THE BOUNDARY PROBLEM OF AN ORDINARY LINEAR DIFFERENTIAL SYSTEM IN THE COMPLEX DOMAIN*

BY

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1. **Introduction.** The subject of this discussion is to be the system of ordinary linear differential equations which is of the form, or is reducible to the form,

$$(1.1)\qquad y_i'(x, \lambda) = \left\{ \lambda r_i(x) + \sum_{\nu=1}^{n} q_{i,\nu}(x, \lambda) \right\} y_\nu(x, \lambda), \qquad i = 1, 2, \cdots, n,$$

the variable x and the parameter λ being complex, $|\lambda|$ being indefinitely large, and the coefficients $q_{i,j}(x, \lambda)$ being bounded.† Specifically the matters to be considered are:

In Part I, the dependence of solutions of the system upon λ, when the modulus of the latter is large, and the domain of x is a suitable finite portion of the complex x plane;

In Part II, the boundary problem which arises when a set of conditions applying at any suitable finite set of points of the x domain is imposed upon the system;

In Part III, the theory of the expansibility of a set of n arbitrary analytic functions of x in series of characteristic solutions of the boundary problem.

These matters have, of course, all been widely investigated before this, and discussions of them are to a large extent classical in the literature. However, these discussions—and the present one is, to be sure, no exception in this respect—are invariably restricted in their scope in one way or another by being of necessity based upon hypotheses which to a greater or less extent delimit the considerations and the applicability of the results. Such restrictive hypotheses may, of course, be essential, in the sense that they serve to delimit the considerations to intrinsically identifiable cases of a problem of

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† The reduction of differential systems

$$u_i'(x, \lambda) = \sum_{\nu=1}^{n} \{ \lambda a_{i,\nu}(x) + b_{i,\nu}(x, \lambda) \} u_\nu(x, \lambda), \qquad i = 1, 2, \cdots, n,$$

to form (1) is considered in G. D. Birkhoff and R. E. Langer, *The boundary problems and developments associated with a system of ordinary linear differential equations of the first order*, Proceedings of the American Academy of Arts and Sciences, vol. 58 (1923), pp. 72–74.

excessive generality. On the other hand, they may be unessential in the sense that they are primarily called forth by shortcomings of the methods used, or by inadequate or otherwise faulty formulations of the problems themselves. It is believed that the present paper contributes something to a removal or relaxation of several hypotheses of the latter category upon which related earlier discussions are dependent.

The features in which the present paper differs most markedly from previous ones include the following.

(a) With only few and fragmentary exceptions the problems dealt with have heretofore been considered only in the cases of a real variable. The discussion here is allocated to the complex plane, and so includes the earlier results as special cases.

(b) The complete dependence of the functional forms of the solutions of a system of the type (1.1) upon the parameter λ, when $|\lambda|$ is large, has been derived heretofore, even for a real variable, only under the heavily restrictive hypothesis that the coefficient functions $r_i(x)$, as complex quantities, are such that their differences $\{r_i(x)-r_j(x)\}$ all maintain constant arguments over the x range considered. The present discussion is not so restricted, and hence materially extends the existing theory, this being so even when the variable is specialized to be real.

(c) The boundary problems which have been studied in connection with the system (1.1) have almost exclusively been such as arise when the boundary conditions apply only at collinear points, that is, generally points of the axis of reals. In the present paper these conditions are permitted to apply at any finite set of points within appropriate regions of the complex plane. This generalization calls for corresponding generalizations of many familiar notions, among them those of the adjoint boundary problem, of the Green's function, of regularity of the boundary problem, and so on; and such generalizations are given.

(d) Heretofore the theory of the expansibility of an arbitrary vector, that is, of a set of n arbitrary functions, in terms of the characteristic solutions of a regular boundary problem, has been given only for the very restricted cases in which the coefficient functions $r_i(x)$, as complex quantities, each maintain a constant argument for all values of x involved. In the present paper this restriction is dispensed with.

The system (1.1) is notationally treated, in the following, in its matrix form. Insofar as deductions of a formal nature are concerned, those here given include as special cases almost all those which have become classical for the cases of a real variable, whether the boundary conditions are taken to apply at just two, or at more than two, points. In a number of instances the present formulations are thought to embody material improvements, even when they are specialized to the ranges of the earlier discussions. This seems to be so particularly in the cases of boundary conditions applying at intermediate points of the x interval as well as at the end points. In the rigorous analysis no attempt has been made to pare the hypotheses down to a minimum, or to sharpen the deductions to any point at which they would include any major portion of the many refined and precise results which are known in the case of a real variable. To do that would have extended the bounds of the paper excessively.

PART I. THE FORMS OF THE SOLUTIONS WHEN $|\lambda|$ IS LARGE

2. **The matrix equation.** If $\mathfrak{Y}(x, y)$ is a matrix* which satisfies the differential matrix equation

$$\mathfrak{Y}'(x, \lambda) = \{\lambda\mathfrak{R}(x) + \mathfrak{Q}(x, \lambda)\}\mathfrak{Y}(x, \lambda), \tag{2.1}$$

in which the prime indicates differentiation with respect to x, and in which the coefficient matrices are

$$\mathfrak{R}(x) \equiv (\delta_{i,j}r_i(x)),\dagger \qquad \mathfrak{Q}(x, \lambda) \equiv (q_{i,j}(x, \lambda)), \tag{2.2}$$

then the elements of any column of $\mathfrak{Y}(x, \lambda)$ comprise a solution of the differential system (1.1). If $\mathfrak{Y}(x, \lambda)$ is nonsingular, its columns are linearly independent and so yield a complete set of solutions of (1.1). The matrix equation (2.1) may, therefore, be chosen to replace completely the scalar system (1.1) as the basis of the discussion. This will henceforth be done, because of the notational advantages which are thereby to be gained.

The equation (2.1) is to be considered with the parameter λ complex and ranging over some suitable region of the λ plane in which $|\lambda|$ is unbounded. The variable x is likewise to be complex, and is to range either over some suitable bounded region, or over some suitable finite curvilinear arc. The term *suitable* as here used needs, of course, to be made precise. In the case of an x region, that is, of a two-dimensional x domain, this is to be done by the definition:

A pair of regions in the x and λ planes will be said to be suitable to the matrix equation (2.1), *if for x and λ within them the coefficients of the equation fulfill the specifications*:

(a) *the functions $r_i(x)$, $(i=1, 2, \cdots, n)$, are analytic and bounded, and their differences $\{r_i(x)-r_j(x)\}$, $(i\neq j)$, are all bounded from zero*;

* Throughout the paper square matrices of order n will be designated by means of German capital letters, and these letters will be used solely in this sense. The elements of a matrix will then generally be designated by the corresponding lower case italic letters, that is, in the manner $\mathfrak{A}(x, \lambda) \equiv (a_{i,j}(x, \lambda))$.

† The symbol $\delta_{i,j}$ will always be used in the sense $\delta_{i,j}=0$, if $i\neq j$; $\delta_{i,j}=1$, if $i=j$.

(b) *the functions* $q_{i,j}(x,\lambda)$, $(i,j=1,2,\cdots,n)$, *are analytic and bounded in* x *and in* λ, *and, when* $|\lambda|$ *is sufficiently large, admit of either actual or asymptotic representations, such that*

$$\mathfrak{Q}(x,\lambda) \sim \sum_{h=0}^{\infty} \lambda^{-h}\mathfrak{Q}^{(h)}(x), \tag{2.3}$$

the elements of the matrices $\mathfrak{Q}^{(h)}(x)$ *being analytic and bounded.*

If the domain of x is one-dimensional, that is, an arc, as it is in the case of a real variable, the definition of the term *suitable* is to be that obtained from the definition above when the term *analytic*, as used relative to x, is replaced by *indefinitely differentiable along the arc.*

To assure the existence of a basis for the entire discussion at hand, this will be assumed as

Hypothesis (i). *The given differential matrix equation* (2.1) *is one for which there exist some suitable regions of the* x *and* λ *planes.*

If in the equation (2.1) the substitution

$$\mathfrak{Y}(x,\lambda) = (\delta_{i,j}e^{\lambda\omega x+\phi_j(x)})\mathfrak{U}(x,\lambda), \tag{2.4}$$

is made, with ω any constant and the $\phi_j(x)$, $(j=1, 2, \cdots, n)$, any analytic functions, the equation satisfied by the matrix $\mathfrak{U}(x,\lambda)$ is found to be of the same form as (2.1), and to differ from the latter by having the functions $\{r_j(x)-\omega\}$ in the place of the $r_j(x)$, and the functions $\{q_{i,j}(x,\lambda)-\phi_j'(x)\}\exp\{\phi_j(x)-\phi_i(x)\}$ in the place of the $q_{i,j}(x,\lambda)$. From this it may be observed firstly, that since ω can always be chosen so that the determinant of the matrix $(\delta_{i,j}[r_i(x)-\omega])$ is not zero, any given matrix equation (2.1) is transformable into another such equation in which the matrix filling the role of $\mathfrak{R}(x)$ is nonsingular. Secondly, it may be noted that since the functions $\phi_j(x)$ may be chosen so that $\phi_j'(x)\equiv \overset{(0)}{q}_{j,j}(x)$, any given equation (2.1) is always transformable into one in which the elements of the main diagonal of the coefficient $\mathfrak{Q}^{(0)}(x)$ of (2.3) all vanish identically. Of these facts, that concerning $\mathfrak{R}(x)$ yields no immediate advantage, though it will later be referred to. That concerning $\mathfrak{Q}(x,\lambda)$ does yield an advantage, and hence it will be assumed forthwith that (2.1) represents such a transformation of the given matrix equation that in it

$$\overset{(0)}{q}_{j,j}(x) \equiv 0, \qquad j = 1, 2, \cdots, n. \tag{2.5}$$

From classical and familiar existence theorems it is known that an equation of the form (2.1) possesses solutions which are nonsingular matrices whose elements are analytic functions of x and λ. Moreover, if any particular such solution is designated by $\mathfrak{Y}^{(p)}(x, \lambda)$, then the general solution of the equation is obtained from the formula

$$(2.6) \qquad \mathfrak{Y}(x, \lambda) = \mathfrak{Y}^{(p)}(x, \lambda)\mathfrak{C},$$

by permitting $\mathfrak{C}$ to represent any matrix whose elements are independent of x. The matrix $\mathfrak{C}$ may, of course, depend upon λ.

3. **"Associated" regions and "fundamental" regions.** In any given suitable region of the x plane, a set of analytic functions $R_i(x)$ may be chosen such that their derivatives are

$$(3.1) \qquad R_i'(x) \equiv r_i(x), \qquad i = 1, 2, \cdots, n,$$

and these functions will be bounded. We suppose such a choice to have been made. Then if λ is regarded for the moment as fixed, in some suitable λ region, each one of the relations

$$(3.2) \qquad \xi^{(i,j)} = \lambda\{R_i(x) - R_j(x)\}, \qquad i, j = 1, 2, \cdots, n; \; i \neq i,$$

defines its left-hand member as a complex variable, and maps any closed subregion X of the given suitable x region upon a corresponding closed region $\Xi^{(i,j)}$ in the respective $\xi^{(i,j)}$ plane. Consider now the possibility of such a subregion X containing a set of points $x_*^{(i,j)}$, not necessarily distinct, which have the properties that the point $\xi_*^{(i,j)}$ which lies in the $\xi^{(i,j)}$ plane and corresponds to $x_*^{(i,j)}$ under (3.2), admits of connection with each and every point of the respective region $\Xi^{(i,j)}$ by some curve of bounded length, which lies entirely in $\Xi^{(i,j)}$ and upon which the abscissa is a non-increasing function of the arc length as measured from $\xi_*^{(i,j)}$. This possibility is readily seen to be contingent directly upon the shapes of the regions $\Xi^{(i,j)}$, and hence upon the shape of the region X. When such points $x_*^{(i,j)}$ do exist, they evidently lie upon the boundary of the region X, and the points $\xi_*^{(i,j)}$ are clearly boundary points of maximum abscissa of the respective regions $\Xi^{(i,j)}$.

If λ is now allowed to vary, it is clear at once from (3.2), that all changes in $|\lambda|$ produce in the several $\xi^{(i,j)}$ planes merely changes of scale. Such changes cannot, therefore, influence either the existence or the location of any point $x_*^{(i,j)}$. On the other hand, any change in arg λ produces a rotation of each $\xi^{(i,j)}$ plane, and hence, in particular, of each of the regions $\Xi^{(i,j)}$. Such a rotation may deprive the points of an existing set $x_*^{(i,j)}$ of their characteristic properties. However, it does not necessarily do so, it being possible for a set $x_*^{(i,j)}$ to remain independent of λ and retain its properties under a variation of arg λ over some specific range. This will be shown below. The possibility is again contingent upon the shape of the region X, but also upon the range of arg λ which is in question. We make the definition

A closed subregion X of a suitable x region and a subregion Λ of a suitable λ region will be termed "associated" regions if there exists in X a set of points $x_*^{(i,j)}$, $(i, j=1, 2, \cdots, n; i\neq j)$, not necessarily distinct, but fixed as to λ, having the properties described above, and retaining them for all λ in the region Λ.

A given suitable λ region may not admit of any region of the x plane being associated with it. It may, however, still admit of being completely covered by subregions each of which admits of association with some x region. The x regions here in question may, moreover, in some cases have a part in common. We make the definition

A closed region of the x plane will be designated as a fundamental region relative to a given suitable λ region, if it is included in each of a finite number of regions X, which are associated with regions Λ completely covering the suitable λ region in question.

Finally it may be observed that if a given suitable λ region is bounded by lines along which $\arg \lambda$ is constant, that is, if it is a sector (or the part of a sector in which $|\lambda| > N$†), then since only $\arg \lambda$ comes into question, any subregion Λ which is associated with an x region may also be taken as a sector (or the part of it in which $|\lambda| > N$).

4. **The existence of associated and fundamental regions.** Inasmuch as regions to be termed associated or fundamental have been defined only in terms of properties prescribed for them, the question of their existence must be considered. In this connection the following will be shown

If x_0 and λ_0 are arbitrarily chosen interior points of a suitable two-dimensional x region and a suitable λ region, respectively, then there exist associated regions of which they are likewise interior points, and there exist x regions which are fundamental relative to the suitable λ region and having x_0 in their interiors.

In the case $n=2$ these facts are evident almost by inspection. For in this case the only variables defined by (3.2) are $\xi^{(1,2)}$ and $\xi^{(2,1)}$, and these are negatives of each other. Let X, therefore, be taken as any such part of the given suitable x region as contains x_0 in its interior, and as maps in the $\xi^{(1,2)}$ plane, under (3.2), with $\lambda=\lambda_0$, upon a convex polygon with no side parallel to the axis of imaginaries. This polygon is then the region $\Xi^{(1,2)}$, and clearly the region $\Xi^{(2,1)}$ is also such a polygon. The extreme right-hand vertices of these polygons evidently fill, respectively, the specifications upon the points

† The symbol $|\lambda| > N$, which there will be frequent occasion to use, is to be read as a mere abbreviation of the phrase "when $|\lambda|$ is sufficiently large." The letter N is, therefore, not to be regarded as designating always one and the same number, but as designating in each case *some* number, possibly different ones in different recurrences of the symbol. The precise magnitude of N is generally left undiscussed as not germane to the argument.

$\xi_*^{(1,2)}$ and $\xi_*^{(2,1)}$. If λ is now allowed to vary, the resulting rotations under which each polygon maintains some one vertex in the extreme right-hand position determine ranges of $\arg\lambda$, and hence subregions of the given λ region, which are associated with the region X. Of these, one contains λ_0 in its interior. Since any given suitable λ region may clearly be covered by a finite number of such subregions, the region X which was chosen is seen to be a fundamental one relative to any suitable portion of the λ plane.

If $n>2$ the reasoning may be fashioned as follows. With any choice of a real number τ_0, the interval $(0, \pi)$ is divided into at most $n(n-1)$ subintervals by those of its points which are congruent, modulo π, to the points of the set

$$\tau_0^{(i,j)} = \tau_0 + \arg\lambda_0 + \arg\{r_i(x_0) - r_j(x_0)\}, \qquad i, j = 1, 2, \cdots, n;\ i \neq j. \tag{4.1}$$

Of these subintervals at least one is of a length 2δ, with $\delta \geqq \pi/2n(n-1)$, and with a proper choice of τ_0 this subinterval is bisected by the point $\pi/2$. We suppose τ_0 so chosen. Then each of the points (4.1) is congruent, modulo π, to some point of the closed interval $(-\pi/2+\delta, \pi/2-\delta)$. Let ϵ_1, ϵ_2, and ϵ_3 be chosen as positive constants subject to the restriction

$$\epsilon_1 + \epsilon_2 + \epsilon_3 < \delta, \tag{4.2}$$

but otherwise arbitrary.

Consider now any curve C in the given x region, which

(a) lies in a neighborhood of the point x_0 in which the relations

$$\left|\arg\{r_i(x) - r_j(x)\} - \arg\{r_i(x_0) - r_j(x_0)\}\right| < \epsilon_1, \qquad i, j = 1, 2, \cdots, n;\ i \neq j, \tag{4.3}$$

are all fulfilled, and

(b) has a continuously turning tangent whose inclination τ satisfies the condition

$$|\tau - \tau_0| < \epsilon_2. \tag{4.4}$$

Finally let $\arg\lambda$ be restricted by the relation

$$|\arg\lambda - \arg\lambda_0| \leqq \epsilon_3. \tag{4.5}$$

For any set of indices (i, j), the arc C corresponds under (3.2) to an arc $\Gamma^{(i,j)}$ in the plane of the variable $\xi^{(i,j)}$. If the inclination of the tangent line to $\Gamma^{(i,j)}$ is denoted by $\tau^{(i,j)}$, it follows from (3.2) that $\tau^{(i,j)} = \tau + \arg\lambda + \arg\{r_i(x) - r_j(x)\}$, and hence, from (4.4), (4.5), (4.3), (4.2), that $|\tau^{(i,j)} - \tau_0^{(i,j)}| < \delta$. Thus $\tau^{(i,j)}$ is bounded from becoming congruent, modulo π, with either of the values $-\pi/2$ or $\pi/2$; that is, the slope of $\Gamma^{(i,j)}$ is bounded.

Let X be chosen now as any subregion of the given suitable x region which contains x_0 in its interior, and which is bounded by a pair of arcs of the type C described. The corresponding region $\Xi^{(i,j)}$, for each (i, j), is then bounded by a pair of arcs of the type $\Gamma^{(i,j)}$. These arcs intersect, and one of their intersections, the extreme right-hand point of $\Xi^{(i,j)}$, fills the specifications on the point $\xi_*^{(i,j)}$. It does this, moreover, for all values of λ which are admitted by (4.5). Thus (4.5) determines a λ subregion which is associated with the region X chosen.

Since the constant ϵ_3 is not dependent upon λ_0, it is clear that in any suitable λ region a finite number of points may be chosen so that they fill the role of λ_0 above, and such that the entire λ region is covered by the corresponding subregions (4.5). Each of these subregions, it has been shown, is associated with some region X which contains x_0 in its interior. The part common to these regions X is seen at once to be a fundamental region relative to the given suitable λ region.

The discussion thus given was based explicitly upon the assumption that the suitable x region containing x_0 was a two-dimensional one. If the x region is one-dimensional, that is, an arc, the discussion is not generally applicable, and no association of x and λ regions may be possible. Exceptional in this respect is the case in which some segment of the x domain maps under each of the transformations (3.2) upon a straight segment, that is, if on such a segment

$$\arg\{R_i(x) - R_j(x)\} \equiv \text{constant}, \qquad i, j = 1, 2, \cdots, n;\ i \neq j. \tag{4.6}$$

In this case the argument given above serves without modification to show that the x segment in question is a fundamental region relative to any suitable λ region.

The conditions (4.6) will be recognized as an important part of the hypotheses upon which the discussions analogous to that of this paper, but applying to the real variable x, have classically been based. The motivation for this is thus seen to lie in the need of having the basic domain of the variable be a fundamental region.

5. **The solution of an approximating equation.** If x and λ are taken in any suitable regions, and the matrices $\mathfrak{Q}^{(h)}(x)$ are those of (2.3), the formulas

$$\begin{aligned} p_{i,j}^{(0)} &\equiv \delta_{i,j}, \qquad\qquad i, j = 1, 2, \cdots, n; \\ p_{i,j}^{(l)}(x) &\equiv \{r_i(x) - r_j(x)\}^{-1}\left\{p_{i,j}^{(l-1)\prime}(x) - \sum_{h=0}^{l-1}\sum_{\nu=1}^{n} q_{i,\nu}^{(l-h-1)}(x)p_{\nu,j}^{(h)}(x)\right\}, \quad i \neq j; \\ p_{i,i}^{(l)\prime}(x) &\equiv \sum_{h=0}^{l}\sum_{\nu=1}^{n} q_{i,\nu}^{(l-h)}(x)p_{\nu,i}^{(h)}(x), \end{aligned} \tag{5.1}$$

together with any choice of constants of integration, define in succession for $l=0, 1, 2, 3$, and so on, a sequence of matrices $\mathfrak{P}^{(l)}(x)$. These matrices are analytic and bounded, and satisfy the matrix equations

$$\mathfrak{P}^{(l+1)}(x)\mathfrak{R}(x) - \mathfrak{R}(x)\mathfrak{P}^{(l+1)}(x) + \mathfrak{P}^{(l)\prime}(x) - \sum_{h=0}^{l} \mathfrak{Q}^{(l-h)}(x)\mathfrak{P}^{(h)}(x) = \mathfrak{O}, \quad l = 0, 1, 2, \cdots. \tag{5.2}$$

Let any natural number k be chosen, then, and let the formulas

$$\mathfrak{P}_k(x, \lambda) \equiv \sum_{l=0}^{k} \lambda^{-l}\mathfrak{P}^{(l)}(x), \tag{5.3}$$

$$\mathfrak{E}(x, \lambda) \equiv (\delta_{i,j}e^{\lambda R_i(x)}), \tag{5.4}$$

define their left-hand members. The functions $R_i(x)$ are those of (3.2). Then in virtue of the relations (2.3) and (5.2), it is found directly that the matrix

$$\mathfrak{S}(x, \lambda, k) \equiv \mathfrak{P}_k(x, \lambda)\mathfrak{E}(x, \lambda), \tag{5.5}$$

is such that

$$\mathfrak{S}' - \{\lambda\mathfrak{R} + \mathfrak{Q}\}\mathfrak{S} \sim \left\{\lambda^{-k}\mathfrak{P}^{(k)\prime} - \sum_{h=k}^{\infty} \lambda^{-h}\sum_{l=0}^{k} \mathfrak{Q}^{(h-l)}\mathfrak{P}^{(l)}\right\}\mathfrak{E}.$$

When $|\lambda| > N$, this can be written, since the matrix (5.3) is then certainly nonsingular, in the form

$$\mathfrak{S}' = \{\lambda\mathfrak{R}(x) + \mathfrak{Q}(x, \lambda) + \lambda^{-k}\mathfrak{A}(x, \lambda, k)\}\mathfrak{S}, \tag{5.6}$$

in which the coefficient matrix $\mathfrak{A}(x, \lambda, k)$ is one which admits of a representation

$$\mathfrak{A}(x, \lambda, k) \sim \left\{\mathfrak{P}^{(k)\prime}(x) - \sum_{h=k}^{\infty} \lambda^{-h+k}\sum_{l=0}^{k} \mathfrak{Q}^{(h-l)}(x)\mathfrak{P}^{(l)}(x)\right\}\mathfrak{P}_k^{-1}(x, \lambda). \tag{5.7}$$

The equation (5.6) is a matrix differential equation which in an obvious sense approximates the given equation (2.1) when $|\lambda|$ is large. The matrix (5.5) is thus seen to be a nonsingular analytic solution of an equation which approximates the given equation when $|\lambda| > N$.

Let $\mathfrak{Y}^{(p)}(x, \lambda)$ designate any particular nonsingular analytic solution of the equation (2.1). Then in virtue of the equation (5.6) the relation

$$\{\mathfrak{S}^{-1}\mathfrak{Y}^{(p)}\}' = -\lambda^{-k}\mathfrak{S}^{-1}\mathfrak{A}\mathfrak{Y}^{(p)} \tag{5.8}$$

is readily found to be an identity. In terms of the matrices defined by the formulas

$$\mathfrak{J}_{h,l} = (\delta_{i,h}\delta_{l,j}), \qquad h, l = 1, 2, \cdots, n, \tag{5.9}$$

in each of which one element is unity and all others are zero, the relation (5.8) may be written in the alternative form

$$\{\mathfrak{S}^{-1}\mathfrak{Y}^{(p)}\}' + \lambda^{-k}\sum_{h,l=1}^{n} \mathfrak{J}_{h,h}\mathfrak{S}^{-1}\mathfrak{A}\mathfrak{Y}^{(p)}\mathfrak{J}_{l,l} = \mathfrak{O}. \tag{5.10}$$

This form is convenient for the use to which the relation is to be put below.

6. **The solutions of the given equation.** For the formal deductions of the preceding section it sufficed to regard x and λ as in any suitable regions. Let them be restricted now to any pair of associated regions X and Λ. There exists then in X a set of points $x_*^{(h,l)}$ as described in §3, and these points do not depend upon λ. By an appropriate integration based upon these points, the relation (5.10) may be given the form

$$\mathfrak{S}^{-1}(x)\mathfrak{Y}^{(p)}(x) + \lambda^{-k}\sum_{h,l=1}^{n} \int_{x_*^{(h,l)}}^{x} \mathfrak{J}_{h,h}\mathfrak{S}^{-1}(x_1)\mathfrak{A}(x_1)\mathfrak{Y}^{(p)}(x_1)\mathfrak{J}_{l,l}dx_1 = \mathfrak{K}(\lambda), \tag{6.1}$$

and this may be looked upon as defining its right-hand member as ananalytic matrix independent of x.

Let $\mathfrak{C}(\lambda)$ be a matrix which is unspecified, except that $\mathfrak{C}(\lambda) \neq \mathfrak{O}$, and let

$$\mathfrak{V}(x, \lambda) \equiv \mathfrak{Y}^{(p)}(x)\mathfrak{C}(\lambda)\mathfrak{S}^{-1}(x). \tag{6.2}$$

If it is observed that by (5.9)

$$\sum_{l=1}^{n} \mathfrak{J}_{l,l}\mathfrak{C}(\lambda) = \sum_{l=1}^{n} \mathfrak{C}(\lambda)\mathfrak{J}_{l,l},$$

it is found that on multiplication by $\mathfrak{S}(x)$ on the left, and by $\mathfrak{C}(\lambda)\mathfrak{S}^{-1}(x)$ on the right, the relation (6.1) becomes

$$\begin{aligned}\mathfrak{V}(x) + \lambda^{-k}\sum_{h,l=1}^{n} \mathfrak{S}(x)\mathfrak{J}_{h,h}\mathfrak{S}^{-1}(x_1)\mathfrak{A}(x_1)\mathfrak{V}(x_1)\mathfrak{S}(x_1)\mathfrak{J}_{l,l}\mathfrak{S}^{-1}(x)dx_1 \\ = \mathfrak{S}(x)\mathfrak{K}(\lambda)\mathfrak{C}(\lambda)\mathfrak{S}^{-1}(x).\end{aligned} \tag{6.3}$$

Now from (5.5) and (5.9) it is seen that

$$\mathfrak{S}(x)\mathfrak{J}_{h,h}\mathfrak{S}^{-1}(x_1) \equiv \mathfrak{P}_k(x)\mathfrak{J}_{h,h}\mathfrak{P}_k^{-1}(x_1) \exp\,[\lambda\{R_h(x) - R_h(x_1)\}],$$

and

$$\mathfrak{V}(x_1) = \sum_{\alpha,\beta=1}^{n} \mathfrak{J}_{\alpha,\beta}v_{\alpha,\beta}(x_1).$$

If $\xi_1^{(i,j)}$ is the value which corresponds under (3.2) to x_1, it follows that (6.3) finally takes the form

$$\mathfrak{V}(x, \lambda) + \lambda^{-k}\mathfrak{H}(x, \lambda) = \mathfrak{S}(x)\mathfrak{K}(\lambda)\mathfrak{C}(\lambda)\mathfrak{S}^{-1}(x), \tag{6.4}$$

in which

$$\mathfrak{H}(x, \lambda) = \sum_{h,l,\alpha,\beta=1}^{n} \int_{x_*^{(h,l)}}^{x} \mathfrak{P}_k(x)\mathfrak{J}_{h,h}\mathfrak{P}_k^{-1}(x_1)\mathfrak{A}(x_1)\mathfrak{J}_{\alpha,\beta}\mathfrak{P}_k(x_1)\mathfrak{J}_{l,l}\mathfrak{P}_k^{-1}(x) \cdot e^{\xi^{(h,l)}-\xi_1^{(h,l)}} v_{\alpha,\beta}(x_1)dx_1. \tag{6.5}$$

The elements of the matrix (6.2) are analytic in X, and this region, is by definition, closed. Its elements, therefore, take on numerical maxima; hence there exists a scalar $m(\lambda)$ independent of x, such that

$$| v_{i,j}(x) | \leqq m(\lambda), \qquad i, j = 1, 2, \cdots, n, \tag{6.6}$$

and the equality holds for some index pair (i, j), at some point x. Moreover $m(\lambda) > 0$, since by hypothesis $\mathfrak{C}(\lambda) \neq \mathfrak{O}$.

Let the path of integration from the point $x_*^{(h,l)}$ to x in (6.5) be taken now as a curve along which the real part of $\xi_1^{(h,l)}$ is non-increasing. It is precisely the characteristic property of the point $x_*^{(h,l)}$ that, whatever x may be, there exists such a path. During the integration, then, it is clear that

$$| e^{\xi^{(h,l)}-\xi_1^{(h,l)}} | \leqq 1.$$

Finally since the matrices in the integrands of (6.5) are obviously bounded, both as to x and as to λ, when $|\lambda| > N$, there exists an absolute scalar constant M such that the elements of (6.5) satisfy the relation

$$| h_{i,j}(x, \lambda) | \leqq Mm(\lambda), \qquad i, j = 1, 2, \cdots, n. \tag{6.7}$$

For that pair of indices, and that point x, for which the equality holds in (6.6), therefore,

$$m(\lambda)\left\{1 - \frac{M}{|\lambda|^k}\right\} \leqq \left| v_{i,j}(x, \lambda) + \frac{h_{i,j}(x, \lambda)}{\lambda^k} \right|. \tag{6.8}$$

Since the left-hand member of this is positive, when $|\lambda| > N$, it follows that the matrix on the left of (6.4) is not the zero matrix. This must, therefore, be so for the matrix on the right, namely, $\mathfrak{K}(\lambda)\mathfrak{C}(\lambda) \neq \mathfrak{O}$. Since this follows with $\mathfrak{C}(\lambda)$ unspecified except for $\mathfrak{C}(\lambda) \neq \mathfrak{O}$, it must be concluded that the matrix $\mathfrak{K}(\lambda)$ is nonsingular.

With the existence of the matrix $\mathfrak{K}^{-1}(\lambda)$ thus established, we may now choose $\mathfrak{C}(\lambda) = \mathfrak{K}^{-1}(\lambda)$. The right-hand member of (6.4), and hence the left-hand member, reduces then to the unit matrix. Since the right-hand member of (6.8) is thus at most unity, it follows that the function $m(\lambda)$ is bounded, when $|\lambda| > N$. Then by (6.7) the elements of the matrix $\mathfrak{H}(x, \lambda)$ are bounded. From (6.4) and (6.2), lastly,

$$\mathfrak{Y}^{(p)}(x, \lambda)\mathfrak{K}^{-1}(\lambda) = \{\mathfrak{J} - \lambda^{-k}\mathfrak{H}(x, \lambda)\}\mathfrak{S}(x, \lambda, k), \tag{6.9}$$

and in this the left-hand member is a nonsingular analytic solution of the given equation (2.1). The existence of a solution of this form (6.9) is what was to be established. The result may be formulated thus:

If x and λ are restricted to any pair of associated regions, there exists an analytic solution of the equation (2.1) *which is of the form*

$$\mathfrak{Y}(x, \lambda) = \mathfrak{P}(x, \lambda)\mathfrak{E}(x, \lambda), \tag{6.10}$$

the matrix $\mathfrak{P}(x, \lambda)$ being of the form

$$\mathfrak{P}(x, \lambda) = \mathfrak{I} + \sum_{h=1}^{k-1} \lambda^{-h}\mathfrak{P}^{(h)}(x) + \lambda^{-k}\mathfrak{B}_k(x, \lambda). \tag{6.11}$$

In this k may be taken as any natural number, and the elements of the matrix $\mathfrak{B}_k(x, \lambda)$ are bounded when $|\lambda| > N$.

This result may be extended at once to the case in which λ is restricted merely to some suitable region, while x, on the other hand, remains in a region which is fundamental relative to the λ region in question. In this case the λ region may be subdivided into a finite number of subregions, in each of which some solution of the equation maintains the form (6.10), (6.11). The solutions which respectively maintain these forms in different λ subregions will in general be different. In virtue of (2.6) it is clear that in any λ subregion the general solution of the equation (2.1) is of the form

$$\mathfrak{Y}(x, \lambda) = \mathfrak{P}(x, \lambda)\mathfrak{E}(x, \lambda)\mathfrak{C}(\lambda),$$

with $\mathfrak{P}(x, \lambda)$ as given by (6.11).

PART II. THE BOUNDARY PROBLEM

7. **Definition and qualitative aspects of the problem.** If any finite set of points $\eta_1, \eta_2, \cdots, \eta_m$ with $m \geqq 2$ is chosen in any x region suitable to the differential system (1), and the variable is restricted to this domain, while the parameter is restricted to some suitable λ region, the solution of the differential system may be conditioned relative to this set of points by a set of relations

$$\sum_{\mu=1}^{m} \sum_{\nu=1}^{n} w_{i,\nu}^{(\mu)} y_\nu(\eta_\mu, \lambda) = 0, \qquad i = 1, 2, \cdots, n.$$

Such relations are then termed boundary conditions, and the differential system together with such boundary conditions is said to constitute a boundary problem. The coefficients $w_{i,j}^{(\mu)}$ involved in the boundary conditions may be constants, or may more generally depend analytically upon the parameter λ. They are, of course, independent of x.

The functions which together make up any solution of the differential system are, as has been seen, n in number, constituting the elements of a column of some matrix solution of the differential equation (2.1). They may, therefore, be considered as a vector $\mathfrak{y}(x)$.† If they satisfy the boundary conditions, this vector then satisfies the relations

$$\mathfrak{Y}'(x, \lambda) = \{\lambda\mathfrak{R}(x) + \mathfrak{Q}(x, \lambda)\}\mathfrak{Y}(x, \lambda), \tag{7.1a}$$

$$\sum_{\mu=1}^{m} \mathfrak{W}^{(\mu)}(\lambda)\mathfrak{Y}(\eta_\mu, \lambda) = \mathfrak{O}. \tag{7.1b}$$

The boundary problem is thus formulated as that of finding a vector solution of the problem (7.1).

In §2 it was observed that the general solution of the matrix equation (2.1), that is, of (7.1a), is expressible in terms of any particular nonsingular analytic solution $\mathfrak{Y}(x, \lambda)$ by means of the formula (2.6). A solution is, therefore, a vector if and only if $\mathfrak{C}(\lambda)$ is a vector, and the general vector solution of (7.1a) is thus given by the formula

$$\mathfrak{y}(x, \lambda) \equiv \mathfrak{Y}(x, \lambda)\mathfrak{c}(\lambda), \tag{7.2}$$

the vector $\mathfrak{c}(\lambda)$ being arbitrary. If such a vector is to satisfy the relation (7.1b), it follows that the equation

$$\mathfrak{D}(\lambda)\mathfrak{c}(\lambda) = \mathfrak{o}, \tag{7.3}$$

with

$$\mathfrak{D}(\lambda) \equiv \sum_{\mu=1}^{m} \mathfrak{W}^{(\mu)}(\lambda)\mathfrak{Y}(\eta_\mu, \lambda), \tag{7.4}$$

must be fulfilled. Now the solution (7.2) is evidently trivial, that is, $\mathfrak{y}(x) \equiv \mathfrak{o}$, if the vector $\mathfrak{c}(\lambda)$ is trivial, that is, $\mathfrak{c}(\lambda) = \mathfrak{o}$. Hence a necessary and sufficient condition for a non-trivial solution of the boundary problem, is the existence of a non-trivial vector $\mathfrak{c}(\lambda)$, which satisfies the equation (7.3). Such familiarly exists if and only if the matrix $\mathfrak{D}(\lambda)$ is singular, namely if

$$D(\lambda) = 0, \tag{7.5}$$

where $D(\lambda)$ designates the determinant of the matrix (7.4).

† The use of lower case German letters will be reserved to the designation of vectors of n elements or components, and such vectors are to be regarded freely, as may be convenient, as matrices of one row and n columns, or vice versa. This will lead to no ambiguity if it is agreed, and it shall hereby be so agreed, that all multiplications between matrices and vectors, or of vectors by vectors, is to be understood as being in the matrix sense. A vector is, therefore, to be regarded as a matrix of one row and n columns whenever it appears as a left-hand factor, and as a matrix of n rows and one column if it appears as a right-hand factor.

If at any specified value of λ the condition (7.5) is not fulfilled, the boundary problem admits of no solution and is therefore said to be incompatible. On the other hand, if for a specified λ the condition (7.5) is satisfied, that is, if λ is a root of the equation (7.5), the equation (7.3) does admit a non-trivial solution. More explicitly, if at this λ the rank of the determinant $D(\lambda)$ is $(n-r)$, the equation (7.3) is satisfied by r distinct vectors $\mathfrak{c}(\lambda)$, and these lead through (7.2) to precisely r linearly independent solutions of the problem. The latter is therefore said in this case to be *compatible to the order r*, the term *simply compatible* being used interchangeably with compatible to the order 1. The roots of the equation (7.5), which thus appear as the λ values for which the boundary problem is solvable, are known as *characteristic values*, and the non-trivial solutions of the problem which exist at these values are called *characteristic* solutions. A characteristic value at which the problem is compatible to the order r is said to be of the *index r*. On the other hand, a characteristic value will be said to be of the *multiplicity s* if it is an s-fold zero of the determinant $D(\lambda)$. It will be seen below that the index of a characteristic value cannot exceed its multiplicity.

It must be observed that although the characteristic values, which are intrinsic to the boundary problem, are obtained from the determinant $D(\lambda)$, neither this determinant, nor the corresponding matrix $\mathfrak{D}(\lambda)$, is uniquely determined by the boundary problem. This is due in part to the fact that the solution $\mathfrak{Y}(x, \lambda)$ of the equation (7.1a) to be used in (7.4) was specified only to the extent that it be analytic and nonsingular, and also in part to the fact that the content of the equation (7.1b) is unchanged if it is multiplied on the left by any analytic nonsingular matrix $\mathfrak{C}_1(\lambda)$. Since when $\mathfrak{Y}(x, \lambda)$ is any eligible solution, all such solutions are expressible by (2.6) in the form $\mathfrak{Y}(x, \lambda)\mathfrak{C}_2(\lambda)$, with the matrix $\mathfrak{C}_2(\lambda)$ analytic and nonsingular, it is clear that the role of the matrix $\mathfrak{D}(\lambda)$ may be given at will to any matrix of the form

$$\mathfrak{C}_1(\lambda)\mathfrak{D}(\lambda)\mathfrak{C}_2(\lambda), \tag{7.6}$$

with $\mathfrak{C}_1$ and $\mathfrak{C}_2$ analytic and nonsingular. Conversely, of course, the matrix (7.6) is the most general by which $\mathfrak{D}(\lambda)$ may be replaced. Since the determinant of the matrix (7.6) differs from $D(\lambda)$ only by nonvanishing factors, it is clear that (7.5) as an equation is invariant.

The inverse of the matrix $\mathfrak{D}(\lambda)$ is familiarly given by the formula

$$\mathfrak{D}^{-1}(\lambda) \equiv \left(\frac{D_{j,i}(\lambda)}{D(\lambda)}\right), \tag{7.7}$$

with $D_{i,j}(\lambda)$ denoting the cofactor of the element in the ith row and jth column of $D(\lambda)$. It is clear from this that the elements of $\mathfrak{D}^{-1}(\lambda)$ are analytic except possibly at the characteristic values, where they may have poles.

8. **The adjoint boundary problem.** Let η_0 be chosen arbitrarily as a point of the x region, either distinct from the points $\eta_1, \eta_2, \cdots, \eta_m$, or coincident with any one of them. Then with the various matrices involved identified as those which are similarly denoted in (7.1), and with any specific value of λ, there may or may not exist a *parametric matrix* $\mathfrak{A}(\lambda)$ which is independent of x, and is such that the system of relations

$$(8.1\text{a})\qquad \mathfrak{Z}^{(h)\prime}(x,\lambda) = -\mathfrak{Z}^{(h)}(x,\lambda)\{\lambda\mathfrak{R}(x) + \mathfrak{Q}(x,\lambda)\},$$

$$(8.1\text{b})\qquad \begin{aligned} &\mathfrak{Z}^{(h)}(\eta_h,\lambda) + \mathfrak{A}(\lambda)\mathfrak{W}^{(h)}(\lambda) = \mathfrak{O}, \qquad h = 1, 2, \cdots, m,\\ &\sum_{\mu=1}^{m} \mathfrak{Z}^{(\mu)}(\eta_0,\lambda) = \mathfrak{O}, \end{aligned}$$

admits of solution by a set of m matrices $\mathfrak{Z}^{(h)}(x,\lambda)$, $(h=1, 2, \cdots, m)$. It is immediately evident that with $\mathfrak{A}(\lambda)=\mathfrak{O}$, the system is uniquely solved, irrespective of the value of λ, by $\mathfrak{Z}^{(h)}(x,\lambda)\equiv\mathfrak{O}$, $(h=1, 2, \cdots, m)$, and conversely. This solution is trivial. It is, therefore, requisite for a non-trivial solution that $\mathfrak{A}(\lambda)\neq\mathfrak{O}$, and this will accordingly be generally assumed henceforth. If the parametric matrix $\mathfrak{A}(\lambda)$ is one having a single row, that is, is a vector, any eventual solution of the system will obviously also consist of a set of vectors, and vice versa a vector solution can exist only in connection with a parametric vector. Such a solution of the system (8.1) by vectors is the matter of immediate interest to the discussion, and this problem will be referred to henceforth as the boundary problem *adjoint to* the problem (7.1).

The differential matrix equation (8.1a) is familarly solved by the inverse of any nonsingular solution of the equation (2.1), and its general solution is therefore given by $\mathfrak{C}(\lambda)\mathfrak{Y}^{-1}(x,\lambda)$, in which $\mathfrak{C}(\lambda)$ is arbitrary and $\mathfrak{Y}(x,\lambda)$ may, in particular, be understood to be that analytic solution of (2.1), that is, of (7.1a), which was used in the deductions of the preceding section. Any vector solutions of the equations (8.1a) are, therefore, of the form

$$(8.2)\qquad \mathfrak{z}^{(h)}(x,\lambda) = \mathfrak{c}^{(h)}(\lambda)\mathfrak{Y}^{-1}(x,\lambda), \qquad h = 1, 2, \cdots, m.$$

With these the relations (8.1b) become

$$(8.3)\qquad \begin{aligned} &-\mathfrak{c}^{(h)}(\lambda) = \mathfrak{a}(\lambda)\mathfrak{W}^{(h)}(\lambda)\mathfrak{Y}(\eta_h,\lambda), \qquad h = 1, 2, \cdots, m,\\ &\sum_{\mu=1}^{m} \mathfrak{c}^{(\mu)}(\lambda) = \mathfrak{o}, \end{aligned}$$

and are to be solved by choice of the vectors $\mathfrak{c}^{(h)}(\lambda)$. If these relations are summed, and the formula (7.4) is recalled, the result is found to be

$$(8.4)\qquad \mathfrak{a}(\lambda)\mathfrak{D}(\lambda) = \mathfrak{o}.$$

A solution of the adjoint boundary problem can exist, therefore, only in connection with a parametric vector $\mathfrak{a}(\lambda)$ which satisfies the equation (8.4). A necessary condition for this, since $\mathfrak{a}(\lambda)$ must differ from $\mathfrak{o}$, is that the matrix $\mathfrak{D}(\lambda)$ be singular, that is, that λ be a root of the equation (7.5). Since the roots of (7.5) are the characteristic values of the boundary problem (7.1), it follows that whenever the latter is incompatible the adjoint problem is insolvable, that is, is likewise incompatible. Conversely, if $\mathfrak{D}(\lambda)$ is singular, and is, say, of the rank $(n-r)$, the equation (8.4) is solvable and determines precisely r distinct parametric vectors $\mathfrak{a}(\lambda)$. Each of these leads through the formulas (8.3) to a set of vectors $\mathfrak{c}^{(h)}(\lambda)$, and by (8.2) r linearly independent solutions of the problem (8.1) are then determined. The adjoint problem is thus appropriately described as compatible to the order r. Since in this case λ is a characteristic value for which the problem (7.1) is compatible to precisely the order r, the result may be formulated thus:

A boundary problem and its adjoint problem have the same characteristic values, and at any characteristic value are compatible to the same order.

The boundary conditions (7.1b) and (8.1b) are so interrelated that if $\mathfrak{U}(x, \lambda)$ is any matrix satisfying the former, and the matrices $\mathfrak{V}^{(h)}(x, \lambda)$, $(h=1, 2, \cdots, m)$, together with a parametric matrix $\mathfrak{A}(\lambda)$ satisfy the latter, then

$$\sum_{\mu=1}^{m} \int_{\eta_0}^{\eta_\mu} d\{\mathfrak{V}^{(\mu)}(x, \lambda)\mathfrak{U}(x, \lambda)\} = \mathfrak{O}. \tag{8.5}$$

In the classical case of a real variable x and two-point boundary conditions, this will be recognized as a familiar relation; indeed one upon which the definition of the adjoint problem is sometimes based. After performance of the integrations, the left-hand member of the relation takes the form

$$\sum_{\mu=1}^{m} \mathfrak{V}^{(\mu)}(\eta_\mu, \lambda)\mathfrak{U}(\eta_\mu, \lambda) - \sum_{\mu=1}^{m} \mathfrak{V}^{(\mu)}(\eta_0, \lambda)\mathfrak{U}(\eta_0, \lambda).$$

The second of these sums vanishes by (8.1b), while the first may be written

$$-\mathfrak{A}(\lambda) \sum_{\mu=1}^{m} \mathfrak{W}^{(\mu)}(\lambda)\mathfrak{U}(\eta_\mu, \lambda).$$

This vanishes by (7.1b).

An alternative definition of the problem adjoint to (7.1), and one which avoids the introduction of the parametric matrix, may be given by choosing the point η_0 in coincidence with one of the points of the set $\eta_1, \eta_2, \cdots, \eta_m$, say with η_r. This is the following:

$$
\begin{aligned}
&\mathfrak{Z}_1^{(h)\prime}(x, \lambda) = -\,\mathfrak{Z}_1^{(h)}(x, \lambda)\{\lambda\mathfrak{R}(x) + \mathfrak{Q}(x, \lambda)\}, && h = 1, 2, \cdots, m,\\
(8.6)\qquad &\mathfrak{Z}_1^{(h)}(\eta_h, \lambda) + \sum_{\mu=1}^{m} \mathfrak{Z}_1^{(\mu)}(\eta_r, \lambda)\mathfrak{W}^{(h)}(\lambda) = \mathfrak{O}, && h \neq r,\\
&\mathfrak{Z}_1^{(r)}(\eta_r, \lambda) + \sum_{\mu=1}^{m} \mathfrak{Z}_1^{(r)}(\eta_r, \lambda)\{\mathfrak{W}^{(r)}(\lambda) - \mathfrak{I}\} = \mathfrak{O},
\end{aligned}
$$

the equations to be solved by a set of vectors $\mathfrak{z}_1^{(h)}(x, \lambda)$, $(h=1, 2, \cdots, m)$, which are not all identically zero. The problem in this formulation is amenable to precisely the same deductions and conclusions as were drawn above from the form (8.1). The passage from the one formulation to the other is easily made by means of the relations

$$
\mathfrak{a}(\lambda) = \sum_{\mu=1}^{m} \mathfrak{z}_1^{(\mu)}(\eta_r, \lambda), \quad \mathfrak{z}^{(h)}(x, \lambda) \equiv \mathfrak{z}_1^{(h)}(x, \lambda), \qquad h \neq r,
$$

$$
\mathfrak{z}^{(r)}(x, \lambda) \equiv -\sum_{\mu \neq r} \mathfrak{z}_1^{(\mu)}(x, \lambda).
$$

The form (8.1) was preferred above because of its greater symmetry.

9. **The Green's matrices.** If $\mathfrak{f}(x, \lambda)$ is any vector that is analytic in the chosen suitable x and λ regions, the equations

$$
(9.1a)\qquad \mathfrak{u}'(x, \lambda) = \{\lambda\mathfrak{R}(x) + \mathfrak{Q}(x, \lambda)\}\mathfrak{u}(x, \lambda) + \mathfrak{f}(x, \lambda),
$$

$$
(9.1b)\qquad \sum_{\mu=1}^{m} \mathfrak{W}^{(\mu)}(\lambda)\mathfrak{u}(\eta_\mu, \lambda) = \mathfrak{o},
$$

define a vector boundary problem which is related to the problem (7.1), being evidently a nonhomogeneous generalization of it. The solution of this problem is expressible in terms of any nonsingular analytic solution of the matrix differential equation (2.1), and may be deduced as follows.

It is verifiable by actual substitution that the formula

$$
(9.2)\qquad \mathfrak{u}^{(p)}(x, \lambda) = \int_{\eta_0}^{x} \mathfrak{Y}(x, \lambda)\mathfrak{Y}^{-1}(x_1, \lambda)\mathfrak{f}(x_1, \lambda)dx_1,
$$

yields a particular solution of the vector differential equation (9.1a). The general solution of this equation is, therefore, given by

$$
(9.3)\qquad \mathfrak{u}(x, \lambda) = \mathfrak{u}^{(p)}(x, \lambda) + \mathfrak{Y}(x, \lambda)\mathfrak{c}(\lambda),
$$

with the vector $\mathfrak{c}(\lambda)$ arbitrary. With this evaluation, and in virtue of the formula (7.4), the relation (9.1b) becomes

$$
(9.4)\qquad \sum_{\mu=1}^{m} \mathfrak{W}^{(\mu)}(\lambda)\mathfrak{u}^{(p)}(\eta_\mu, \lambda) + \mathfrak{D}(\lambda)\mathfrak{c}(\lambda) = \mathfrak{o}.
$$

Thus the solvability of the problem (9.1) depends upon the possibility of a choice of the vector $\mathfrak{c}(\lambda)$ to satisfy the equation (9.4). Such a choice is evidently possible and unique provided the matrix $\mathfrak{D}(\lambda)$ is nonsingular; that is, provided λ is not a characteristic value. If this is so, the vector $\mathfrak{c}(\lambda)$ determined by (9.4) yields through (9.3) the solution of the problem (9.1). The result may be explicitly written

$$
\begin{aligned}
\mathfrak{u}(x, \lambda) = \int_{\eta_0}^{x} \mathfrak{Y}(x, \lambda)\mathfrak{Y}^{-1}(x_1, \lambda)\mathfrak{f}(x_1, \lambda)dx_1 \\
- \sum_{\mu=1}^{m} \int_{\eta_0}^{\eta_\mu} \mathfrak{G}^{(\mu)}(x, x_1, \lambda)\mathfrak{f}(x_1, \lambda)dx_1,
\end{aligned}
\tag{9.5}
$$

with

$$
\mathfrak{G}^{(h)}(x, x_1, \lambda) \equiv \mathfrak{Y}(x, \lambda)\mathfrak{D}^{-1}(\lambda)\mathfrak{W}^{(h)}(\lambda)\mathfrak{Y}(\eta_h, \lambda)\mathfrak{Y}^{-1}(x_1, \lambda), \quad h = 1, 2, \cdots, m.
\tag{9.6}
$$

The matrices $\mathfrak{G}^{(h)}(x, x_1, \lambda)$, for which the formulas (9.6) are definitive, thus serve for the solution of the problem (9.1) independently of the vector $\mathfrak{f}(x, \lambda)$ which may be involved therein. They are to be known henceforth as the *Green's matrices*, and are best regarded as associated with the boundary problem (7.1), since they are constructed solely from matrices involved in the latter. They exist and are evidently analytic whenever the associated boundary problem is incompatible. Moreover, they are unique, for though $\mathfrak{Y}(x, \lambda)$ and the $\mathfrak{W}^{(\mu)}(\lambda)$ may at will be replaced by $\mathfrak{Y}(x, \lambda)\mathfrak{C}_2(\lambda)$ and $\mathfrak{C}_1(\lambda)\mathfrak{W}^{(\mu)}(\lambda)$, respectively, as was observed in §7, such a change would call for the replacement of $\mathfrak{D}(\lambda)$ by the matrix (7.6). The formulas (9.6) are evidently invariant under such substitutions.

For subsequent use it may be recorded that the Green's matrices satisfy the relations

$$
\sum_{\mu=1}^{m} \mathfrak{G}^{(\mu)}(x, x_1, \lambda) \equiv \mathfrak{Y}(x, \lambda)\mathfrak{Y}^{-1}(x_1, \lambda),
\tag{9.7a}
$$

$$
\mathfrak{G}^{(h)}(x, \eta_h, \lambda) \equiv \mathfrak{Y}(x, \lambda)\mathfrak{D}^{-1}(\lambda)\mathfrak{W}^{(h)}(\lambda),
\tag{9.7b}
$$

$$
\sum_{\mu=1}^{m} \mathfrak{W}^{(\mu)}(\lambda)\mathfrak{G}^{(h)}(\eta_\mu, x_1, \lambda) \equiv \mathfrak{W}^{(h)}(\lambda)\mathfrak{Y}(\eta_h, \lambda)\mathfrak{Y}^{-1}(x_1, \lambda), \quad h = 1, 2, \cdots, m.
\tag{9.7c}
$$

These are readily deduced directly from the formulas (9.6). The formula (9.7a) makes possible the reduction of (9.5) to an alternative and more compact form. Since the integrands involved in (9.5) are all analytic, the paths of integration may be chosen at pleasure, and hence may in particular be chosen to lie in coincidence from the point η_0 to the point x. With this choice, and in virtue of (9.7a), the formula reduces to

$$\mathfrak{u}(x, \lambda) = \sum_{\mu=1}^{m} \int_{\eta_\mu}^{x} \mathfrak{G}^{(\mu)}(x, x_1, \lambda)\mathfrak{f}(x_1, \lambda)dx_1. \tag{9.8}$$

The nonhomogeneous vector boundary problem which generalizes the adjoint problem (8.1) in a manner similar to the above, is evidently given by the equations

$$\mathfrak{v}^{(h)\prime}(x, \lambda) = -\mathfrak{v}^{(h)}(x, \lambda)\{\lambda\mathfrak{R}(x) + \mathfrak{Q}(x, \lambda)\} + \mathfrak{f}(x, \lambda), \tag{9.9a}$$

$$\begin{aligned} \mathfrak{v}^{(h)}(\eta_h, \lambda) + \mathfrak{a}_1(\lambda)\mathfrak{W}^{(h)}(\lambda) &= \mathfrak{o}, \qquad h = 1, 2, \cdots, m, \\ \sum_{\mu=1}^{m} \mathfrak{v}^{(\mu)}(\eta_0, \lambda) &= \mathfrak{o}. \end{aligned} \tag{9.9b}$$

Its solution is obtainable by reasoning similar to that used above. The general solution of the equation (9.9a) is of the form

$$\mathfrak{v}^{(h)}(x, \lambda) = \mathfrak{c}^{(h)}(\lambda)\mathfrak{Y}^{-1}(x, \lambda) + \int_{\eta_0}^{x} \mathfrak{f}(x_1, \lambda)\mathfrak{Y}(x_1, \lambda)\mathfrak{Y}^{-1}(x, \lambda)dx_1, \tag{9.10}$$

and with this the conditions (9.9b) take the form

$$\begin{aligned} -\mathfrak{c}^{(h)}(\lambda) - \int_{\eta_0}^{\eta_h} \mathfrak{f}(x_1, \lambda)\mathfrak{Y}(x_1, \lambda)dx_1 &= \mathfrak{a}_1(\lambda)\mathfrak{W}^{(h)}(\lambda)\mathfrak{Y}(\eta_h, \lambda), \\ \sum_{\mu=1}^{m} \mathfrak{c}^{(\mu)}(\lambda) &= \mathfrak{o}, \qquad h = 1, 2, \cdots, m. \end{aligned} \tag{9.11}$$

An addition of these leads to the evaluation of the parametric vector

$$\mathfrak{a}_1(\lambda) = -\sum_{\mu=1}^{m} \int_{\eta_0}^{\eta_\mu} \mathfrak{f}(x_1, \lambda)\mathfrak{Y}(x_1, \lambda)dx_1\mathfrak{D}^{-1}(\lambda), \tag{9.12}$$

and in terms of this the vectors $\mathfrak{c}^{(h)}(\lambda)$ are given by (9.11). The solution, by (9.10), is then explicitly

$$\begin{aligned} \mathfrak{v}^{(h)}(x, \lambda) = &\int_{\eta_h}^{x} \mathfrak{f}(x_1, \lambda)\mathfrak{Y}(x_1, \lambda)\mathfrak{Y}^{-1}(x, \lambda)dx_1 \\ &+ \sum_{\mu=1}^{m} \int_{\eta_0}^{\eta_\mu} \mathfrak{f}(x_1, \lambda)\mathfrak{G}^{(h)}(x_1, x, \lambda)dx_1, \qquad h = 1, 2, \cdots, m. \end{aligned} \tag{9.13}$$

The problem (9.9), like the problem (9.1), is thus solvable, irrespective of $\mathfrak{f}(x, \lambda)$, for all values of λ other than the characteristic values.

10. **The Green's matrix for a linear x domain.** In the classical case of a real variable x, the region of the variable, being a segment of the axis, is not a

two-dimensional domain but a one-dimensional one. It is of some interest on this account to consider somewhat further the more immediate generalization of this case to that in which the domain of x consists of a set of curves in the complex plane which respectively join a point η_0 to the points $\eta_1, \eta_2, \cdots, \eta_m$. In the absence of any requirement that these curves be distinct, the configuration is seen to be immediately specializable to the case of a real x and boundary conditions which apply at more than two points of a given interval. The adaptation of the discussion already made to this case of a general curvilinear x domain calls for no modification of the deductions of §7. It permits, however, of an interesting reformulation of the matter of §8, and of an extension of the considerations of §9.

Let the boundary problem adjoint to (7.1) be defined in this case by the equations

$$\mathfrak{z}'(x, \lambda) = -\mathfrak{z}(x, \lambda)\{\lambda\mathfrak{R}(x) + \mathfrak{Q}(x, \lambda)\}, \tag{10.1a}$$

$$\begin{aligned} &\mathfrak{z}(\eta_h, \lambda) + \mathfrak{a}(\lambda)\mathfrak{W}^{(h)}(\lambda) = \mathfrak{o}, \qquad h = 1, 2, \cdots, m, \\ &\sum_{\mu=1}^{m} \mathfrak{z}(\eta_0 + 0\eta_\mu, \lambda) = \mathfrak{o}, \end{aligned} \tag{10.1b}$$

the solution to exist for a suitable parametric vector $\mathfrak{a}(\lambda)$, and to consist of a vector $\mathfrak{z}(x, \lambda)$ which satisfies the conditions (10.1b) and solves the equation (10.1a) along each one of the curves constituting the x domain. The symbol $\mathfrak{z}(\eta_0+0\eta_h, \lambda)$ is to be interpreted as designating the limit of $\mathfrak{z}(x, \lambda)$ as $x \to \eta_0$, the approach being along the x curve from η_h. The solution vector $\mathfrak{z}(x, \lambda)$ will in general be discontinuous at the point η_0, that is, the vectors $\mathfrak{z}(x_0+0\eta_h, \lambda)$ will not in general coincide for all h. The deductions of §8 are adapted to this formulation of the adjoint problem without difficulty, being made, in fact, by merely identifying the vector $\mathfrak{z}^{(h)}(x, \lambda)$ of the solution of (8.1), as the solution $\mathfrak{z}(x, \lambda)$ of the problem (10.1) when the variable is on the respective curve from η_0 to η_h.

If x and x_1 are regarded as independent variables, both confined to the given set of curves, a matrix $\mathfrak{G}_1(x, x_1, \lambda)$ is defined by the formula

$$\mathfrak{G}_1(x, x_1, \lambda) = \pm \tfrac{1}{2}\mathfrak{Y}(x, \lambda)\mathfrak{Y}^{-1}(x_1, \lambda), \tag{10.2}$$

if it is agreed that the plus sign is to apply when x_1 lies on the curve segment which is terminated by the points η_0 and x, while the minus sign is to apply otherwise. The formula

$$\mathfrak{G}(x, x_1, \lambda) \equiv \mathfrak{G}_1(x, x_1, \lambda) - \mathfrak{Y}(x, \lambda)\mathfrak{D}^{-1}(\lambda)\sum_{\mu=1}^{m} \mathfrak{W}^{(\mu)}(\lambda)\mathfrak{G}_1(\eta_\mu, x_1, \lambda), \tag{10.3}$$

then defines its left-hand member, which will be designated briefly as *the*

Green's matrix. This matrix $\mathfrak{G}(x, x_1, \lambda)$ is related in several ways to the Green's matrices previously defined by the formulas (9.6). Thus, in particular, it will be seen that

$$\mathfrak{G}(\eta_\mu, x_1, \lambda) = -\mathfrak{G}^{(h)}(\eta_\mu, x_1, \lambda) + \delta_{\mu,h}\mathfrak{Y}(\eta_\mu, \lambda)\mathfrak{Y}^{-1}(x_1, \lambda), \qquad \mu, h = 1, 2, \cdots, m, \tag{10.4}$$

whenever x_1 lies on the curve from η_0 to η_h, whereas when x lies on that curve, then

$$\mathfrak{G}(x, \eta_0 + 0\eta_\mu, \lambda) = -\mathfrak{G}^{(\mu)}(x, \eta_0, \lambda) + \delta_{\mu,h}\mathfrak{Y}(x, \lambda)\mathfrak{Y}^{-1}(\eta_0, \lambda). \tag{10.5}$$

The matrix $\mathfrak{G}(x, x_1, \lambda)$ evidently depends upon x solely by virtue of the occurrence of the matrix $\mathfrak{Y}(x, \lambda)$ as a left-hand factor in the formula (10.3). It follows from this that as a function of x (that is, when x_1 is regarded as fixed) this matrix satisfies the equation (7.1a) along each of the arcs into which the domain of x is divided by the points η_0 and x_1. Beyond that it is clear from the formula (10.4) and the relation (9.7b) that

$$\sum_{\mu=1}^{m} \mathfrak{W}^{(\mu)}(\lambda)\mathfrak{G}(\eta_\mu, x_1, \lambda) \equiv \mathfrak{O}, \tag{10.6}$$

namely, that as a function of x it satisfies the condition (7.1b). Formally, therefore, the Green's matrix as a function of its first argument solves the boundary problem (7.1). It fails of being a true solution of that problem because of a discontinuity inherent in it at the point $x = x_1$, for, as is easily verified,

$$\mathfrak{G}(x_1 + 0\eta_h, x_1, \lambda) - \mathfrak{G}(x_1 + 0\eta_0, x_1, \lambda) \equiv \mathfrak{J}, \tag{10.7}$$

for any x_1 on the curve from η_0 to η_h.

In an entirely similar manner it will be observed that $\mathfrak{G}(x, x_1, \lambda)$ depends upon x_1 solely by virtue of the presence of the matrix $\mathfrak{Y}^{-1}(x_1, \lambda)$ as a right-hand factor in the formula (10.3). Hence as a function of x_1 (that is, with x fixed) it formally solves the equation (8.1a) along each of the arcs into which the domain of the variable is divided by the points η_0 and x. Since from (10.3) together with (10.5) and (9.7a)

$$\begin{gathered} \mathfrak{G}(x, \eta_h, \lambda) + \mathfrak{Y}(x, \lambda)\mathfrak{D}^{-1}(\lambda)\mathfrak{W}^{(h)}(\lambda) = \mathfrak{O}, \qquad h = 1, 2, \cdots, m, \\ \sum_{\mu=1}^{m} \mathfrak{G}(x, \eta_0 + 0\eta_\mu, \lambda) = \mathfrak{O}, \end{gathered} \tag{10.8}$$

it is seen that as a function of its second argument the Green's matrix is formally a solution of the boundary problem (10.1), with the matrix $\mathfrak{Y}(x, \lambda)\mathfrak{D}^{-1}(\lambda)$ in the role of the parametric matrix. In this instance, as before, however, it fails to be a true solution because of its discontinuity.

The nonhomogeneous boundary problem (9.1) and also the problem

$$
\begin{aligned}
&\mathfrak{v}'(x, \lambda) = -\mathfrak{v}(x, \lambda)\{\lambda\mathfrak{R}(x) + \mathfrak{Q}(x, \lambda)\} + \mathfrak{f}(x, \lambda), \\
&\mathfrak{v}(\eta_h, \lambda) + \mathfrak{a}_1(\lambda)\mathfrak{W}^{(h)}(\lambda) = \mathfrak{o}, \qquad h = 1, 2, \cdots, m, \\
&\sum_{\mu=1}^{m} \mathfrak{v}(\eta_0 + 0\eta_\mu, \lambda) = \mathfrak{o},
\end{aligned}
\tag{10.9}
$$

which is the reformulation of (9.9), may now be considered, with the $\mathfrak{f}(x, \lambda)$ as any vectors which are defined merely over the curves of the x domain, and are integrable over these curves. It is easily verified that the solutions of these problems are then given respectively, by the formulas

$$
\begin{aligned}
\mathfrak{u}(x, \lambda) &= \sum_{\mu=1}^{m} \int_{\eta_0}^{\eta_\mu} \mathfrak{G}(x, x_1, \lambda)\mathfrak{f}(x_1, \lambda)dx_1, \\
\mathfrak{v}(x, \lambda) &= -\sum_{\mu=1}^{m} \int_{\eta_0}^{\eta_\mu} \mathfrak{f}(x_1, \lambda)\mathfrak{G}(x_1, x, \lambda)dx_1.
\end{aligned}
\tag{10.10}
$$

This result is, of course, entirely familiar in its specialization to the case of a real variable with boundary conditions applying at just two points. It seems, however, to be more explicit and compact than any that has heretofore been given even for the case of a real variable, when the boundary conditions are taken to apply at intermediate points as well as at the end points of the interval.

11. **On the characteristic values when the boundary conditions apply in a fundamental region.** Returning to the discussion as it was left in §9, it will be observed that the deductions of that and the two preceding sections were in the main qualitative, or of a formal nature only. The derivation of more quantitative results requires as a basis some more specific assumptions than those which have heretofore been made. A consideration of the distribution of the characteristic values in the remote part of the λ region, which is now to be undertaken, is, therefore, to be based upon the following addition to the hypotheses of the discussion.

HYPOTHESIS (ii). (a) *The points $\eta_1, \eta_2, \cdots, \eta_m$, at which the boundary conditions apply, lie in some fundamental x region, while* (b) *in the part $|\lambda| > N$ of the relative suitable λ region, the matrices $\mathfrak{W}^{(h)}(\lambda)$, $(h=1, 2, \cdots, m)$, which define the boundary conditions, are analytic and admit of either actual or asymptotic representations of the form*

$$
\mathfrak{W}^{(h)}(\lambda) \sim \lambda^{\sigma} \sum_{k=0}^{\infty} \lambda^{-k}\mathfrak{W}^{(h,k)},
\tag{11.1}
$$

in which the matrices $\mathfrak{W}^{(h,k)}$ *are constant, and* σ *is an integer (positive, negative, or zero) such that* $\mathfrak{W}^{(h,0)} \neq \mathfrak{O}$ *for some index h.*

By the definition of a fundamental x region and the deductions of §6, the related suitable λ region may be covered by a finite number of subregions Λ, such that while λ remains in any one such, some solution of the matrix equation (2.1) maintains the form (6.10) for all x concerned. With the use of this solution $\mathfrak{Y}(x, \lambda)$, the formula (7.4) yields for the matrix $\mathfrak{D}(\lambda)$ the form

$$\mathfrak{D}(\lambda) = \left(\sum_{\mu=1}^{m} \sum_{\nu=1}^{n} w_{i,\nu}^{(\mu)}(\lambda) p_{\nu,j}(\eta_\mu, \lambda) e^{\lambda R_j(\eta_\mu)} \right). \tag{11.2}$$

The determinant $D(\lambda)$, when expanded, is, therefore, given by a formula

$$D(\lambda) = \sum_{\alpha} A_\alpha(\lambda) e^{\lambda \Omega_\alpha}, \tag{11.3}$$

in which

(a) the index α covers some finite range;

(b) the symbols Ω_α stand for distinct complex constants, which are all included in the set of values which may be obtained from the formula $\sum_{\nu=1}^{n} R_\nu(\eta_{\mu_\nu})$ by giving to each index μ_ν independently one of the values $1, 2, \cdots, m$;

(c) for the coefficient functions,

$$A_\alpha(\lambda) \not\equiv 0, \qquad \text{for each } \alpha. \tag{11.4}$$

The representability of the coefficient functions $A_\alpha(\lambda)$ in a form

$$A_\alpha(\lambda) \sim \lambda^{\rho_\alpha} \sum_{k=0}^{\infty} A_{\alpha,k} \lambda^{-k}, \tag{11.5}$$

follows from (11.1) and (6.11). In this the coefficients $A_{\alpha,k}$ are constants, and the exponents ρ_α are integers such that $A_{\alpha,0} \neq 0$, for each α.

The evaluation (11.3) thus obtained depends upon a choice of the solution $\mathfrak{Y}(x, \lambda)$ of the equation (2.1), and the result is valid for a subregion Λ since the form of the solution was specific to such a subregion. As has been previously observed, however, in §7, the determinants $D(\lambda)$ formed from different solutions $\mathfrak{Y}(x, \lambda)$ differ among each other only by factors which are nonvanishing. It may readily be inferred from this that the forms of any specific $D(\lambda)$, formed from a specific solution $\mathfrak{Y}(x, \lambda)$, in different subregions Λ differ from (11.3) at most by such factors. Thus the characteristic values in the entire originally given λ region are simply the zeros of (11.3), that is, the roots of the equation

$$\sum_{\alpha} A_\alpha(\lambda) e^{\lambda \Omega_\alpha} = 0. \tag{11.6}$$

The left-hand member of this equation may, depending upon the various elements involved in the boundary problem, consist of no terms at all, of a single term, or of more terms than one. The third of these possibilities is that of the greatest interest; the first two are readily disposed of. If the sum consists of no terms at all, the equation (11.6) is vacuous, and imposes no restriction at all upon λ. The boundary problem is accordingly compatible for all λ of the given region. From the formula (11.2) it may be observed that this case inevitably maintains whenever the rank of the matrix

$$\left\|\mathfrak{W}^{(1)}(\lambda), \mathfrak{W}^{(2)}(\lambda), \cdots, \mathfrak{W}^{(m)}(\lambda)\right\|$$

is less than n. Phrased relatively to the scalar differential system (1), this is merely the statement that the boundary problem is compatible for all λ if the independent boundary conditions are less than n in number. If the number of terms in (11.6) is just one, there are no characteristic values in the domain $|\lambda| > N$. The boundary problem is incompatible for all such λ.

If the left-hand member of (11.6) consists of two or more terms, it is functionally of a structure which is known as an *exponential sum*. The zeros of such a sum are discrete. Their distribution in the λ plane is known,* and may be briefly described as follows. In the complex plane let the points $\bar{\Omega}_\alpha$ (the complex conjugates of the Ω_α) be plotted, and let $\bar{P}$ designate the smallest convex polygon which contains them all in its interior or upon its perimeter. The characteristic values in the λ region in question are all located within a finite number of strips of that region, each strip being bounded by two curves which have asymptotes that are parallel to each other and normal to a side of the polygon $\bar{P}$. With each of these strips there is associated a pair of constants γ and δ, such that for any choices of $|\lambda_0|$ and Δ, the number of characteristic values which lie in the strip and between the arcs $|\lambda| = |\lambda_0|$, and $|\lambda| = |\lambda_0| + \Delta$, is between $\gamma\Delta - \delta$ and $\gamma\Delta + \delta$.

Part III. The representation of arbitrary vectors

12. **Further hypotheses; contours in the λ plane.** The considerations which have been set forth in the preceding sections have been based, insofar as they have depended upon the parameter, upon an assumption of the existence merely of some suitable λ region. The results have bearing, therefore, only relative to such regions, even though in specific instances these may constitute but minor portions of the entire λ plane. This does not suffice for the considerations with which the discussion is to continue. For these it is essential, rather, that some qualitative facts be available for all values of λ,

* Cf. R. E. Langer, *On the zeros of exponential sums and integrals*, Bulletin of the American Mathematical Society, vol. 37 (1931), p. 213.

and that quantitative results be generally applicable to the entire remote portion of the plane, that is, for $|\lambda| > N$. To insure this, the basis of hypotheses must be enlarged, and this is to be done by addition of the following:

HYPOTHESIS (iii). (a) *The differential matrix equation* (7.1a) *is one for which the entire* λ *plane is a suitable region*; (b) *the points* $\eta_1, \eta_2, \cdots, \eta_m$, *at which the boundary conditions apply, lie in an* x *region which is fundamental relative to the entire* λ *plane*; (c) *the elements of the matrices* $\mathfrak{W}^{(h)}(\lambda)$, $(h=1, 2, \cdots, m)$, *of* (7.1b) *are rational functions of* λ; (d) *the boundary conditions* (7.1b) *are such that the expression* (11.3) *for the determinant* $D(\lambda)$ *consists of at least two terms.*

Several observations are in order with respect to this hypothesis. To begin with, it will be noted that by virtue of part (a) the further discussion will be restricted to boundary problems of the type (7.1) in which the coefficient matrix $\mathfrak{Q}$ of the differential equation does not involve the parameter λ. This follows from the fact that as a function of λ this matrix has been restricted to be both analytic and bounded over the entire complex plane. In connection with part (c) of the hypothesis, it will be noted that under it the matrices $\mathfrak{W}^{(h)}(\lambda)$ may without any loss of generality be taken to be polynomials in λ. This inference follows from the fact observed in §7 that the matrices $\mathfrak{W}^{(h)}(\lambda)$ may at will be replaced by $\mathfrak{C}_1(\lambda)\mathfrak{W}^{(h)}(\lambda)$ without thereby affecting the content of the boundary problem. The matrix $\mathfrak{C}_1(\lambda)$ can, however, be chosen so that the elements are polynomials in λ, and such as to remove all poles which the elements of the matrices $\mathfrak{W}^{(h)}(\lambda)$ may have in points of the finite λ plane, that is, such that the elements of the matrices $\mathfrak{C}_1(\lambda)\mathfrak{W}^{(h)}(\lambda)$ are integral rational functions of λ. It may be assumed in virtue of this, and it will henceforth be assumed, that such a formal adjustment has been made, and that, therefore, the formulas (11.1) are hereinafter superseded by

$$\mathfrak{W}^{(h)}(\lambda) = \lambda^{\sigma} \sum_{k=0}^{\sigma} \lambda^{-k} \mathfrak{W}^{(h,k)}, \qquad h = 1, 2, \cdots, m, \tag{12.1}$$

the matrices $\mathfrak{W}^{(h,k)}$ being still constant, and σ being now a nonnegative integer such that $\mathfrak{W}^{(h,0)} \neq \mathfrak{O}$ for at least one index value h.

Under Hypothesis (iii) the matrix $\mathfrak{D}(\lambda)$ is analytic over the entire λ plane. The determinant $D(\lambda)$ is, therefore, likewise analytic; hence its zeros, that is, the characteristic values, in any bounded region of the λ plane are finite in number. This applies in particular to the region $|\lambda| \leqq N$, whatever the constant N may be. Since for an appropriately large value of N the distribution of the characteristic values in the domain $|\lambda| > N$ is, by virtue of part (d) of the hypothesis, such as is obtained by applying the results of §11 to the whole λ plane, it is seen, in particular, that these values have no finite limit point. They are, therefore, enumerable, and may, in particular, be so enumerated that $|\lambda_1| \leqq |\lambda_2| \leqq |\lambda_3| \leqq \cdots$. It will be assumed in the following that such an enumeration has been made and will be retained.

Because the characteristic values in the region $|\lambda| > N$ lie in a finite number of strips of the plane, and their densities in these strips are bounded, as was remarked at the end of §11, it is possible to draw in the λ plane certain closed contours which encircle the origin, pass through no characteristic value, and coincide with circles on which $|\lambda|$ is constant, except possibly where they traverse the strips containing the characteristic values. There exists, moreover, an unending sequence of such contours, of which each encloses its predecessor in the sequence, and such that no one of the sequence passes within less than some specifiable positive distance of any characteristic value. Every such contour encloses, of course, only a finite number of characteristic values. If the contours are designated by Γ_κ, with the index κ so assigned as to denote the number of characteristic values enclosed, the following can be shown. There exists a sequence of simple closed contours Γ_κ as partially described above, for which

(a) the sequence of index values κ is an unbounded increasing sequence of positive integers;

(b) the ratio of κ to the shortest distance from the origin to the contour Γ_κ is bounded;

(c) the ratio of the length of the contour Γ_κ to κ is bounded.

If ρ is used to designate the smallest of the integers ρ_α which occurs in the formulas (11.5) and for which $\overline{\Omega}_\alpha$ is one of the vertices of the polygon $\overline{P}$ described in §11, the function

$$\lambda^{-\rho} D(\lambda) e^{-\lambda \Omega_\alpha} \tag{12.2}$$

is an exponential sum whose coefficients are each asymptotic to some nonnegative power of λ. The zeros of this sum, moreover, are simply the characteristic values. Now it is known of such sums* that they remain uniformly bounded from zero for all values of the variable which are uniformly bounded from the roots of the sum. Since λ is so bounded from the characteristic values when it is restricted to vary over the contours of the set Γ_κ as described above, it must be inferred that for λ on such a set of contours the reciprocals of all the functions (12.2) are bounded.

13. **The generalized relation of biorthogonality.** As has been variously remarked above, and particularly in §7, the matrix $\mathfrak{D}(\lambda)$ is not uniquely determined by the boundary problem, it being in fact a mere matter of adjustment to replace any specific $\mathfrak{D}(\lambda)$ by the matrix (7.6) with any pre-

* Cf. R. E. Langer, *The asymptotic location of the roots of a certain transcendental equation*, these Transactions, vol. 31 (1929), p. 837.

scribed nonsingular analytic matrices $\mathfrak{C}_1$ and $\mathfrak{C}_2$. Deductions which are intrinsic to the boundary problem are, of course, independent of such adjustment. Their derivation may, however, be simpler with a fortunate adjustment than with a contrary one, and this is the case in the discussions of the present and the following sections. There are, in other words, advantages of simplicity to be gained by a suitable normalization of the matrix $\mathfrak{D}(\lambda)$.

Let λ_β be any characteristic value, and for generality let its multiplicity be denoted by s. Whatever the adjustment of the problem, the matrix $\mathfrak{D}(\lambda)$, being analytic, has elements which are expansible in power series in $(\lambda-\lambda_\beta)$. The initial segments of these series, extending to the terms in $(\lambda-\lambda_\beta)^s$, are polynomials in $(\lambda-\lambda_\beta)$ of degree s. If their matrix is designated by $\mathfrak{P}(\lambda)$, it is clear that

$$\mathfrak{D}(\lambda) \equiv \mathfrak{P}(\lambda) + (\lambda - \lambda_\beta)^{s+1}\mathfrak{D}^{(1)}(\lambda), \tag{13.1}$$

with $\mathfrak{D}^{(1)}(\lambda)$ analytic at the value λ_β. Now since $\mathfrak{P}$ is a polynomial matrix, it will, under a suitable adjustment in the sense above, appear in its canonical form

$$\mathfrak{P}(\lambda) \equiv (\delta_{i,j}p_i(\lambda)), \tag{13.2}$$

in which each element p_i is a polynomial in $(\lambda-\lambda_\beta)$, with unity as the coefficient of the lowest power of $(\lambda-\lambda_\beta)$ which is actually present, and each element p_i a factor of its successor p_{i+1}.* The adjustment of the problem for which (13.1) and (13.2) obtain will be assumed throughout the immediately following discussion. The matrix $\mathfrak{Y}(x, \lambda)$ is thereby in part determined.

For generality let it be assumed now that the index of the characteristic value λ_β is r. Then $(n-r)$ is the rank of the matrix $\mathfrak{D}(\lambda_\beta)$, and since it is clear from (13.1) that $\mathfrak{P}(\lambda_\beta)$ is of the same rank, it follows that

$$p_i(\lambda_\beta) = \begin{cases} 1, & \text{for } i \leqq n - r, \\ 0, & \text{for } i > n - r. \end{cases} \tag{13.3}$$

Since the zero of $D(\lambda)$ at λ_β is of the same multiplicity as the zero of the determinant of $\mathfrak{P}(\lambda)$, that is, of $\prod_{i=1}^{n} p_i(\lambda)$, whereas each of the factors $p_i(\lambda)$, $(i=n-r+1, \cdots, n)$, has a zero at λ_β, it is clear that the multiplicity s of this value is at least as great as its index r, a fact which was stated in §7. Now due to the rank of $\mathfrak{D}(\lambda_\beta)$ there are precisely r linearly independent vectors $\mathfrak{c}$ which satisfy the equation (7.3) at λ_β, and each of these leads through (7.2) to a characteristic solution of the boundary problem. However, since all elements in the last r columns of $\mathfrak{D}(\lambda_\beta)$ are zero, it is seen at once that each of the

* Cf. M. Bôcher, *Introduction to Higher Algebra.*

vectors which, with j fixed at one of the values $n-r+1, \cdots, n$, has the components $\delta_{i,j}$, does serve as a solution $\mathfrak{c}$ of the equation (7.3). The formula (7.2), therefore, gives as characteristic solutions

$$\mathfrak{y}^{(k,\beta)}(x) \equiv \mathfrak{Y}(x, \lambda_\beta)(\delta_{i,n-r+k}), \qquad k = 1, 2, \cdots, r, \tag{13.4}$$

and these solutions are thus seen to be given precisely by the last r columns of the matrix $\mathfrak{Y}(x, \lambda_\beta)$.

Again, at λ_β there are precisely r linearly independent vectors $\mathfrak{a}$ which solve the equation (8.4), and due to the fact that all elements in the last r rows of the matrix $\mathfrak{D}(\lambda_\beta)$ are zeros, it is clear that with i fixed at any one of the values $n-r+1, \cdots, n$, the vector with components $\delta_{i,j}$ is such a one. Through the formulas (8.3) and (8.2), it follows then that the formulas

$$\begin{aligned} \mathfrak{z}^{(k,\beta,h)}(x) \equiv -(\delta_{n-r+k,j})\mathfrak{W}^{(h)}(\lambda_\beta)\mathfrak{Y}(\eta_h, \lambda_\beta)\mathfrak{Y}^{-1}(x, \lambda_\beta), \\ h = 1, 2, \cdots, m, \end{aligned} \tag{13.5}$$

yield, for each k on the range $1, 2, \cdots, r$, a characteristic solution of the problem (8.1). These solutions are thus given by the last r rows of the matrices

$$-\mathfrak{W}^{(h)}(\lambda_\beta)\mathfrak{Y}(\eta_h, \lambda_\beta)\mathfrak{Y}^{-1}(x, \lambda_\beta), \qquad h = 1, 2, \cdots, m. \tag{13.6}$$

Let λ be regarded now as distinct from λ_β, and let $\mathfrak{z}^{(k,\beta,h)}(x)$ be any one of the characteristic solutions (13.5). The obvious relation

$$\begin{aligned} \sum_{\mu=1}^{m} \int_{\eta_0}^{\eta_\mu} \left\{ \mathfrak{z}^{(k,\beta,h)}(x)\mathfrak{Y}'(x, \lambda) + \mathfrak{z}^{(k,\beta,h)\prime}(x)\mathfrak{Y}(x, \lambda) \right\} dx \\ = \sum_{\mu=1}^{m} \left\{ \mathfrak{z}^{(k,\beta,h)}(\eta_\mu)\mathfrak{Y}(\eta_\mu, \lambda) - \mathfrak{z}^{(k,\beta,h)}(\eta_0)\mathfrak{Y}(\eta_0, \lambda) \right\}, \end{aligned}$$

assumes, then, because of the relations (8.1) and (7.1a), the form

$$\begin{aligned} (\lambda - \lambda_\beta) \sum_{\mu=1}^{m} \int_{\eta_0}^{\eta_\mu} \mathfrak{z}^{(k,\beta,h)}(x_1)\mathfrak{R}(x_1)\mathfrak{Y}(x_1, \lambda)dx_1 = -\mathfrak{a}^{(k,\beta)} \sum_{\mu=1}^{m} \mathfrak{W}^{(\mu)}(\lambda_\beta)\mathfrak{Y}(\eta_\mu, \lambda), \\ \mathfrak{a}^{(k,\beta)} = (\delta_{n-r+k,j}). \end{aligned} \tag{13.7}$$

Now since the matrices $\mathfrak{W}^{(\mu)}(\lambda)$ are polynomials of degree σ, as shown by (12.1), it is clear that with an arbitrary choice of $(\tau+1)$ as a nonnegative integer, the left-hand member of the formula

$$\left(\frac{\lambda}{\lambda_\beta}\right)^{\tau+1} \left\{ \mathfrak{W}^{(\mu)}(\lambda) - \mathfrak{W}^{(\mu)}(\lambda_\beta) \right\} = (\lambda - \lambda_\beta) \sum_{h=\tau+1}^{\tau+\sigma} \lambda^h \mathfrak{V}^{(\mu,h)}(\lambda_\beta) \tag{13.8}$$

is a polynomial in λ which vanishes at λ_β. Its structure is, therefore, such as is shown on the right of (13.8), and this relation may be looked upon as defining the matrices

$$\mathfrak{V}^{(\mu,h)}(\lambda), \qquad h = \tau + 1, \tau + 2, \cdots, \tau + \sigma.$$

It is likewise seen that

$$\left\{\left(\frac{\lambda}{\lambda_\beta}\right)^{\tau+1} - 1\right\}\mathfrak{W}^{(\mu)}(\lambda_\beta) = (\lambda - \lambda_\beta)\sum_{h=0}^{\tau} \lambda^h \mathfrak{V}^{(\mu,h)}(\lambda_\beta), \tag{13.9}$$

with

$$\mathfrak{V}^{(\mu,h)}(\lambda) \equiv \lambda^{-h-1}\mathfrak{W}^{(\mu)}(\lambda), \qquad h = 0, 1, \cdots, \tau. \tag{13.10}$$

If the relations (13.8) and (13.9) are added, and the sum is multiplied on the right by $\mathfrak{Y}(\eta_\mu, \lambda)$, it is found, on recalling (7.4) that

$$\left(\frac{\lambda}{\lambda_\beta}\right)^{\tau+1}\mathfrak{D}(\lambda) - \sum_{\mu=1}^{m} \mathfrak{W}^{(\mu)}(\lambda_\beta)\mathfrak{Y}(\eta_\mu, \lambda) = (\lambda - \lambda_\beta)\sum_{\mu=1}^{m}\sum_{h=0}^{\tau+\sigma} \mathfrak{V}^{(\mu,h)}(\lambda_\beta)\lambda^h\mathfrak{Y}(\eta_\mu, \lambda).$$

In virtue of this the relation (13.7) may be written

$$\begin{aligned}\sum_{\mu=1}^{m}\left\{-\int_{\eta_0}^{\eta_\mu} \mathfrak{z}^{(k,\beta,\mu)}(x_1)\mathfrak{R}(x_1)\mathfrak{Y}(x_1, \lambda)dx_1 + \sum_{h=0}^{\tau+\sigma} \mathfrak{a}^{(k,\beta)}\mathfrak{V}^{(\mu,h)}(\lambda_\beta)\lambda^h\mathfrak{Y}(\eta_\mu, \lambda)\right\} \\ = \left(\frac{\lambda}{\lambda_\beta}\right)^{\tau+1} \frac{1}{\lambda - \lambda_\beta}\mathfrak{a}^{(k,\beta)}\mathfrak{D}(\lambda).\end{aligned} \tag{13.11}$$

Let λ be taken now as any characteristic value, say λ_γ, distinct from λ_β, and let $\mathfrak{y}^{(q,\gamma)}(x)$ be any corresponding characteristic solution of the problem (7.1). There exists then a vector $\mathfrak{c}$ which satisfies the equation (7.3), and for which the left-hand member of (7.2) is $\mathfrak{y}^{(q,\gamma)}(x)$. If the relation (13.11) is multiplied by this vector $\mathfrak{c}$ upon the right it is found as a result that

$$\begin{aligned}\sum_{\mu=1}^{m}\left\{-\int_{\eta_0}^{\eta_\mu} \mathfrak{z}^{(k,\beta,\mu)}(x_1)\mathfrak{R}(x_1)\mathfrak{y}^{(q,\gamma)}(x_1)dx_1\right. \\ \left. + \sum_{h=0}^{\tau+\sigma} \mathfrak{a}^{(k,\beta)}\mathfrak{V}^{(\mu,h)}(\lambda_\beta)\lambda_\gamma{}^h\mathfrak{y}^{(q,\gamma)}(\eta_\mu)\right\} = 0.\end{aligned} \tag{13.12}$$

The vector $\mathfrak{a}^{(k,\beta)}\mathfrak{D}(\lambda)$ which occurs on the right of (13.11) is represented simply by the $(n-r+k)$th row in the matrix $\mathfrak{D}(\lambda)$. Every element of this row has a zero at λ_β. The right-hand member of (13.11) is, therefore, analytic at λ_β if properly defined there. Consider now the case in which the index r of the value λ_β equals its multiplicity s. Since the zero of the product of r factors $\prod_{i=n-r+1}^{n} p_i(\lambda)$ is precisely of the multiplicity r, each factor has a zero of precisely the first order, that is, each of the elements $p_i(\lambda)$ of (13.2) for which $i > n-r$ is a polynomial in $(\lambda - \lambda_\beta)$ of which the term of lowest degree is precisely $(\lambda - \lambda_\beta)$. It is clear from this, in virtue of (13.1), that

$$\lim_{\lambda \to \lambda_\beta} \frac{1}{\lambda - \lambda_\beta} \mathfrak{a}^{(k,\beta)}\mathfrak{D}(\lambda) = \mathfrak{a}^{(k,\beta)}. \tag{13.13}$$

Since

$$\mathfrak{a}^{(k,\beta)}(\delta_{i,n-r+q}) = \delta_{k,q},$$

it is seen that if the relation (13.11) is multiplied on the right by the vector $(\delta_{i,n-r+q})$ with $q=1, 2, \cdots, r$, and λ is allowed to approach λ_β, the limiting form is, in virtue of (13.4),

$$\sum_{\mu=1}^{m}\left\{-\int_{\eta_0}^{\eta_\mu} \mathfrak{z}^{(k,\beta,\mu)}(x_1)\mathfrak{R}(x_1)\mathfrak{y}^{(q,\beta)}(x_1)dx_1 + \sum_{h=0}^{\tau+\sigma} \mathfrak{a}^{(k,\beta)}\mathfrak{V}^{(\mu,h)}(\lambda_\beta)\lambda_\beta{}^h\mathfrak{y}^{(q,\beta)}(\eta_\mu)\right\} = \delta_{k,q}. \tag{13.14}$$

This deduction does not follow if the multiplicity s of the value λ_β exceeds its index r. For, in that case at least one of the elements $p_i(\lambda)$, with $i>n-r$, has at λ_β a zero of order higher than the first. For at least one value of k, therefore, the left-hand member of the relation (13.13) is zero. The relation (13.14) is, therefore, invalid, its left-hand member being zero irrespective of q when k has certain values.

The results of this section, as involved in the formulas (13.12) and (13.14) may be formulated as follows: If λ_β is any characteristic value whose index equals its multiplicity, then

$$\sum_{\mu=1}^{m}\left\{-\int_{\eta_0}^{\eta_\mu} \mathfrak{z}^{(k,\beta,\mu)}(x_1)\mathfrak{R}(x_1)\mathfrak{y}^{(q,\gamma)}(x_1)dx_1 + \sum_{h=0}^{\tau+\sigma} (\delta_{n-r+k,j})\mathfrak{V}^{(\mu,h)}(\lambda_\beta)\lambda_\gamma{}^h\mathfrak{y}^{(q,\gamma)}(\eta_\mu)\right\} = \delta_{\beta,\gamma}\cdot\delta_{k,q}, \tag{13.15}$$

where r is the index of λ_β, and τ is any integer not less than -1. If the multiplicity of λ_β exceeds its index, the relation (13.15) fails for at least one value of k, the right-hand member of the relation for such k being 0 for all γ and q.

This result must evidently be looked upon as the generalization of the relation of ordinary or weighted biorthogonality which is familiarly a property of the set of characteristic solutions of adjoint boundary problems in the classical specialized cases. Evidently the set may be normalized, in the sense that for $\gamma=\beta$ and $q=k$ the right-hand member of (13.15) is unity, whenever the characteristic values all have indices equal to their multiplicities, whereas complete normalization is impossible when this condition is not fulfilled.

14. **The residues of the Green's matrices.** The matrix $\mathfrak{D}^{-1}(\lambda)$, as has been observed, is analytic over the λ plane except at the characteristic values, where

it has poles. It is appropriate at this point to turn the considerations to the deduction of the residues of the Green's matrices (9.6) at these characteristic values. For this purpose the residue of a matrix, say of $\mathfrak{G}^{(h)}(x, x_1, \lambda)$, at the pole λ_β, will be designated by the symbol $\operatorname{res}_\beta \mathfrak{G}^{(h)}(x, x_1)$. For convenience the choice of the matrix $\mathfrak{Y}(x, \lambda)$ and the adjustment of the boundary problem will be taken, as in the preceding section, to be such that the matrix $\mathfrak{D}(\lambda)$ is in the canonical form (13.1), (13.2). The discussion will be concerned with any characteristic value λ_β whose multiplicity and index are equal.

If the index of λ_β is r, the matrix $\mathfrak{D}(\lambda)$, as has been seen, has $(\lambda-\lambda_\beta)$ as a multiple factor of each element not upon its principal diagonal, while this function is a simple factor of the diagonal elements of the last r columns and is not a factor of the diagonal elements of the first $(n-r)$ columns. The coefficient of the lowest power of $(\lambda-\lambda_\beta)$ occurring in any diagonal element, it will be recalled, is unity. From this it is seen at once that with proper definition at λ_β, and in terms of the matrices (5.9), the matrix

$$\mathfrak{H}(\lambda) \equiv \mathfrak{D}(\lambda)\left\{ \sum_{l=1}^{n-r} \mathfrak{J}_{l,l} + \frac{1}{\lambda - \lambda_\beta} \sum_{l=n-r+1}^{n} \mathfrak{J}_{l,l} \right\} \tag{14.1}$$

is analytic and nonsingular at λ_β. Its elements are polynomials in $(\lambda-\lambda_\beta)$ of which the term of zero degree is precisely $\delta_{i,j}$. The formulas $\mathfrak{H}^{-1}(\lambda_\beta)=\mathfrak{J}$, and

$$\mathfrak{D}^{-1}(\lambda) \equiv \left\{ \sum_{l=1}^{n-r} \mathfrak{J}_{l,l} + \frac{1}{\lambda - \lambda_\beta} \sum_{l=n-r+1}^{n} \mathfrak{J}_{l,l} \right\} \mathfrak{H}^{-1}(\lambda), \tag{14.2}$$

lead directly to the result

$$\operatorname{res}_\beta \mathfrak{D}^{-1} = \sum_{l=n-r+1}^{n} \mathfrak{J}_{l,l}. \tag{14.3}$$

Now from the formula (9.6) and the fact that the poles of $\mathfrak{D}^{-1}(\lambda)$ as shown by (14.2) are of the first order, it follows that

$$\begin{gathered} \operatorname{res}_\beta \mathfrak{G}^{(h)}(x, x_1) = \mathfrak{Y}(x, \lambda_\beta)\left\{\operatorname{res}_\beta \mathfrak{D}^{-1}\right\}\mathfrak{W}^{(h)}(\lambda_\beta)\mathfrak{Y}(\eta_h, \lambda_\beta)\mathfrak{Y}^{-1}(x_1, \lambda_\beta), \\ h = 1, 2, \cdots, m. \end{gathered} \tag{14.4}$$

However, the formulas (13.4) and (13.5) yield readily the fact that for $l>n-r$

$$\mathfrak{Y}(x, \lambda_\beta)\mathfrak{J}_{l,l}\mathfrak{W}^{(h)}(\lambda_\beta)\mathfrak{Y}(\eta_h, \lambda_\beta)\mathfrak{Y}^{-1}(x_1, \lambda_\beta) \equiv -\left(y_i^{(l+r-n,\beta)}(x) z_j^{(l+r-n,\beta,h)}(x_1)\right),$$

in which the components of the characteristic vectors (13.4) and (13.5) have been designated, respectively, by $y_i^{(k,\beta)}(x)$, $(i=1, 2, \cdots, n)$ and $c_j^{(k,\beta,h)}(x_1)$, $(j=1, 2, \cdots, n)$. It follows, on substituting (14.3) into (14.4) that

$$\operatorname{res}_\beta \mathfrak{G}^{(h)}(x, x_1) \equiv -\sum_{k=1}^{r} \left(y_i^{(k,\beta)}(x) z_j^{(k,\beta,h)}(x_1)\right), \qquad h = 1, 2, \cdots, m. \tag{14.5}$$

The residues of the Green's matrices have thus been explicitly evaluated for all characteristic values whose multiplicities and indices are the same. This, of course, includes in particular all the simple characteristic values.

15. **The formal expansion of an arbitrary vector.** Let the consideration be turned now to an infinite series

$$\sum_{\gamma=1}^{\infty}\sum_{q=1}^{r_\gamma} f_{q,\gamma}\mathfrak{y}^{(q,\gamma)}(x), \tag{15.1}$$

in which r_γ is the index of the characteristic value λ_γ; the $\mathfrak{y}^{(q,\gamma)}(x)$ are characteristic solutions; the $f_{q,\gamma}$ are scalar constants; and x varies over some fundamental region of the x plane that contains the points $\eta_0, \eta_1, \cdots, \eta_m$, and in which the coefficient matrix $\mathfrak{R}(x)$ in the equation (7.1a) is nonsingular. The existence of such an x region is an assumption. If the coefficients $f_{q,\gamma}$ are such that the series converges uniformly (a tentative heuristic assumption), say to the vector $f(x)$, the term by term differentiation of the series is permissible, and with the use of the equation (7.1a) a process is evident which by repetition leads to the sequence of relations

$$\sum_{\gamma=1}^{\infty}\sum_{q=1}^{r_\gamma} f_{q,\gamma}\lambda_\gamma^{h}\mathfrak{y}^{(q,\gamma)}(x) = \mathfrak{f}^{(h)}(x), \qquad h = 0, 1, 2, \cdots, \tag{15.2}$$

in which

$$\mathfrak{f}^{(0)}(x) \equiv \mathfrak{f}(x), \quad \mathfrak{f}^{(h)}(x) \equiv \mathfrak{R}^{-1}(x)\left\{\mathfrak{f}^{(h-1)\prime}(x) - \mathfrak{Q}(x)\mathfrak{f}^{(h-1)}(x)\right\}, \quad h = 1, 2, \cdots. \tag{15.3}$$

In particular, it follows from this that

$$\sum_{\gamma=1}^{\infty}\sum_{q=1}^{r_\gamma} f_{q,\gamma}\lambda_\gamma^{h}\mathfrak{y}^{(q,\gamma)}(\eta_\mu) = \mathfrak{f}^{(\mu,h)}, \quad \mu = 1, 2, \cdots, m;\ h = 0, 1, 2, \cdots, \tag{15.4}$$

in which, evidently,

$$\mathfrak{f}^{(\mu,h)} = \mathfrak{f}^{(h)}(\eta_\mu). \tag{15.5}$$

By the relations (15.2) and (15.4) and the fact that the series involved are integrable term by term, it is seen, then, that

$$\begin{aligned}\sum_{\mu=1}^{m}\Big\{-\int_{\eta_0}^{\eta_\mu}\mathfrak{z}^{(k,\beta,\mu)}(x_1)\mathfrak{R}(x_1)\mathfrak{f}(x_1)dx_1 + \sum_{h=0}^{r+\sigma}(\delta_{n-r+k,j})\mathfrak{B}^{(\mu,h)}(\lambda_\beta)\mathfrak{f}^{(\mu,h)}\Big\} \\ = \sum_{\gamma=1}^{\infty}\sum_{q=1}^{r_\gamma} f_{q,\gamma}\sum_{\mu=1}^{m}\Big\{-\int_{\eta_0}^{\eta_\mu}\mathfrak{z}^{(k,\beta,\mu)}(x_1)\mathfrak{R}(x_1)\mathfrak{y}^{(q,\gamma)}(x_1)dx_1 \\ + \sum_{h=0}^{r+\sigma}(\delta_{n-r+k,j})\mathfrak{B}^{(\mu,h)}(\lambda_\beta)\lambda_\gamma^{h}\mathfrak{y}^{(q,\gamma)}(\eta_\mu)\Big\},\end{aligned}$$

for any choice of k and β. In this relation, however, the series on the right reduces to the single term given by $\gamma=\beta$ and $q=k$, in virtue of (13.15). The excepted term is by (13.15) precisely $f_{k,\beta}$, provided the index and multiplicity of the characteristic value λ_β are equal. If the index is less than the multiplicity, on the other hand, then for at least one value of k this excepted term is zero like the rest. In the former case the result is

$$
(15.6)\qquad
\begin{aligned}
f_{k,\beta} = \sum_{\mu=1}^{m}\Big\{ &-\int_{\eta_0}^{\eta_\mu} \mathfrak{z}^{(k,\beta,\mu)}(x_1)\mathfrak{N}(x_1)\mathfrak{f}(x_1)dx_1 \\
&+ \sum_{h=0}^{r+\sigma} (\delta_{n-r+k,j})\mathfrak{W}^{(\mu,h)}(\lambda_\beta)\mathfrak{f}^{(\mu,h)}\Big\},
\end{aligned}
$$

while in the latter case the scheme leads to no evaluation of $f_{k,\beta}$.

In the case of a boundary problem for which each characteristic value is of index equal to its multiplicity—and in the following deduction this case will be assumed—the series (15.1) is completely explicit in virtue of the formula (15.6). The terms of this series may be expressed as residues, as will now be shown.

If with l designating any nonnegative integer the formula (15.6) is multiplied on the right by $\lambda_\beta{}^l\mathfrak{y}^{(k,\beta)}(x)$, then in virtue of the relations

$$
\begin{aligned}
\mathfrak{z}^{(k,\beta,\mu)}(x_1)\mathfrak{N}(x_1)\mathfrak{f}(x_1)\mathfrak{y}^{(k,\beta)}(x) &\equiv (y_i^{(k,\beta)}(x)z_j^{(k,\beta,\mu)}(x_1))\mathfrak{N}(x_1)\mathfrak{f}(x_1), \\
(\delta_{n-r+k,j})\mathfrak{W}^{(\mu,h)}(\lambda_\beta)\mathfrak{f}^{(\mu,h)}\mathfrak{y}^{(k,\beta)}(x) &\equiv \mathfrak{Y}(x,\lambda_\beta)\mathfrak{J}_{n-r+k,n-r+k}\mathfrak{W}^{(\mu,h)}(\lambda_\beta)\mathfrak{f}^{(\mu,h)},
\end{aligned}
$$

the result obtained is

$$
\begin{aligned}
f_{k,\beta}\lambda_\beta{}^l\mathfrak{y}^{(k,\beta)}(x) \equiv \sum_{\mu=1}^{m}\lambda_\beta{}^l\Big\{ &-\int_{\eta_0}^{\eta_\mu}(y_i^{(k,\beta)}(x)z_j^{(k,\beta,\mu)}(x_1))\mathfrak{N}(x_1)\mathfrak{f}(x_1)dx_1 \\
&+ \mathfrak{Y}(x,\lambda_\beta)\mathfrak{J}_{n-r+k,n-r+k}\sum_{h=0}^{r+\sigma}\mathfrak{W}^{(\mu,h)}(\lambda_\beta)\mathfrak{f}^{(\mu,h)}\Big\}.
\end{aligned}
$$

Because of (14.5) and (14.3) this leads to

$$
(15.7)\qquad
\begin{aligned}
\sum_{k=1}^{r_\beta} f_{k,\beta}\lambda_\beta{}^l\mathfrak{y}^{(k,\beta)}(x) \equiv \mathrm{res}_\beta\sum_{\mu=1}^{m}\lambda^l\Big\{ &\int_{\eta_0}^{\eta_\mu}\mathfrak{G}^{(\mu)}(x,x_1,\lambda)\mathfrak{N}(x_1)\mathfrak{f}(x_1)dx_1 \\
&+ \mathfrak{Y}(x,\lambda)\mathfrak{D}^{-1}(\lambda)\sum_{h=0}^{r+\sigma}\mathfrak{W}^{(\mu,h)}(\lambda)\mathfrak{f}^{(\mu,h)}\Big\}.
\end{aligned}
$$

The series (15.1) may, therefore, be looked upon as an infinite series of residues which are contributed by poles at the characteristic values.

The deductions of this section thus far were based upon certain assumptions, that were signalized as tentative, concerning the coefficients of the series (15.1), and concerning the characteristic values. They were also based upon the formulas (15.5) and (15.3). Quite independently of the deduction given, however, and with any set of vectors

$$\mathfrak{f}(x), \quad \mathfrak{f}^{(\mu,h)}, \qquad h = 0, 1, 2, \cdots, \tau+\sigma; \mu = 1, 2, \cdots, m,$$

of which the first is analytic and the others constant, the right-hand member of (15.7) is specific. This is so, in particular, if the choice

$$\begin{aligned} \mathfrak{f}^{(\mu,h)} &= \mathfrak{f}^{(h)}(\eta_\mu), & h &= 0, 1, 2, \cdots, \tau, \\ \mathfrak{f}^{(\mu,h)} &= \mathfrak{o}, & h &= \tau+1, \cdots, \tau+\sigma, \end{aligned}$$

is made. In this case the series of right-hand members of (15.7) reduces, because of (13.10), and (9.7b) to

$$\tag{15.8} \begin{aligned} \mathfrak{s}^{(l)}(x) \equiv \sum_{\beta=0}^{\infty} \operatorname{res}_\beta \sum_{\mu=1}^{m} \lambda^l \Big\{ \int_{\eta_0}^{\eta_\mu} \mathfrak{G}^{(\mu)}(x, x_1, \lambda)\mathfrak{R}(x_1)\mathfrak{f}(x_1)dx_1 \\ + \mathfrak{G}^{(\mu)}(x, \eta_\mu, \lambda) \sum_{h=0}^{\tau} \lambda^{-h-1}\mathfrak{f}^{(h)}(\eta_\mu) \Big\}, \qquad l = 0, 1, 2, \cdots. \end{aligned}$$

The series of this set, $\mathfrak{s}^{(0)}(x)$, $\mathfrak{s}^{(1)}(x)$, and so on, will be referred to briefly hereinafter as the formal expansions of the vector $\mathfrak{f}(x)$. It will be observed that to specify them the integer τ must be given, for inasmuch as the chosen set of vectors depends upon τ, the result (15.8) does so likewise. Arrived at in this manner, the questions of convergence of the formal expansions, or of their values in the event of convergence, remain, of course, entirely open. The continuing discussion is designed to bear upon them.

In §12 an ordering of the characteristic values in an order of non-decreasing absolute magnitude was agreed upon, and the existence of the sequence of contours in the λ plane was deduced, the contour Γ_κ of this sequence enclosing the origin and precisely the first κ of the characteristic values. If the symbol res_0 in (15.8) is interpreted to signify the residue at the origin, it is at once clear that with any fixed l the vector

$$\tag{15.9} \begin{aligned} \mathfrak{s}_\kappa^{(l)}(x) = \frac{1}{2\pi i}\int_{\Gamma_\kappa} \sum_{\mu=1}^{m} \Big\{ \int_{\eta_0}^{\eta_\mu} \mathfrak{G}^{(\mu)}(x, x_1, \lambda)\mathfrak{R}(x_1)\mathfrak{f}(x_1)dx_1 \\ + \mathfrak{G}^{(\mu)}(x, \eta_\mu, \lambda) \sum_{h=0}^{\tau} \lambda^{-h-1}\mathfrak{f}^{(h)}(\eta_\mu) \Big\} \lambda^l d\lambda \end{aligned}$$

represents the sum of the first $\kappa+1$ terms of the respective series (15.8). The form of the right-hand member of (15.9) may be somewhat modified, with advantage to the analysis which is to be applied to it. Since the integrand is analytic, the several paths of integration as to x_1 may be chosen at pleasure, and hence may, in particular, be chosen to pass through the point x, and to coincide from the point η_0 to x. The integrations over this common path contribute to the formula (15.9) the value

$$\frac{1}{2\pi i}\int_{\Gamma_\kappa}\int_{\eta_0}^{x}\sum_{\mu=1}^{m}\mathfrak{G}^{(\mu)}(x, x_1, \lambda)\mathfrak{R}_1(x_1)\mathfrak{f}(x_1)dx_1\lambda^l d\lambda.$$

This, however, is zero, since by the relation (9.7a) the integrand is seen to be analytic everywhere within the contour Γ_k. The formula (15.9) may, therefore, be written alternatively as

$$\tag{15.10} \begin{aligned} \mathfrak{s}_k^{(l)}(x) = \frac{1}{2\pi i}\int_{\Gamma_\kappa}\sum_{\mu=1}^{m}\Big\{-\int_{\eta_\mu}^{x}\mathfrak{G}^{(\mu)}(x, x_1, \lambda)\mathfrak{R}(x_1)\mathfrak{f}(x_1)dx_1 \\ + \mathfrak{G}^{(\mu)}(x, \eta_\mu, \lambda)\sum_{h=0}^{\tau}\lambda^{-h-1}\mathfrak{f}^{(h)}(\eta_\mu)\Big\}\lambda^l d\lambda, \qquad l = 0, 1, 2, \cdots . \end{aligned}$$

16. **Regularity of a boundary problem.** Under Hypothesis (iii) of §12, there exists for the differential equation (7.1a) a fundamental x region relative to the entire λ plane, and this region contains the points $\eta_1, \eta_2, \cdots, \eta_m$, at which the boundary conditions of the problem (7.1) apply. The variable x has been taken in such a region. Hence the λ plane may be thought of as covered by a finite number of λ sectors, in each of which some solution $\mathfrak{Y}(x, \lambda)$ of the equation (7.1a) maintains the form (6.10), (6.11). When formed from that solution in the respective sector, the matrix $\mathfrak{D}(\lambda)$ has the form (11.2). From this the structure of the determinant $D(\lambda)$, or of any of its minors, may be deduced. The former has already been done in §11, the result (11.3) being valid under present hypotheses in the appropriate λ sector.

Consider then $D_{r,c}(\lambda)$, the cofactor of the element in the rth row and cth column of $D(\lambda)$. From the formula (11.2) this is seen to be, when completely expanded,

$$D_{r,c}(\lambda) = \sum_{\mu_1=1}^{m}\cdots\sum_{\mu_{c-1}=1}^{m}\sum_{\mu_{c+1}=1}^{m}\cdots\sum_{\mu_n=1}^{m} b_{c,r}(\lambda, \mu_1, \cdots, \mu_{c-1}, \mu_{c+1}, \cdots, \mu_n) \cdot\exp\Big\{\sum_{\substack{h=1\\h\neq c}}^{n} R_h(\eta_{\mu_h})\Big\},$$

in which $b_{c,r}(\lambda, \mu_1, \cdots, \mu_{c-1}, \mu_{c+1}, \cdots, \mu_n)$ is the cofactor of the (r, c)th element in the matrix

$$\Big(\sum_{\nu=1}^{n} w_{i,\nu}^{(\mu_j)}(\lambda)p_{\nu,j}(\eta_{\mu_j}, \lambda)\Big).$$

This expression when arranged as a simple sum is clearly of the form

$$(16.1)\qquad D_{r,c}(\lambda) = \sum_{\beta} b_{c,r}^{(\beta)}(\lambda)e^{\lambda\Phi_c^{(\beta)}},$$

in which the index β covers a finite range and the symbols $\Phi_c^{(\beta)}$ represent complex constants, the set of them with a specific subscript c being the set of values given by the expressions

$$\sum_{\substack{h=1\\h\neq c}}^{n} R_h(\eta_{\mu_h}),\qquad \begin{array}{r} \mu_1 = 1, 2, \cdots, m, \\ \cdots\cdots\cdots, \\ \mu_n = 1, 2, \cdots, m, \\ c = 1, 2, \cdots, n. \end{array}$$

The matrices $\mathfrak{B}^{(\beta)}(\lambda)$, namely $(b_{i,j}^{(\beta)}(\lambda))$, are clearly analytic, and when $|\lambda| > N$, admit of representations

$$(16.2)\qquad \mathfrak{B}^{(\beta)}(\lambda) \sim \lambda^{\rho'}\sum_{\gamma=1}^{\infty}\lambda^{-\gamma}\mathfrak{B}^{(\beta,\gamma)},$$

in which ρ' is an integer and the matrices $\mathfrak{B}^{(\beta,\gamma)}$ are constant.

The substitution of the result (16.1) into the formula (7.6) yields the evaluation

$$(16.3)\qquad \mathfrak{D}^{-1}(\lambda) = \frac{1}{D(\lambda)}\sum_{\beta}(\delta_{i,j}e^{\lambda\Phi_i^{(\beta)}})\mathfrak{B}^{(\beta)}(\lambda).$$

If this is multiplied on the left by

$$\mathfrak{Y}(x, \lambda) \equiv \sum_{l=1}^{n}\mathfrak{P}(x, \lambda)\mathfrak{J}_{l,l}e^{\lambda R_l(x)},$$

the evaluation being obtained from the formulas (6.10) and (5.9), and on the right by

$$\begin{aligned}&\mathfrak{W}^{(\mu)}(\lambda)\mathfrak{Y}(\eta_\mu, \lambda)\mathfrak{Y}^{-1}(x_1, \lambda)\\ &\qquad\equiv \sum_{k=1}^{n}\mathfrak{W}^{(\mu)}(\lambda)\mathfrak{P}(\eta_\mu, \lambda)\mathfrak{J}_{k,k}\mathfrak{P}^{-1}(x_1, \lambda)\exp\left[\lambda\{R_k(\eta_\mu) - R_k(x_1)\}\right],\end{aligned}$$

the result is

$$(16.4)\quad \mathfrak{G}^{(\mu)}(x, x_1, \lambda) = \sum_{\beta}\sum_{l,k=1}^{n}\mathfrak{H}_{\beta,l,k}^{(\mu)}(x, x_1, \lambda)\frac{\exp\left[\lambda\{\Psi_{\beta,l,k}^{(\mu)}(x) - R_k(x_1)\}\right]}{D(\lambda)},$$

in which

$$\Psi^{(\mu)}_{\beta,l,k}(x) \equiv \Phi^{(\beta)}_l + R_l(x) + R_k(\eta_\mu), \tag{16.5}$$

$$\mathfrak{H}^{(\mu)}_{\beta,l,k}(x, x_1, \lambda) \equiv \mathfrak{P}(x, \lambda)\mathfrak{J}_{l,l}\mathfrak{B}^{(\beta)}(\lambda)\mathfrak{W}^{(\mu)}(\lambda)\mathfrak{P}(\eta_\mu, \lambda)\mathfrak{J}_{k,k}\mathfrak{P}^{-1}(x_1, \lambda). \tag{16.6}$$

From (16.2), together with the formulas (6.11) and (12.1), it is seen that when $|\lambda| > N$, the matrices (16.6) admit of representations of a form

$$\mathfrak{H}^{(\mu)}_{\beta,l,k}(x, x_1, \lambda) \sim \lambda^\theta \sum_{\gamma=1}^{\infty} \lambda^{-\gamma} \mathfrak{H}^{(\mu,\gamma)}_{\beta,l,k}(x, x_1), \tag{16.7}$$

in which θ is an integer and the matrices on the right are analytic in x and x_1.

The formula (16.4) was derived on the assumption that λ remains in a sector of the λ plane. However, since the Green's matrices are independent of the choice of the solution $\mathfrak{Y}(x, \lambda)$ from which they are formed, the result is independent of the sector, that is, the formula is valid for all λ.

If x is thought of now as fixed, and x_1 is taken as the variable, it is conceivable that for some choices of μ and k the values given by (16.5) with different indices β, l may not all be distinct. In such case the same exponential occurs in different terms of certain of the sums of (16.4), and a simplification of the respective formulas is achievable by collecting such terms, and omitting from the results any such collections of terms of which the resultant coefficients reduce to the matrix $\mathfrak{O}$.

Let ζ be taken as a complex variable, and in the plane of ζ let the points Ω_α which are defined by the formula (11.3) be plotted. Then let P designate the smallest convex polygon in the ζ plane, which contains all of these points in its interior or upon its perimeter. For any chosen and fixed value of x, the relations

$$\zeta = \Psi^{(\mu)}_{\beta,l,k}(x) - R_k(x_1) \tag{16.8}$$

define, for each set of indices μ, β, l, k, an analytic map of any configuration in the x_1 plane upon a corresponding configuration in the ζ plane. This latter may or may not in any specific instance fall into the interior of the polygon P, and since x enters into the definition of the transformation in the role of a parameter, this will depend to some extent upon the value x which is in question. With this in mind, the following will be made as a definition.

A boundary problem (7.1) *will be defined to be regular as to the point x if* (a) *the matrix $\mathfrak{R}(x)$ is nonsingular, and if* (b) *for each set of indices μ, β, l, k, to which there corresponds a term of the* (*simplified*) *sums* (16.4), *there exists in the fundamental region which is the domain of the variable x_1, some curve joining the point η_μ with the point x which maps under the respective transformation* (16.8) *into a locus no point of which lies outside of the polygon P.*

The condition (a) for regularity is obviously fulfilled at all points of some neighborhood of any point at which it holds, due to the analyticity of the matrix $\mathfrak{R}(x)$. On the other hand, the condition (b) may apparently be fulfilled relative to a point but not relative to neighboring points. This would be so in the case that its fulfillment at x is ascribable to a simplification of the sums in (16.4); for such simplifications are evidently possible only for isolated x values. In suitable cases, however, the condition (b) also may be fulfilled relative to all points of a region. We therefore agree that:

A boundary problem (7.1) *is to be designated as regular as to a region of the plane if it is regular as to each point of that region.*

17. **The convergence of the formal expansions at points of regularity.** If x is taken as a point relative to which the boundary problem is regular, the arbitrarily chosen analytic vector $\mathfrak{f}(x)$ has associated with it the set of vectors $\mathfrak{f}^{(h)}(x)$, $(h=1, 2, \cdots)$, given by the recurrence formula (15.3). Since the Green's matrices as functions of x_1 all satisfy the differential equation (8.1a) it is easily verified that the relations

$$(17.1)\quad \begin{aligned} \mathfrak{G}^{(\mu)}(x, x_1, \lambda)\mathfrak{R}(x_1)\mathfrak{f}(x_1) \equiv & -\frac{\partial}{\partial x_1}\left\{\mathfrak{G}^{(\mu)}(x, x_1, \lambda)\sum_{h=0}^{\tau}\lambda^{-h-1}\mathfrak{f}^{(h)}(x_1)\right\} \\ & + \lambda^{-\tau-1}\mathfrak{G}^{(\mu)}(x, x_1, \lambda)\mathfrak{R}(x_1)\mathfrak{f}^{(\tau+1)}(x_1) \end{aligned}$$

are identities, and are valid with any choice of τ as a nonnegative integer. They lead, with the use of the formula (9.7a), to the evaluation

$$(17.2)\quad \begin{aligned} & \sum_{\mu=1}^{m}\left\{-\int_{\eta_\mu}^{x}\mathfrak{G}^{(\mu)}(x, x_1, \lambda)\mathfrak{R}(x_1)\mathfrak{f}(x_1)dx_1\right. \\ & = \sum_{h=0}^{\tau}\lambda^{-h-1}\mathfrak{f}^{(h)}(x) - \sum_{\mu=1}^{m}\left\{\mathfrak{G}^{(\mu)}(x, \eta_\mu, \lambda)\sum_{h=0}^{\tau}\lambda^{-h-1}\mathfrak{f}^{(h)}(\eta_\mu)\right. \\ & \left. + \int_{\eta_\mu}^{x}\lambda^{-\tau-1}\mathfrak{G}^{(\mu)}(x, x_1, \lambda)\mathfrak{R}(x_1)\mathfrak{f}^{(\tau+1)}(x_1)dx_1\right\}. \end{aligned}$$

If in this relation the index τ is chosen to coincide with that of the formal expansions (15.10), and in these latter the evaluations (17.2) are substituted, the formulas reduce to

$$(17.3)\quad \begin{aligned} \mathfrak{s}_\kappa^{(l)}(x) = & \frac{1}{2\pi i}\int_{\Gamma_\kappa}\sum_{h=0}^{\tau}\mathfrak{f}^{(h)}(x)\frac{d\lambda}{\lambda^{h-l+1}} \\ & - \frac{1}{2\pi i}\int_{\Gamma_\kappa}\sum_{\mu=1}^{m}\int_{\eta_\mu}^{x}\lambda^{l-\tau-1}\mathfrak{G}^{(\mu)}(x, x_1, \lambda)\mathfrak{R}(x_1)\mathfrak{f}^{(\tau+1)}(x_1)dx_1 d\lambda. \end{aligned}$$

Consider the final member on the right of this relation. By the formula (16.4) its integrand consists of a finite number of terms, of which

$$(17.4) \qquad \int_{\eta_\mu}^{x} \overset{(\mu)}{\mathfrak{H}}_{\beta,l,k}(x, x_1, \lambda) \frac{\lambda^{l-\tau-1} \exp\left[\lambda\{\overset{(\mu)}{\Psi}_{\beta,l,k}(x) - R_k(x_1)\}\right]}{D(\lambda)} d\lambda$$

is a typical one. By the conditions of regularity of the boundary problem as to the chosen x, there exists in the x_1 plane a curve which may be taken as the path of integration in (17.4), and which maps under the transformation (16.7) upon a locus no point of which lies outside of the polygon P. Inasmuch as the vertices of the polygon P all lie by construction at points of the set Ω_α, it follows that whatever λ may be, there corresponds to it a choice of the index α such that the real part of $\lambda\Omega_\alpha$ at least equals the real part of the exponent in (17.4), that is, such that

$$\left| \exp\left[\lambda\{\overset{(\mu)}{\Psi}_{\beta,l,k}(x) - R_k(x_1)\} - \lambda\Omega_\alpha\right] \right| \leqq 1$$

uniformly in x_1.

Now it was observed in §12 that the reciprocals of the expressions (12.2) for all choices of α are bounded when λ is restricted to the contours of a set Γ_κ, as may be assumed in the present discussion. The scalar factor in the integrand of (17.4) is, therefore, of the order of the $(l-\tau-\rho-1)$th power of λ when $|\lambda| > N$. In virtue of the relation (16.7), the order of the entire integrand exceeds that by no more than the θth power of λ, and since the result is uniform as to x_1 on the path of integration, that is true for the integral itself. Thus the final member of the relation (17.3) calls for the integral over the contour Γ_κ of a function which is of the order of λ to the power $(l+\theta-\rho-1-\tau)$. Of the integers θ, ρ, l, τ, which thus come into question, the first two are determined by the boundary problem, and the third is merely indicative of which of the expansions (15.8) is in question. The integer τ, though it is definitive for the formal expansions under consideration, has thitherto remained unspecified. Let it be chosen now as nonnegative and at least equal to the integer $(\theta-\rho+1)$, and let the larger of the numbers τ and $\tau-(\theta-\rho+1)$ be denoted by l_1. Then for any index l such that $l \leqq l_1$, the first member on the right of the relation (17.3), which is directly integrable, has the value $\mathfrak{f}^{(l)}(x)$. The second member, having an integrand of at most the order of λ^{-2}, converges to zero as $\kappa \to \infty$, by virtue of the configurations of the contours Γ_κ. Thus at the point x,

$$(17.5) \qquad \lim_{\kappa\to\infty} \mathfrak{s}_\kappa^{(l)}(x) = \mathfrak{f}^{(l)}(x), \qquad l = 0, 1, 2, \cdots, l_1,$$

and, since the value $l=0$ is at all events included, the formal expansion (15.8) of the chosen vector $\mathfrak{f}(x)$ itself converges to this vector.

If, by the choice of τ, the case is one in which $l_1 \geqq 1$, the series obtained by the term by term differentiation of $\mathfrak{s}^{(l)}(x)$, with $l<l_1$, is found to be identical with the series $\mathfrak{R}(x)\mathfrak{s}^{(l+1)}(x)+\mathfrak{Q}(x)\mathfrak{s}^{(l)}(x)$, and, since this converges, to have the value $\mathfrak{R}(x)\mathfrak{f}^{(l+1)}(x)+\mathfrak{Q}(x)\mathfrak{f}^{(l)}(x)$, a value which by (15.3) reduces to $\mathfrak{f}^{(l)\prime}(x)$. This follows from the fact, which was observed in §13, that the residues of the Green's matrices involved in the terms of $\mathfrak{s}^{(l)}(x)$ satisfy the differential equation (7.1a). Thus every expansion (15.8) for which $l<l_1$ is differentiable term by term at the point x, and by iteration it is seen at once that the expansion for the vector $\mathfrak{f}(x)$ itself admits of term by term differentiation to the order l_1.

Throughout the foregoing discussion it has been assumed only that the boundary problem is regular as to the point x. If it is assumed now that the problem is regular as to a connected region of the x plane, and x is taken in this region, it will be verified without difficulty that the results of the discussion are at each stage valid uniformly as to x. The convergence of the formal expansions indicated by (17.5) is thus uniform, and this applies, in particular, to the expansion of $\mathfrak{f}(x)$ itself if τ is merely chosen as the larger of the numbers 0 and $(\theta-\rho+1)$. Because of the uniformity of the convergence, the term by term differentiability of the expansion necessarily follows in this case, and, by a reversal of the reasoning employed above, it may be inferred therefrom that, with the index τ which was fixed upon, the expansions (15.8), for all l, converge uniformly to the respective vectors $\mathfrak{f}^{(l)}(x)$.

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