CONVERGENCE PROPERTIES OF ANALYTIC FUNCTIONS OF FOURIER-STIELTJES TRANSFORMS*

BY

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1. **Introduction.** Wiener and Pitt† have given conditions under which the reciprocal of an absolutely convergent Fourier-Stieltjes integral is again an absolutely convergent Fourier-Stieltjes integral. It is the purpose of this paper to generalize this result in two directions. We replace reciprocals by general analytic functions which may even be multiple-valued, and we replace absolute convergence by finiteness of certain more general norms. These norms are of two types, both of which depend on a parameter θ, $(0<\theta\leqq 1)$; and both reduce to total variation when $\theta=1$.

Following the notation of (WP), we let $f(x)$ denote a function of bounded variation in $(-\infty, \infty)$ for which $2f(x)=f(x+0)+f(x-0)$, and let

$$F(x) = \int_{-\infty}^{\infty} e^{-iyx} df(y).$$

We write

$$f(x) = h(x) + g(x) + s(x),$$

where $h(x)$ is a step-function, $g(x)$ is absolutely continuous, and $s(x)$ is continuous and has a zero derivative almost everywhere. We refer to $h(x)$, $g(x)$, $s(x)$ as the discrete, smooth, and singular parts of $f(x)$, and to their Fourier-Stieltjes transforms $H(x)$, $G(x)$, $S(x)$ as the almost periodic, transient, and unpredictable parts of $F(x)$; of course we have $F(x)=H(x)+G(x)+S(x)$. Moreover $h(x)$, $g(x)$, $s(x)$ are each of bounded variation and essentially uniquely determined by $f(x)$, while $H(x)$, $G(x)$, $S(x)$ are uniquely determined by $F(x)$.

We define for $0<\theta\leqq 1$,

$$T_\theta^*\{F(x)\} = 2\sum_{n=-\infty}^{\infty}\left[\int_n^{n+1} | df(y) |\right]^\theta$$

and‡

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† *On absolutely convergent Fourier-Stieltjes transforms*, Duke Mathematical Journal, vol. 4 (1938), pp. 420–436. This paper will be referred to as (WP).

‡ We use the symbol $\int |dh(y)|^\theta$ to mean the sum of the θ powers of the jumps of $h(y)$. By a jump we mean the whole jump $|h(y+0)-h(y-0)|$, not a half jump $|h(y+0)-h(y)|$.

$$T_\theta^{**}\{F(x)\} = 2\int_{-\infty}^{\infty} |dh(y)|^\theta + 2\sum_{n=-\infty}^{\infty}\left[\int_n^{n+1} |dg(y) + ds(y)|\right]^\theta;$$

and we say that $F(x) \varepsilon A_\theta^*$ *or* $F(x) \varepsilon A_\theta^{**}$ *if* $T_\theta^*[F(x)] < \infty$ *or* $T_\theta^{**}[F(x)] < \infty$. *We use the symbol* T_θ *to stand for* T_θ^* *or* T_θ^{**} *as* $\pm$ *stands for* $+$ *or* $-$; *and similarly we use* A_θ *for* A_θ^* *or* A_θ^{**}. Thus, the previous statement might have been written $F(x) \varepsilon A_\theta$ if $T_\theta[F(x)] < \infty$. We will also suppress the θ when no confusion will be caused.

We now state the main theorem of this paper:

THEOREM I. *Let* $F(x) \varepsilon A_\theta$, *let* R *be the closure of the set of values of* $F(x)$, *and let* R^* *be the set of complex numbers whose distance from* R *is not greater than* $\{T_\theta[S(x)]\}^{1/\theta}$. *Let* $\mathcal{J}(z)$ *be a multiple-valued function defined on an open set* $\mathfrak{R}$ *containing* R^*; *and let* $\mathcal{J}(z)$ *consist of exactly* n *distinct nonintersecting analytic sheets in the neighborhood of each point of* $\mathfrak{R}$. *Let the* n *continuous branches of* $\mathcal{J}[F(x)]$ *and* $\mathcal{J}[H(x)]$ *be denoted in some arbitrary order by* $[\mathcal{J}(F(x))]_j$, $(j=1, \cdots, n)$, *and by* $[\mathcal{J}(H(x))]_j$, $(j=1, \cdots, n)$. *Then there exist two permutations* $p_1, \cdots, p_n$ *and* $p_1', \cdots, p_n'$ *(each unique) of the numbers* $1, 2, \cdots, n$ *such that*

$$\lim_{N\to\infty} \frac{1}{N}\int_0^N |[\mathcal{J}(F(x))]_j - [\mathcal{J}(H(x))]_{p_j}|^2 dx = 0, \tag{1.1}$$

$$\lim_{N\to\infty} \frac{1}{N}\int_{-N}^0 |[\mathcal{J}(F(x))]_j - [\mathcal{J}(H(x))]_{p_j'}|^2 dx = 0. \tag{1.2}$$

Moreover if for any particular j *we have* $p_j = p_j'$, *then* $[\mathcal{J}(F(x))]_j \varepsilon A_\theta$.

2. **Properties of the norms.** We must first establish the fact that the norms $T_\theta\{F(x)\}$ satisfy the axioms

I. $T(F_1) + T(F_2) \geqq T(F_1 + F_2)$;
II. $T(F_1)T(F_2) \geqq T(F_1F_2)$;
III. $|a|^\theta T_\theta(F) = T_\theta(aF)$.

The first of these relations follows immediately from the inequality $a^\theta + b^\theta \geqq (a+b)^\theta$, which holds whenever $a \geqq 0$, $b \geqq 0$, and $0 < \theta \leqq 1$ (as we can readily see by choosing $a > b$ and considering the function $(b/a)^\theta + 1 - (b/a+1)^\theta$ and its θ derivative). The third relation is obvious; and it therefore only remains to prove axiom II. Assuming therefore that $F(x) = F_1(x)F_2(x)$, we obtain

$$f(y) = \int_{-\infty}^{\infty} f_1(y-u)\,df_2(u)$$

except at a countable set of points. Then

$$\begin{aligned} T_\theta^*\{F_1(x)F_2(x)\} &= 2\sum_{n=-\infty}^{\infty}\left[\int_n^{n+1}\left|d_y\int_{-\infty}^{\infty}f_1(y-u)df_2(u)\right|\right]^\theta \\ &= 2\sum_{n=-\infty}^{\infty}\left[\int_n^{n+1}\left|d_y\sum_{m=-\infty}^{\infty}\int_m^{m+1}f_1(y-u)df_2(u)\right|\right]^\theta \\ &\leqq 2\sum_{n=-\infty}^{\infty}\left[\sum_{m=-\infty}^{\infty}\int_n^{n+1}\left|d_y\int_m^{m+1}f_1(y-u)df_2(u)\right|\right]^\theta \\ &\leqq 2\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\left[\int_m^{m+1}|df_2(u)|\int_{n-m-1}^{n-m+1}|df_1(u)|\right]^\theta \\ &= 2\sum_{m=-\infty}^{\infty}\left[\int_m^{m+1}|df_2(u)|\right]^\theta\sum_{n=-\infty}^{\infty}\left[\int_{n-1}^{n+1}|df_1(y)|\right]^\theta \\ &\leqq T_\theta^*[F_2(x)]\cdot T_\theta^*[F_1(x)]. \end{aligned}$$

Thus II holds for T_θ^*; and since T_θ^* and T_θ^{**} are identical for functions with zero almost periodic part, II holds for T_θ^{**} applied to functions of the form $F(x)=G(x)+S(x)$. Since I holds for T_θ^{**}, we need merely show that II holds if $F_1(x)=H_1(x)$ and $F_2(x)=H_2(x)$ and also holds if $F_1(x)=H_1(x)$ and $F_2(x)=G_2(x)+S_2(x)$. But $H(x)$ is merely an infinite sum of terms of the form $ae^{i\lambda x}$, and since I can be extended to infinite sums, we need merely show that II holds for products of the form $a_1e^{i\lambda_1x}a_2e^{i\lambda_2x}$ and $ae^{i\lambda x}[G(x)+S(x)]$. Direct substitution takes care of the first of these products; and the proof is completed by noting that if m is the greatest integer less than λ,

$$\begin{aligned} T_\theta^{**}\{ae^{i\lambda x}[G(x)+S(x)]\} &= T_\theta^{**}\left\{a\int_{-\infty}^{+\infty}e^{ixy}d[g(y-\lambda)+ds(y-\lambda)]\right\} \\ &= 2|a|^\theta\sum_{n=-\infty}^{\infty}\left[\int_n^{n+1}|dg(y-\lambda)+ds(y-\lambda)|\right]^\theta \\ &\leqq 2|a|^\theta\sum_{n=-\infty}^{\infty}\left[\int_{n-m-1}^{n-m+1}|dg(y)+ds(y)|\right]^\theta \\ &\leqq T_\theta^{**}\{ae^{i\lambda x}\}\cdot T_\theta^{**}\{G(x)+S(x)\}. \end{aligned}$$

3. **Functions of small norm.** We begin to prove Theorem I by first proving that the special case of it in which $\mathcal{J}(z)$ is single-valued and $F(x)$ is a constant plus a function of small norm.

LEMMA 1. *Let $F(x)\ \varepsilon\ A_\theta$, let $F(x)=\tau+F_1(x)$ where $T_\theta[F_1(x)]<K^\theta$, and let $\mathcal{J}(z)$ be analytic in a circle about τ of radius K. Then $\mathcal{J}\{F(x)\}\ \varepsilon\ A_\theta$.*

For $\mathcal{J}(z)$ has a Taylor's series $\mathcal{J}(z)=\sum_{n=0}^{\infty}a_n(z-\tau)^n$ converging when

$|z-\tau|<K$. Then if $T_\theta[F_1(x)]<K_1^\theta<K^\theta$, $\sum_{n=0}^\infty|a_n|K_1^n$ converges and $|a_n|K_1^n$ is bounded in n. Hence $\sum_{n=0}^\infty|a_n|^\theta W^n$ converges when $0<W<K_1^\theta$. Thus

$$T_\theta[\mathcal{J}(F(x))] = T_t\left[\sum_{n=0}^\infty a_n(F(x)-\tau)^n\right]$$
$$= T_\theta\left[\sum_{n=0}^\infty a_n(F_1(x))^n\right] \leqq \sum_{n=0}^\infty T_\theta[a_n(F_1(x))^n]$$
$$\leqq \sum_{n=0}^\infty |a_n|^\theta\{T_\theta[F_1(x)]\}^n,$$

and since $T_\theta[F_1(x)]<K_1^\theta$, it follows that the last sum is finite. Thus $\mathcal{J}(F(x))\ \varepsilon\ A_\theta$.

4. **The space** $C\mathfrak{T}_n$. *Let $\mathfrak{T}_n$ be the set of points each of which consists of n ordered numbers, each reduced modulo 2π. Let C be the set of all real numbers, together with one special symbol ∞. Let $C\mathfrak{T}_n$ be the product space of C and $\mathfrak{T}_n$.*

The set of points $(x_1, \cdots, x_n)$ of $\mathfrak{T}_n$ which satisfy

$$|x_j - x_j'| < \epsilon \pmod{2\pi}, \qquad j = 1, \cdots, n,$$

is called the ϵ-neighborhood of $(x_1', \cdots, x_n')$. The set of points x of C which satisfy $|x-x'|<\epsilon$ is called the ϵ-neighborhood of x. The set of finite points x which satisfy $|x|>1/\epsilon$ together with ∞, is called the ϵ-neighborhood of ∞. Product neighborhoods such as the ϵ-neighborhood of $(x_1, \cdots, x_n; x)$ or $(x_1, \cdots, x_n; \infty)$ are defined in the usual way. The ϵ-neighborhood of $(x_1, \cdots, x_n; \infty)$ will be called an infinite $C\mathfrak{T}_n$ neighborhood, and $(x_1, \cdots, x_n; \infty)$ will be called an infinite point of $C\mathfrak{T}_n$. It is obvious that the Heine-Borel theorem holds for the whole space.

5. **Finiteness of the norm a local property.** *A function $F(x)$ is called locally of finite norm in a finite C neighborhood N if there exists a function $F^*(x)$ which is of finite norm and equals $F(x)$ when x is in N. A function $F(x)$ is called locally of finite norm with respect to $\lambda_1, \cdots, \lambda_n$ in a $C\mathfrak{T}_n$ neighborhood N if there exists a function $F^*(x)$ which is of finite norm and equals $F(x)$ when $(\lambda_1 x, \cdots, \lambda_n x; x)$ is in N.*

LEMMA 2. *Let $\lambda_1, \cdots, \lambda_n$ be given. Then a necessary and sufficient condition that a function $f(x)$ be of finite norm is that it be locally of finite norm in a C neighborhood of each finite point of C and locally of finite norm with respect to $\lambda_1, \cdots, \lambda_n$ in a $C\mathfrak{T}_n$ neighborhood of each infinite point of $C\mathfrak{T}_n$.*

The necessity of the condition is obvious; so we need only prove sufficiency. We note at the outset that the hypothesis implies that $f(x)$ is locally of finite norm with respect to $\lambda_1, \cdots, \lambda_n$ in a $C\mathfrak{T}_n$ neighborhood of every point of $C\mathfrak{T}_n$. For a function which equals $f(x)$ in the ϵ-neighborhood of the point x_0 of C necessarily equals $f(x)$ when $(\lambda_1 x, \cdots, \lambda_n x; x)$ is in the ϵ-neighborhood of the point $(x_1, \cdots, x_n; x_0)$ of $C\mathfrak{T}_n$. Thus for each point P of $C\mathfrak{T}_n$ there is an $\epsilon_P > 0$ and a function $f_P(x)$ which is of finite norm and which equals $f(x)$ when $(\lambda_1 x, \cdots, \lambda_n x; x)$ is in the ϵ_P-neighborhood $\mathfrak{N}_P$ of P. Then by the Heine-Borel theorem there is a finite number of points $P_1, \cdots, P_q$, such that the $\epsilon/2$-neighborhoods of P_j cover $C\mathfrak{T}_n$. Choose an integer

$$N > 2\pi[\min(\epsilon_{P_1}, \cdots, \epsilon_{P_q})]^{-1} \max [|\lambda_1|, \cdots, |\lambda_n|; 1].$$

Let $\Phi^*(x)$ be an even function which is zero on $|x| > 1$, unity at $x = 0$, and is continuous and has derivatives of all orders everywhere and satisfies $\Phi^*(x) + \Phi^*(1-x) = 1$ on $0 \leqq x \leqq 1$. Thus, to be specific, we may define

$$\Phi^*(x) = \begin{cases} \dfrac{1}{2} - \dfrac{\int_0^{2|x|-1} e^{(\xi^2-1)^{-1}} d\xi}{2\int_0^1 e^{(\xi^2-1)^{-1}} d\xi} & \text{if } 0 < |x| \leqq 1, \\ 0 & \text{if } 1 \leqq |x|. \end{cases}$$

We obviously have

$$\sum_{k=-N^2}^{N^2} \Phi(x-k) = \begin{cases} 1 & \text{if } |x| \leqq N^2, \\ 0 & \text{if } |x| > (N+1)^2. \end{cases}$$

Moreover if

$$\begin{cases} \Phi(x) = \Phi^*\left(\dfrac{Nx}{\pi}\right), & |x| < \pi, \\ \Phi(x) = \Phi(x + 2\pi), & \text{for all } x, \end{cases}$$

then

$$\sum_{k=1}^{2N} \Phi\left(x - \frac{\pi k}{N}\right) = 1, \quad \text{for all } x.$$

Thus

$$\sum_{k_1, \cdots, k_n=1}^{2N} \Phi\left(\lambda_1 x - \frac{\pi}{N} k_1\right) \cdots \Phi\left(\lambda_n x - \frac{\pi}{N} k_n\right) = 1 \quad \text{for all } x,$$

and

$$\begin{aligned} f(x) &= \sum_{k=-N^2}^{N^2} \Phi^*(Nx - k) f(x) \\ &\quad + \left[1 - \sum_{k=-N^2}^{N^2} \Phi^*(Nx-k)\right] \sum_{k_1, \cdots, k_n=1}^{2N} \Phi\left(\lambda_1 x - \frac{\pi}{N} k_1\right) \\ &\quad \cdots \Phi\left(\lambda_n x - \frac{\pi}{N} k_n\right) f(x) \end{aligned}$$

for all x. Thus if we show that $\Phi^*(Nx-k)f(x)$ and

$$\left[1-\sum_{k=-N^2}^{N^2}\Phi^*(Nx-k)\right]\Phi\left(\lambda_1 x-\frac{\pi}{N}k_1\right)\cdots\Phi\left(\lambda_n x-\frac{\pi}{N}k_n\right)f(x)$$

are of finite norm for all $k, k_1, \cdots, k_n$, it follows that $f(x)$ is of finite norm.

To show this for $\Phi^*(Nx-k)$, consider the point P_j whose $\epsilon_{P_j}/2$-neighborhood covers

$$\left(\frac{k}{N}\lambda_1, \cdots, \frac{k}{N}\lambda_n; \frac{k}{N}\right).$$

The ϵ_{P_j}-neighborhood $\mathfrak{N}_j$ of this point covers the $\epsilon_{P_j}/2$-neighborhood of

$$\left(\frac{k}{N}\lambda_1, \cdots, \frac{k}{N}\lambda_n; \frac{k}{N}\right).$$

Thus when $|Nx-k|<1$, $(\lambda_1 x, \lambda_2 x, \cdots, \lambda_n x; x)$ is in $\mathfrak{N}_j$, and when $\Phi^*(Nx-k)$ is not zero, $f_{P_j}(x)$ equals $f(x)$ and for all x

$$T^*(Nx-k)f(x) = T^*(Nx-k)f_{P_j}(x),$$

which is of finite norm.

Again, consider the point P_l whose $\epsilon_{P_l}/2$-neighborhood covers the point $(k_1\pi/N, \cdots, k_n\pi/N; \infty)$. The ϵ_{P_l}-neighborhood $\mathfrak{N}_l$ of P_l covers the $\epsilon_{P_l}/2$-neighborhood of $(k_1\pi/N, \cdots, k_n\pi/N; \infty)$. Thus when

$$|\lambda_1 x - k_1\pi/N| < \pi/N, \cdots, |\lambda_n x - k_n\pi/N| < \pi/N, \qquad |x| > N,$$

$(\lambda_1 x, \lambda_2 x, \cdots, \lambda_n x, x)$ is in $\mathfrak{N}_l$ and

$$\left[1-\sum_{k=-N^2}^{N^2}\Phi^*(Nx-k)\right]\Phi\left(\lambda_1 x-\frac{\pi}{N}k_1\right)\cdots\Phi\left(\lambda_n x-\frac{\pi}{N}k_n\right)f(x)$$
$$=\left[1-\sum_{k=-N^2}^{N^2}\Phi^*(Nx-k)\right]\Phi\left(\lambda_1 x-\frac{\pi}{N}k_1\right)\cdots\Phi\left(\lambda_n x-\frac{\pi}{N}k_n\right)f_{P_l}(x).$$

Thus $f(x)$ is of finite norm.

6. **Periodic functions with assigned derivatives and small norm.** Section 5 makes it necessary only to show that $\mathcal{J}[F(x)]$ is locally of finite norm corresponding to all infinite points $C\mathfrak{T}_n$ and all finite points of C; and §3 shows that this will be established if we show how to replace functions by others locally equivalent but of sufficiently small norm. We begin by replacing exponential functions by locally equivalent functions of small norm.

The first step is to find functions of small norm having derivatives at the origin equal to those of $e^{ix}-1$.

Such a function is given by

$$\Psi_{m,p}(x) = e^{(i/p)\psi_m(px)} - 1$$

where

$$\psi_m(x) = x - \frac{2\pi\int_0^x \sin^{2m}\xi\, d\xi}{\int_0^{2\pi} \sin^{2m}\xi\, d\xi}; \tag{6.1}$$

for $\psi_m(x)$ is obviously periodic and hence is an exponential polynomial. Thus $\psi_m(px)$ has norms independent of p (for p an integer greater than 1) and the norms of $(1/p)\psi_m(px) = O(1/p^\theta)$. Thus for fixed m the norm of $\Psi_{m,p}(x)$ can be made arbitrarily small by making p sufficiently great; and since $\psi_m(x) = x + O(x^{2m+1})$ at the origin, it follows that $\Psi_{m,p}(x) = e^{ix} - 1 + O(x^{2m+1})$, and $\Psi_{m,p}(x)$ and $e^{ix}-1$ have the same first $2m$ derivatives at $x=0$ and points congruent (mod 2π).

7. **Locally exponential functions of small norm.** Now to obtain a function of small norm which is actually equal to $e^{ix}-1$ in the neighborhood of points congruent to zero (mod 2π) we introduce the function

$$\Omega(\epsilon, x) = \begin{cases} 1, & 0 \leqq |x| \leqq \epsilon \pmod{2\pi}, \\ \dfrac{1}{2} - \dfrac{\int_0^{2|x|/\epsilon - 3} e^{(\xi^2-1)^{-1}} d\xi}{2\int_0^1 e^{(\xi^2-1)^{-1}} d\xi}, & \epsilon \leqq |x| \leqq 2\epsilon \pmod{2\pi}, \\ 0, & 2\epsilon \leqq |x| \leqq \pi \pmod{2\pi} \end{cases}$$

which is obviously of finite norm for all θ and of period 2π.

Let $H(x)$ be a periodic function of period 2π whose first m derivatives are continuous everywhere. Then if $H(0) = H'(0) = \cdots = H^{(m)}(0) = 0$, and $1/m < \theta \leqq 1$, we shall show that

$$\lim_{\epsilon \to 0} T_\theta\{H(x)\Omega(\epsilon, x)\} = 0. \tag{7.1}$$

For if $H(x)\Omega(\epsilon, x) = P(\epsilon, x) = \sum_{n=-\infty}^{\infty} p_n(\epsilon)e^{inx}$, we have by integration by parts

$$p_n = \frac{1}{2\pi}\int_0^{2\pi} P(\epsilon, x)e^{-inx}dx = \frac{1}{2\pi(in)^m}\int_0^{2\pi} P^{(m)}(\epsilon, x)e^{-inx}dx \quad \text{for} \quad n \neq 0.$$

Thus if $L(\epsilon)$ is the greatest value taken on by either $|P(\epsilon, x)|$ or $|P^{(m)}(\epsilon, x)|$ for all x, we have $|p_n(\epsilon)| \leqq 2L(\epsilon)/(1+|n|^m)$ for all n and

$$T_\theta\{P(\epsilon, x)\} \leqq 2[L(\epsilon)]^\theta \sum_{n=-\infty}^{\infty} \frac{2}{(1+|n|^m)^\theta} \quad \text{for all } m.$$

Since $m\theta > 1$, this sum converges, and we need merely show that $L(\epsilon) \to 0$ to establish (7.1). But $\max_x |P(\epsilon, x)| \to 0$ as $\epsilon \to 0$ since $H(x)$ is continuous and vanishes at zero and $\Omega(\epsilon, x)$ is bounded and is zero when $|x| \geqq 2\epsilon$ (mod 2π). Moreover

$$P^{(m)}(\epsilon, x) = \sum_{j=0}^{m} C_{m,j} H^{(m-j)}(x)\Omega^{(j)}(\epsilon, x),$$

and we shall show that each term of this sum approaches zero uniformly as $\epsilon \to 0$. In the interval $|x| < 2\epsilon$ the function $\Omega(\epsilon, x)$ is a function of x/ϵ alone, and hence its j'th derivative with respect to x is less than $C_j\epsilon^{-j}$, where C_j is independent of ϵ and x. Moreover $H^{(m-j)}(x)x^{-j} \to 0$ as $x \to 0$; so

$$\max_{|x| \leqq 2\epsilon} \frac{|H^{(m-j)}(x)|}{\epsilon^j} \to 0 \quad \text{as } \epsilon \to 0.$$

Thus

$$\begin{aligned} \max_x |H^{(m-j)}(x)\Omega^{(j)}(\epsilon, x)| &= \max_{|x| \leqq 2\epsilon} |H^{(m-j)}(x)\Omega^{(j)}(\epsilon, x)| \\ &\leqq \left[\max_{|x| \leqq 2\epsilon} \frac{|H^{(m-j)}(x)|}{\epsilon^j}\right] \cdot C_j \to 0 \end{aligned}$$

and $L(\epsilon) \to 0$ as $\epsilon \to 0$, and (7.1) is established.

We now define

$$E_{m,p}(\epsilon, x) = \Omega(\epsilon, x)\{e^{ix} - 1 - \Psi_{m,p}(x)\} + \Psi_{m,p}(x)$$

and have, for fixed m and θ such that $2m\theta > 1$,

$$\lim_{p \to \infty} \left[\lim_{\epsilon \to 0} T_\theta\{E_{m,p}(\epsilon, x)\}\right] = 0,$$

while for $|x| < \epsilon$ (mod 2π), $E_{m,p}(\epsilon, x) = e^{ix} - 1$.

8. ***G*-functions with assigned derivatives and small norm.** We now seek to replace a G-function by a function of small norm C-locally equivalent to it. We begin by finding a function of small norm having the same derivatives as the given function at the origin.

Let $G(x)$ be an entire function which vanishes at the origin and m a positive integer. Then $G[\psi_m(px)/p]$ (where $\psi_m(x)$ is the function defined in (6.1)) has its first $2m$ derivatives at $x=0$ equal to those of $G(x)$, and the norm of $G[\psi_m(px)/p]$ approaches zero as $p \to \infty$.

9. ***G*-functions locally of small norm.** *Let $G(x)$ be a G-function which is also an entire function having $G(0) = G'(0) = \cdots = G^{(m)}(0) = 0$. Let*

$$\Omega^*(\epsilon, x) = \begin{cases} \Omega(\epsilon, x), & 0 \leqq |x| \leqq \pi, \\ 0, & |x| \geqq \pi. \end{cases}$$

Then if $1/m<\theta\leqq 1$, *we shall show that*

$$\lim_{\epsilon\to 0} T_\theta\{\Omega^*(\epsilon, x)G(x)\} = 0. \tag{9.1}$$

This statement is proved in much the same way as the corresponding statement for H-functions.

Let

$$\Omega^*(\epsilon, x)G(x) = P^*(\epsilon, x) = \int_{-\infty}^{\infty} p^*(\epsilon, \xi)e^{-i\xi x}d\xi,$$

so that

$$p^*(\epsilon, \xi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P^*(\epsilon, x)e^{i\xi x}dx.$$

Integrating by parts we have

$$p^*(\epsilon, \xi) = \frac{1}{(-i\xi)^m 2\pi}\int_{-\pi}^{\pi} P^{*(m)}(\epsilon, x)e^{i\xi x}dx \quad \text{if} \quad \xi\neq 0.$$

Thus

$$|\, p^*(\epsilon, \xi)\,| < \frac{2}{|\,\xi\,|^m + 1} L^*(\epsilon) \quad \text{for all } \xi,$$

where $L^*(\epsilon)$ is the greater of the upper bounds of $|P^*(\epsilon, x)|$ and $|P^{*(m)}(\epsilon, x)|$ on $|x|\leqq\pi$. Now as $\epsilon\to 0$, the bounds of $|P^*(\epsilon, x)|$ and $|P^{*(m)}(\epsilon, x)|$ approach zero for the same reason that $|P(\epsilon, x)|$ and $|P^{(m)}(\epsilon, x)|$ approached zero in §7. Thus $\lim_{\epsilon\to 0} L^*(\epsilon)=0$, and

$$\begin{aligned} T_\theta[P^*(\epsilon, x)] &= 2\sum_{n=-\infty}^{\infty}\left[\int_n^{n+1} |\, p^*(\epsilon, \xi)\,|\, d\xi\right]^\theta \\ &\leqq 2[L^*(\epsilon)]^\theta \sum_{n=-\infty}^{\infty}\left[\int_n^{n+1} \frac{2d\xi}{(1+|\,\xi\,|^m)}\right]^\theta; \end{aligned}$$

and if $m\theta>1$, (9.1) holds.

We now define for any G-function which is also an entire function and vanishes at the origin,

$$\Gamma_{m,p}(G\,|\,\epsilon, x) = \Omega^*(\epsilon, x)\{G(x) - G[\psi_m(px)/p]\} + G[\psi_m(px)/p],$$

and we have for fixed m and θ such that $2m\theta>1$,

$$\lim_{p\to\infty}\lim_{\epsilon\to 0} T_\theta\{\Gamma_{m,p}(G\,|\,\epsilon, x)\} = 0,$$

while for $|x|<\epsilon$, $\Gamma_{m,p}(G|\,\epsilon, x)=G(x)$.

10. **G-functions of small norm at ∞.** Turning now to the C-neighborhood of ∞, we seek to replace an entire G-function by a G-function of small norm equal to the given function near ∞. Let $G(x)$ be an entire G-function given by

$$G(x) = \int_{-A}^{A} e^{-iux} g(u)du$$

(where $g(u)=0$ if $|u|>A$), and let

$$G_\delta(x) = \int_{-\infty}^{\infty} e^{-iux} g_\delta(u)du$$

where

$$g_\delta(u) = \frac{1}{\delta^3}\int_u^{u+\delta}\int_{\xi_3}^{\xi_3+\delta}\int_{\xi_2}^{\xi_2+\delta} g(\xi_1)d\xi_1 d\xi_2 d\xi_3$$

is the triple smoothing of $g(u)$.

Then

$$T_\theta[G(x) - G_\delta(x)] = 2\sum_{n=-\infty}^{\infty}\left[\int_n^{n+1} | g(u) - g_\delta(u) | du\right]^\theta;$$

so

$$\lim_{\delta\to 0} T_\theta[G(x) - G_\delta(x)] = 0.$$

Now let

$$[1 - \Omega^*(N, x)]G_\delta(x) = P_\delta(N, x) = \int_{-\infty}^{\infty} p_\delta(N, \xi)e^{-i\xi x}d\xi.$$

Since $g_\delta(u)$ has two continuous derivatives, if $m>1/\theta$, $G_\delta^{(k)}(x)=o(1/x^2)$ and $P_\delta^{(k)}(N, x)=o(1/x^2)$ at $\pm\infty$ for $k=0, 1, \cdots, m$. But

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} P_\delta(N, x)e^{iux}dx = p_\delta(N, u),$$

and integrating by parts, we have

$$p_\delta(N, u) = \frac{1}{(-iu)^m 2\pi}\int_{-\infty}^{\infty} e^{iux}P_\delta^{(m)}(N, x)dx.$$

Thus if $L_\delta(N)$ is the greater of the upper bounds of

$$| (x^2 + 1)P_\delta(N, x) |, \qquad | (x^2 + 1)P_\delta^{(m)}(N, x) |,$$

we have

$$| p_\delta(N, u) | \leq \min\left(1, \frac{1}{| u |^m}\right)\frac{\pi}{2\pi} L_\delta(N) \leq \frac{L_\delta(N)}{1 + | u |^m};$$

so that

$$T_\theta[P_\delta(N, x)] = 2\sum_{n=-\infty}^{\infty}\left[\int_n^{n+1} |p_\delta(N, \xi)|\, du\right]^\theta$$
$$\leqq 2[L_\delta(N)]^\theta \sum_{n=-\infty}^{\infty}\left[\int_n^{n+1} \frac{du}{1+|u|^m}\right]^\theta,$$

where the latter sum converges if $m\theta>1$. But $1-\Omega^*(N, x)$ is never numerically greater than 1 and is zero when $|x|<N$; so $\overline{\mathrm{Bd}}\ P_\delta(N, x)\to 0$ as $N\to\infty$. Moreover for $N>1$, each derivative of $1-\Omega^*(N, x)$ is bounded in N and x, and is zero when $|x|<N$. Thus each term of Leibnitz' expansion of

$$(1+x^2)\frac{d^m}{dx^m}[(1-\Omega^*(N, x))G_\delta(x)]$$

has its upper bounds approach zero as $N\to\infty$. Hence $L_\delta(N)\to 0$ as $N\to\infty$, and $\lim_{N\to\infty} T_\theta[P_\delta(N, x)]=0$.

Finally, we define

$$\Gamma^*_{N,\delta}(G\,|\,x) = G(x) - G_\delta(x) + P_\delta(N, x)$$

and have

$$\lim_{\delta\to 0}\lim_{N\to\infty} T_\theta[\Gamma^*_{N,\delta}(G\,|\,x)] = 0$$

and $\Gamma^*_{N,\delta}(G|x)=G(x)$ for $x\geqq 2N$.

11. **Proof of Theorem** I. Returning now to the proof of Theorem I, we note that the fact that $\mathcal{J}[F(x)]$ consists of n continuous functions is obvious, as is also the same fact for $\mathcal{J}(H(x))$, since each value of $H(x)$ belongs to R. Now the symmetric functions of the n values of $\mathcal{J}(H(x))$ are single-valued analytic functions of $\mathcal{J}(H(x))$ which are therefore almost periodic; and by Walther's theorem on algebraic functions of almost periodic functions, it follows that each branch of $\mathcal{J}(H(x))$ is almost periodic.

Since the closed set R^* is contained in the open set $\mathcal{R}$, there is a positive number η such that every number within a distance η of R^* is in $\mathcal{R}$. Choose L so great that $|G(x)|<\eta/2$ when $|x|\geqq L$. Then $|F(x)-H(x)|<\eta/2+|S(x)|$, so that if once chosen on the same branch, $F(x)$ and $H(x)$ remain on the same branch of $\mathcal{J}(z)$ when $x\geqq L$ and also when $x\leqq -L$. Thus given $\epsilon>0$, we can find $\delta>0$ such that if $x>L$ and $|F(x)-H(x)|<\delta$, then $|[\mathcal{J}(F(x))]_j - [\mathcal{J}(H(x))]_{p_j}|<\epsilon$, where p_j is so chosen that $[\mathcal{J}(F(x))]_j$ and $[\mathcal{J}(H(x))]_{p_j}$ are on the same branch for $x>L$. From this (1.1) and (1.2) readily follow. But two almost periodic functions never have their mean square difference zero, and hence p_j and p_j' are unique.

Now suppose that $p_j=p_j'$; and write simply $\mathcal{J}(F(x))$ and $\mathcal{J}(H(x))$ for

$[\mathcal{J}(F(x))]_j$ and $[\mathcal{J}(H(x))]_{p_j}$. Then whenever $|x| \geqq L$, $F(x)$ and $H(x)$ are to be taken on the same branch of $\mathcal{J}(z)$.

In the closed region $\mathcal{R}^*_{\eta/2}$ of points within $\eta/2$ of R^*, there is a minimum distance γ between the branches of $\mathcal{J}(z)$, so that for all z in $\mathcal{R}^*_{\eta/2}$, $|[\mathcal{J}(z)]_j - [\mathcal{J}(z)]_k| \geqq \gamma$ for all pairs of branches. Now if M is the least common module of $\mathcal{J}(H(x))$ and $H(x)$, we can find a finite number of elements $\mu_1, \cdots, \mu_q$ of M and a positive number ϵ_1 such that whenever

$$|\mu_j h| < \epsilon_1 \pmod{2\pi}, \qquad j = 1, \cdots, q,$$

then

$$|\mathcal{J}(H(x+h)) - \mathcal{J}(H(x))| < \gamma/2 \quad \text{for all } x,$$

and $|H(x+h) - H(x)|$ is uniformly so small that if $\mathcal{J}$ were taken on the same branch for both of these values, then $|\mathcal{J}(H(x+h)) - \mathcal{J}(H(x))|$ would also be less than $\gamma/2$. Thus $\mathcal{J}(H(x+h))$ and $\mathcal{J}(H(x))$ are on the same branch for all x.

Choose $\bar{\delta}$ so small that $[4\bar{\delta} + T_\theta[S(x)]]^{1/\theta} + (2\bar{\delta})^{1/\theta} < T_\theta[S(x)]^{1/\theta} + \eta/2$. Let $H(x) = \sum_{k=1}^{\infty} h_k e^{i\lambda_k x}$, and let q' be chosen so that if $H^*(x) = \sum_{k=q'+1}^{\infty} h_k e^{i\lambda_k x}$, then $T_\theta(H^*(x)) < \bar{\delta}$. We shall show that $\mathcal{J}(F(x))$ is locally of finite norm with respect to $\mu_1, \cdots, \mu_q, \lambda_1, \cdots, \lambda_{q'}$ in the $\mathfrak{T}_{q+q'}C$ neighborhood of every infinite point of $\mathfrak{T}_{q+q'}C$; and hence after a similar argument for finite C neighborhoods that $\mathcal{J}[F(x)]$ is of finite norm.

Let us consider the $\mathfrak{T}C$ neighborhood of $(\mu_1\xi_1, \cdots, \mu_q\xi_q, \lambda_1\xi_1', \cdots, \lambda_{q'}\xi_{q'}'; \infty)$. Let

$$S(x) = \int_{-\infty}^{\infty} e^{-i\xi x} ds(\xi), \qquad G(x) = \int_{-\infty}^{\infty} e^{-i\xi x} g(\xi) d\xi;$$

and let $G^*(x) = \int_{-A}^{A} e^{-i\xi x} g(\xi) d\xi$, where A is so great that $T_\theta[G(x) - G^*(x)] < \bar{\delta}$. Let $\Gamma^*_{N',\delta'}[G^*|x] = G^{**}(x)$, where N', δ' are chosen so that $T_\theta[G^{**}(x)] < \bar{\delta}$. Then $G^{**}(x) = G^*(x)$ when $|x| > 2N'$. Let

$$H^{**}(x) = \sum_{k=1}^{q'} h_k E_{m,p_k}(\epsilon^{(k)}, \lambda_k x - \lambda_k \xi_k) e^{i\lambda_k \xi_k},$$

where m, p_k, $\epsilon^{(k)}$ are so chosen that

$$T_\theta[h_k E_{m,p_k}(\epsilon^{(k)}, \lambda_k x - \lambda_k \xi_k)] < \bar{\delta}/2q'.$$

Then $T_\theta[H^{**}(x)] < \bar{\delta}$, and

$$H^{**}(x) = H(x) - H^*(x) - \sum_{k=1}^{q'} h_k e^{i\lambda_k \xi_k}$$

whenever $|\lambda_k(x-\xi_k)|<\epsilon^{(k)}$ (mod 2π), ($k=1,\cdots,q'$). Let

$$\epsilon = \min(\epsilon_1/2,\ 1/2N',\ 1/L,\ \epsilon^{(1)},\cdots,\epsilon^{(q')}),$$

and consider the ϵ-neighborhood of $(\mu_1\xi_1,\cdots,\mu_q\xi_q,\lambda_1\xi_1',\cdots,\lambda_{q'}\xi_{q'}';\ \infty)$. We shall show that the function $\mathcal{J}(F(x))$ is locally of finite norm with respect to $\mu_1,\cdots,\mu_q,\lambda_1,\cdots,\lambda_{q'}$ in this neighborhood. We say $x\ \varepsilon\ \mathcal{N}$ if $(x\mu_1,\cdots,x\mu_q,x\lambda_1,\cdots,x\lambda_{q'},x)$ is in this neighborhood.

If $x\ \varepsilon\ \mathcal{N}$, we have $G^{**}(x)=G^*(x)$ and

$$H^{**}(x) = H(x) - H^*(x) - \tau \tag{11.1}$$

where $\tau=\sum_{k=1}^{q'}h_k e^{i\lambda_k\xi_k}$; and if x_1, x_2 are any two points in $\mathcal{N}$,

$$|(x_1-x_2)\mu_j| < \epsilon_1 \ (\text{mod } 2\pi), \qquad j=1,\cdots,q,$$

and

$$|\mathcal{J}(H(x_1)) - \mathcal{J}(H(x_2))| < \gamma/2,$$

and $\mathcal{J}[H(x_1)]$ and $\mathcal{J}[H(x_2)]$ are on the same branch of $\mathcal{J}(z)$. Thus $\mathcal{J}(z)$ is a single-valued analytic function over the set of values of $H(x)$ for x in $\mathcal{N}$, and since there is no branch point or singularity within $[T_\theta(S(x))]^{1/\theta}+\eta/2$ of these points, it follows that for x in $\mathcal{N}$, $\mathcal{J}(z)$ is analytic and single-valued for all $z=F(x)$. Moreover $\mathcal{J}(z)$ has an analytic and single-valued branch for all points within $\eta/2+[T_\theta(S(x))]^{1/\theta}$ of points for which $z=F(x)$ with x in $\mathcal{N}$, and on this branch $\mathcal{J}[F(x)]$ has the values we have agreed to denote by $\mathcal{J}(F(x))$.

For x in $\mathcal{N}$, $F(x)=F^*(x)$, where

$$F^*(x) = \tau + H^*(x) + H^{**}(x) + [G(x) - G^*(x)] + G^{**}(x) + S(x).$$

Moreover

$$T_\theta\{F^*(x) - \tau\} < 4\bar{\delta} + T_\theta[S(x)], \tag{11.2}$$

and by (11.1) the point τ is within $(2\bar{\delta})^{1/\theta}$ of R; and by (11.2) and the definition of $\bar{\delta}$, we find that $\mathcal{J}(z)$ is analytic and single-valued throughout a circle of radius greater than $\{T_\theta[F^*(x)-\tau]\}^{1/\theta}$ about τ. Thus by Lemma 1, $\mathcal{J}[F^*(x)]\ \varepsilon\ A^*$, and since $\mathcal{J}[F(x)]=\mathcal{J}[F^*(x)]$ in $\mathcal{N}$, $\mathcal{J}[F(x)]$ is locally of finite norm with respect to $(\mu_1,\cdots,\mu_q,\lambda_1,\cdots,\lambda_{q'})$ in a $C\mathfrak{T}$ neighborhood of $(\mu_1\xi_1,\cdots,\mu_q\xi_q,\lambda_1\xi_1',\cdots,\lambda_{q'}\xi_{q'}';\ \infty)$. The simpler fact that $\mathcal{J}[F(x)]$ is locally of finite norm in the neighborhood of every finite point of C can be proved in a similar but simpler manner; and we therefore conclude that $\mathcal{J}[F(x)]\ \varepsilon\ A_\theta$.

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