $(z, y_1, \cdots, y_n, u)$*-space the functions* $a_0, a_1, \cdots, a_n$ *are continuously differentiable and the derivatives of* a_0 *satisfy a Lipschitz condition of exponent* 1. *Assume that* n *continuously differentiable functions* $g_{1\mu}(x), \cdots, g_{n\mu}(x)$ *are defined in an interval* I $(0\leqq x\leqq X)$, *that they tend to continuous functions* $g_1(x), \cdots, g_n(x)$, *and the total variations* $T[g_{i\mu} - g_i]\to 0$ *as* $\mu\to\infty$, *and that* $g_{i\mu}(0) = 0$. *Assume furthermore that continuously differentiable functions* $f_\mu(x)$ *converge to a function* $f(x)$ *in* I *and that for all* μ *we have* $|f_\mu(x)| < F$. *Then the solution* $u_\mu(x)$ *of* (E_μ) *exists in a sufficiently small interval* $0\leqq x\leqq X'$ *which does not depend on* μ, *and converges there to a limit function* $u(x)$ *provided* F *is sufficiently small.*

The existence Theorems 3 and 3′ can be supplemented by a study of the manner in which the solution $u(x)$ depends on a parameter α on which the known functions in (E) may be supposed to depend. Usual methods of proving

† See Adams and Clarkson, loc. cit.

the continuity of $u(x)$, considered as a function of x and α, from that of the known functions could be carried through with only slight modifications.

Instead of (E) a system of differential equations of the form

$$(19)\qquad du_h(x) = a_{0h}(f(x), g_i(x), u_l(x))df(x) + \sum_{i=1}^{n} a_{ih}dg_i(x), \qquad u_h(0) = 0,$$

where $h=1, 2, \cdots, m$, may be studied, and the existence of a solution $u_h(x)$, $h=1, 2, \cdots, m$, can be concluded by a method analogous to that used in the proofs of Theorems 2 and 3. In view of the similarity of the procedure we shall not carry out these generalizations.

3. **Double integrals.** We denote by $T[\alpha, \beta]$ a triangle bounded by the line $\alpha=\beta$ of an (α, β)-plane and the parallels to the axes through (α, β). Similarly $t[f, g]$ designates a triangle of an (f, g)-plane, bounded by $f=g$ and the parallels to the f- and g-axes through (f, g). The elements of area $d\alpha d\beta$ and $dfdg$ are to be counted positive. By $f(\alpha)$ and $g(\beta)$ we understand continuously differentiable mappings of the α-axis on the f-axis and of the β-axis on the g-axis, which are, but for the elements used, identical with each other; $f(\alpha)=g(\beta)$ if $\alpha=\beta$. The range of the four variables α, β, f, g is the domain D with origin as center

$$(D)\qquad |\alpha| \leqq \omega, \quad |\beta| \leqq \omega, \quad |f| \leqq \omega, \quad |g| \leqq \omega,$$

and the function $f(\alpha)$ (and consequently $g(\beta)$) is such that for α and β satisfying (D) the point $(\alpha, f(\alpha), \beta, g(\beta))$ belongs to D. Furthermore there are defined in D three functions a, b, c of α, f, β, g having continuous derivatives up to the fourth order.

We introduce three functions X, Y, Z in D by the relations

$$(20)\qquad X_{fg} = ca_f b_g,$$

$$(21)\qquad Y_f = ca_f b_\beta, \qquad Z_g = ca_\alpha b_g,$$

and the initial conditions

$$\left.\begin{array}{ll}(22) & X = X_f = X_g = 0,\\ & Y = 0,\\ (23) & Z = 0,\end{array}\right\} \quad \text{if} \quad f = g.$$

We find

$$X(\alpha, f, \beta, g) = \iint_{t[f,g]} ca_f b_g(\alpha, f', \beta, g')df'dg'.$$

Here, as in all integrals that follow, care has been taken to indicate the arguments of the integrand at least in one of the factors of the integrand, to denote the variables of integration by a prime and to denote by subscripts the partial derivatives, while we reserve the symbols $d/d\alpha$ and $d/d\beta$ for total derivatives with respect to these variables.

We are going to study the function

$$\begin{aligned} I(\alpha, f, \beta, g) = X(\alpha, f, \beta, g) &- \int_\beta^\alpha (X_\alpha(\alpha', f(\alpha'), \beta, g) - Z)d\alpha' \\ &+ \int_\beta^\alpha (X_\beta(\alpha, f, \beta', g(\beta')) - Y)d\beta' \\ &+ \iint_{T[\alpha,\beta]} (-X_{\alpha\beta}+Z_\beta+Y_\alpha-ca_\alpha b_\beta(\alpha', f(\alpha'), \beta', g(\beta')))d\alpha' d\beta'. \end{aligned} \tag{24}$$

In order to abbreviate as much as possible, we write $A \sim B$ if $A - B$ is expressible as a polynomial in a, b, c, their first partial derivatives and their second partial derivatives of the type $\partial^2/\partial\alpha\partial\beta$, $\partial^2/\partial\alpha\partial g$, $\partial^2/\partial\beta\partial f$, $\partial^2/\partial f\partial g$. We write $A \cong B$ if $A - B$ is expressible as an integral over a function which itself is ~ 0.

Thus we find

$$I_f = X_f + \int_\beta^\alpha (X_{\beta f}(\alpha, f, \beta', g(\beta')) - Y_f)d\beta', \tag{25}$$

$$I_g = X_g - \int_\beta^\alpha (X_{\alpha g}(\alpha', f(\alpha'), \beta, g) - Z_g)d\alpha', \tag{26}$$

$$\begin{aligned} I_\alpha = {} & X_\alpha(\alpha, f, \beta, g) - X_\alpha(\alpha, f(\alpha), \beta, g) + Z(\alpha, f(\alpha), \beta, g) \\ &+ X_\beta(\alpha, f, \alpha, f(\alpha)) - Y(\alpha, f, \alpha, f(\alpha)) \\ &+ \int_\beta^\alpha (-X_{\alpha\beta} + Z_\beta + Y_\alpha - ca_\alpha b_\beta(\alpha, f(\alpha), \beta', g(\beta')))d\beta', \\ &+ \int_\beta^\alpha (X_{\alpha\beta}(\alpha, f, \beta', g(\beta')) - Y_\alpha)d\beta', \end{aligned} \tag{27}$$

$$\begin{aligned} I_\beta = {} & X_\beta(\alpha, f, \beta, g) - X_\beta(\alpha, f, \beta, g(\beta)) + Y(\alpha, f, \beta, g(\beta)) \\ &+ X_\alpha(\beta, g(\beta), \beta, g) - Z(\beta, g(\beta), \beta, g) \\ &- \int_\beta^\alpha (X_{\alpha\beta}(\alpha', f(\alpha'), \beta, g) - Z_\beta)d\alpha' \\ &- \int_\beta^\alpha (-X_{\alpha\beta} + Z_\beta + Y_\alpha - ca_\alpha b_\beta(\alpha', f(\alpha'), \beta, g(\beta)))d\alpha', \end{aligned} \tag{28}$$

$$
(29)\quad
\begin{aligned}
I_{fg} &= ca_f b_g,\\
I_{f\beta} &= X_{f\beta}(\alpha, f, \beta, g) - X_{f\beta}(\alpha, f, \beta, g(\beta)) + Y_f(\alpha, f, \beta, g(\beta)),\\
I_{g\alpha} &= X_{g\alpha}(\alpha, f, \beta, g) - X_{g\alpha}(\alpha, f(\alpha), \beta, g) + Z_g(\alpha, f(\alpha), \beta, g),\\
I_{\alpha\beta} &= X_{\alpha\beta}(\alpha, f, \beta, g) - X_{\alpha\beta}(\alpha, f(\alpha), \beta, g) + Z_\beta(\alpha, f(\alpha), \beta, g)\\
&\quad - Z_\beta(\alpha, f(\alpha), \beta, g(\beta)) + X_{\alpha\beta}(\alpha, f(\alpha), \beta, g(\beta)) - X_{\alpha\beta}(\alpha, f, \beta, g(\beta))\\
&\quad + Y_\alpha(\alpha, f, \beta, g(\beta)) - Y_\alpha(\alpha, f(\alpha), \beta, g(\beta)) + ca_\alpha b_\beta(\alpha, f(\alpha), \beta, g(\beta)).
\end{aligned}
$$

Hence

$$
\begin{aligned}
I_{f\beta} &= \int_{g(\beta)}^{g} (X_{f\beta g} - Y_{fg}(\alpha, f, \beta, g'))dg' + Y_f(\alpha, f, \beta, g)\\
&= \int_{g(\beta)}^{g} [(ca_f b_g)_\beta - (ca_f b_\beta)_g]dg' + ca_f b_\beta\\
&= \int_{g(\beta)}^{g} [(ca_f)_\beta b_g - (ca_f)_g b_\beta]dg' + ca_f b_\beta,
\end{aligned}
$$

$$(30)\qquad I_{f\beta} \cong ca_f b_\beta.$$

Similarly

$$(31)\qquad I_{g\alpha} \cong ca_\alpha b_g.$$

Moreover,

$$
\begin{aligned}
X_{\alpha\beta}(\alpha, f, \beta, g) - X_{\alpha\beta}(\alpha, f(\alpha), \beta, g) + X_{\alpha\beta}(\alpha, f(\alpha), \beta, g(\beta))&\\
- X_{\alpha\beta}(\alpha, f, \beta, g(\beta)) = \int_{f(\alpha)}^{f}\int_{g(\beta)}^{g} X_{\alpha\beta f g}(\alpha, f', \beta, g')df'dg'&.
\end{aligned}
$$

But

$$
\begin{aligned}
X_{\alpha\beta f g} &= (ca_f b_g)_{\alpha\beta} = (c_\alpha a_f b_g + ca_{f\alpha} b_g + ca_f b_{g\alpha})_\beta\\
&\sim c_\alpha a_f b_{g\beta} + c_\beta a_{\alpha f} b_g + ca_{f\alpha\beta} b_g + ca_{f\alpha} b_{g\beta} + ca_f b_{g\alpha\beta}\\
&\sim (c_\alpha a_f b_\beta)_g + (c_\beta a_\alpha b_g)_f + (ca_{\alpha\beta} b_g)_f + (ca_f b_{\alpha\beta})_g + ca_{\alpha f} b_{\beta g},\\
ca_{\alpha f} b_{\beta g} &= (ca_\alpha b_{\beta g})_f - c_f a_\alpha b_{\beta g} - ca_\alpha b_{g\beta f}\\
&\sim (ca_\alpha b_\beta)_{gf} - [(ca_\alpha)_g b_\beta]_f - (c_f a_\alpha b_\beta)_g - (ca_\alpha b_{\beta f})_g.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\int_{f(\alpha)}^{f}\int_{g(\beta)}^{g} X_{\alpha\beta f g}(\alpha, f', \beta, g')df'dg' &\cong ca_\alpha b_\beta(\alpha, f, \beta, g) - ca_\alpha b_\beta(\alpha, f(\alpha), \beta, g)\\
&\quad + ca_\alpha b_\beta(\alpha, f(\alpha), \beta, g(\beta)) - ca_\alpha b_\beta(\alpha, f, \beta, g(\beta)).
\end{aligned}
$$

Furthermore,

$$Z_\beta(\alpha, f(\alpha), \beta, g) - Z_\beta(\alpha, f(\alpha), \beta, g(\beta)) = \int_{g(\beta)}^{g} Z_{\beta g}(\alpha, f(\alpha), \beta, g')dg'$$
$$= \int_{g(\beta)}^{g} [ca_\alpha b_g(\alpha, f(\alpha), \beta, g')]_\beta dg'$$
$$\cong \int_{g(\beta)}^{g} [ca_\alpha b_\beta(\alpha, f(\alpha), \beta, g')]_g dg'$$
$$\cong ca_\alpha b_\beta(\alpha, f(\alpha), \beta, g) - ca_\alpha b_\beta(\alpha, f(\alpha), \beta, g(\beta)).$$

Similarly,

$$Y_\alpha(\alpha, f, \beta, g(\beta)) - Y_\alpha(\alpha, f(\alpha), \beta, g(\beta)) \cong ca_\alpha b_\beta(\alpha, f, \beta, g(\beta)) - ca_\alpha b_\beta(\alpha, f(\alpha), \beta, g(\beta)).$$

Finally

$$I_{\alpha\beta}(\alpha, f, \beta, g) \cong ca_\alpha b_\beta(\alpha, f, \beta, g). \tag{32}$$

In view of (29), we find

$$\frac{d^2 I}{d\alpha d\beta}(\alpha, f(\alpha), \beta, g(\beta)) = ca_\alpha b_\beta + ca_f b_\beta \frac{df(\alpha)}{d\alpha} + ca_\alpha b_g \frac{dg(\beta)}{d\beta} + ca_f b_g \frac{df(\alpha)}{d\alpha}\frac{dg(\beta)}{d\beta}$$
$$= c\frac{da(\alpha, f(\alpha), \beta, g(\beta))}{d\alpha}\frac{db(\alpha, f(\alpha), \beta, g(\beta))}{d\beta}$$

and

$$I(\alpha, f(\alpha), \beta, g(\beta)) = 0, \qquad \frac{dI}{d\alpha} = 0, \qquad \frac{dI}{d\beta} = 0,$$

for $\alpha = \beta$. Thus

$$I(\alpha, f(\alpha), \beta, g(\beta)) = -\iint_{T[\alpha,\beta]} c\frac{da}{d\alpha'}\frac{db}{d\beta'}(\alpha', f(\alpha'), \beta', g(\beta'))d\alpha' d\beta'. \tag{33}$$

In order to transform $I_f(\alpha, f, \beta, g)$ we calculate

$$\int_\beta^\alpha X_{\beta f}(\alpha, f, \beta', g(\beta'))d\beta' = \int_\beta^\alpha d\beta'\left[\int_f^{g(\beta')} X_{\beta f g}(\alpha, f, \beta', g')dg' + X_{\beta f}(\alpha, f, \beta', f)\right]$$
$$= \int_\beta^\alpha d\beta' \int_f^{g(\beta')} (ca_f b_g(\alpha, f, \beta', g'))_\beta dg'$$
$$\cong \int_\beta^\alpha d\beta' \int_f^{g(\beta')} (ca_f b_\beta(\alpha, f, \beta', g'))_g dg'$$
$$\cong \int_\beta^\alpha d\beta' [ca_f b_\beta(\alpha, f, \beta', g(\beta')) - ca_f b_\beta(\alpha, f, \beta', f)]$$

since, by (22), $X_{\beta f}(\alpha, f, \beta', f)$ vanishes.

Consequently

$$I_f(\alpha, f, \beta, g) \cong 0. \tag{34}$$

Similarly

$$I_g(\alpha, f, \beta, g) \cong 0. \tag{35}$$

Also we have

$$\begin{aligned} X_\alpha(\alpha, f, \beta, g) &= -\iint_{t[f,g]} (ca_f b_g(\alpha, f', \beta, g'))_\alpha df' dg' \\ &\cong -\iint_{t[f,g]} (ca_\alpha b_g(\alpha, f', \beta, g'))_f df' dg' \\ &\cong -\int ca_\alpha b_g(\alpha, f', \beta, g') dg', \end{aligned}$$

where the integration is to be extended over the boundary of $t[f, g]$; whence

$$X_\alpha(\alpha, f, \beta, g) \cong 0, \qquad X_\beta(\alpha, f, \beta, g) \cong 0.$$

Thus

$$I_\alpha(\alpha, f, \alpha, g) \cong 0, \tag{36}$$

as obviously $Y \cong 0$, $Z \cong 0$.

On writing

$$I_\alpha(\alpha, f, \beta, g) = I_\alpha(\alpha, f, \alpha, g) + \int_\alpha^\beta I_{\alpha\beta}(\alpha, f, \beta', g) d\beta',$$

we find by (36) and (32)

$$I_\alpha(\alpha, f, \beta, g) \cong 0. \tag{37}$$

Similarly,

$$I_\beta(\alpha, f, \beta, g) \cong 0. \tag{38}$$

Finally

$$\begin{aligned} &I(\alpha, f, \beta, g) = I(\alpha, f, \alpha, g) + \int_\alpha^\beta I_\beta(\alpha, f, \beta', g) d\beta', \\ &I(\alpha, f, \beta, g) \cong 0. \end{aligned} \tag{39}$$

The formulas (29), (30), (31), (32), (34), (35), (37), (38), and (39) prove that $I(\alpha, f, \beta, g)$, its first derivatives and its second derivatives of the type

$\partial^2/\partial\alpha\partial\beta$, $\partial^2/\partial\alpha\partial g$, $\partial^2/\partial\beta\partial f$, $\partial^2/\partial f\partial g$ may be expressed in terms of a, b, c, their first and second derivatives of the same type, and integrals over products of such functions.

Henceforth the definition (24) *of I is to be replaced by the explicit formula whose abbreviated equivalent is* (39), and which retains sense in the case that a, b, c admit only of continuous first derivatives and of continuous second derivatives of the indicated type. If we uniformly approximate a, b, c and said derivatives by polynomials in α, f, β, g and their respective derivatives, we may easily see that all of the formulas (29)–(32), (34), (35), (37)–(39) remain valid under the new assumptions and that $I(\alpha, f, \beta, g)$ still retains continuous first and second derivatives of said type.

Moreover a study of the dependence of I on the function $f(\alpha)$ shows that convergence of a sequence of continuously differentiable functions $f_\mu(\alpha)$ to a limit function $f_0(\alpha)$ implies the convergence of the corresponding functionals $I_\mu(\alpha, f, \beta, g)$ to a limit functional $I_0(\alpha, f, \beta, g)$, and uniform convergence of $f_\mu(\alpha)$ to $f_0(\alpha)$ entails uniform convergence of I_μ to I_0. In fact, $f_\mu(\alpha)$ appears in the definition of $I_\mu(\alpha, f, \beta, g)$ only in limits of integration with respect to f' or g', which implies the convergence mentioned of I_μ to I.

From the formulas (29)–(32), (34), (35), (37)–(39) we can derive estimates for $I(\alpha, f, \beta, g)$ and its derivatives. Suppose first that in $|\alpha|$, $|f|$, $|\beta|$, $|g| \leqq \omega$

$$|a|,\ |b|,\ |c| \leqq K, \quad |a_\alpha|, \cdots, |c_g| \leqq K', \quad |a_{\alpha\beta}|, \cdots, |c_{fg}| \leqq K''.$$

The terms suppressed in the above formulas by the use of the symbol $\cong$ are simple, double, triple, and quadruple integrals of polynomials of third degree in $a, b, c, a_\alpha, \cdots, c_{fg}$ with ranges of integration, respectively, $\leqq \rho\omega$, $\rho\omega^2$, $\rho\omega^3$, $\rho\omega^4$, where ρ denotes a sufficiently large number, for instance, 64. On the other hand, any one of the polynomials to be integrated is numerically smaller than a suitable polynomial in K, K', K'' of third degree with positive coefficients. Hence there exists a polynomial p with positive coefficients and of third degree in K, K', K'', such that for any one of formulas (29), $\cdots$, (39) the terms suppressed by the symbol $\cong$ are numerically less than or equal to

$$p(K, K', K'')(\omega + \omega^2 + \omega^3 + \omega^4).$$

We apply these estimates to the case where a, b, c *depend only indirectly* on α, f, β, g and are functions of $\psi_1, \cdots, \psi_n(\alpha, f, \beta, g)$ having third derivatives with respect to $\psi_1, \cdots, \psi_n$. We assume that in $|\alpha|$, $|f|$, $|\beta|$, $|g| \leqq \omega$, the following quantities exist and satisfy

$$(40) \qquad |\psi_1|, \cdots, |\psi_n| \leqq k,$$

$$(41) \qquad |\psi_{1\alpha}|, \cdots, |\psi_{ng}| \leqq k', \quad |\psi_{1\alpha\beta}|, \cdots, |\psi_{ngf}| \leqq k'', \quad (k, k', k'' > 0),$$

and that for ψ_i satisfying (40)

$$
\begin{aligned}
&|a|,\ |b|,\ |c| \leqq L,\\
&\left|\frac{\partial a}{\partial \psi_i}\right|,\ \left|\frac{\partial b}{\partial \psi_i}\right|,\ \left|\frac{\partial c}{\partial \psi_i}\right| \leqq L',\\
&\left|\frac{\partial^2 a}{\partial \psi_i \partial \psi_j}\right|,\ \cdots,\ \left|\frac{\partial^2 c}{\partial \psi_i \partial \psi_j}\right| \leqq L'',\\
&\left|\frac{\partial^3 a}{\partial \psi_i \partial \psi_j \partial \psi_l}\right|,\ \cdots,\ \left|\frac{\partial^3 c}{\partial \psi_i \partial \psi_j \partial \psi_l}\right| \leqq L'''.
\end{aligned}
\tag{42}
$$

We then write

$$I(\alpha, f, \beta, g) = I\begin{pmatrix}\alpha, f, \beta, g\\ \psi\end{pmatrix}.$$

In order to utilize the estimates found, it is legitimate to replace K by L, K' by $nL'k'$, K'' by $nLk''+n^2L''k'^2$. Thus from (29), $\cdots$, (39) we obtain

$$
\begin{aligned}
&\left|I\begin{pmatrix}\alpha, f, \beta, g\\ \psi\end{pmatrix}\right|,\ |I_\alpha|,\ \cdots,\ |I_g|\\
&\qquad \leqq (\omega + \omega^2 + \omega^3 + \omega^4) q(L, L', L'', k, k', k'')\\
&|I_{\alpha\beta}|,\ |I_{\alpha g}|,\ |I_{\beta f}|,\ |I_{fg}|\\
&\qquad \leqq n^2 L L'^2 k'^2 + (\omega + \omega^2 + \omega^3 + \omega^4) q(L, \cdots, k''),
\end{aligned}
\tag{43}
$$

where q is a suitable polynomial with positive coefficients.

We next state bounds for the difference between

$$I\begin{pmatrix}\alpha, f, \beta, g\\ \psi\end{pmatrix} \quad \text{and} \quad I\begin{pmatrix}\alpha, f, \beta, g\\ \psi'\end{pmatrix}$$

and its derivatives, assuming ψ_i' to be another system of continuously differentiable functions of α, f, β, g which satisfy the same inequalities (40), (41) as the ψ_i themselves do. For the sake of simplicity we suppose $\omega < \Omega$, with $\Omega > 0$ and denote by u and v

$$u = \sum_{i=1}^n \max |\psi_i - \psi_i'| + \sum_{i=1}^n \max |\psi_{i\alpha} - \psi_{i\alpha}'| + \cdots + \sum_{i=1}^n \max |\psi_{ig} - \psi_{ig}'|,$$

$$
\begin{aligned}
v = &\sum_{i=1}^n \max |\psi_{i\alpha\beta} - \psi_{i\alpha\beta}'| + \sum_{i=1}^n \max |\psi_{i\alpha g} - \psi_{i\alpha g}'|\\
&+ \sum_{i=1}^n \max |\psi_{i\beta f} - \psi_{i\beta f}'| + \sum_{i=1}^n \max |\psi_{ifg} - \psi_{ifg}'|,
\end{aligned}
$$

the maxima to be taken for the domain $|\alpha|$, $|f|$, $|\beta|$, $|g| \leqq \omega$.

We find by a procedure similar to the one previously used,

$$
\text{(44.1)}\qquad \left| I \begin{pmatrix}\alpha, f, \beta, g\\ \psi\end{pmatrix} - I \begin{pmatrix}\alpha, f, \beta, g\\ \psi'\end{pmatrix}\right| \leqq (u+v)C\omega,
$$

$$
\left| I_\alpha \begin{pmatrix}\alpha, f, \beta, g\\ \psi\end{pmatrix} - I_\alpha \begin{pmatrix}\alpha, f, \beta, g\\ \psi'\end{pmatrix}\right| \leqq (u+v)C\omega,
$$

$$
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
$$

$$
\left| I_g \begin{pmatrix}\alpha, f, \beta, g\\ \psi\end{pmatrix} - I_g \begin{pmatrix}\alpha, f, \beta, g\\ \psi'\end{pmatrix}\right| \leqq (u+v)C\omega,
$$

$$
\text{(44.2)}\qquad \left| I_{\alpha\beta} \begin{pmatrix}\alpha, f, \beta, g\\ \psi\end{pmatrix} - I_{\alpha\beta} \begin{pmatrix}\alpha, f, \beta, g\\ \psi'\end{pmatrix}\right| \leqq uC + vC\omega,
$$

$$
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
$$

$$
\left| I_{fg} \begin{pmatrix}\alpha, f, \beta, g\\ \psi\end{pmatrix} - I_{fg} \begin{pmatrix}\alpha, f, \beta, g\\ \psi'\end{pmatrix}\right| \leqq uC + vC\omega,
$$

with C depending on Ω, k, k', k'', L, L', L'', L'''. In an analogous way the difference between

$$
I\begin{pmatrix}\alpha, f, \beta, g\\ \psi\end{pmatrix} \quad \text{and} \quad \bar I\begin{pmatrix}\alpha, f, \beta, g\\ \psi'\end{pmatrix}
$$

may be estimated under the assumption that the functions used in the formation of $\bar I$, namely $\bar a$, $\bar b$, $\bar c(\psi_1, \cdots, \psi_n)$, and their first and second derivatives, differ by less than ϵ from those used for

$$
I\begin{pmatrix}\alpha, f, \beta, g\\ \psi\end{pmatrix}.
$$

We get

$$
\left| I \begin{pmatrix}\alpha, f, \beta, g\\ \psi\end{pmatrix} - \bar I \begin{pmatrix}\alpha, f, \beta, g\\ \psi'\end{pmatrix}\right| \leqq (u+v+\epsilon)C\omega,
$$

$$
\text{(45)}\qquad \left| I_\alpha \begin{pmatrix}\alpha, f, \beta, g\\ \psi\end{pmatrix} - \bar I_\alpha \begin{pmatrix}\alpha, f, \beta, g\\ \psi'\end{pmatrix}\right|, \cdots, \leqq C(u+v+\epsilon)\omega,
$$

$$
\left| I_{\alpha\beta} \begin{pmatrix}\alpha, f, \beta, g\\ \psi\end{pmatrix} - \bar I_{\alpha\beta} \begin{pmatrix}\alpha, f, \beta, g\\ \psi'\end{pmatrix}\right|, \cdots,
$$

$$
\left| I_{\beta g} \begin{pmatrix}\alpha, f, \beta, g\\ \psi\end{pmatrix} - \bar I_{fg} \begin{pmatrix}\alpha, f, \beta, g\\ \psi'\end{pmatrix}\right| \leqq C(\epsilon + u) + vC\omega.
$$

We now are able to formulate the main theorem of this section. Denote by

$$I_{ijl}\begin{pmatrix}\alpha, f, \beta, g\\ \psi\end{pmatrix} \quad \text{the functional} \quad I\begin{pmatrix}\alpha, f, \beta, g\\ \psi\end{pmatrix}$$

formed for

$$a = \psi_j, \qquad b = \psi_l, \qquad c = c_{ijl}.$$

THEOREM 4. *Denote by $\omega > 0$ a number $< \Omega$ and by D the domain $|\alpha|$, $|f|$, $|\beta|$, $|g| \leqq \omega$. Suppose c_{ijl} to be continuously differentiable up to the third order with respect to its arguments $\psi_1, \cdots, \psi_n$, and that for ψ_i in* (40), *$a=\psi_j$, $b=\psi_l$, $c=c_{ijl}$ the relations* (42) *hold. Suppose furthermore that in D the functions $\psi_i^0(\alpha, f, \beta, g)$ are continuously differentiable and admit of continuous second derivatives of the type mentioned such that*

$$|\psi_i^0(\alpha, f, \beta, g)| \leqq k/2, \tag{46}$$

$$|\psi^0{}_\alpha|, \cdots, |\psi^0{}_g| \leqq k'/2, \tag{47}$$

$$|\psi_{i\alpha\beta}^0|, |\psi_{i\alpha g}^0|, |\psi_{i\beta f}^0|, |\psi_{igf}^0| \leqq k''/3. \tag{48}$$

Then the system

$$\psi_i(\alpha, f, \beta, g) = \psi_i^0(\alpha, f, \beta, g) + \sum_{j,l=1}^{n} I_{ijl}\begin{pmatrix}\alpha, f, \beta, g\\ \psi\end{pmatrix}, \quad (i = 1, 2, \cdots, n), \tag{49}$$

has a solution $\psi_i(\alpha, f, \beta, g)$ existing and uniquely determined in $|\alpha|$, $|f|$, $|\beta|$, $|g| \leqq \omega'$, where $\omega' > 0$ is a number $\leqq \omega$ that may be determined with the aid of $\Omega, k, k', k'', L, L', L'', L'''$ only. $\psi_{i\alpha}(\alpha, f, \beta, g), \cdots, \psi_{ig}(\alpha, f, \beta, g)$ exist, are continuous and in absolute value $\leqq k'$, and $\psi_{i\alpha\beta}(\alpha, f, \beta, g), \cdots, \psi_{ifg}(\alpha, f, \beta, g)$ exist, are continuous and in absolute value $\leqq \max\,(k'', 12n^5LL'^2k'^2)$.*

We start the proof by increasing, if necessary, k'' so as to satisfy the inequality

$$4n^5LL'^2k'^2 < k''/3.$$

We use successive approximations:

$$\psi_i^1(\alpha, f, \beta, g) = \psi_i^0(\alpha, f, \beta, g) + \sum_{k,l=1}^{n} I_{ikl}\begin{pmatrix}\alpha, f, \beta, g\\ \psi^0\end{pmatrix},$$

and generally for $m \geqq 0$

$$\psi_i^{m+1}(\alpha, f, \beta, g) = \psi_i^0(\alpha, f, \beta, g) + \sum_{k,l=1}^{n} I_{ikl}\begin{pmatrix}\alpha, f, \beta, g\\ \psi^m\end{pmatrix}. \tag{49.1}$$

* In particular ω' does not depend on the function $f(\alpha)$ used in the definition of I_{ijl}.

Denoting, as before, by C a constant depending only on Ω, k, k', k'', L, L', L'', by ∂ a generic first derivative with respect to α, f, β, or g, and by ∂^2 a generic derivative of type $\partial^2/\partial\alpha\partial\beta$, $\partial^2/\partial\alpha\partial g$, $\partial^2/\partial\beta\partial f$, $\partial^2/\partial f\partial g$, we set, for $m>0$,

$$u_m = \sum_{i,j,l=1}^{n} \max \left| I_{ijl}\binom{\alpha, f, \beta, g}{\psi^m} - I_{ijl}\binom{\alpha, f, \beta, g}{\psi^{m-1}} \right| + \sum_{i,j,l=1}^{n} \sum_{\partial} \max \left| \partial(I_{ikl}(\psi^m) - I_{ijl}(\psi^{m-1})) \right|$$

$$v_m = \sum_{i,j,l=1}^{n} \sum_{\partial^2} \max \left| \partial^2\left(I_{ijl}\binom{\alpha, f, \beta, g}{\psi^m} - I_{ijl}\binom{\alpha, f, \beta, g}{\psi^{m-1}} \right) \right|,$$

where the maxima are to be taken in $|\alpha|$, $|f|$, $|\beta|$, $|g| \leqq \omega'$ with $\omega' < \Omega$ to be determined later. On putting

$$u_0 = \sum_{i,j,l=1}^{n} \max \left| I_{ijl}\binom{\alpha, f, \beta, g}{\psi^0} \right| + \sum_{i,j,l=1}^{n} \sum_{\partial} \max \left| \partial I_{ijl}\binom{\alpha, f, \beta, g}{\psi^0} \right|,$$

$$v_0 = \sum_{i,j,l=1}^{n} \sum_{\partial^2} \max \left| \partial^2 I_{ijl}\binom{\alpha, f, \beta, g}{\psi^0} \right|,$$

we find, by (43),

$$u_0 \leqq C\omega', \tag{50}$$

$$v_0 \leqq 4n^5 LL'^2 k'^2 + C\omega', \tag{51}$$

and, for u_m and v_m the recursion formulas, in view of (44.1) and (44.2),

$$u_{m+1} \leqq C\omega'(u_m + v_m), \tag{52.1}$$

$$v_{m+1} \leqq Cu_m + C\omega' v_m, \qquad (m = 0, 1, 2, \cdots) \tag{52.2}$$

provided, however, that we can choose $\omega' > 0$ so as to make sure the existence of *all* successive approximations $|\alpha|$, $|f|$, $|\beta|$, $|g| \leqq \omega'$ in the common domain D' ($|\alpha|$, $|f|$, $|\beta|$, $|g| \leqq \omega'$). Now determine $\omega' < \Omega$ so small that

$$1 - C\omega' > 0, \qquad (1 - C\omega')^2 - C^2\omega' > 0,$$

$$U \equiv \frac{C\omega'(1 - C\omega') + (4n^5 LL'^2 k'^2 + C\omega')C\omega'}{(1 - C\omega')^2 - C^2\omega'} \leqq \frac{\min(k, k')}{2}, \tag{53}$$

$$V \equiv \frac{C^2\omega' + (1 - C\omega')(4n^5 LL'^2 k'^2 + C\omega')}{(1 - C\omega')^2 - C^2\omega'} \leqq \frac{2k''}{3} \cdot$$

Note that

$$(1 + U + V)C\omega' \leqq U$$

and

$$4n^5LL'^2k'^2 + C\omega' + CU + C\omega'V \leqq V.$$

Now the conditions (46), (47), and (48) permit the construction of

$$I_{ikl}\begin{pmatrix}\alpha, f, \beta, g\\ \psi^0\end{pmatrix}$$

in D, hence a fortiori in D' ($|\alpha|, |f|, |\beta|, |g| \leqq \omega'$), and we certainly have, by (53),

$$u_0 \leqq U,$$
$$v_0 \leqq V.$$

Suppose that we could construct, throughout D', the mth approximation $\psi_i^m(\alpha, f, \beta, g)$ and that

$$\sum_0^m u_j \leqq U, \tag{54}$$

$$\sum_0^m v_j \leqq V. \tag{55}$$

We are then able to prove that we can construct the $(m+1)$st approximation and that

$$\sum_0^{m+1} u_j \leqq U, \qquad \sum_0^{m+1} v_j \leqq V.$$

In fact, we have, in view of (54) and (55), (46), (47), and (48),

$$\begin{aligned} |\psi_i^{m+1}(\alpha, f, \beta, g)| &\leqq |\psi_i^0(\alpha, f, \beta, g)| + \sum_0^m u_j \\ &\leqq |\psi_i^0(\alpha, f, \beta, g)| + U \leqq k, \end{aligned} \tag{56}$$

$$|\partial\psi_i^{m+1}(\alpha, f, \beta, g)| \leqq |\partial\psi_i^0(\alpha, f, \beta, g)| + \sum_0^m u_j \leqq k', \tag{57}$$

$$|\partial^2\psi_i^{m+1}(\alpha, f, \beta, g)| \leqq |\partial^2\psi_i^0(\alpha, f, \beta, g)| + \sum_0^m v_j \leqq k'', \tag{58}$$

and by (52),

$$\sum_0^{m+1} u_j \leqq u_0 + C\omega'\left(\sum_0^m u_j + \sum_0^m v_j\right) \leqq C\omega'(1 + U + V) \leqq U,$$

$$\sum_0^{m+1} v_j \leqq v_0 + C\sum_0^m u_j + C\omega'\sum_0^m v_j \leqq 4n^5LL'^2k'^2 + C\omega' + CU + C\omega'V \leqq V.$$

Thus (54) and (55) hold for all $m \geqq 0$, and we conclude the uniform convergence of $\psi_i^m(\alpha, f, \beta, g)$, $\partial\psi_i^m$, $\partial^2\psi_i^m$ to limit functions $\psi_i(\alpha, f, \beta, g)$ and their corresponding derivatives $\partial\psi_i$, $\partial^2\psi_i$. The continuity relation (45) finally proves

$$I\begin{pmatrix}\alpha, f, \beta, g\\ \psi^m\end{pmatrix} \to I\begin{pmatrix}\alpha, f, \beta, g\\ \psi\end{pmatrix}.$$

Hence by passage to the limit in (49.1) we obtain (49).

The uniqueness follows similarly from the relations analogous to (44.1) and (44.2),

$$u \leqq C\omega'(u + v) \tag{59.1}$$

$$v \leqq Cu + C\omega' v, \tag{59.2}$$

which yield $u(1 - C\omega') \leqq C\omega' v \leqq C\omega' \cdot Cu/(1 - C\omega')$, $u((1 - C\omega')^2 - C^2\omega') \leqq 0$, $u = 0$, $v = 0$, with

$$u = \sum_{i,j,l=1}^{n} \max \left| I_{ijl}\begin{pmatrix}\alpha, f, \beta, g\\ \psi\end{pmatrix} - I_{ijl}\begin{pmatrix}\alpha, f, \beta, g\\ \psi'\end{pmatrix}\right| + \sum_{i,j,l}\sum_{\partial} \max \left| \partial(I_{ijl}(\psi) - I_{ijl}(\psi'))\right|,$$

$$v = \sum_{i,j,l}\sum_{\partial^2} \max \left| \partial^2(I_{ijl}(\psi) - I_{ijl}(\psi'))\right|,$$

$\psi_i(\alpha, f, \beta, g)$ and $\psi_i'(\alpha, f, \beta, g)$ being solutions of (49).

COROLLARY 1. *If, in Theorem* 4, $f(\alpha)$, $\psi_1^0(\alpha, f, \beta, g), \cdots, \psi_n^0(\alpha, f, \beta, g)$, $\partial\psi_1^0(\alpha, f, \beta, g), \cdots, \partial\psi_n^0(\alpha, f, \beta, g)$, $\partial^2\psi_1^0(\alpha, f, \beta, g), \cdots, \partial^2\psi_n^0(\alpha, f, \beta, g)$, *and* $c_{ijl}(\psi_1, \cdots, \psi_n)$ *and its derivatives up to the third order depend on a parameter* μ *and converge uniformly as* $\mu \to \infty$, *then* $\psi_1(\alpha, f, \beta, g), \cdots, \psi_n(\alpha, f, \beta, g)$, $\partial\psi_1, \cdots, \partial\psi_n, \partial^2\psi_1, \cdots, \partial^2\psi_n$ *converge uniformly, as* $\mu \to \infty$, *in* $|\alpha|$, $|f|$, $|\beta|$, $|g| \leqq \omega'' \leqq \omega'$, *where* ω'' *depends only on* $\Omega, k, k', k'', L, L', L''$.

Denote by Δ the operation of taking the difference for two sufficiently large values of μ, and put, in the successive approximations of the proof of Theorem 4,

$$u_m = \sum_{i=1}^{n} \left| \Delta\psi_i^m(\alpha, f, \beta, g)\right| + \sum_{i=1,\partial}^{n} \left| \Delta\partial\psi_i^m\right|,$$

$$v_m = \sum_{i=1,\partial^2}^{n} \left| \Delta\partial^2\psi_i^m(\alpha, f, \beta, g)\right|.$$

Observing that $f(\alpha)$ enters in the functional I only as a limit of integration, as has been remarked earlier, we may use (45) and find, with some

$C=C(\Omega, k, k', k'', L, L', L'', L''')$ and a new and smaller value ω'' of ω', satisfying (53) with the new C:

$$u_m \leqq C(u_{m-1} + v_{m-1} + \epsilon)\omega'' + C\epsilon + u_0,$$
$$v_m \leqq Cu_{m-1} + Cv_{m-1}\omega'' + C\epsilon + v_0,$$
$$u_0 \leqq \epsilon, \qquad v_0 \leqq \epsilon.$$

Hence, for $m\to\infty$, $\lim u_m = u$, $\lim v_m = v$

$$u \leqq C(u + v + \epsilon)\omega'' + C\epsilon + u_0,$$
$$v \leqq Cu + Cv\omega'' + C\epsilon + v_0,$$
$$u(1 - C\omega'') \leqq (C + 1)\epsilon + \epsilon C\omega'' + C\omega'' v,$$
$$C\omega'' v \leqq \frac{C\omega''}{1 - C\omega''} Cu + \frac{(C+1)\epsilon C\omega''}{1 - C\omega''}\cdot$$

Hence u and v are $\leqq C'\epsilon$, with $C'(C, \omega'')$, which proves the corollary.

COROLLARY 2. *If, in Theorem 4, $f(\alpha)$, $\psi_i^0(\alpha, f, \beta, g)$, C_{ijl} depend on a parameter μ, and if $f(\alpha)$ converges uniformly as $\mu\to\mu_0$, then there exists a subsequence of μ, such that $\psi_i(\alpha, f, \beta, g)$ and also $\psi_i(\alpha, f(\alpha), \beta, g(\beta))$ converge uniformly.*

For by (57), $|\partial\psi_i(\alpha, f, \beta, g)| \leqq k'$ and hence the $\psi_i(\alpha, f, \beta, g)$ are equicontinuous and bounded, in view of (56). Hence there exists a uniformly convergent subsequence, and the corollary follows. From (33) we conclude under the hypothesis of the theorem, that

$$(60)\quad \frac{d^2}{d\alpha d\beta}\psi_i(\alpha, f(\alpha), \beta, g(\beta)) = \frac{d^2}{d\alpha d\beta}\psi_i^0(\alpha, f(\alpha), \beta, g(\beta)) + \sum_{j,l=1}^{n} c_{ijl}(\psi)\frac{d\psi_j}{d\alpha}\frac{d\psi_l}{d\beta},$$

and, for $\alpha=\beta$, $f=f(\alpha)$, $g=f(\alpha)$

$$\psi_i = \psi_i^0, \qquad \partial\psi_i = \partial\psi_i^0.$$

In the application we intend to make of Theorem 4 and its corollaries, the values of the constants such as Ω, k, k', k'', L, L', L'', L''' are of no importance. What matters, however, is their existence and their interdependence. Therefore, we are led to use the following terminology: we call a function *bounded* if its absolute value is bounded by a positive number irrespective of the values of its arguments and possible other parameters; we call, in a theorem, a quantity *relatively bounded* if its absolute value can be bounded by a positive number which depends only on other bounds *previously* introduced in the theorem; and we use the same term, in a proof, as meaning limitable by bounds, either assumed by the hypotheses of the theorem, or previously introduced in the course of the same proof.

Thus Theorem 4 and Corollary 2 may be formulated as follows:

THEOREM 5. *Suppose c_{ijl} and its derivatives up to the third order with respect to its arguments $\psi_1, \cdots, \psi_n$ bounded for bounded values of ψ_i, and assume that $\psi_i = \psi_i^0(\alpha, f, \beta, g)$, $i = 1, \cdots, n$, has derivatives $\partial\psi_i^0$, $\partial^2\psi_i^0$ which are continuous and bounded when α, f, β, g are bounded. Then the system*

$$\psi_i(\alpha, f, \beta, g) = \psi_i^0(\alpha, f, \beta, g) + \sum_{j,l=1}^{n} I_{ijl}\begin{pmatrix}\alpha, f, \beta, g\\ \psi\end{pmatrix}, \qquad (i = 1, 2, \cdots, n),$$

has a solution $\psi_i(\alpha, f, \beta, g)$, continuous together with $\partial\psi_i$ and $\partial^2\psi_i$. This solution exists, is uniquely determined and is relatively bounded together with the derivatives $\partial\psi_i$, $\partial^2\psi_i$ for relatively bounded α, f, β, g. If, in addition, $f(\alpha)$, $\psi_i^0(\alpha, f, \beta, g)$ and c_{ijl} and its derivatives up to the third order depend on a parameter μ, and if $f(\alpha)$ converges uniformly as $\mu \rightarrow \mu_0$, then there exists a subsequence of μ such that the corresponding functions $\psi_i(\alpha, f, \beta, g)$ and $\psi_i(\alpha, f(\alpha), \beta, g(\beta))$ converge uniformly for relatively bounded values of α, f, β, g.

4. **Hyperbolic systems.** The results of the preceding section may be used for a study of Cauchy's problem* for the system

$$(61) \qquad \begin{aligned} &\sum_{j=1}^{n} a_{ij}(\phi_1, \cdots, \phi_n)\frac{\partial\phi_j(\alpha, \beta)}{\partial\alpha} = 0, && i = 1, 2, \cdots, m < n,\\ &\sum_{j=1}^{n} a_{ij}(\phi_1, \cdots, \phi_n)\frac{\partial\phi_j(\alpha, \beta)}{\partial\beta} = 0, && i = m + 1, \cdots, n, \end{aligned}$$

in which a_{ij} and its partial derivatives up to the fourth order as well as the reciprocal value of the determinant $|a_{ij}|$ are bounded for bounded values of $\phi_1, \cdots, \phi_n$. The initial line is a bounded neighborhood of the origin on the line $\alpha = \beta$, and on it the unknown functions $\phi_1(\alpha, \beta), \cdots, \phi_n(\alpha, \beta)$ assume relatively bounded values $\xi_1(\alpha), \cdots, \xi_n(\alpha)$ which are continuously differentiable.

In view of the applications we subject the $\xi_1(\alpha), \cdots, \xi_n(\alpha)$ to the following condition:

Condition ϑ. $\xi_1(\alpha), \cdots, \xi_n(\alpha)$ depend continuously on α, and there exists a transformation

$$\zeta_i(\alpha) = \sum_{j=1}^{n} \xi_j(\alpha)\gamma_{ij}, \qquad \xi_j(\alpha) = \sum_{i=1}^{n} \Gamma_{ji}\zeta_i(\alpha),$$

* A study of this Cauchy problem with a view to enlarging the class of admissible initial conditions was undertaken by Margaret Gurney in her dissertation, Brown University, 1935 (unpublished).

with constant γ_{ij} of determinant ± 1 such that the derivatives of $\zeta_2(\alpha)$, $\zeta_3(\alpha), \cdots, \zeta_n(\alpha)$ are bounded.

Then there exists a relatively bounded solution $\phi_1(\alpha, \beta), \cdots, \phi_n(\alpha, \beta)$ of (61) in a relatively bounded (α, β)-neighborhood of the origin, assuming the given initial values, continuous in α, β, and continuously differentiable with respect to α and β.

It should be noticed that the essential content of the above statement lies in the fact that the derivative of $\zeta_1(\alpha)$ has no influence on the determination of the domain of existence.

The idea of the proof is to construct instead of functions $\phi_i(\alpha, \beta)$ other functions ψ_i of four arguments α, f, β, g which reduce to the solution of the initial problem in question for $f=\zeta_1(\alpha)$, $g=\zeta_1(\beta)$. In order to conform with the terminology formerly introduced, we henceforth shall identify $\zeta_1(\alpha)$ with $f(\alpha)$.

We try to satisfy the following conditions for functions $\psi_i^0(\alpha, f, \beta, g)$:

$$\text{(i)} \qquad \frac{d^2\psi_i^0}{d\alpha d\beta}(\alpha, f(\alpha), \beta, g(\beta)) = 0,$$

$$\text{(ii)} \qquad \psi_i^0(\alpha, f(\alpha), \alpha, f(\alpha)) = \xi_i(\alpha),$$

$$\text{(iii)} \qquad \begin{aligned} &\sum_{j=1}^n a_{ij}(\xi_1(\alpha), \cdots, \xi_n(\alpha)) \Big[\overset{0}{\psi}_{i\alpha}(\alpha, f(\alpha), \alpha, f(\alpha)) \\ &\qquad\qquad + \psi_{if}(\alpha, f(\alpha), \alpha, f(\alpha)) \frac{df(\alpha)}{d\alpha} \Big] = 0, \quad i \leqq m, \\ &\sum_{j=1}^n a_{ij}(\xi_1(\alpha), \cdots, \xi_n(\alpha)) \Big[\overset{0}{\psi}_{i\beta}(\alpha, f(\alpha), \alpha, f(\alpha)) \\ &\qquad\qquad + \psi_{ig}(\alpha, f(\alpha), \alpha, f(\alpha)) \frac{df(\alpha)}{d\alpha} \Big] = 0, \quad i > m. \end{aligned}$$

We first introduce $\psi_i^0(\alpha, f, \alpha, f)$ by

$$\text{(62)} \qquad \psi_i^0(\alpha, f, \alpha, f) = \sum_{k=2}^n \Gamma_{ik}\zeta_k(\alpha) + \Gamma_{i1} f, \qquad i = 1, 2, \cdots, n,$$

which yields

$$\text{(63)} \qquad \overset{0}{\psi}_{if}(\alpha, f, \alpha, f) + \overset{0}{\psi}_{ig}(\alpha, f, \alpha, f) = \Gamma_{i1}.$$

To determine ψ_{if}^0 and ψ_{ig}^0 we set up the system

$$\text{(64)} \qquad \begin{aligned} &\sum_{k=1}^n a_{ik}(\psi^0(\alpha, f, \alpha, f))\overset{0}{\psi}_{kf}(\alpha, f, \alpha, f) = 0, \qquad i = 1, \cdots, m, \\ &\sum_{k=1}^n a_{ik}(\psi^0(\alpha, f, \alpha, f))\overset{0}{\psi}_{kg}(\alpha, f, \alpha, f) = 0, \qquad i = m+1, \cdots, n, \end{aligned}$$

which together with (63), in view of the boundedness of $|a_{ik}|^{-1}$, determines $\psi^0_{if}(\alpha, f, \alpha, f)$ and $\psi_{ig}^0(\alpha, f, \alpha, f)$ as analytic functions of $a_{ik}(\psi^0(\alpha, f, \alpha, f))$ and thus as relatively bounded functions with relatively bounded and continuous total derivatives with respect to α and f.

From (ii)

$$\psi^0_{i\alpha}(\alpha, f, \alpha, f) + \psi^0_{i\beta}(\alpha, f, \alpha, f) = \sum_{k=2}^{n} \Gamma_{ik} \frac{d\zeta_k(\alpha)}{d\alpha}, \qquad i = 1, 2, \cdots, n,$$

and by (64) and (iii)

$$\sum_{k=1}^{n} a_{ik}(\xi_1(\alpha), \cdots, \xi_n(\alpha))\psi^0_{i\alpha}(\alpha, f(\alpha), \alpha, f(\alpha)) = 0, \qquad i \leqq m;$$
$$\sum_{k=1}^{n} a_{ik}\psi^0_{i\beta}(\alpha, f(\alpha), \alpha, f(\alpha)) = 0, \qquad i > m, \tag{65}$$

which determine $\psi^0_{i\alpha}(\alpha, f(\alpha), \alpha, f(\alpha))$ and $\psi^0_{i\beta}(\alpha, f(\alpha), \alpha, f(\alpha))$ as continuous and relatively bounded functions of α for relatively bounded α.

We now put

$$A_i(\alpha, f) = \int_0^f \psi^0_{ig}(\alpha, f', \alpha, f')df'.$$

Obviously, $A_i(\alpha, f)$ has continuous and relatively bounded derivatives with respect to f and α.

Finally put

$$\Psi_i^0(\alpha, f, \beta, g) = \psi_i^0(\alpha, f, \alpha, f) + A_i(\beta, g) - A_i(\alpha, f)$$
$$- \int_\beta^\alpha [\psi^0_{i\beta}(\alpha', f(\alpha'), \alpha', f(\alpha')) - A_{i\alpha}(\alpha', f(\alpha'))]d\alpha'. \tag{66}$$

The reader will easily verify that the function $\Psi_i^0(\alpha, f, \beta, g)$, as defined by (66), has the following properties:

$$\Psi_i^0(\alpha, f, \alpha, f) = \psi_i^0(\alpha, f, \alpha, f),$$
$$\Psi^0_{ig}(\alpha, f, \alpha, f) = \psi^0_{ig}(\alpha, f, \alpha, f),$$
$$\Psi^0_{if}(\alpha, f, \alpha, f) = \frac{d}{df}\psi_i^0(\alpha, f, \alpha, f) - \frac{dA_i(\alpha, f)}{df} = \psi^0_{if}(\alpha, f, \alpha, f),$$
$$\Psi^0_{i\beta}(\alpha, f(\alpha), \alpha, f(\alpha)) = \psi^0_{i\beta}(\alpha, f(\alpha), \alpha, f(\alpha)),$$
$$\Psi^0_{i\alpha}(\alpha, f(\alpha), \alpha, f(\alpha)) = \psi^0_{i\alpha}(\alpha, f(\alpha), \alpha, f(\alpha)).$$

Hence we are justified in considering $\Psi_i^0(\alpha, f, \beta, g)$ as an extension of those elements of the unknown function $\psi_i^0(\alpha, f, \beta, g)$ which were used in the construction of $\Psi_i^0(\alpha, f, \beta, g)$, and we write $\Psi_i^0(\alpha, f, \beta, g) = \psi_i^0(\alpha, f, \beta, g)$. We have, moreover, $\partial^2\psi_i^0 = 0$ so that (i) is true. From (62) we obtain (ii). Formulas (64) and (65) give (iii).

Evidently, $|\psi_i^0(\alpha, f, \beta, g)|$ is less than an arbitrary positive number ϵ if the bounds of the initial data $\xi_1(\alpha), \cdots, \xi_n(\alpha)$ are sufficiently small and if α, f, β, g are relatively bounded.

On differentiating the first m equations of (61) with respect to β, and the last $(n-m)$ equations with respect to α and solving with respect to the mixed derivatives, we obtain a system of the form

$$\frac{\partial^2\phi_i(\alpha, \beta)}{\partial\alpha\partial\beta} = \sum_{j,l} c_{ijl}(\phi_1, \cdots, \phi_n) \frac{\partial\phi_j(\alpha, \beta)}{\partial\alpha} \frac{\partial\phi_l(\alpha, \beta)}{\partial\beta}, \tag{67}$$

where the $c_{ijl}(\phi_1, \cdots, \phi_n)$ have bounded derivatives up to the third order for bounded $\phi_1, \cdots, \phi_n$. Replacing ϕ by ψ, we solve

$$\psi_i(\alpha, f, \beta, g) = \psi_i^0(\alpha, f, \beta, g) + \sum_{j,l=1}^{n} I_{ijl}\begin{pmatrix}\alpha, f, \beta, g \\ \psi\end{pmatrix}$$

with the aid of Theorem 5. By (60) and (i) we have

$$\frac{d^2\psi_i(\alpha, f(\alpha), \beta, g(\beta))}{d\alpha d\beta} = \sum_{i,l} c_{ijl}(\psi(\alpha, f(\alpha), \beta, g(\beta))) \frac{d\psi_j(\alpha, f(\alpha), \beta, g(\beta))}{d\alpha} \frac{d\psi_l(\alpha, f(\alpha), \beta, g(\beta))}{d\beta}.$$

In view of (ii), $\phi_i(\alpha, \beta) = \psi_i(\alpha, f(\alpha), \beta, g(\beta))$ assumes the given initial values, satisfies (61) on $\alpha = \beta$, and has continuous derivatives with respect to α and β and continuous mixed second derivatives with respect to α and β.

A conclusion, familiar in the theory of hyperbolic equations shows that equations (61) are satisfied identically in α and β.

Thus we have established the following theorem:

THEOREM 6. *If in* (61) a_{ij} *and its partial derivatives up to the fourth order and the reciprocal value of the determinant* $|a_{ij}|$ *are bounded for bounded values of* $\phi_1, \phi_2, \cdots, \phi_n$, *and if the initial values of* $\phi_i(\alpha, \beta)$ *on* $\alpha = \beta$ *are relatively bounded in a bounded neighborhood of* $\alpha = 0$ $(=\beta)$ *and satisfy condition* ϑ, *then Cauchy's problem has a solution existing for all relatively bounded* α, β. *This solution has continuous derivatives with respect to* α *and* β. *If the initial values and the* a_{ij} *depend on a parameter* μ *and converge uniformly as* $\mu \to \mu_0$, *then there exists a subsequence of* μ, *for which the corresponding solutions* $\phi_i(\alpha, \beta)$ *converge uniformly.*

COROLLARY 1. *The solution of Theorem* 6 *is unique.**

Let $|\alpha| \leqq A$, $|\beta| \leqq B$ be the common domain D of existence of two solutions of our initial problem. Denote by $u, v, w(\tau)$

$$u(\tau) = \sum_{i=1}^{n} \max |\Delta\phi_i|, \qquad v(\tau) = \sum_i \max \left| \Delta \frac{\partial\phi_i}{\partial\alpha} \right|,$$

$$w(\tau) = \sum_i \max \left| \Delta \frac{\partial\phi_i}{\partial\beta} \right|,$$

where the operator Δ indicates the difference of the expression following Δ for the two solutions, and the maximum is to be taken on that segment of the line $\tau = |\alpha - \beta|$ which is contained in D. By (67) we have for a suitable constant K

$$u(\tau) \leqq \int_0^\tau (v + w) |d\tau|, \qquad u(0) = 0,$$

$$v(\tau) \leqq K \int_0^\tau (u + v + w) |d\tau|, \qquad v(0) = 0,$$

$$w(\tau) \leqq K \int_0^\tau (u + v + w) |d\tau|, \qquad w(0) = 0.$$

Hence

$$u + v + w \leqq (2K + 1) \int_0^\tau (u + v + w) |d\tau|,$$

and by the well known iteration $u \equiv v \equiv w \equiv 0$.

By reasoning very similar to the preceding it may be shown that the dependence of the initial data on a parameter such that the initial data of ϕ_i and those of $\partial\phi_i/\partial\alpha$, $\partial\phi_i/\partial\beta$ satisfy a Lipschitz condition of exponent 1 in the parameter implies a Lipschitz condition of exponent 1 in the solution. Furthermore, passing to the limit from difference quotient to derivative with respect to the parameter we obtain the following corollary:

COROLLARY 2. *If, in Theorem* 6, *the initial data of* ϕ_i *and those of* $\partial\phi_i/\partial\alpha$, $\partial\phi_i/\partial\beta$ *are continuously differentiable with respect to a parameter, the solution and its first derivatives with respect to* α *and* β *are also continuously differentiable with respect to the parameter, continuity being understood with respect to the parameter and variables.*

* Cf. Hadamard, *Leçons sur le Problème de Cauchy*, Paris, 1932, pp. 488–501.

THE UNIVERSITY OF CALIFORNIA,
BERKELEY, CALIF.