STEINITZ FIELD TOWERS FOR MODULAR FIELDS*

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1. **Introduction.** The systematic study of the most general modular fields of characteristic p appears in its classical form in the famous Steinitz monograph [5]. Very little further analysis of such fields has been undertaken, except that in 1934 Hasse and Schmidt [2] showed that the structure of complete fields with valuations can be discussed in terms of a suitable transfinite but "separable" generation for an arbitrary modular field K. The theorem that such a "separable" generation (with the specific properties quoted below, in §9) must exist for every field K they stated but did not prove. We propose to show that their theorem, as stated, cannot be true. First, certain special cases or modifications of this theorem can be established, as in §§4 and 5, but badly imperfect fields and fields obtained by the adjunction of a denumerable infinitude of algebraically independent elements can be suitably constructed (§§7 and 8) as counter-examples to the general theorem. The most elaborate of our counter-examples, given in §8, seems almost pathological, but actually initiates many problems on the structure of such modular fields, such as the generalization of the lemmas used to analyze such an example or the formulation of other canonical generations for arbitrary fields.

What "separable" generations of a field K are considered? If K can be obtained from a prime field P by the successive adjunction of elements, each one of which is transcendental or separable algebraic over the field previously obtained, then K has a "separating transcendence basis" over the subfield P. When there is no such separating basis, it may still be possible to represent the whole field K as the union of the fields of a tower

$$M_0 \subset M_1 \subset M_2 \subset \cdots \subset K, \tag{1}$$

in which each individual field M_i does have a separating transcendence basis. Such towers of "residue-class fields" M_i appear in the Hasse-Schmidt analysis of a topologically complete field $\mathfrak{K}$ with a discrete valuation. The "residue-class field" K of such a field $\mathfrak{K}$ is obtained just as the Galois field of p elements is obtained by reducing the integers (or the p-adic integers†) modulo p. To construct a complete field $\mathfrak{K}$ with given residue-class field K one seeks to obtain $\mathfrak{K}$ by successive extensions

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† For a discussion of valuations, see, for instance, Albert [1, chaps. 11 and 12].

$$\mathfrak{M}_0 \subset \mathfrak{M}_1 \subset \mathfrak{M}_2 \subset \cdots \subset \mathfrak{K} \tag{2}$$

parallel to the tower (1) under the residue-class field. This parallel construction requires the Hensel-Rychlik irreducibility theorem, which states that a separable polynomial $g(t)$ with a root x in the residue-class field K corresponds to a polynomial over the complete field $\mathfrak{K}$ with a root ξ in $\mathfrak{K}$ (and in the residue class x).* This theorem will construct $\mathfrak{M}_1/\mathfrak{M}_0$ provided M_1 is separable and algebraic over M_0 in (1); hence the desirability of a separable tower (1).

For a perfect field K, Schmidt obtained such a "Steinitz" separating tower. For instance, if $K=P(t, t^{p^{-1}}, t^{p^{-2}}, \cdots)$ is obtained from a perfect subfield P by the adjunction of all pth roots of a single indeterminate t, and if S_e is the subfield $P(t^{p^{-e}})$, then we have a "tower"

$$S_0 \subset S_1 \subset S_2 \subset S_3 \subset \cdots, \qquad K = \sum_{e=0}^{\infty} S_e,$$

where the summation sign used here denotes the "union" or "composite" of the fields S_e indicated. This tower then has additional properties:

(i) Each S_e is separable over the transcendence basis $t^{p^{-e}}$.

(ii) $S_{e-1}=S_e^p$, where S_e^p denotes the field of all pth powers of elements of S_e.

For imperfect fields Schmidt has formulated a generalized Steinitz tower, constructed over a suitable base field L, and having properties similar to (i) and (ii). Our counter-examples† concern this tower, and we show in §§4 and 5 that a modified such tower is possible if K has a finite transcendence basis over L. Our chief tool is Lemma I in §2, which makes it possible to exchange certain elements for other elements in a given transcendence basis, without any loss of separability. This lemma resembles the so-called Steinitz exchange theorem.

The subfields L over which the towers for the "relatively perfect" field K are to be constructed are obtained in §3 by a simple application of Teichmüller's notion of the p-basis of a modular field [6, §3]. The last paragraphs contain a precise statement of the relation of our counter-examples to the theorem of Schmidt.

2. **The exchange lemma.** We consider exclusively fields of characteristic a fixed prime p. If $L \subset K$ are such fields, an element α of K is said to be

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* Cf. Hasse-Schmidt [2, p. 31], or, for the p-adic number case, Albert [1, Lemma, p. 296].

† The results of Hasse-Schmidt in [2] on complete fields with valuations are not called into question, since Witt and Teichmüller have subsequently established them by other methods. See [7], [8], or [4].

separable over L if α satisfies over L an irreducible polynomial equation without multiple roots. The field K is *separable algebraic* over L if every element of K is separable and algebraic over L. If K is not algebraic over L, a *transcendence basis* for K over L is a subset T of K such that K is algebraic over $L(T)$ but not algebraic over $L(T')$ for any proper subset T' of T. Here $L(T)$ denotes the field obtained from L by adjoining all elements of the set T.

We shall be concerned often with a "separating" basis T. A subset T of K is a *separating transcendence basis* (s.t.b.) for K over L if and only if T is a transcendence basis for K over L such that K is separable algebraic over $L(T)$.

THEOREM 2.1. *A field K has a separating transcendence basis over a subfield L if and only if the elements of K can be well ordered in such a way that every element b of K is either transcendental or separable algebraic over the field K_b obtained by adjoining to L all elements prior to b in the well ordering of K.*

For a given s.t.b. T the required well ordering can be constructed by listing first the elements of T in any order and then the remaining elements of K in any order. Conversely, given the well ordering, the corresponding s.t.b. T is simply the set of those elements b which are, respectively, transcendental over the corresponding fields K_b. A field K with such a well ordering has been called by Schmidt [2] a field "separable" over L. Hence K is "separable" over L if and only if it has a s.t.b.

Inseparable equations involve the variables only as pth powers. If a polynomial in the variable y (the coefficients may involve other variables) can be written as $f(y) = \sum a_i y^{ip^e}$ with at least one $a_i \neq 0$ we say that f has *exponent* p^e in y. We recall that an element α inseparable over a field L satisfies an irreducible equation $f(y)=0$ over L in which y has exponent $p^e > 1$; furthermore p^e is the smallest exponent such that α^{p^e} is separable over L. We call p^e the *exponent* of α over L. (In Steinitz' work e itself was known as the exponent.)

The following "exchange" lemma is used repeatedly:

LEMMA I. *If in a field K the elements of a subset $T \subset K$ are algebraically independent* over a perfect subfield P of K, and if the element y of K is separable over the field $P(T)$, while $y^{1/p}$ is not separable over $P(T)$, then there is an element x in the set T such that y is not separable over† $P(T-\{x\}, x^p)$. Any such element x is separable over $P(T-\{x\}, y)$, but not over the field $P(T-\{x\}, y^p)$.*

In effect, the lemma says that the fields $P(T-\{x\}, x)$ and $P(T-\{x\}, y)$ each consist of elements separable over the other field—an exchange of x for y.

* A set of elements is algebraically independent over a field if the elements satisfy no non-trivial polynomial equations with coefficients in the field.

† Here $T-\{x\}$ denotes the set T with the element x deleted.

Proof. The algebraic equation for y over $P(T)$ can be written in the form $g(y, T)=0$, where g has coefficients in P, is of exponent 1 in y, and is irreducible as a polynomial over P in the variables y, T. If g had exponent p or greater in each variable of T, we could take the pth root of each term in $g(y, T)$ to get a separable equation for $y^{1/p}$ over $P(T)$, counter to hypothesis. Therefore, at least one quantity x of T appears in g with exponent 1. If $T'=T-\{x\}$ be the set of the remaining elements in T, the equation $g=0$, in the form $g(y, x, T')=0$, shows that x is algebraic over the field $P(y, T')$. The elements y, T' of the set generating this field are therefore algebraically independent.

By construction, $g(y, x, T')$ is irreducible as a polynomial in the variable x over the ring $P[y, T']$. The Gauss lemma shows that $g(y, x, T')$ is also irreducible as a polynomial in x over $P(y, T')$. Since the polynomial has exponent 1 in x, the root x is separable over the field $P(y, T')$, as asserted. Furthermore, x cannot be separable over the smaller field $P(y^p, T')$, for in that event y would be separable over $P(x, T')$ which in turn would be separable over $P(y^p, T')$, although y manifestly satisfies an inseparable irreducible equation of degree p over $P(y^p, T')$.

The element x so exchanged with y was chosen as any element of T of exponent 1 in the equation $g(y, T)=0$. The assertion of the lemma that it may be chosen as any x such that y is not separable over $P(T-\{x\}, x^p)$ is a result of the following lemma:

LEMMA II. *If the elements of $T\subset K$ are algebraically independent over a perfect subfield P of K, and if an element y in K satisfies a separable polynomial equation $f(y)=0$ with coefficients in $P[T]$ and irreducible over $P[T]$, then an element x of T appears in this equation with exponent* 1 *if and only if y is inseparable over $P(T-\{x\}, x^p)$.*

If x appears in f only with exponent $p^e>1$, then $f(y)=0$ is manifestly an irreducible separable equation for y over $P(T', x^p)$, where $T'=T-\{x\}$. This establishes one half of the lemma. Conversely, suppose that x appears with exponent 1 in $f(y)$. Then $f(y)=g(y, x, T')$ has exponent 1 in x and in y and is irreducible in the ring $P[y, x, T']$, where y, x, and T' are regarded as independent variables. Consider the polynomial

$$g^{(p)}(y^p, x^p, T'^p) = [g(y, x, T')]^p$$

where $g^{(p)}$ denotes the function obtained from g by replacing each coefficient by its pth power. Then $g^{(p)}(y^p, x^p, T'^p)$, which is the pth power of an irreducible polynomial g in $P[y, x, T']$, can be in no way reducible in the smaller ring $P[y, x^p, T']$, which does not contain this irreducible factor $g(y, x, T')$. In other words, $g^{(p)}(y^p, x^p, T'^p)$ is irreducible in $P[y, x^p, T']$ and hence by the

Gauss lemma is irreducible in $P(x^p, T')[y]$. This means that the element y satisfies an equation with exponent p over $P(x^p, T')$, which makes y inseparable over this field $P(x^p, T')$, as asserted.

An element α is said to be *purely inseparable* over a field L if α^{p^m} is in L for some power p^m. If p^m is chosen as the least such power, then $x^{p^m} - \alpha^{p^m} = 0$ is known to be the irreducible equation satisfied by α over L. A field K is purely inseparable over L if every element of K is purely inseparable over L. If an element α of K is both purely inseparable and separable algebraic over L, then α satisfies over L two equations, a separable equation $f(x)$ with no multiple roots and a purely inseparable equation with only one root. The greatest common divisor of these two equations is linear and has the form $x - \alpha = 0$, with a coefficient α in L; hence the useful remark (Teichmüller [6, Theorem 12]):

LEMMA III. *An element α both separable and purely inseparable over a field L lies in that field.*

3. **Relatively perfect intermediate fields.** The *perfect closure* or *least perfect extension* of a field K is the field obtained by adjoining to K all roots x^{1/p^e} of elements x in K, for all integers $e \geqq 0$. If K^{p^e} is taken to denote the field of all elements x^{p^e}, for x in K, then $K^{p^{-1}}$ is the field obtained from K by the adjunction of all pth roots of elements of K, while the perfect closure $K^{p^{-\infty}}$ becomes $K^{p^{-\infty}} = K(K^{p^{-1}}, K^{p^{-2}}, \cdots)$.

F. K. Schmidt has called a field K *relatively perfect* over a subfield L if the perfect closure of K can be obtained by adjoining to K roots of elements in L alone; that is, if $K^{p^{-\infty}} = K(L^{p^{-\infty}})$. Here $K(L^{p^{-\infty}})$ can be considered as the composite $K \cup L^{p^{-\infty}}$ of K and $L^{p^{-\infty}}$ formed within the larger field $K^{p^{-\infty}}$. In particular, K is certainly relatively perfect over L if $K = K^p(L)$; that is, if $K^{p^{-1}} = K(L^{p^{-1}})$. For the construction of field towers we use the existence of such subfields L in the following explicit sense:*

THEOREM 3.1. *If P is a perfect subfield of K, then there exists an intermediate field L with $P \subset L \subset K$ such that $K = K^p(L)$ and such that L has a separating transcendence basis over P and is relatively algebraically closed† in K.*

To establish this theorem, we utilize the notion of p-independence due to Teichmüller [6]. A subset X of K is p-independent in K if $K^p(X')$ is a proper subfield of $K^p(X)$ whenever X' is a proper subset of X. Alternatively, X is p-independent if and only if no element x in X is contained in the field $K^p(X - \{x\})$. A subset X of K is a p-basis of K if X is p-independent in K

* F. K. Schmidt [2] states without proof a similar theorem, omitting the property, essential to our purposes, that L is relatively algebraically closed in K.

† L is relatively algebraically closed in K if and only if every element of K algebraic over L is in L.

and if, in addition, $K = K^p(X)$. It follows readily that X is p-independent in K if and only if each finite subset of X is p-independent. This means, in other words, that the degree $[K^p(x_1, \cdots, x_m):K^p]$ is p^m for any m distinct elements $x_1, \cdots, x_m$ of X. The latter statement was used as a definition of p-independence by Teichmüller [6, §3], so that our definition agrees with his. We next obtained another alternative definition based on the following:

LEMMA 3.2. *If Y is a p-independent subset of K, then $K \cap K^p(Y) = K \cap K^p(Y^{p^{-\infty}})$.*

Here and subsequently $K \cap L$ denotes the intersection of the fields K and L, while $K^p(Y^{p^{-\infty}})$ designates the field $K^p(Y, Y^{p^{-1}}, \cdots)$ obtained by adjoining to K^p all elements $y^{p^{-e}}$, for y in Y and e a positive integer.

Proof. We need only derive a contradiction from the assumption that some x of K not in $K^p(Y)$ is in $K^p(Y^{p^{-\infty}})$. For such an x there is an integer $e>0$ such that x is in $K^p(Y^{p^{-e}})$, but not in the field $M_e = K^p(Y^{p^{-e+1}})$. There then is a finite subset Z of Y such that x is in the field $M_e(Z^{p^{-e}})$. Therefore x has the form $x = f(y_1^{p^{-e}}, \cdots, y_n^{p^{-e}})$ where each y_i is an element of Z, where the polynomial f has coefficients in M_e, has degree less than p in each variable $y_i^{p^{-e}}$, and contains at least one variable, say $y_1^{p^{-e}}$, with an exponent 1. If g is the polynomial obtained from f by replacing each coefficient by its p^eth power, then

$$x^{p^e} - g(y_1, \cdots, y_n) = 0 \tag{1}$$

where g has coefficients in $M_e^{p^e} \subset K^p$, and is of degree less than p in y_1. Hence, over the field $K^p(y_2, \cdots, y_n)$, y_1 satisfies the separable equation (1) as well as the purely inseparable equation $y_1^p = a$, a in K^p. Therefore y_1 lies in $K^p(y_2, \cdots, y_n)$ as in Lemma III, contrary to the assumed p-independence of the set Y.

From this lemma one obtains the following theorems:

THEOREM 3.3. CRITERION FOR INDEPENDENCE. *A subset X of K is p-independent in K if and only if no x in X is contained in the field $K^p(X_0^{p^{-\infty}})$ where $X_0 = X - \{x\}$ is the set X with x deleted.*

THEOREM 3.4. *A subset X of K is a p-basis of K if and only if X is a p-independent subset of K for which $K^{p^{-\infty}} = K(X^{p^{-\infty}})$.*

Proof. If X is a p-basis, then by definition $K = K^p(X)$, so that an application of the isomorphism $a \leftrightarrow a^p$ yields the equation $K^p = K^{p^2}(X^p)$. By induction, we then obtain $K = K^{p^e}(X)$, or, by another isomorphism carrying each element into its p^eth root, $K^{p^{-e}} = K(X^{p^{-e}})$. This yields the conclusion that $K(X^{p^{-\infty}})$ is the perfect closure of K.

Conversely, if $K^{p^{-\infty}} = K(X^{p^{-\infty}})$, then $K^{p^{-\infty}} = K^p(X^{p^{-\infty}})$, $K \subset K^p(X^{p^{-\infty}})$, and hence by Lemma 3.2, $K \subset K^p(X)$. This is exactly the condition used to define a p-basis.

Returning to the existence of relatively perfect subfields, we prove a more explicit form of Theorem 3.1.

THEOREM 3.5. *If P is a perfect subfield of K, if X is any p-basis of K, and if L is the field of all elements of K algebraic over $P(X)$, then X is a separating transcendence basis for L over P and $K = K^p(L)$.*

When this theorem has been established, Theorem 3.1 will be an immediate consequence, for a straightforward argument by transfinite induction can be used to establish the existence of a p-basis X for any field K (Teichmüller [6]).

Proof. The fact that the set X is algebraically independent over P is known (Teichmüller [6, Theorem 15]). If L did not have X as a s.t.b., there would be an element z in L inseparable with exponent p over $P(X)$.

The element $y = z^p$ is therefore separable over $P(X)$, although $y^{1/p}$ is not so separable, as in the hypothesis of Lemma I (§2). The conclusion of that lemma produces an element x in X which is separable over $P(X - \{x\}, z^p)$ and hence over the larger field $K^p(X - \{x\})$. But x is also purely inseparable over $K^p(X - \{x\})$, and therefore x must be contained in the field $K^p(X - \{x\})$, contrary to the assumed p-independence of the set X.

Finally, since $X \subset L$ is a p-basis of K, $K = K^p(X) \subset K^p(L)$ must hold, as stated in the theorem.

4. **The Steinitz field tower.** Throughout this section we shall study the properties of a certain tower of fields over one of the intermediate fields L constructed in the last theorem.

HYPOTHESIS. *P is a perfect subfield of K; X is a p-basis of K; L is the field of elements of K algebraic over $P(X)$.*

For any transcendence basis T of K over L, we consider the set

$$S_n = \mathfrak{S}_n(K; L(T)) = [\text{all } \alpha \text{ in } K \text{ with } \alpha^{p^n} \text{ separable over } L(T)], \tag{1}$$

consisting of all elements of K with exponents p^n or less over $L(T)$. Steinitz [5, §14, Theorem 2] showed that S_n is a field and that K is the union of these fields S_n:

$$S_0 \subset S_1 \subset S_2 \subset \cdots; \qquad K = S_0(S_1, S_2, \cdots). \tag{2}$$

We call this chain of fields a Steinitz field tower for K over L. Steinitz' results also yield (Steinitz [5, §13, Theorem 1]) the following description of this tower:

LEMMA 4.1. *Each field S_n of the tower* (2) *consists of those elements of K of exponent p or less over the previous field S_{n-1}.*

If $K>L$, then T is non-void. Furthermore each inclusion in the tower (2) is a proper inclusion. For were $S_n=S_{n-1}$, there would be no elements of exponent p^n and hence no elements of any larger exponent over $L(T)$. Therefore $K=S_{n-1}$ and $K^{p^{n-1}}\subset S_0$, which means that $K^{p^{n-1}}$ is separable over $L(T)$, while K^{p^n} is separable over $L(T^p)$. Because X is a p-basis of K, the definition of §3 makes $K=K^{p^n}(X)=K^{p^n}(L)=L(K^{p^n})$. This implies that K, like K^{p^n}, is separable over $L(T^p)$, and that any t in T is so separable. But $t=(t^p)^{1/p}$ is also purely inseparable over $L(T^p)$ so that Lemma III requires t to be in $L(T^p)$. This is a contradiction because the set T^p is composed of elements algebraically independent over L. We conclude that

$$K > L \text{ implies } S_n > S_{n-1}, \qquad n = 1, 2, \cdots . \tag{3}$$

In the special case K a perfect field, the structure of the Steinitz tower has been formulated thus by Schmidt:

THEOREM OF F. K. SCHMIDT. *If K is a perfect field containing a perfect field $L=P$ relatively algebraically closed in K and if K has a transcendence basis T over L, then*

(i) *The nth field S_n of the Steinitz tower* (2) *has the separating transcendence basis $T^{p^{-n}}$ over P;*

(ii) $S_n=P(S_{n+1}^p)$.

Proof. In this case, we can assume $L=P$ because L is constructed from a p-basis X, whereas a p-basis of a perfect field is automatically empty. The second conclusion of the theorem can be asserted in the stronger form $S_n=S_{n+1}^p$ because of Lemma 4.1 and because each element of S_k has a pth root in the perfect field and hence in the field S_{n+1}. Furthermore, if y is an element of S_n, then y^{p^n} satisfies a separable irreducible equation with coefficients polynomials from $P[T]$, so that the p^nth root of this equation yields for y itself a separable equation with coefficients in $P[T^{p^{-n}}]$. Therefore $T^{p^{-n}}$, patently contained in S_n, is a s.t.b. for S_n, as asserted.

Our main problem is then the investigation of the two properties (i) and (ii) given for the Steinitz tower in this theorem, in the case K not a perfect field. We consider first the question of separating transcendence bases as in property (i). Our next objective is the following theorem:

THEOREM 4.2. *If K has a finite degree of transcendence over L, then each field S_n of the Steinitz field tower* (2) *has a separating transcendence basis T_n over L and hence also has a separating transcendence basis $X+T_n$ over P. Each basis T_n has the same number of elements as does T.*

Proof. We construct first a transcendence basis for S_1. Suppose that the finite transcendence basis T has exactly m elements which are pth powers in K, so that

$$T = U + W^p, \qquad W^p = \{w_1^p, \cdots, w_m^p\}, \tag{4}$$

while no element of U is in K^p. Then $U+W$, where W is the set $\{w_1, \cdots, w_m\}$, consists of elements of the field S_1. If this set $U+W$ is not already a s.t.b. for S_1, there is an element z in S_1 not separable over $L(U+W)$. We seek a modified basis T^* containing $y=z^p$. By hypothesis, the element z has exponent p over $L(T)=L(U, W^p)$ and also over $L(U, W)$. Since L is separable over $P(X)$, K/P has the transcendence basis $X+U+W$, and, by the transitivity of separability, z has exponent p over $P(X, U, W^p)$ and $P(X, U, W)$.

Let $f(z)=0$ be the irreducible equation for z over the polynomial ring $P[X, U, W]$. Then f must have exponent p in z; but no element of W can appear in f with an exponent 1, for otherwise Lemma II would imply that z^p is inseparable over $P(X, U, W^p)$, contrary to hypothesis. Suppose that all the variables of U appear with exponent at least p in f. As f is irreducible and inseparable in z, at least one of the elements of $X+W+U$ has exponent 1 in f. This must then be an element x of X. Since $f(z)$ is irreducible over $P[X, U^p, W^p]$, Lemma II implies that $y=z^p$ is inseparable over the field $P(X_0, U^p, W^p, x^p)$ where $X_0=X-\{x\}$. Therefore, by Lemma I, x is separable over $P(X_0, U^p, W^p, z^p)$, and hence over $K^p(X_0)$. This contradicts the assumed p-independence of X.

There must then be an element u from U with exponent 1 in $f(y)$; in particular, we know that U is not void. Another application of the exchange lemma to the polynomial $f(y)$ shows that u is separable over $P(X, W^p, U_0, z^p)$, where $U_0=U-\{u\}$. In other words, the transcendence basis

$$T^* = U_0 + W^p + \{z^p\}, \qquad U_0 = U - \{u\}, \tag{5}$$

for K over L has exactly $m+1$ pth powers, one more than T, and T^* is separably equivalent to T in the sense that $L(T)$ is separable over $L(T^*)$ and conversely. Consequently $\mathfrak{S}_n(K; L(T))=\mathfrak{S}_n(K; L(T^*))$ for every n, so T and T^* yield the same Steinitz towers (2).

Repeated applications of this transition from T to T^* whenever $U+W$ is not already a s.t.b. for S_1 will, after a finite number of steps, either yield a s.t.b. for S_1 or a new transcendence basis $T_r=W_r^p$ for K/L consisting only of pth powers. In this case the remark above that $U\neq 0$ shows that W_r must be a s.t.b. T_1 for all z in S_1.

This construction of a basis T_1 for S_1 yields by induction a similar s.t.b. for each S_n, for according to Lemma 4.1, S_n consists of elements of exponent p

or less over S_{n-1}, just as S_1 consists of elements of exponent p or less over S_0. The theorem is thus established.

For a subsequent use in §5 we need the following lemma:

LEMMA 4.3. *If $C_n = S_n - S_{n-1}$ is, for $n > 0$, the set of all elements of K of exponent exactly p^n over $L(T)$, then, when $T \neq 0$,*

$$L(C_n^{\ p}) = L(S_n^{\ p}), \qquad L(C_n) = L(S_n).$$

Proof. By (3), there exists an element x in C_n with exponent p^n over $L(T)$. If y is an arbitrary element of S_n not in C_n, then y has an exponent p^m, $(m < n)$, over $L(T)$. Hence y lies in S_{n-1} and xy must be in $C_n = S_n - S_{n-1}$. Since x is in C_n, y is in $L(C_n)$; therefore $L(S_n) \subset L(C_n)$. Similarly

$$(xy)^p \,\varepsilon\, C_n^{\ p}, \qquad x^p \,\varepsilon\, C_n^{\ p},$$

$$y^p = (xy)^p/x^p \,\varepsilon\, L(C_n^{\ p}), \qquad L(S_n^{\ p}) \subset L(C_n^{\ p}).$$

5. **Modified towers of fields.** When K is itself a perfect field, the Steinitz field tower (§4, (2)) has the useful property (ii) of Schmidt's theorem (§4): $S_n = P(S_{n+1}^p)$. Though we cannot assert this fact for every Steinitz field tower, we can in certain cases obtain another tower with an analogous property by omitting certain of the fields from the Steinitz tower.

THEOREM 5.1. *If, in the hypothesis of §4, the transcendence basis T for K over L is finite, then there exists a set of subfields M_k of K,*

$$(1) \qquad M_0 \subset M_1 \subset M_2 \subset M_3 \subset \cdots \subset K, \qquad K = \sum_{k=0}^{\infty} M_k,$$

where $\sum$ denotes the union of the fields M_k, such that

(i) *Each M_k has a separating transcendence basis T_k' over L,*

(ii) *$M_k \subset L(D_{k+1}^p)$ where $D_{k+1} = M_{k+1} - M_k$ is the set of elements in M_{k+1} but not in M_k for $k = 0, 1, 2, \cdots$.*

More explicitly, we shall show that every M_k can be picked as a field $M_k = S_{e_k}$ from the Steinitz tower (§4, (1)). In other words, we shall exhibit integers $0 = e_0 < e_1 < e_2 < \cdots$ such that the conditions (i) and (ii) above obtain. That such fields S_{e_k} form a tower (1) is trivial, while (i) follows from Theorem 4.2. To establish (ii), we shall show by induction that if the integers $e_0 < e_1 < e_2 < \cdots < e_k$ have already been chosen, there is an integer $e_{k+1} > e_k$ such that $M_k = S_{e_k} \subset L(D_{k+1}^p)$. Here

$$D_{k+1} = S_{e_{k+1}} - S_{e_k} \supset S_{e_{k+1}} - S_{e_{k+1}-1} = C_{e_{k+1}},$$

where $C_n = S_n - S_{n-1}$, as in Lemma 4.3. Hence it will suffice to demonstrate $S_{e_k} \subset L(C_{e_{k+1}}^p)$. This is a consequence of the following lemma:

LEMMA 5.2. *For any integer* $e \geqq 0$, *there exists an integer* $m > e$ *so that in the Steinitz tower* $S_e \subset L(C_m^p)$ *where* $C_m = S_m - S_{m-1}$.

The proof will depend essentially upon the finiteness of T and the "relative perfection" of K over L. This latter property we assume in the form (cf. Theorem 3.5) $K = K^p(L) = L(K^p)$. Let T_e be a separating transcendence basis, obtained as in Theorem 4.2, for S_e over L. The basis T_e is finite because T is, while $T_e \subset L(K^p)$; so there is a finite set R of elements of K such that $T_e \subset L(R^p)$. Since $K = \sum S_e$, each element of R is in some one Steinitz field S_e, so that there is a finite integer $m > e$ such that $R \subset S_m$. Combining these conclusions, we have $T_e \subset L(S_m^p)$.

Consider now any element z in S_e. By the construction of T_e, z is separable over $L(T_e)$ and hence over $L(S_m^p)$. But z is also in S_e, hence in S_m since $m > e$. Therefore z^p is in $L(S_m^p)$, so that z is also purely inseparable over $L(S_m^p)$. This implies that z is in $L(S_m^p)$, so that Lemma 4.3 gives

$$S_e \subset L(S_m^p) = L(C_m^p) \tag{2}$$

as required for the lemma.

Theorem 5.1 is now established under the essential hypothesis that the transcendence basis is finite. Examples readily show that the same method cannot be used when T is infinite. However T will certainly be finite when the transcendence degree of K over its subfield P is finite. This special case we reformulate as follows:

THEOREM 5.3. *If* K *has a finite transcendence degree over a perfect subfield* P, *then there exists a tower of subfields*

$$L \subset M_0 \subset M_1 \subset M_2 \subset \cdots \subset K, \qquad K = \sum_{k=0}^{\infty} M_k, \tag{3}$$

all containing P, *such that*

(i) *L has a separating transcendence basis over P;*

(ii) *Each field M_k has a separating transcendence basis over L;*

(iii) *L is relatively algebraically closed in K, and $K = K^p(L)$;*

(iv) *$M_k \subset L(D_{k+1}^p)$ where $D_{k+1} = M_{k+1} - M_k$, for $k = 0, 1, \cdots$.*

6. **Exponent lemmas.** The difficulties in the way of proving properties (i) and (ii) of Schmidt's theorem for arbitrary Steinitz towers will be subsequently illustrated by elaborate examples, which require as a preliminary the structure of the Steinitz tower for a purely transcendent extension of a perfect field.

LEMMA 6.1. *If T is a set of elements algebraically independent over the field F, and if $K = F(T^{p^{-\infty}})$ is the field obtained from F by the adjunction of all elements*

$t^{p^{-e}}$ *for e any integer and t in T, then for any e*

$$(1)\qquad \mathfrak{S}_e(K; F(T)) = F(T^{p^{-e}}).$$

In other words, $F(T^{p^{-e}})$ *is exactly the set of the elements* α *of* K *such that* α^{p^e} *is separable over* $F(T)$. *A Steinitz field tower for* K *over* $F(T)$ *is then*

$$(2)\qquad F(T) \subset F(T^{p^{-1}}) \subset F(T^{p^{-2}}) \subset \cdots .$$

Proof. $S_e = \mathfrak{S}_e(K; F(T))$ denotes the set of all α in K such that α^{p^e} is separable over $F(T)$. That $S_e \supset F(T^{p^{-e}})$ results immediately, so that we need only prove the converse $F(T^{p^{-e}}) \supset S_e$. Since any element of S_e depends algebraically on but a finite number of the elements of T, it suffices to give a proof for the case when T is finite. We treat this case by an induction on the number of elements in T.

Case 1. T has one element t_1. Over the field S_e any power $z = t^{p^{-m}}$, with $m > e$ satisfies an equation $z^{p^{m-e}} = t^{p^{-e}}$. This equation is irreducible over S_e because otherwise the pth root of $t^{p^{-e}}$ is in S_e. This would imply that $t^{p^{-1}}$ is in S_0 and hence is separable as well as purely inseparable over $F(t)$. Consequently, $t^{p^{-1}}$ is in $F(t)$, an impossibility. Therefore $z^{p^{m-e}} = t^{p^{-e}}$ is irreducible over S_e, and the degree of $z = t^{p^{-m}}$ is

$$(3)\qquad [S_e(t^{p^{-m}}):S_e] = p^{m-e}, \qquad m > e.$$

Suppose now that an element α of S_e is not in $F(t^{p^{-e}})$. For a sufficiently large m, $\alpha \in F(t^{p^{-m}})$. As $S_e \supset F(t^{p^{-e}}, \alpha) > F(t^{p^{-e}})$, we have according to (3) the following degree relations:

$$[F(t^{p^{-m}}):F(t^{p^{-e}}, \alpha)] < [F(t^{p^{-m}}):F(t^{p^{-e}})] = p^{m-e},$$
$$[F(t^{p^{-m}}):F(t^{p^{-e}}, \alpha)] \geqq [S_e(t^{p^{-m}}):S_e] = p^{m-e},$$

a contradiction. We have proven $\mathfrak{S}_e(K, F(t)) = F(t^{p^{-e}})$.

Case 2. Suppose next that the lemma is known when the transcendence basis has $n-1$ elements, and let $T = T_0 + \{t\}$ have n elements, so that T_0 has $n-1$ elements. K contains a subfield $F' = F(T_0^{p^{-\infty}})$ and $K = F'(t^{p^{-\infty}})$. Any α in K with α^{p^e} separable over $F(T)$ has α^{p^e} also separable over $F'(t)$ so that α is contained in $F'(t^{p^{-e}}) = F(t^{p^{-e}}, T_0^{p^{-\infty}})$ by the proof of the previous case. If we set $F_0 = F(t^{p^{-e}})$, then α is in $F_0(T_0^{p^{-\infty}})$ and has α^{p^e} separable over $F_0(T_0)$. Therefore, by the induction assumption, α is in

$$F_0(T_0^{p^{-e}}) = F(t^{p^{-e}}, T_0^{p^{-e}}) = F(T^{p^{-e}}),$$

as required in the assertion (1).

7. **Irregular Steinitz field towers.** We shall now show that the field tower M_k of Theorem 5.1 with the special property

$$M_k \subset L(D_{k+1}^p), \qquad D_{k+1} = M_{k+1} - M_k,$$

can be taken to be a Steinitz field tower itself whenever the transcendence basis T has only one element, but not always in other cases.

THEOREM 7.1. *If the basis T of the hypothesis of* §4 *consists of exactly one element, then*

$$(1) \qquad S_k = L(S_{k+1}^p), \qquad k = 0, 1, 2, \cdots .$$

Proof. By Theorem 4.2 each field S_e has over L a s.t.b. consisting of one element t_e. Thus each S_e consists of the elements of K of exponent p or less over $L(t_{e-1})$. By reason of this symmetry it patently suffices to prove our conclusion $S_{e-1}=L(S_e^p)$ only for the case $e=1$. Since t_1 is in S_1 and not S_0, it has exponent p over $L(t_0)$ and also over $P(X, t_0)$. By the exchange lemma, some element of $\{t_0\}+X$ can be exchanged with t_1^p. If t_0 is not so exchangeable, this means, as in the exchange lemma, that t_1^p is separable over $P(X, t_0^p)$, and that some x in X can be here exchanged with t_1^p. This exchange makes x separable over $P(X-\{x\}, t_0^p, t_1^p)$ and thus over $K^p(X-\{x\})$. Hence (Lemma III) x lies in $K^p(X-\{x\})$, counter to the p-independence of the set X.

It must then be possible to exchange t_0 with t_1^p. Hence t_0 is separable over $P(t_1^p, X) \subset L(t_1^p)$. By the transitivity of separability, every element of S_0 is then separable over $L(t_1^p) \subset L(S_1^p)$. Every element of S_0 is in S_1 and hence is also purely inseparable over $L(S_1^p)$. Combining these facts (Lemma III), we conclude that $S_0 \subset L(S_1^p)$, as required in the theorem.

We now show by an example that this theorem is not always true when T has more than one element. Over a perfect field P construct the field

$$(2) \qquad K = P(x, y^{p^{-\infty}}, z^{p^{-\infty}}) = P(x, y, z, y^{p^{-1}}, z^{p^{-1}}, y^{p^{-2}}, z^{p^{-2}}, \cdots)$$

where x, y, and z are algebraically independent over P. The element x is by inspection a p-basis of K (cf. Teichmüller [6, Theorem 18]). Furthermore $P(x)$ is relatively algebraically closed in $P(x, y^{p^{-\infty}}, z^{p^{-\infty}})$ because any field is relatively algebraically closed in a purely transcendental extension. $P(x)$ is then also relatively algebraically closed in K, so that the field L of our hypothesis (cf. §4), consisting of all elements algebraic over $P(x)$, here becomes $P(x)$ itself. If we now introduce the quantity

$$(3) \qquad u = xz^{p^{-1}} + y^{p^{-1}}, \qquad u^p = x^p z + y,$$

then $T=\{u, z\}$ is a transcendence basis for K over L, because $y=u^p-x^p z$. For the Steinitz field tower relative to this basis T, we shall demonstrate

$$(4) \qquad S_0 = P(x, u, z), \qquad S_1^p = P(x^p, y, z).$$

From these equations it is clear that $L(S_1^p)=P(x, y, z)<S_0$, unlike (1), so that here Theorem 5.1 certainly does not hold.

The first equation of (4) will be established if we show $S_0 \subset P(x, u, z)$. Since K is a purely inseparable extension of $L(y, z)$, K is also a purely inseparable extension of the larger field $P(x, u, z) \supset L(y, z)$, and any element of K is either in $P(x, u, z)$ or is purely inseparable over $P(x, u, z) = L(u, z)$. Therefore the field S_0 of separable elements of K is $P(x, y, z)$, as in (4).

The crux of the example is the second equation of (4). Note first that

$$S_1^p = S_0 \cap K^p. \tag{5}$$

Introduce the additional subfields

$$F = P(x^p), \qquad B = P(x^p, y, z) = F(y, z),$$

so that the terms of (5) become

$$S_0 = B(x, u) = B((x^p)^{1/p}, (x^p z + y)^{1/p}), \qquad K^p = B(y^{p^{-\infty}}, z^{p^{-\infty}}). \tag{6}$$

Any element α of the intersection $S_0 \cap K^p$ is in S_0 and so has a pth power α^p separable over $B=F(y, z)$, by (6). But y and z are algebraically independent over F, so that by Lemma 6.1, applied to α and to the field K^p, α must be in $F(y^{p^{-1}}, z^{p^{-1}})$. The expression (5) can then be rewritten as

$$S_1^p = S_0 \cap F(y^{p^{-1}}, z^{p^{-1}}) = B((x^p)^{1/p}, (x^p z + y)^{1/p}) \cap B(y^{1/p}, z^{1/p}). \tag{7}$$

The generators x^p, y, and z of B are algebraically independent over P. Under these conditions the intersection on the right of (7) has been shown to be B itself.* This establishes the second half of (4).

The field K of this counter-example has a relatively simple structure, for K is simply $P'(x)$ where $P'=P(y^{p^{-\infty}}, z^{p^{-\infty}})$ is the maximal perfect subfield of K. The field K has a s.t.b. over this field P'. The example, however, can be so modified that this simple alternative description of its structure is not possible. We now construct such a modification in which the base field P is itself the maximal perfect subfield of K.

Over the perfect field P, consider four denumerable sets of quantities

$$Y = \{y_1, y_2, \cdots\}, Z = \{z_1, z_2, \cdots\}, V = \{v_1, v_2, \cdots\}, W = \{w_1, w_2, \cdots\}.$$

Let the elements of the set $V+W+\{y_1, z_1\}$ be algebraically independent over P. Define the remaining elements by the equations

$$y_{k+1}^p = v_k + y_k, \qquad z_{k+1}^p = w_k + z_k, \qquad k = 1, 2, 3, \cdots, \tag{8}$$

* Mac Lane [3, §6]. The intersection was computed to show that the lattice of all fields between B and $B^{1/p}$ is not a modular lattice in the sense of G. Birkhoff; that is, is not a Dedekind structure in the terminology of O. Ore.

and construct the field $K=P(V, W, Y, Z)$. Then one can show that $X=V+W$ is a p-basis of K, that the corresponding field L is $P(V, W)$, and that $T=\{u, z_1\}$ is a transcendence basis for K over L, where u is defined by $u=v_1z_2+y_2$. Relative to this transcendence basis, the Steinitz field tower begins with the field $S_0=L(u, z_1)$. By an extension of the argument of the last example we compute $S_1{}^p=P(V^p, W^p, y_2{}^p, z_2{}^p)$ and hence find that $L(S_1{}^p)=P(V, W, y_1, z_1)$. This field does not contain the element u of S_0, for

$$u^p = (v_1z_2+y_2)^p = v_1^p(w_1+z_1)+v_1+y_1$$

is an irreducible equation for u over $L(S_1{}^p)$. Hence S_0 is not contained in $L(S_1{}^p)$, and the conclusion of Theorem 7.1 does not hold for this example.

Furthermore, in this case the maximal perfect subfield K^{p^∞} of K is the base field P. For $P(Y, Z)$ contains all elements v_k and w_k by (8) and hence is the whole field K. Furthermore, the set $Y+Z$ is algebraically independent over P, and it can be readily seen* that the maximal perfect subfield of such a purely transcendental extension $K=P(Y, Z)$ is simply the base field P itself. In conclusion, we can state the theorem:

THEOREM 7.2. *If the field K of the hypothesis of* §4 *has the transcendence degree* 2 *or more over the intermediate field L, then the fields of the Steinitz towers do not always satisfy the condition $S_0=L(S_1{}^p)$ of Theorem* 7.1. *Specifically, there exist such fields K with maximal perfect subfield P and $S_0>L(S_1{}^p)$.*

8. **Inseparable Steinitz field towers.** If the field K under consideration does not have a finite transcendence degree over its subfield L, as assumed in the treatment of §4, then the fields S_k of the Steinitz field tower need not all have separating transcendence bases over L. This we shall show by an example (which is summarized below in Theorem 8.6).

Let P be any perfect field, and consider two denumerable sets of elements

$$T=\{t_0, t_1, t_2, \cdots\}, \qquad Y=\{y_2, y_3, \cdots\};$$

let the elements of the set T be algebraically independent over P, define the elements y of Y by the equations

$$y_n^p = t_{n-2}+t_{n-1}t_n^p, \qquad n=2, 3, 4, \cdots, \tag{1}$$

and take K to be the field

$$K=P(T, Y^{p^{-\infty}})=P(T, Y, Y^{p^{-1}}, Y^{p^{-2}}, \cdots).$$

LEMMA 8.1. *The set X composed of t_0 alone is a p-basis of K.*

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* Added in proof: A proof is given in S. Mac Lane, *Modular fields*, I, *Separating transcendence bases*, Duke Mathematical Journal, vol. 5 (1939). See Theorem 19, Corollary 1.

Proof. The defining algebraic equations (1) can be rewritten as

$$t_n = (y_{n+1}^p - t_{n-1})/t_{n+1}^p, \qquad n \geqq 1. \tag{2}$$

An induction on n proves that each t_n is in $K^p(t_0)$, so that $T \subset K^p(t_0)$, $Y^{p^{-\infty}} \subset K^p$, and hence $K = K^p(t_0)$. This is the first condition that t_0 be a p-basis. On the other hand, t_0 is p-independent; that is to say, t_0 is not in K^p. For suppose that $t_0 \in K^p$. The $n+1$ algebraically independent elements $t_0, t_1, \cdots, t_n$ are algebraic over $P(t_0, t_1, y_2, \cdots, y_n)$ by the equations (1). Consequently the $n+1$ elements $t_0, t_1, y_2, \cdots, y_n$ must themselves be independent (algebraically) over P. Hence $Y + \{t_0, t_1\}$ is a set of elements independent over P, and $\{t_0, t_1\}$ are likewise independent over the subfield $P(Y^{p^{-\infty}})$ of K. Introduce the additional subfields $K_n = P(Y^{p^{-\infty}}, t_0, t_1, \cdots, t_n)$, with $K = \sum K_n$. By the equations (1) the t's in this field K_n can be expressed rationally in terms of the y's and the last two t's. Hence $K_n = P(Y^{p^{-\infty}}, t_{n-1}, t_n)$, and $\{t_{n-1}, t_n\}$ is a set algebraically independent over $P(Y^{p^{-\infty}})$. Suppose now that t_0 is in K^p. Since $K = \sum K_n$, t_0 is in some field

$$K_n^p = P(Y^{p^{-\infty}}, t_{n-1}^p, t_n^p) = P(Y^{p^{-\infty}}, t_0^p, t_1^p, \cdots, t_n^p).$$

A successive application of the equations (2) then shows that $t_1, t_2, \cdots$, and finally t_{n-1} are also in K_n^p. But the elements t_{n-1}^p, t_n^p are known to be algebraically independent over $P(Y^{p^{-\infty}})$, so that the extended field $K_n^p = P(Y^{p^{-\infty}}, t_{n-1}^p, t_n^p)$ certainly cannot contain a pth root $t_{n-1} = (t_{n-1}^p)^{1/p}$. This contradiction shows that t_0 is p-independent.

LEMMA 8.2. *The field L of all elements algebraic over $P(t_0)$ is $L = P(t_0)$.*

Proof. By Theorem 3.5, any element α in L is separable algebraic over $P(t_0)$ and so over $P(T)$. But K is obtained from $P(T)$ by the successive adjunction of pth roots, which means that K is purely inseparable over $P(T)$. Therefore (Lemma III) the elements α of L all lie in $P(T)$. The remaining elements of T are algebraically independent of t_0; so L must be $P(t_0)$, as asserted.

We now choose for K over L the transcendence basis $T_1 = \{t_1, t_2, \cdots\}$.

LEMMA 8.3. *The Steinitz field $S_1 = \mathfrak{S}(K; L(T_1))$ of all elements of K of exponent p or less over $L(T_1)$ is the field $S_1 = L(T_1, Y) = P(t_0, T_1, Y)$.*

The defining equations (1) for the elements y make each y of exponent p over $P(t_0, T_1)$. Hence $L(T_1, Y) \subset S_1$. Conversely, S_1 consists of certain elements of $K^{p^{-\infty}} = P(T^{p^{-\infty}})$ of exponent p or less over $P(T)$. Therefore, by Lemma 6.1, $S_1 \subset P(T^{p^{-1}})$. In other words, S_1 satisfies

$$M_1 \subset S_1 \subset M_2, \qquad M_1 = L(T_1, Y), \qquad M_2 = P(T^{p^{-1}}). \tag{3}$$

The equations (2) show that M_2 can also be generated as

$$M_2 = P(t_0^{p^{-1}}, T_1^{p^{-1}}) = P(T, Y, t_0^{p^{-1}}) = M_1(t_0^{p^{-1}}).$$

Therefore the field M_2 of (3) has degree p or 1 over M_1, so that S_1 is necessarily M_1 or M_2. If $S_1=M_2$, then $t_0^{p^{-1}}$ is in $S_1 \subset K$; hence t_0 is in K^p, contrary to the result of Lemma 8.1. Therefore $S_1=M_1=L(T_1, Y)$.

LEMMA 8.4. *The field S_1^p contains neither t_n nor t_n/t_{n+1} for any integer $n \geqq 0$.*

Proof. If t_n were in S_1^p, the equation (2) solved for t_{n-1} shows that t_{n-1} is in S_1^p. A repetition of this argument shows that t_{n-2}, t_{n-3}, and finally t_0 are in $S_1^p \subset K^p$, in contradiction to Lemma 8.1.

On the other hand, if t_n/t_{n+1} is in S_1^p, the equation (1) written in the form $y_{n+2}^p/t_{n+1}=t_n/t_{n+1}+t_{n+2}^p$ would imply that $1/t_{n+1}$ and hence t_{n+1} are in S_1^p, contrary to the already established part of the lemma.

LEMMA 8.5. *The first field $S_1=L(T_1, Y)$ of the Steinitz tower does not have a separating transcendence basis over L.*

If there were such a basis over $L=P(t_0)$, the adjunction of t_0 to this basis would yield an enlarged s.t.b. $Z=\{z_1, z_2, \cdots\}$ for S_1 over P. We shall show that this leads to a contradiction by finding a single z the adjunction of which would simultaneously make y_k and t_k separable, in conflict with the form of the inseparable defining equation (1). The argument depends on a reduction to a finite subset of Z. Specifically, both t_0 and t_1 are separable over $P(Z)$, so that there must be a finite subset $Z_m=\{z_1, \cdots, z_m\}$ so large that t_0 and t_1 are separable over $P(Z_m)$. All of the independent elements of T cannot be dependent on this subset Z_m, so that there must be an integer $n \geqq 2$, such that $t_0, t_1, \cdots, t_{n-1}$ are algebraic and hence separable over $P(Z_m)$, while the next element t_n is not so algebraic over Z_m. However, t_n will be algebraic over a larger set of z's, so that there is a set $Z_k=\{z_1, \cdots, z_k\}$, $(k \geqq m)$, for which t_n is algebraic over $P(Z_k, z_{k+1})$, but not over $P(Z_k)$. The equations (1) make (a) y_n algebraic over $P(t_{n-2}, t_{n-1}, t_n)$, (b) t_n algebraic over $P(t_{n-2}, t_{n-1}, y_n)$. Since both t_{n-2} and t_{n-1} are already algebraic over $P(Z_k) \supset P(Z_m)$, neither t_n nor y_n can be algebraic over $P(Z_k)$, and both t_n and y_n must be algebraic over $P(Z_k, z)$, where $z=z_{k+1}$.

From the equations for t_n and y_n over $P(Z_k, z)$, we can, by Lemma II, pick the largest integers e and f such that

(4) t_n is separable over $P(Z_k, z^{p^e})$; y_n is separable over $P(Z_k, z^{p^f})$.

By the exchange lemma, we then have

(5) z^{p^e} separable over $P(Z_k, t_n)$; z^{p^f} separable over $P(Z_k, y_n)$.

If $e \geqq f$, the first statement of (4) and the second statement of (5) imply that

t_n is separable over $P(Z_k, y_n)$. Let N denote the field of all elements of S_1 separable over $P(Z_k)$. By construction, t_{n-2} and t_{n-1} are in N so that (1) makes t_n purely inseparable over $N(y_n)$. Therefore $t_n \varepsilon N(y_n)$. In other words, t_n is a rational function

$$t_n = f(y_n)/g(y_n), \qquad f(y_n),\ g(y_n) \text{ in } N[y_n],$$

where we can assume that the coefficients $f(0)$, $g(0)$ are not both 0. This value of t_n substituted in (1) yields

$$y_n^p[g(y_n)]^p = t_{n-2}[g(y_n)]^p + t_{n-1}[f(y_n)]^p.$$

Here the variable y_n over N can be replaced by 0 with the result

$$-t_{n-2}[g(0)]^p = t_{n-1}[f(0)]^p.$$

One and consequently both of $f(0)$, $g(0)$ are different from 0. Therefore $t_{n-2}/t_{n-1} = -[f(0)/g(0)]^p$ is in S_1^p, in contradiction to Lemma 8.4.

In the remaining case, when $e<f$, a similar argument proves $y_n \varepsilon N(t_n)$ and hence $t_{n-2} \varepsilon N^p \subset K^p$, another contradiction. We have therefore constructed a Steinitz field tower in which one of the fields S_1 has no s.t.b. over the ground field L.

THEOREM 8.6. *There is a modular field K with maximal perfect subfield P, a p-basis X, and a transcendence basis T over the subfield L of elements algebraic over $P(X)$, such that some field of the Steinitz tower for K relative to T over L does not have a separating transcendence basis over L.*

The example given establishes this theorem except for the hypothesis that P is the maximal perfect subfield of K; for the maximal perfect subfield of the field used above manifestly includes $P(Y^{p^{-\infty}})$. The following modification of the example will complete this point.

Choose sets of elements

$$T = \{t_k\}, \qquad X = \{x_{ij}\}, \qquad Y = \{y_{ij}\},$$
$$k = 0, 1, 2, \cdots;\ i = 2, 3, 4, \cdots;\ j = 0, 1, 2, \cdots,$$

where the elements of $T+X$ are to be viewed as algebraically independent over a perfect field P, and where the elements y_{ij} are algebraic over $P(T, X)$ in accord with the equations

$$y_{i0}^p = t_{i-2} + t_{i-1}t_i^p, \qquad i = 2, 3, \cdots, \tag{6}$$

$$y_{ij+1}^p = x_{ij} + y_{ij}, \qquad i = 2, 3, \cdots;\ j = 0, 1, 2, \cdots. \tag{7}$$

Equations (6) are analogous to the defining equations (1) of the previous example, while equations (7) differ from the repeated pth roots $Y^{p^{-e}}$ of the previous example only in the presence of the x_{ij}, which will insure that P is the maximal perfect subfield. The field K to be considered is $K=P(T, X, Y)$.

LEMMA 8.61. (Compare Lemma 8.1.) *The set* $X+\{t_0\}$ *is a p-basis of* K.

That $K=K^p(X, t_0)$, one sees by inspection of the equations (6) and (7). Conversely, to prove the p-independence of $X+\{t_0\}$ it suffices to prove that each $X_n+\{t_0\}$ is p-independent, where X_n is the first of the "truncated" sets

$$X_n = \{x_{ij}\}, \qquad Y_n = \{y_{ij}\}, \qquad i = 2, \cdots, n; j = 0, \cdots, n.$$

The field K is approximated by a tower of fields

$$K_n = P(t_0, t_1, \cdots, t_n, X_n, Y_n). \tag{8}$$

Since any p-dependence will occur at some stage in this tower, it will suffice to prove $X_n+\{t_0\}$ p-independent in K_n. The equations (7) allow rational computations of y_{ij} with $j<n$ in terms of y_{in}, while according to (6), $t_2, \cdots, t_n$ are algebraic over $Y_n+\{t_0, t_1\}$. Hence K_n has the transcendence basis

$$U = X_n + \{t_0, t_1, y_{2n}, \cdots, y_{nn}\}.$$

Specifically, over $P(U)$, K_n is the algebraic extension $K_n=P(U, t_2, \cdots, t_n)$, of degree $[K_n:P(U)]\leqq p^{n-1}$. $P(U)$ has a p-basis of $m+n+1$ elements, where m is the number of elements in X_n, so that K_n, as a finite purely inseparable extension of $P(U)$, has a p-basis of the same number* of elements. By the definition of p-independence in terms of degrees this means that $[K_n: K_n^p]$ $=p^{m+n+1}$. Hence

$$[K_n:K_n^p(t_0, X_n)]\cdot[K_n^p(t_0, X_n):K_n^p] = p^{m+n+1}. \tag{9}$$

On the other hand, $F_n=K_n^p(t_0, X_n)$ contains all pth powers from K_n, while by repeated applications of (6) it must contain $t_1, t_2, \cdots, t_{n-1}$. But K_n is generated over P by $X_n, t_0, \cdots, t_n$ and $y_{2n}, \cdots, y_{nn}$, so that

$$K_n = [K_n^p(t_0, X_n)](t_n, y_{2n}, \cdots, y_{nn}).$$

Each element adjoined on the right is purely inseparable of exponent p or 1; hence $[K_n: K_n^p(t_0, X_n)]\leqq p^n$. Combined with (9), this yields the inequality $[K_n^p(t_0, X_n):K_n^p]\geqq p^{m+1}$, where $m+1$ is the number of elements in $X_n+\{t_0\}$. Therefore $X_n+\{t_0\}$ is p-independent in K_n, as required for Lemma 8.61.

Using this p-basis, denote by L the field of those elements of K algebraic over $P(t_0, X)$, and consider the transcendence basis $T_1=\{t_1, t_2, \cdots\}$ for K over L.

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* By a theorem (unpublished) due to Dr. M. Becker, or by direct computation in this case.

LEMMA 8.62. *The first field $S_1=\mathfrak{S}_1(K; L(T_1))$ of the Steinitz tower relative to $L(T_1)$ is $S_1=P(T, X, y_{20}, y_{30}, y_{40}, \cdots)$.*

Proof. That S_1 includes the quantities indicated is manifest from the defining equations; so the conclusion could be false only in the presence of an element w not in $P(T, X, y_{20}, \cdots)$ but in S_1. The pth power w^p is then separable over $L(T_1)$ and hence over $P(t_0, X, T_1)$, by Theorem 3.5. Choose n so that w is in K_n of (8) and so that w^p is separable over the field

$$D_n = P(t_0, t_1, \cdots, t_n, X_n). \tag{10}$$

The defining equations (7) for y_{in} can be combined as

$$y_{in}^{p^{n+1}} = x_{in-1}^{p^n} + x_{in-2}^{p^{n-1}} + \cdots + x_{i0}^{p} + t_{i-1}t_i^{p} + t_{i-2}. \tag{11}$$

These equations have the form $y_{in}^{p^{n+1}}=u_{i-2}$, where the quantities u on the right lie in D_n and can be successively exchanged with the corresponding t_{i-2} in (10) to yield the generation

$$D_n = P(u_0, u_1, \cdots, u_{n-2}, t_{n-1}, t_n, X_n).$$

The field $K_n=P(U, t_2, \cdots, t_n)$ of (8) becomes

$$K_n = P(X_n, t_0, t_1, \cdots, t_n, y_{2n}, \cdots, y_{nn}),$$

and hence is generated by adjoining to D_n the roots $y_{in}=u_{i-2}^{p^{-n-1}}$:

$$K_n = D_n(u_0^{p^{-n-1}}, u_1^{p^{-n-1}}, \cdots, u_{n-2}^{p^{-n-1}}).$$

The element w of K_n of exponent 1 over D_n must then by Lemma 6.1 (applied with $F=P(t_{n-1}, t_n, X_n)$) lie in the field $D_n(u_0^{1/p}, u_1^{1/p}, \cdots, u_{n-2}^{1/p})$. By the expansions for the u's on the right of (11), this is the field $D_n(y_{20}, y_{30}, \cdots, y_{n0})$. This field is contained in the field $P(T, X, y_{20}, y_{30}, \cdots)$ of the lemma, counter to the assumption that w does not lie in this field. This field is therefore equal to S_1, as asserted in the lemma.

This field S_1 may be briefly described as the field $S_1=P(T, X, y_{20}, y_{30}, \cdots)$ generated by the adjunction of the independent variables T, X, and the roots y_{i0} of the equations (6). It differs from the field S_1 of the previous example only in the presence of certain variables X which nowhere figure in the defining equations (6). A reapplication of the arguments used in the previous case (Lemmas 8.1, 8.2, and 8.5) then establishes the following lemma:

LEMMA 8.63. *The first field S_1 of the Steinitz tower does not have a separating transcendence basis over L.*

This completes the counter-example, with the following additional property not present in the previous example:

LEMMA 8.64. *The field K above has P as its maximal perfect subfield.*

Proof. Embed K in the field $K'=K(s_0, s_1, \cdots)$, where $s_i=t_i^{1/p}$. If Y' is the set of elements y_{ij} with $i=2, 3, \cdots$ and $j=1, 2, \cdots$, then $K'=P(Y', s_0, s_1, \cdots)$, by the defining equations (6). Furthermore, the generators $Y'+\{s_0, s_1, \cdots\}$ are algebraically independent. For the set of elements $\{t_0, t_1, \cdots, t_m, x_{ij}\}$ with $i=2, \cdots, m$ and $j=0, \cdots, n-1$ consists of $(m+1)+(m-1)n$ elements and is known to be algebraically independent, but is algebraically dependent upon the set $\{s_0, \cdots, s_m, y_{ij}\}$ with $i=2, \cdots, m$ and $j=1, \cdots, n$, which has the same number of elements. Therefore this subset and the whole set $Y'+\{s_0, s_1, \cdots\}$ are algebraically independent. The purely transcendental field $K'=P(Y', s_0, s_1, \cdots)$ therefore has P as maximal perfect subfield, as asserted.

9. **Separating linear orders of the Steinitz field tower.** F. K. Schmidt has considered the possibility of "separating" orders for fields. Let a set K which is a field have a linear order given by a relation $<$. For any element b in K let K_b denote the subfield of K generated by the set of all elements c with $c<b$. The given linear order is said to be a *separating order* if every element b of K is either transcendental or separable and algebraic over the corresponding K_b. The elements b algebraic over their respective fields K_b are said to be *algebraic* in the given order. Schmidt [2, pp. 16, 46] now considers the following situation:* K is a field which has no separating transcendence basis over its prime field P; L is a subfield of K with a separating transcendence basis over P such that $K^{p^{-\infty}}=K(L^{p^{-\infty}})$; T is any transcendence basis for K over L, and S_n is again the Steinitz field composed of all elements α of K such that α^{p^n} is separable over $L(T)$. This situation includes, in particular,† the situation described in the hypothesis in our §4, provided we suppose that the P used there is the prime field $GF[p]$ and that K has no separating transcendence basis over P. (Both of these assumptions can be made in the examples of fields constructed in §§7 and 8.)

Given any such situation, Schmidt now asserts *without proof* [2, p. 46] that "there exists a separating order '$<$' of K such that (i) '$<$' induces in each field S_n a separating normal order W_n (well ordering); (ii) if the elements of K are written down in the order specified by '$<$', then one obtains an additive representation

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* The notation has been changed thus:

F. K. Schmidt:	$\mathfrak{K}$	$\mathfrak{K}'$	$\mathfrak{S}$	$\mathfrak{K}^{(i)}$	$\mathfrak{O}$
S. Mac Lane:	K	L	T	S_i	$<$

† Schmidt does not assume that his field L can be constructed from a p-basis X; examples can be given of a field L which cannot be so constructed and which still has the properties specified by Schmidt.

$$K = L + \sum_{n=\infty}^{0} C_n, \qquad C_n = S_n - S_{n-1}, \qquad C_0 = S_0 - L.$$

In other words, the elements of L precede all other elements, and the elements of each complement C_n preceded those of the complement $C_{n-1}, \cdots$. (iii) Every element b of S_n algebraic in the order '$<$' of K is separable and algebraic in the separating normal order W_{n+1} of the subfield S_{n+1}. Furthermore, the coefficients of the irreducible separable polynomial $G(x)$ satisfied by b in the order W_{n+1} (that is, satisfied by b over $(S_{n+1})_b$) are present in the field $S_{nb}(C_{n+1}^p)$ where S_{nb} is the smallest subfield of S_n containing all elements of S_n which precede b in the order W_n."

The separating order W_n obtained here means that S_n contains no element inseparable and algebraic over the intermediate field L. In other words, K can contain no such element. The hypotheses stated for L are not in themselves sufficient to insure this condition. Certainly an additional hypothesis is intended, such as the assumption that L is relatively algebraically closed in K or the assumption that the elements of K algebraic over L are separable over L.

The conclusion (iii) formulated above can be further reduced. For any b in S_n the field $(S_{n+1})_b$ which contains all elements of S_{n+1} preceding b must by (ii) contain L and C_{n+1}, and, therefore, by Lemma 4.3, also contains $L(C_{n+1}) = L(S_{n+1}) = S_{n+1}$. In other words, b is contained in $(S_{n+1})_b$; the irreducible equation $G(x)$ is $x-b$, and condition (iii) becomes

$$b \in S_{nb}(C_{n+1}^p). \tag{1}$$

We now show from (iii) by transfinite induction that every b of S_n is in $L(C_{n+1}^p)$. The first b of S_n lies, by condition (ii), in L and hence in $L(C_{n+1}^p)$. Suppose now that our assertion has been established for all predecessors of b in the normal order of W_n of S_n. The field S_{nb} is then generated by elements $c<b$ which, by assumption, are all in $L(C_{n+1}^p)$; hence by (1), b is also in $L(C_{n+1}^p)$. Since K is supposed to have no separating transcendence basis, $K>L$ and Lemma 4.3 applies. It shows that $L(C_{n+1}^p) = L(S_{n+1}^p)$, so that the conclusion obtained can be stated thus:

LEMMA 9.1. *Conditions* (i), (ii), *and* (iii) *above imply that* $S_n \subset L(S_{n+1}^p)$.

This conclusion cannot always be true, as indicated in Theorem 7.2. Therefore the conclusion (iii) must be dropped. On the other hand, (i) means, as in Theorem 2.1, that each S_n has a separating transcendence basis over P. That this cannot always be the case was shown in Theorem 8.1. Schmidt's conclusions can then only be taken in some restricted form, as in our Theorems 4.2 and 5.1, or perhaps by stating that for a field K there exists a specifically selected field L and transcendence basis T for which the conclusions are true. A restricted theorem of this latter type, if demonstrable, would be satisfactory for the applications to the structure of perfect fields envisaged by Schmidt.

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