ON A GENERALIZATION OF THE STIELTJES MOMENT PROBLEM*

BY

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Introduction. The moment problem of Stieltjes is the problem of determining the non-decreasing solutions $\alpha(t)$ of the set of equations

$$(1) \qquad \mu_n = \int_0^\infty t^n d\alpha(t), \qquad n = 0, 1, 2, \cdots;$$

the phrase "a moment problem" is also used to describe the system (1) itself. If a solution of (1) is known, there arises the further question of whether or not the function $\alpha(t)$ is unique.‡ It is this question which we shall discuss for a generalized moment problem, namely

$$(2) \qquad \mu_n = \int_0^\infty t^{\lambda_n} d\alpha(t), \quad 0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n \to \infty.$$

If (2) has a unique solution $\alpha(t)$, we say that (2) is determined; otherwise (2) is said to be undetermined.

The various classical methods for the study of (1) seem not to apply to (2), since they depend too much on special properties of the sequence $\{\lambda_n\} = \{n\}$. We shall discuss the determination problem for (2) by considering the function

$$(3) \qquad f(z) = \int_0^\infty t^z d\alpha(t),$$

which is analytic for $\Re(z) > 0$, and takes the values μ_n at the points λ_n; since $\alpha(t)$ is non-decreasing, the growth of $f(z)$ is governed by the growth of the μ_n. We obtain sufficient conditions for (2) to be determined by applying a fundamental theorem of T. Carleman concerning the growth of functions analytic in a half-plane.

The criteria obtained in this way are probably not the best possible; when $\lambda_n = n$, they are certainly not, since we obtain

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‡ Two solutions $\alpha(t)$ of (1) are considered the same if they have the same "normalization," determined by $\alpha(0) = 0$, $\alpha(t) = [\alpha(t+) + \alpha(t-)]/2$, $t > 0$.

$$\mu_n^{1/(2n)} = o(n) \tag{4}$$

as a sufficient condition for (1) to be determined; this is much weaker than Carleman's criterion, $\sum_{n=1}^{\infty}\mu_n^{-1/(2n)}$ divergent. On the other hand, we obtain what may be regarded as new criteria for the case $\lambda_n = n$, since we shall show that (4) is still a sufficient condition for (1) to be determined if we disregard a set of integers n_k such that

$$\sum_{k=1}^{\infty} \frac{1}{n_k} < \infty, \qquad \liminf_{r\to\infty} \frac{\Delta(r)}{r^{1/2}} < \infty,$$

where $\Delta(r)$ is the maximum number of consecutive integers which are n_k's for $n_k \leqq r$ (for example, $n_k = k^2$).

Another interesting case is that where

$$\limsup_{n\to\infty} | \lambda_n - n | < \infty .$$

In this case, (4) is again a sufficient condition for (2) to be determined.

In general, the denser the λ_n, the less we have to restrict the growth of the μ_n to be sure that (2) will be determined. On the other hand, if the λ_n are so sparse that $\sum_1^{\infty} 1/\lambda_n < \infty$, there are presumably no criteria for determination depending only on the order of magnitude of the μ_n. For, since even the moment problem for a finite interval,

$$\mu_n = \int_0^1 t^{\lambda_n} d\alpha(t), \tag{5}$$

may be undetermined in this case,* we could only hope to show that (2) would be determined if the μ_n approached zero with extreme rapidity. But, if $\alpha(t)$ has a point of increase $t_0 > 0$, we necessarily have

$$\mu_n \geqq t_0^{\lambda_n}[\alpha(\infty) - \alpha(t_0 -)],$$

and so a lower limit to the rate of decrease of the μ_n.

1. Let

$$\lambda_0 = 0, \qquad \lambda_1 \geqq 1, \qquad \lambda_n \uparrow \infty \text{ as } n \uparrow \infty . \tag{1.1}$$

We write

* In fact, if $d\alpha(t) = a(t)dt$ and $a(t) \geqq \delta > 0$, $(0 \leqq t \leqq 1)$, then (5) is undetermined. For, by Müntz's theorem (see, for example, R. E. A. C. Paley and N. Wiener, *Fourier Transforms in the Complex Domain*, American Mathematical Society Colloquium Publications, vol. 19, 1934, p. 36), there is a continuous $b(t)$ such that $\int_0^1 t^{\lambda_n} b(t)dt = 0$, $(n = 0, 1, 2, \cdots)$, and $b(t) \not\equiv 0$. We may suppose $b(t) \leqq \delta$; then $\int_0^1 t^{\lambda_n}[a(t) - b(t)]dt = \int_0^1 t^{\lambda_n} a(t)dt$, and $a(t) - b(t) \geqq 0$. Cf. F. Hallenbach, *Zur Theorie der Limitierungsverfahren von Doppelfolgen*, Dissertation, Bonn, 1933, p. 94.

$$(1.2)\qquad d(\lambda_n) = \lambda_n - \lambda_{n-1}; \qquad \Delta(r) = \max_{\lambda_n \leqq r} d(\lambda_n);$$

$$(1.3)\qquad \xi_0 = 0, \qquad \xi_n = \lambda_n - 1, \qquad n \geqq 1;$$

and $z=x+iy=re^{i\theta}$. Throughout the paper, A denotes a constant, depending on the data of the problem in hand, and not necessarily the same at each appearance.

In this section we estimate the expression

$$(1.4)\qquad M(r) = \frac{1}{2}\int_{-\pi/2}^{\pi/2} \log |f(re^{i\theta})| \cos\theta \, d\theta,$$

formed with a function $f(z)$, analytic for $x \geqq 0$, and subject to a limitation of the form

$$(1.5)\qquad |f(x+iy)| \leqq A\mu_n, \qquad \xi_{n-1} < x \leqq \xi_n, \qquad n = 1, 2, \cdots;$$

we suppose that $\mu_n \geqq A > 0$, or (without loss of generality) $\mu_n \geqq 1$. In the applications to moment problems, the μ_n and λ_n will be the μ_n and λ_n of the introduction, and $f(z)$ will be essentially the difference of the functions (3) formed for two solutions of the moment problem under consideration. The relevance of the expression (1.4) is clear from inspection of Carleman's theorem (quoted in §2).

THEOREM 1. *Let $f(z)$ be analytic for $x \geqq 0$, let $f(z)$ satisfy* (1.4), *and let*

$$(1.6)\qquad 0 \leqq \log \mu_n \leqq 2\lambda_n G(\lambda_n),$$

where $G(r)$ is a non-decreasing function. Then for any $\epsilon > 0$ there is an m_ϵ such that for $m > m_\epsilon$

$$(1.7)\qquad \frac{M(\xi_m)}{\xi_m} \leqq \frac{A}{\lambda_m} + G(\lambda_m)\left\{\frac{\pi}{2} + 2^{3/2}(1+\epsilon)d(\lambda_m)^{1/2}\lambda_m^{-1/2} + 2^{1/2}(1+\epsilon)\Delta(\lambda_m)d(\lambda_m)^{-1/2}\lambda_m^{-1/2}\right\}.$$

We have, from (1.5),

$$(1.8)\qquad \log |f(re^{i\theta})| \leqq A + \log \mu_n, \qquad \xi_{n-1} < r\cos\theta \leqq \xi_n.$$

We then have

$$(1.9)\qquad \frac{M(\xi_m)}{\xi_m} = \frac{1}{\xi_m}\int_0^{\phi_{m-1}} (A + \log \mu_m)\cos\theta\, d\theta + \frac{1}{\xi_m}\sum_{k=1}^{m-1}\int_{\phi_k}^{\phi_{k-1}} (A + \log\mu_k)\cos\theta\, d\theta,$$

where $\phi_k = \cos^{-1}(\xi_k/\xi_m)$, $(k=0, 1, \cdots, m-1)$. That is,

$$\frac{M(\xi_m)}{\xi_m} \leqq \frac{A}{\xi_m} + \frac{\log \mu_m}{\xi_m}\left(1 - \frac{\xi_{m-1}^2}{\xi_m^2}\right)^{1/2}$$
$$+ \frac{1}{\xi_m}\sum_{k=1}^{m-1}\left\{\left(1 - \frac{\xi_{k-1}^2}{\xi_m^2}\right)^{1/2} - \left(1 - \frac{\xi_k^2}{\xi_m^2}\right)^{1/2}\right\} \log \mu_k$$
$$= \frac{A}{\xi_m} + P_m + 2\sum_{k=1}^{m-1} R_k \log \mu_k,$$

say.

Using (1.6), we have

$$P_m \leqq 2\lambda_m \xi_m^{-2} G(\lambda_m)(\xi_m^2 - \xi_{m-1}^2)^{1/2} \leqq 2^{3/2}\lambda_m \xi_m^{-3/2} G(\lambda_m)(\xi_m - \xi_{m-1})^{1/2}$$
$$= 2^{3/2}\lambda_m(\lambda_m - 1)^{-3/2} G(\lambda_m) d(\lambda_m)^{1/2} \leqq 2^{3/2}(1+\epsilon)\lambda_m^{-1/2} G(\lambda_m) d(\lambda_m)^{1/2},$$

for m sufficiently large.

Again,

$$R_k \log \mu_k \leqq \frac{1}{2}\,\frac{\xi_k d(\lambda_k)\log \mu_k}{\xi_m^2(\xi_m^2 - \xi_k^2)^{1/2}} \leqq \frac{\lambda_k \xi_k G(\lambda_k) d(\lambda_k)}{\xi_m^2(\xi_m^2 - \xi_k^2)^{1/2}},$$

and since $G(r)$ is non-decreasing,

$$\sum_{k=1}^{m-1} R_k \log \mu_k \leqq \frac{G(\lambda_m)}{\xi_m^2}\sum_{k=1}^{m-1} g(\xi_k)(\xi_k - \xi_{k-1}),$$

where $g(x) = x(x-1)(\xi_m^2 - x^2)^{-1/2}$.

Now we have*

$$\sum_{k=1}^{m-1} g(\xi_k)(\xi_k - \xi_{k-1}) = \xi_{m-1} g(\xi_{m-1}) - \sum_{k=0}^{m-2} \xi_k[g(\xi_{k+1}) - g(\xi_k)]$$
$$= \xi_{m-1} g(\xi_{m-1}) - \sum_{k=0}^{m-2}\int_{\xi_k}^{\xi_{k+1}} \xi_k g'(x)dx$$
$$= \xi_{m-1} g(\xi_{m-1}) - \int_0^{\xi_{m-1}} \langle x\rangle g'(x)dx,$$

where $\langle x\rangle$ denotes the largest ξ_k not exceeding x. Since

$$\int_0^{\xi_{m-1}} x g'(x)dx = \xi_{m-1} g(\xi_{m-1}) - \int_0^{\xi_{m-1}} g(x)dx,$$

we obtain

* Cf. the derivation of Euler's summation formula: K. Knopp, *Theory and Application of Infinite Series*, 1928, p. 522.

$$\sum_{k=1}^{m-1} g(\xi_k)(\xi_k - \xi_{k-1}) = \int_0^{\xi_{m-1}} g(x)dx + \int_0^{\xi_m} (x - \langle x \rangle) g'(x)dx.$$

Now

$$\int_0^{\xi_{m-1}} g(x)dx = \int_0^{\xi_{m-1}} \frac{x(x-1)dx}{(\xi_m^2 - x^2)^{1/2}} \leqq \int_0^{\xi_m} \frac{x^2 dx}{(\xi_m^2 - x^2)^{1/2}} = \frac{\pi \xi_m^2}{4}.$$

And, since $g(x)$ is an increasing function,

$$\left| \int_0^{\xi_{m-1}} (x - \langle x \rangle) g'(x)dx \right| \leqq \frac{\xi_{m-1}\lambda_{m-1}\Delta(\xi_m)}{(\xi_m^2 - \xi_{m-1}^2)^{1/2}} \leqq \frac{(1+\epsilon)\xi_m^{3/2}\Delta(\xi_m)}{2^{1/2} d(\lambda_m)^{1/2}},$$

for m sufficiently large.

Collecting results, we obtain

$$\frac{M(\xi_m)}{\xi_m} \leqq \frac{A}{\xi_m} + \frac{2^{3/2}(1+\epsilon)G(\lambda_m)d(\lambda_m)^{1/2}}{\lambda_m^{1/2}} + G(\lambda_m)\left\{\frac{\pi}{2} + \frac{2^{1/2}(1+\epsilon)\Delta(\xi_m)}{d(\lambda_m)^{1/2}\xi_m^{1/2}}\right\},$$

for sufficiently large m; this is (1.7).

2. We now consider the moment problem

$$\mu_n = \int_0^\infty t^{\lambda_n} d\alpha(t), \tag{2.1}$$

where $\alpha(t)$ is non-decreasing, $\lambda_0 = 0$, $\lambda_1 \geqq 1$, $\lambda_n \uparrow \infty$, and

$$\sum_{n=1}^\infty \frac{1}{\lambda_n} \quad \text{diverges.} \tag{2.2}$$

We may then suppose that $\mu_n \to \infty$, since otherwise $\alpha(t)$ would be constant outside (0, 1), and (2.1) would be determined.* Hence we may (and shall) suppose that $\mu_n \geqq 1$, $(n = 0, 1, 2, \cdots)$.

It is reasonable to suppose that the μ_n satisfy an inequality of the form

$$\mu_n^{1/(2\lambda_n)} \leqq e^{G(\lambda_n)}, \qquad G(r) \uparrow \infty \text{ as } r \uparrow \infty; \tag{2.3}$$

or, more conveniently written,

$$\log \mu_n \leqq 2\lambda_n G(\lambda_n). \tag{2.4}$$

We define the expression $Q(r)$ by

$$Q(r) = \sum_{\lambda_n \leqq r} \left(\frac{1}{\lambda_n} - \frac{\lambda_n}{r^2} \right), \tag{2.5}$$

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* F. Hausdorff, *Summationsmethoden und Momentfolgen*, II, Mathematische Zeitschrift, vol. 9 (1921), pp. 280–299; p. 287.

and define $d(\lambda_n)$, $\Delta(r)$, and ξ_n by relations (1.2), (1.3). We shall prove

THEOREM 2. *If* (2.1) *is undetermined, then for any* $\epsilon > 0$ *and* m *sufficiently large,*

$$Q(\xi_m) \leqq A + G(\lambda_m)\left\{1 + \frac{C_1 d(\lambda_m)^{1/2}}{\lambda_m^{1/2}} + \frac{C_2\Delta(\lambda_m)}{\lambda_m^{1/2} d(\lambda_m)^{1/2}}\right\}, \tag{2.6}$$

*where** $C_1 = 2^{5/2}\pi^{-1}(1+\epsilon)$, $C_2 = 2^{3/2}\pi^{-1}(1+\epsilon)$.

We may state less forbidding special cases of (2.6) if we suppose that the growth of the sequence $\{\lambda_n\}$ is very regular. Thus we have

COROLLARY 2.1. *If* $d(\lambda_n)$ *increases and* $d(\lambda_n) = o(\lambda_n)$, *then, if* (2.1) *is undetermined,*

$$Q(\xi_m) \leqq G(\lambda_m)(1 + o(1)). \tag{2.7}$$

COROLLARY 2.2. *If* $d(\lambda_n)$ *decreases and* $d(\lambda_n) = o(1/\lambda_n)$, *then, if* (2.1) *is undetermined,* (2.7) *holds.*

From Theorem 2 it follows that any condition which makes $G(\lambda_n)$ so small that (2.6) is impossible is a sufficient condition for (2.1) to be determined; in §3 we shall give examples of such conditions for special sequences $\{\lambda_n\}$.

We derive Theorem 2 from the estimate of Theorem 1 applied to

CARLEMAN'S THEOREM.† *Let* $f(z)$ *be analytic for* $x \geqq 0$, *and let* $r_n e^{i\theta_n}$, $(r_1 \leqq r_2 \leqq \cdots)$, *denote the zeros of* $f(z)$ *for* $x \geqq 0$, *each counted according to its multiplicity. Then if* $R > \rho > 0$,

$$\sum_{\rho < r_n \leqq R}\left[\frac{1}{r_n} - \frac{r_n}{R^2}\right]\cos\theta_n = \frac{2M(R)}{\pi R} + A(R) + O(1), \tag{2.8}$$

where

$$A(R) = \frac{1}{2\pi}\int_\rho^R \left\{\frac{1}{r^2} - \frac{1}{R^2}\right\}\log\{\,|f(iy)f(-iy)|\,\}dy,$$

$$M(r) = \frac{1}{2}\int_{-\pi/2}^{\pi/2}\log|f(re^{i\theta})|\cos\theta\, d\theta, \tag{2.9}$$

and the term $O(1)$ *depends on* ρ *and is bounded as* $R \to \infty$ *for fixed* ρ.

Under the hypotheses of Theorem 2, there are two solutions of (2.1); let $\gamma(t)$ be their difference. Consider the function

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* The precise values of C_1 and C_2 do not seem very important.

† See, for example, E. C. Titchmarsh, *The Theory of Functions*, 1932, p. 130.

$$f(z) = \frac{1}{2}\int_0^\infty t^{z+1} d\gamma(t). \tag{2.10}$$

Then $f(z)$ is analytic for $x \geqq 0$, and has zeros at least at the points $\xi_n = \lambda_n - 1$, $(n = 1, 2, \cdots)$.

Since $\mu_n \to \infty$, we have for $\xi_{n-1} < x \leqq \xi_n$, $(n = 0, 1, 2, \cdots)$,

$$\begin{aligned} |f(x+iy)| &\leqq \frac{1}{2}\int_0^1 t^{\lambda_{n-1}} |d\gamma(t)| + \frac{1}{2}\int_1^\infty t^{\lambda_n} |d\gamma(t)| \\ &\leqq \mu_n + A \leqq A\mu_n. \end{aligned} \tag{2.11}$$

Now

$$f(iy) = \frac{1}{2}\int_0^\infty t^{1+iy} d\gamma(t), \qquad |f(iy)| \leqq \frac{1}{2}\int_0^\infty t\,|d\gamma(t)| \leqq \mu_1;$$

consequently $A(R) = O(1)$, $R \to \infty$.

If we apply Carleman's theorem to $f(z)$, taking $R = \xi_m$ and ρ sufficiently large, use the estimate of Theorem 1 for $M(R)$, and neglect possible zeros of $f(z)$ other than those at the ξ_n (which would only increase the left-hand side of (2.8)), we obtain Theorem 2.

3. We now illustrate Theorem 2 by applying it to a number of specific sequences $\{\lambda_n\}$.

EXAMPLE 1. Let $\lambda_n = n$.

Here $Q(r) = \log r + O(1)$, $r \to \infty$; $d(\lambda_n) = \Delta(r) \equiv 1$; and (2.6) becomes

$$\log(m-1) \leqq A + G(m)(1 + O(m^{-1/2})),$$

which is impossible if $G(r) = \log r + \log \sigma(r)$, where $\liminf_{r\to\infty} \sigma(r) = 0$. Consequently, the moment problem (2.1) is determined if $\lambda_n = n$ and

$$\lim_{n\to\infty} n^{-1} \mu_n^{1/(2n)} = 0. \tag{3.1}$$

EXAMPLE 2. Let λ_n run through the positive integers with the exception of a set $\{n_k\}$ for which $\sum_{k=1}^\infty 1/n_k < \infty$, and such that $\lim_{r\to\infty} \Delta(r) r^{-1/2} < \infty$.

Then $Q(r) = \log r + O(1)$, $d(\lambda_n) \geqq 1$, and from (2.6) we see that (2.1) is determined if

$$\lim_{n\to\infty} \lambda_n^{-1} \mu_n^{1/(2\lambda_n)} = 0. \tag{3.2}$$

Moreover, as we stated in the introduction, (2.1) is determined even if (3.1) is satisfied. In fact, we may write (3.2) in the form

$$\lim_{n\to\infty} \left(\frac{1}{2n} \cdot \frac{n}{\lambda_n} \log \mu_n - \log n - \log \frac{\lambda_n}{n} \right) = -\infty. \tag{3.3}$$

If (3.1), or $\lim_{n\to\infty} [(2n)^{-1}\log \mu_n - \log n] = -\infty$, is satisfied, (3.3) is certainly satisfied if $\lambda_n/n = O(1)$ as $n\to\infty$. The difference $\lambda_n - n$ is $N(\lambda_n)$, the number of $n_k \leqq \lambda_n$; consequently $0<\delta\leqq n/\lambda_n \leqq 1$ unless $N(\lambda_n)\sim\lambda_n$, $n\to\infty$. But if $N(\lambda_n)\geqq c\lambda_n$, we have

$$\sum_{n_k\leqq\lambda_n} \frac{1}{n_k} \geqq \sum_{(1-c)\lambda_n}^{\lambda_n} \frac{1}{k} \sim \log\frac{1}{1-c}, \qquad n\to\infty,$$

so that, since $\sum_{k=1}^{\infty} 1/n_k < \infty$, we must have $N(\lambda_n)\leqq c\lambda_n$, $c<1$, for all sufficiently large n, and hence $\lambda_n/n = O(1)$.

EXAMPLE 3. Let

$$|\lambda_n - n| < A, \qquad n = 1, 2, \cdots. \tag{3.4}$$

Then $Q(r) = \log r + O(1)$, $\limsup_{n\to\infty} d(\lambda_n) > 0$, and $\Delta(r) \leqq 2A$. Consequently, (2.1) is determined if (3.1) is satisfied. Condition (3.4) can, of course, be considerably weakened.

EXAMPLE 4. Let $\lambda_n = n^a$, $(0<a<1)$.

Then

$$Q(r) = \frac{1}{1-a} r^{(1-a)/a} + O(1), \qquad \Delta(\lambda_n) \leqq a\rho^{(a-1)/a},$$

$$a\lambda_n^{(a-1)/a}(1 - o(1)) \leqq d(\lambda_n) \leqq a\lambda_n^{(1-a)/a}.$$

Consequently, for an undetermined moment problem we must have, with $r = \lambda_n = n^a$,

$$\frac{1}{1-a}(r-1)^{(1-a)/a} \leqq A + G(r)\{1 + C_1 a^{1/2} r^{-(a^2-a+1)/(2a)} + C_2 a^{1/2}\rho^{(a-1)/a} r^{-(a^2+2a-2)/(2a)}\}.* \tag{3.5}$$

It is clear that we must expect somewhat different results for different values of a.

(i) Let $1>a>2(2^{1/2}-1)$, so that $a^2+4a-4>0$. If we suppose that

$$G(r) \leqq (1-a)^{-1} r^{(1-a)/a} + \log\sigma(r), \qquad \sigma(r) = o(1), \tag{3.6}$$

the right-hand side of (3.5) does not exceed

$$A + (1-a)^{-1} r^{(1-a)/a} + \log\sigma(r) + C_1(1-a)^{-1} a^{1/2} r^{-(a^2+a-1)/(2a)} + C_2 a^{1/2}(1+a)^{-1}\rho^{(a-1)/a} r^{-(a^2+4a-4)/(2a)}.$$

Since $a^2+a-1 > a^2+4a-4 > 0$, if the moment problem is undetermined and $G(r)$ satisfies (3.6), we must have

* We have absorbed the factor $1/(1-o(1))$ into C_2, which already contained a factor $(1+\epsilon)$, $(\epsilon>0)$.

$$(3.7)\qquad (1-a)^{-1}\left[(r-1)^{(1-a)/a} - r^{(1-a)/a}\right] \leqq \log \sigma(r) + O(1).$$

This, however, is impossible, since the left-hand side of formula (3.7) is $O(r^{(1-2a)/a}) = O(1)$ if $a \geqq \frac{1}{2}$. Hence (2.1) is determined if

$$\mu_n^{1/(2n^a)} = o\left\{\exp\left(\frac{n^{1-a}}{1-a}\right)\right\}.$$

(ii) Let $2(2^{1/2}-1) > a > 3^{1/2}-1$, so that $a^2+2a-2>0$. If we suppose that

$$G(r) \leqq \frac{1-\eta}{1-a} r^{(1-a)/a} + \log \sigma(r), \qquad \sigma(r) = o(1),$$

for some $\eta > 0$, the right-hand side of (3.5) does not exceed

$$A + \frac{1-\eta}{1-a} r^{(1-a)/a} + \log \sigma(r) + C_1 \frac{1-\eta}{1-a} a^{1/2} r^{-(a^2+a-1)/(2a)} + C_2 a^{1/2} \frac{1-\eta}{1-a} \rho^{(a-1)/a} r^{-(a^2+4a-4)/(2a)}.$$

Since $(4-4a-a^2)/(2a) < (1-a)/a$ if $a^2+2a-2>0$, and since $a^2+a-1 > a^2+2a-2$, this expression does not exceed

$$\left(\frac{1-\eta}{1-a} + o(1)\right) r^{(1-a)/a} + \log \sigma(r) + O(1);$$

consequently, (2.1) is determined if for some $\eta > 0$

$$\mu_n^{1/(2n^a)} = o\left\{\exp\left(\frac{1-\eta}{1-a} n^{1-a}\right)\right\}.$$

(iii) Let $3^{1/2}-1 > a > 0$. If we suppose that

$$(3.8)\qquad G(r) \leqq B r^{a/2},$$

for some $B > 0$, the right-hand side of (3.5) does not exceed

$$A + Br^{a/2} + C_1 a^{1/2} r^{(a-1)/(2a)} + BC_2 a^{1/2} \rho^{(a-1)/a} r^{(1-a)/a} = r^{(1-a)/a}(BC_2 a^{-1/2} \rho^{(a-1)/a} + o(1)) + O(1).$$

If ρ is so large that $BC_2 a^{1/2} \rho^{(a-1)/a} < 1/(1-a)$, (3.5) is clearly impossible for large r; consequently, (2.1) is determined if (3.8) is satisfied; that is, if

$$\mu_n^{1/(2n^a)} \leqq \exp\left\{O(n^{a^2/2})\right\}.$$

Conditions for the cases $a = 2(2^{1/2}-1)$ or $a = 3^{1/2}-1$ are easily written.

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