NON-COMMUTATIVE RESIDUATED LATTICES*

BY

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Introduction and summary. In the theory of non-commutative rings certain distinguished subrings, one-sided and two-sided ideals, play the important roles. Ideals combine under crosscut, union and multiplication and hence are an instance of a lattice over which a non-commutative multiplication is defined.† The investigation of such lattices was begun by W. Krull (Krull [3]) who discussed decomposition into isolated component ideals. Our aim in this paper differs from that of Krull in that we shall be particularly interested in the lattice structure of these domains although certain related arithmetical questions are discussed.

In Part I the properties of non-commutative multiplication and residuation over a lattice are developed. In particular it is shown that under certain general conditions each operation may be defined in terms of the other.

The second division of the paper deals with the structure of non-commutative residuated lattices in the vicinity of the unit element. It is found that this structure may be characterized to a large extent in terms of special types of distributive lattices (arithmetical and semi-arithmetical lattices). The next division contains a discussion of the arithmetical properties of non-commutative residuated lattices. In particular decompositions into primary and semi-primary elements are discussed.

Finally we investigate the case where both the ascending and descending chain conditions hold and prove some structure theorems which are analogous to the structure theorems of hypercomplex systems.

I. Multiplication and residuation

1. **Definitions and notations.** The fixed lattice of elements $a, b, c, \cdots$ will will be denoted by $\mathfrak{S}$. Sublattices will be denoted by German capitals, and Latin capitals will denote subsets of $\mathfrak{S}$ which are not necessarily sublattices. $(,)$, $[,]$, $\supset$ will denote union, crosscut, and lattice division respectively. If $a \neq b$ and $a \supset x \supset b$ implies either $x = a$ or $x = b$, a is said to *cover* b and we write

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† Lattices with a commutative multiplication have been investigated by Professor Morgan Ward and the author in a previous paper (Ward-Dilworth [7]).

$a > b$. If $\mathfrak{S}$ has a unit element u, the elements covered by u are called *divisor-free* elements of $\mathfrak{S}$. If $\mathfrak{S}$ has a null element it will be denoted by z.

$\mathfrak{S}$ is said to satisfy the ascending chain condition if every chain $a_1 \subset a_2 \subset a_3 \subset \cdots$ has only a finite number of distinct elements. Similarly if every descending chain $a_1 \supset a_2 \supset a_3 \supset \cdots$ has only a finite number of distinct elements, $\mathfrak{S}$ is said to satisfy the descending chain condition. $\mathfrak{S}$ is called *archimedian* if both the ascending and descending chain conditions hold.

The direct product (Birkhoff [1]) of lattices $\mathfrak{L}_1, \mathfrak{L}_2, \cdots, \mathfrak{L}_n$ is defined to be the set of vectors $a = \{a_1, a_2, \cdots, a_n\}$, $a_i \varepsilon \mathfrak{L}_i$ with division defined by $a \supset b$ if and only if $a_i \supset b_i$. Union and crosscut are given by $(a, b) = \{(a_1, b_1), \cdots, (a_n, b_n)\}$, $[a, b] = \{[a_1, b_1], \cdots, [a_n, b_n]\}$.

2. **Multiplication.** A one-valued, binary operation xy is called a *multiplication* over $\mathfrak{S}$ if the following postulates are satisfied:

M_1. *ab lies in $\mathfrak{S}$ whenever a and b lie in $\mathfrak{S}$.*

M_2. *$a = b$ implies $ac = bc$, $ca = cb$.*

M_3. $a(b, c) = (ab, ac)$, $(a, b)c = (ac, bc)$.

M_4. $a(bc) = (ab)c$.

From M_2 and M_3 we have

(2.1) $a \supset b$ implies $ac \supset bc$ and $ca \supset cb$;

(2.2) $[ab, ac] \supset a[b, c]$, $[ac, bc] \supset [a, b]c$.

If in addition to M_1–M_4, postulate M_5 below is satisfied, $\mathfrak{S}$ is said to be a *left ideal* lattice.

M_5. $a \supset ba$.

In a similar manner if $M_{5'}$ is satisfied, $\mathfrak{S}$ is said to be a *right ideal* lattice.

$M_{5'}$. $a \supset ab$.

If a lattice is both a left and right ideal lattice, it is called a *two-sided ideal* lattice, or simply *ideal* lattice.

Consider a lattice with unit element u over which a multiplication satisfying M_1–M_4 is defined and for which M_6 holds.

M_6. $ua = au = a$.

Then by M_3, M_5 and $M_{5'}$ hold so that $\mathfrak{S}$ is an ideal lattice. A lattice with unit element in which M_6 holds we call an *ideal lattice with unit.*

$\mathfrak{S}$ is said to be commutative if it satisfies M_7.

M_7. $ab = ba$.

3. **Residuation.** Consider now an ideal lattice $\mathfrak{S}$ in which the ascending

chain condition* holds. Let a and b be two elements of $\mathfrak{S}$. Then the set X of all elements $x \varepsilon \mathfrak{S}$ such that $a \supset xb$ is non-empty and closed with respect to union. Hence by the ascending chain condition X has a unit element $a \cdot b^{-1}$ which we call the *left residual* of b with respect to a. The left residual $a \cdot b^{-1}$ has the fundamental properties:

R_1. $a \supset (a \cdot b^{-1})b$.
R_2. $a \supset xb \rightarrow a \cdot b^{-1} \supset x$.†

In a similar manner the *right residual* $b^{-1} \cdot a$ is defined by the following properties:

$R_{1'}$. $a \supset b(b^{-1} \cdot a)$.
$R_{2'}$. $a \supset bx \rightarrow b^{-1} \cdot a \supset x$.

The two residuals are connected by the relation

$$a^{-1} \cdot (b \cdot c^{-1}) = (a^{-1} \cdot b) \cdot c^{-1}. \tag{3.1}$$

The residuals are connected with the multiplication by the formulas

$$(ab) \cdot b^{-1} \supset a, \qquad a^{-1} \cdot (ab) \supset b, \tag{3.2}$$

$$a \cdot (bc)^{-1} = (a \cdot c^{-1}) \cdot b^{-1}, \qquad (ab)^{-1} \cdot c = b^{-1} \cdot (a^{-1} \cdot c). \tag{3.3}$$

Some of the more important properties of the residuals are the following:

$$a \cdot (b^{-1} \cdot a)^{-1} \supset (a, b), \qquad (a \cdot b^{-1})^{-1} \cdot a \supset (a, b); \tag{3.4}$$

$$[a, b] \cdot c^{-1} = [a \cdot c^{-1}, b \cdot c^{-1}], \qquad a^{-1} \cdot [b, c] = [a^{-1} \cdot b, a^{-1} \cdot c]; \tag{3.5}$$

$$a \cdot (b, c)^{-1} = [a \cdot b^{-1}, a \cdot c^{-1}], \qquad (a, b)^{-1} \cdot c = [a^{-1} \cdot c, b^{-1} \cdot c]; \tag{3.6}$$

$$(a, b) \cdot c^{-1} \supset (a \cdot c^{-1}, b \cdot c^{-1}), \qquad a^{-1} \cdot (b, c) \supset (a^{-1} \cdot b, a^{-1} \cdot c); \tag{3.7}$$

$$a \supset b \rightarrow a \cdot c^{-1} \supset b \cdot c^{-1},\ c^{-1} \cdot a \supset c^{-1} \cdot b; \tag{3.8}$$

$$a \supset b \rightarrow c \cdot b^{-1} \supset c \cdot a^{-1},\ b^{-1} \cdot c \supset a^{-1} \cdot c; \tag{3.9}$$

$$a \cdot b^{-1} \supset a,\ b^{-1} \cdot a \supset a; \tag{3.10}$$

$$a \cdot b^{-1} \supset c \rightleftarrows c^{-1} \cdot a \supset b. \tag{3.11}$$

On the other hand, if we start with a lattice $\mathfrak{S}$ in which the descending chain condition‡ holds and over which left and right residuals are defined having the properties given above, then we may define a multiplication over $\mathfrak{S}$ satisfying M_1–$M_{5'}$. For let a and b be two elements of $\mathfrak{S}$ and let X be the

* This condition may be replaced by the weaker condition that every set S of elements of $\mathfrak{S}$ have a union $u(S)$ and that $u(S)c = u(Sc)$.

† The symbol $\rightarrow$ indicates formal implication.

‡ As in the previous case this condition may be weakened.

set of elements x such that $x \cdot b^{-1} \supset a$. Then X is non-empty and closed with respect to crosscut, and hence by the descending chain condition has a null element ab. It can be shown that the product so defined satisfies M_1–$M_{5'}$ and moreover is equal to the product similarly defined in terms of the right residual.

II. Residuated lattices with unit

4. **Lattice structure.** Throughout this and the following section we shall assume that $\mathfrak{S}$ is a lattice in which the ascending chain condition holds and having a multiplication satisfying $M_1, \cdots, M_6$. As a consequence of M_6 the residuals have the following properties:

$$a \supset b \rightleftarrows a \cdot b^{-1} = u \rightleftarrows b^{-1} \cdot a = u; \tag{4.1}$$

$$a \cdot u^{-1} = u^{-1} \cdot a = a; \tag{4.2}$$

$$(a, b) = u \rightarrow a \cdot b^{-1} = a,\ b^{-1} \cdot a = a. \tag{4.3}$$

Conversely, if we start with residuals having property (4.1) and define multiplication in terms of the residuals as in §3, then it is readily verified that the multiplication satisfies M_6.

Of particular importance in the proofs that follow are the properties:

$$(b, c) = u \rightarrow (a, [b, c]) = [(a, b), (a, c)]; \tag{4.4}$$

$$(b, c) = u \rightarrow ([a, b], [a, c]); \tag{4.5}$$

$$(a, b) = u, (a, c) = u \rightarrow (a, [b, c]) = u. \tag{4.6}$$

As a consequence of (4.4) and (4.6) we have the following property:

(4.7) If $a_1, \cdots, a_n$ are coprime in pairs, then

$$(c, [a_1, \cdots, a_n]) = [(c, a_1), \cdots, (c, a_n)].$$

Two sublattices $\mathfrak{A}$ and $\mathfrak{B}$ are said to be *coprime* if $a \in \mathfrak{A}$ and $b \in \mathfrak{B}$ imply $(a, b) = u$. We have then

Lemma 4.1. *Let $\mathfrak{A}$ be the sublattice generated by the sublattices $\mathfrak{A}_1, \mathfrak{A}_2, \cdots, \mathfrak{A}_n$ each of which contains u. Then $\mathfrak{A}$ is the direct product of $\mathfrak{A}_1, \cdots, \mathfrak{A}_n$ if and only if $\mathfrak{A}_1, \cdots, \mathfrak{A}_n$ are coprime in pairs.*

From the definitions of §1 it follows directly that $\mathfrak{A}_1, \cdots, \mathfrak{A}_n$ are coprime in pairs if $\mathfrak{A}$ is the direct product of $\mathfrak{A}_1, \cdots, \mathfrak{A}_n$. Let now $\mathfrak{A}_1, \cdots, \mathfrak{A}_n$ be coprime in pairs and let L denote the set of crosscuts $[a_1, \cdots, a_n]$ where $a_i \in \mathfrak{A}_i$. We have clearly

$$[[a_1, \cdots, a_n], [a_1', \cdots, a_n']] = [[a_1, a_1'], [a_2, a_2'], \cdots, [a_n, a_n']].$$

Furthermore

$$([a_1, \cdots, a_n], [a_1', \cdots, a_n']) = [(a_1, [a_1', \cdots, a_n']), \cdots, (a_n, [a_1', \cdots, a_n'])]$$
$$= [(a_1, a_1'), \cdots, (a_n, a_n')]$$

by (4.7). Hence L is a sublattice and is thus equal to $\mathfrak{A}$. If $[a_1, \cdots, a_n] = [a_1', \cdots, a_n']$, then

$$a_i = (a_i, [a_1', \cdots, a_n']) = [(a_i, a_1'), \cdots, (a_i, a_n')] = (a_i, a_i').$$

Whence $a_i \supset a_i'$. Similarly $a_i' \supset a_i$ and hence $a_i = a_i'$. This completes the proof.

If the sublattices $\mathfrak{A}_1, \cdots, \mathfrak{A}_n$ have minimal elements, the conditions of Lemma 4.1 may be simplified.

COROLLARY. *If the sublattices $\mathfrak{A}_1, \cdots, \mathfrak{A}_n$ of Lemma 4.1 have minimal elements $m_1, \cdots, m_n$, then $\mathfrak{A}$ is the direct product of $\mathfrak{A}_1, \cdots, \mathfrak{A}_n$ if and only if $m_1, \cdots, m_n$ are coprime in pairs.*

From Lemma 4.1 we have immediately

LEMMA 4.2. *Any finite set of divisor-free elements generates a finite Boolean algebra.*

If there are only a finite number of divisor-free elements in $\mathfrak{S}$, we may speak of *the* Boolean algebra generated by the divisor-free elements. This is certainly the case when the descending chain condition holds in $\mathfrak{S}$, for we have

LEMMA 4.3. *If the descending chain condition holds in $\mathfrak{S}$, then there are only a finite number of divisor-free elements.*

Let $p_1, p_2, \cdots, p_n, \cdots$ be an infinite sequence of distinct divisor-free elements, and form the descending chain $a_1 \supset a_2 \supset a_3 \supset \cdots$ where $a_i = [p_1, p_2, \cdots, p_i]$. If $a_i = a_i + 1$, then $[p_1, \cdots, p_i] = [p_1, \cdots, p_{i+1}]$ and hence

$$p_{i+1} = (p_{i+1}, [p_1, \cdots, p_i]) = [(p_{i+1}, p_1), \cdots, (p_{i+1}, p_i)] = u,$$

which is impossible. Thus $a_1 \supset a_2 \supset a_3 \supset \cdots$ is an infinite descending chain.

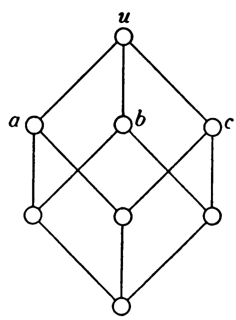
5. We turn now to the study of the structure of a residuated lattice in the vicinity of the unit element and prove first the fundamental

THEOREM 5.1. *Let $\mathfrak{S}$ be a residuated lattice with unit having only a finite number of divisor-free elements $p_1, p_2, \cdots, p_n$. Moreover let $\mathfrak{L}$ be the direct product of chain lattices $\mathfrak{L}_1, \cdots, \mathfrak{L}_n$ where $\mathfrak{L}_i$ is the chain $u \supset p_i \supset a_i \supset \cdots \supset m_i$. Then if $m_k > b$ and b does not belong to $\mathfrak{L}$, the sublattice generated by the elements of $\mathfrak{L}$ and the element b is the direct product of the chain lattices $\mathfrak{L}_1, \cdots, \mathfrak{L}_k', \cdots, \mathfrak{L}_n$ where $\mathfrak{L}_k'$ is the chain lattice $u \supset p_k \supset a_k \supset \cdots \supset m_k \supset b$.*

Proof. In view of the corollary to Lemma 4.1 it is sufficient to show that $(b, m_i) = u, i \neq k$. If $(b, m_i) \neq u$, there exists a divisor-free element p such that

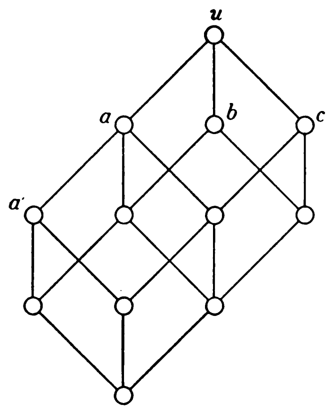
$p \supset (b, m_i)$. Since $p \supset m_i$, we have $p = p_i$. Now $m_k \supset [m_k, p_i] \supset b$ since $p \supset b$. But $m_k \neq [m_k, p_i]$ since otherwise $p_i \supset m_k$ while $(p_i, m_k) = u$. Hence $b = [m_k, b_i]$ and b is contained in $\mathfrak{L}$ which is contrary to assumption. Thus $(b, m_i) = u$, $i \neq k$.

This theorem enables us to construct certain characteristic sublattices with very simple properties. For let $\mathfrak{B}$ be the Boolean algebra generated by the divisor-free elements of $\mathfrak{S}$. If a divisor-free element p of $\mathfrak{B}$ covers an element a_1, not contained in $\mathfrak{B}$, then $\mathfrak{B}$ and a_1 generate a sublattice $\mathfrak{L}_1$ which is a direct product of chain lattices. If a_1 covers a_2 and a_2 does not belong to $\mathfrak{L}_1$, then $\mathfrak{L}_1$ and a_2 generate a sublattice $\mathfrak{L}_2$ which is again a direct product of chain lattices. We may continue in this manner as long as we obtain elements a_i not contained in $\mathfrak{L}_{i-1}$. Having obtained a sublattice $\mathfrak{L}_k$ in this manner, we may further extend it by building chains from other divisor-free elements. Thus if we call lattices which are direct products of chain lattices, *arithmetical* (Ward [5]), we see that the structure of a residuated lattice in the vicinity of the unit element is characterized to a large extent in terms of arithmetical lattices.



$\mathfrak{B}$

FIG. 1



$\mathfrak{L}$

FIG. 2

This principle is very useful in constructing examples of residuated lattices. For example, suppose we wish to construct a residuated lattice containing three divisor-free elements. We start then with the Boolean algebra $\mathfrak{B}$ of Fig. 1.

Now if we wish to add an element a' covered by a, by Theorem 5.1 we have immediately the sublattice $\mathfrak{L}$ of Fig. 2.

The condition of Theorem 5.1 that each divisor-free element be a member of one of the chain lattices is essential for the truth of the theorem as may be seen by simple examples. However in general a residuated lattice will have an infinite number of divisor-free elements and Theorem 5.1 will no longer apply. It may be generalized as follows:

THEOREM 5.2. *Let $\mathfrak{L}$ be the direct product of chain lattices $\mathfrak{L}_1, \cdots, \mathfrak{L}_n$ of a residuated lattice $\mathfrak{S}$, and let $\mathfrak{B}$ be the lattice generated by $\mathfrak{L}$ and the set of divisor-free elements p which divide at least one element of $\mathfrak{L}$. Furthermore let $m_k > b$. Then either b lies in $\mathfrak{B}$ or the lattice generated by $\mathfrak{L}$ and b is the direct product of the chain lattices $\mathfrak{L}_1, \cdots, \mathfrak{L}_k', \cdots, \mathfrak{L}_n$ where $\mathfrak{L}_k' = \{\mathfrak{L}_k, b\}$.*

Proof. If $(b, m_i) \neq u$, $i \neq k$, there exists a divisor-free element p such that $p \supset (b, m_i)$. Now $m_k \supset [m_k, p] \supset b$ and $m_k \neq [m_k, p]$ since otherwise $p \supset m_k$ while $(p, m_k) = u$. Hence $b = [m_k, b]$ and $b \in \mathfrak{B}$. Hence if $b \notin \mathfrak{B}$, $(b, m_i) = u$, $i \neq k$, and the theorem follows by Lemma 4.

The structure of the lattice $\mathfrak{B}$ of Theorem 5.2 is comparatively simple. We shall study its properties in terms of the notion of *semi-arithmetical* lattices introduced by Morgan Ward (Ward [5]). We make the

DEFINITION 5.1. *A distributive lattice $\mathfrak{D}$ is said to be semi-arithmetical if the indecomposable elements divisible by a given divisor-free element form a chain lattice.*

A semi-arithmetical lattice in which the ascending chain condition holds may be characterized as follows:

LEMMA 5.1. *A distributive lattice $\mathfrak{D}$ in which the ascending chain condition holds is semi-arithmetical if and only if the indecomposables occurring in the reduced representation of an element as a crosscut of indecomposables are coprime in pairs.*

From Definition 5.1 it follows trivially that an arithmetical lattice is semi-arithmetical.

We shall show now that the lattice $\mathfrak{B}$ of Theorem 5.2 is semi-arithmetical and to that end prove the

THEOREM 5.3. *Let $\mathfrak{L}$ be a semi-arithmetical sublattice of a residuated lattice $\mathfrak{S}$ and let $\mathfrak{L}$ contain the unit element u. Then if p is a divisor-free element of $\mathfrak{S}$, the sublattice $\mathfrak{L}'$ generated by p and the sublattice $\mathfrak{L}$ is semi-arithmetical.*

Proof. If p is contained in $\mathfrak{L}$, the theorem is trivial and we may thus assume that $p \notin \mathfrak{L}$. Now let U be the set of all elements of the form a or $[p, a]$ where $a \in \mathfrak{L}$. The set U is clearly closed with respect to crosscut. We show that U is also closed with respect to union. Let x and y be two members of the set U. If both x and y are contained in $\mathfrak{L}$, (x, y) is obviously in U. Let $x = [p, x_1]$, $p \not\supset x_1$ and $y \in \mathfrak{L}$. Let $x_1 = [q_1, \cdots, q_s]$ where the q_i are indecomposables and $(q_i, q_j) = u$, $i \neq j$. Then since $p \not\supset x_1$, $(p, q_i) = u$ $(i = 1, \cdots, s)$. Hence

$$(x, y) = (y, [p, q_1, \cdots, q_s]) = [(y, p), (y, q_1), \cdots, (y, q_s)]$$

by (4.3). But (y, p) is either p or u hence (x, y) is contained in U. If $x=[p, x_1]$, $p \not\supset x_1$ and $y=[p, y_1]$, $p \not\supset y_1$, then

$$\begin{aligned}(x, y) &= ([p, q_1, \cdots, q_s], [p, q_1', \cdots, q_{s'}'])\\ &= [p, (q_1, p), \cdots, (q_s, p), \cdots, (q_{s'}', p), (q_1, q_1'), \cdots, (q_s, q_{s'}')] = [p, a]\end{aligned}$$

where $a \in \mathfrak{L}$. Hence U is identical with $\mathfrak{L}'$.

Now let a, b, c be contained in U. Then in exactly the same manner as above we find that $(a, [b, c] = [(a, b), (a, c)]$. For example, if $b=[p, b_1]$, $p \not\supset b_1$ and $c \in \mathfrak{L}$, then

$$\begin{aligned}(a, [b, c]) &= (a, [p, q_1, \cdots, q_s, q_1', \cdots, q_{s'}'])\\ &= [(a, p), (a, q_1), \cdots, (a, q_{s'}')] = [(a, b), (a, c)]\end{aligned}$$

if $p \not\supset c$; and if $p \supset c$, then

$$\begin{aligned}(a, [b, c]) &= (a, [q_1, \cdots, q_s, q_1', \cdots, q_{s'}'])\\ &= [(a, q_1), \cdots, (a, q_s), (a, q_1'), \cdots, (a, q_{s'}')]\\ &= [(a, q_1), \cdots, (a, q_s), (a, c)] = [(a, p), (a, q_1), \cdots, (a, q_s), (a, c)]\\ &= [(a, b), (a, c)].\end{aligned}$$

Hence $\mathfrak{L}'$ is distributive.

Finally let $x \in \mathfrak{L}'$; then either $x \in \mathfrak{L}$ or $x=[p, x_1]$ where $p \not\supset x_1$. If $x \in \mathfrak{L}$, then $x=[q_1, \cdots, q_r]$ where the q_i are indecomposable and $(q_i, q_j)=u$, $i \neq j$. If $x=[p, x_1]$ then $x=[p, q_1, \cdots, q_r]$ where $p, q_1, \cdots, q_r$ are indecomposable and $(q_i, q_j)=u$, $i \neq j$; $(p, q_i)=u$ $(i=1, \cdots, r)$. Thus $\mathfrak{L}'$ is semi-arithmetical by Lemma 5.1 and the proof is complete.

Now since $\mathfrak{B}$ is obtained from an arithmetical lattice $\mathfrak{L}$ by a successive adjunction of divisor-free elements and since at each stage a semi-arithmetical sublattice is obtained, $\mathfrak{B}$ itself is semi-arithmetical. We have thus proved

THEOREM 5.4. *The lattice $\mathfrak{B}$ of Theorem 5.2 is a semi-arithmetical sublattice of $\mathfrak{S}$.*

In forming the sublattice $\mathfrak{B}$ from the arithmetical lattice $\mathfrak{L}$ only divisor-free elements which are divisors of some element of $\mathfrak{L}$ are considered. If we adjoin a divisor-free element which does not divide any of the elements of $\mathfrak{L}$, the results are even simpler; for we have

THEOREM 5.5. *Let $\mathfrak{L}$ be a direct product of the chain lattices $\mathfrak{L}_1, \cdots, \mathfrak{L}_n$, and let p denote a divisor-free element not contained in $\mathfrak{L}$. Then if p does not divide any of the elements of $\mathfrak{L}$, the sublattice generated by p and $\mathfrak{L}$ is the direct product $\mathfrak{L}'$*

of the chain $\{u, p\}$ *and the chain lattices of* $\mathfrak{L}$. *Furthermore if* $\mathfrak{L}$ *is dense in* $\mathfrak{S}$, *then* $\mathfrak{L}'$ *is dense in* $\mathfrak{S}$.

Proof. Since p does not divide a_i if $a_i \varepsilon \mathfrak{L}_i$, $(a^i, p) = u$. Hence the first part of the theorem follows. Let now $x \supset [p, a_1, \cdots, a_n]$. Then $x = [(x, p), (x, a_1), \cdots, (x, a_n)]$. Now (x, p) is clearly in $\mathfrak{L}'$ and (x, a_i) is in $\mathfrak{L}$ by hypothesis. Hence $x \varepsilon \mathfrak{L}'$.

We conclude this section with

THEOREM 5.6. *Let* $\mathfrak{L}$ *be the direct product of the chain lattices* $\mathfrak{L}_1, \cdots, \mathfrak{L}_n$ *of a residuated lattice* $\mathfrak{S}$ *and let* $m_k > b$ *where* b *is indecomposable. Then* $\mathfrak{L}$ *and* b *generate a sublattice* $\mathfrak{L}'$ *which is the direct product of the chain lattices* $\mathfrak{L}_1, \cdots, \{\mathfrak{L}_k, b\}, \cdots, \mathfrak{L}_n$. *Furthermore if* $\mathfrak{L}$ *is dense in* $\mathfrak{S}$, *then* $\mathfrak{L}'$ *is dense in* $\mathfrak{S}$.

Proof. The first part follows directly from Theorem 5.2. Let now $x \supset [m_1, m_2, \cdots, b, \cdots, m_n]$. Then $x = [(x, m_1), \cdots, (x, b), \cdots, (x, m_n)]$. Since $\mathfrak{L}$ is dense by hypothesis, $(x, m_1), \cdots, (x, m_n)$ are contained in $\mathfrak{L}$. Now either $(x, b) = b$ in which case $x \varepsilon \mathfrak{L}'$ or $(x, b) \supset m_i$ since b is indecomposable. But then $(x, b) \varepsilon \mathfrak{L}$ and x is contained in $\mathfrak{L}'$.

III. ARITHMETICAL PROPERTIES OF IDEAL LATTICES

6. Assume that $\mathfrak{S}$ is an ideal lattice in which the ascending chain condition holds.

DEFINITION 6.1. *An element* $p \varepsilon \mathfrak{S}$ *is said to be a prime if* $p \supset ab$ *and* $p \not\supset a$ *implies* $p \supset b$.

DEFINITION 6.2. *An element* $q \varepsilon \mathfrak{S}$ *is said to be right primary if* $q \supset ab$ *and* $q \not\supset a$ *implies* $q \supset b^s$ *for some whole number* s.

In the theory of commutative residuated lattices a residuated lattice in which the ascending chain condition holds is said to be a *Noether* lattice (Ward-Dilworth [7]) if every irreducible is primary. It is then shown that every element of a Noether lattice may be represented as a simple* crosscut of a finite number of primaries each of which is associated with a different prime. The primes themselves and the total number of primaries are uniquely determined by the element. This result also holds for the non-commutative case although there are certain complications due to the non-commutativity of the multiplication. We shall show how these complications may be avoided.

Let $\mathfrak{S}$ be a non-commutative Noether lattice; that is, assume that every irreducible is right primary. If a and b are elements of $\mathfrak{S}$, the product ab then has the form $ab = [q_1, \cdots, q_r]$ where the q_i are right primary. Let $q_i \supset a$

* A crosscut representation is said to be simple if omitting any one of the terms changes the representation.

$(i=1, \cdots, l)$, $q_i \not\supset a$ $(i=l+1, \cdots, r)$. Then since $q_i \supset ab$ we have $q_i \supset b^{s_i}$ $(i=l+1, \cdots, r)$. If we then set $s = \max(s_{l+1}, \cdots, s_r)$, we have

$$ab \supset [a, b^s] \supset b^s a. \tag{6.1}$$

Let q be right primary and consider the union p of all elements x such that $q \supset x^s$ for some whole number s. Then $q \supset p^t$ for some whole number t by the ascending chain condition. Furthermore p is a prime. For if $p \supset ab$, then $q \supset p^t \supset (ab)^t \supset a^r b^t$ by (6.1). If $q \supset a^r$, then $p \supset a$. If $q \not\supset a^r$, then $q \supset b^{ts}$ and $p \supset b$. Hence either $p \supset a$ or $p \supset b$. This prime is clearly unique and is called the prime element associated with the right primary q. We have moreover

Lemma 6.1. *The crosscut of two right primaries associated with the same prime p is also a right primary associated with p.*

Let $[q, q'] \supset ab$, $[q, q'] \not\supset a$. Then either q or q', say q, does not divide a and hence $q \supset b^s$. But then $p \supset b$ and hence $q' \supset b^t$. Hence $[q, q'] \supset b^{s'}$ where $s' = \max(s, t)$. Obviously $[q, q']$ is associated with p.

Lemma 6.2. *Let q and q' be right primaries associated with p and p' respectively. Then if $p \not\supset p'$, $q \cdot q'^{-1} = q$.*

For $q \supset (q \cdot q'^{-1})q'$. Hence either $q = q \cdot q'^{-1}$ or $q \supset q'^s$. But if $q \supset q'^s$, then $p \supset p'^t$ and hence $p \supset p'$ contrary to hypothesis.

Note that Lemma 6.2 holds only for the right residual. If we were considering left primaries, the left residual would replace the right residual.

The proof from this point on is exactly analogous to the proof in classical ideal theory and will be omitted. We thus obtain

Theorem 6.1. *Let $\mathfrak{S}$ be a non-commutative Noether lattice. Then every element of $\mathfrak{S}$ may be represented as a simple crosscut of a finite number of right primaries. The primes and the total number of right primaries are uniquely determined by the element.*

The following theorem proved in Ward-Dilworth [7] for the commutative case holds also for non-commutative residuated lattices and is proved in exactly the same manner.

Theorem 6.2. *The following two conditions are sufficient that $\mathfrak{S}$ be a Noether lattice:*

(i) *$\mathfrak{S}$ is modular,*

(ii) *$ab \supset [a, b^s]$.*

The distinction between left and right primaries may be removed by weakening the condition of Definition 6.2. We adopt the name *semi-primaries* for these new elements.

DEFINITION 6.3. *An element $a \varepsilon \mathfrak{S}$ is said to be semi-primary if $a \supset bc$ and $a \not\supset b^s$ for all s implies $a \supset c^t$ for some whole number t.*

Let $\mathfrak{S}$ be an ideal lattice in which every element may be represented as a crosscut of a finite number of semi-primaries. Moreover let x and y be any two elements of $\mathfrak{S}$. Then $xy = [a_1, \cdots, a_r]$ where the a_i are semi-primary. Let $a_i \supset x^{s_i}$ for $i=1, \cdots, l$ and $a_i \supset y^{t_i}$, $i = l+1, \cdots, r$. Then $xy \supset [x^s, y^t]$ where $s = \max(s_1, \cdots, s_l)$ and $t = \max(t_{l+1}, \cdots, t_r)$. We thus have

THEOREM 6.3. *If every element of a residuated lattice $\mathfrak{S}$ is expressible as a crosscut of a finite number of semi-primaries, then for every x and y in $\mathfrak{S}$, there exist whole numbers s and t such that*

$$xy \supset [x^s, y^t]. \tag{6.2}$$

If (6.2) holds in a residuated lattice, the semi-primary elements may be simply characterized as follows:

THEOREM 6.4. *Let $\mathfrak{S}$ be a residuated lattice in which* (6.2) *holds. Then an element a is semi-primary if and only if a prime p exists such that $p \supset a \supset p^s$ for some whole number s.*

Proof. Let a be semi-primary, and let p denote the union of all elements x such that $a \supset x^r$ for some r. Then $a \supset p^t$ for some t. Now let $p \supset xy$. Then $a \supset xy \supset x^m y^n$ for some integers m and n by (6.2). Hence $a \supset x^s$ for some s or $a \supset y^t$ for some t. Hence either $p \supset x$ or $p \supset y$. Clearly $p \supset a \supset p^s$ for some s.

Conversely let $p \supset a \supset p^s$ and suppose that $a \supset bc$. Then $p \supset bc$, and hence either $p \supset a$ or $p \supset b$. Hence either $a \supset b^s$ or $a \supset c^s$.

The converse to Theorem 6.3 does not hold in general. However under the assumption of the distributive law we have

THEOREM 6.5. *The following two conditions are sufficient that every element of a residuated lattice $\mathfrak{S}$ satisfying the ascending chain condition be expressible as a crosscut of a finite number of semi-primaries.*

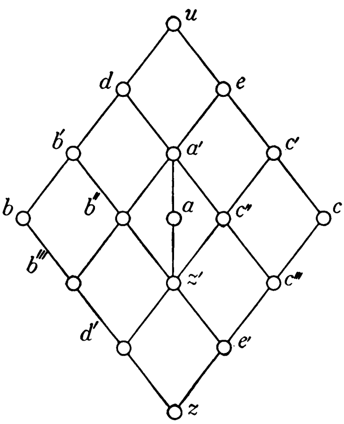
(i) *$\mathfrak{S}$ is distributive,*

(ii) *$xy \supset [x^s, y^t]$ for suitable s and t.*

Every element of $\mathfrak{S}$ is clearly expressible as a crosscut of a finite number of indecomposables. Hence it is sufficient to show that every indecomposable is semi-primary. Let a be indecomposable, and let $a \supset bc$, $a \not\supset b^s$, for any s. Then $a \supset [b^s, c^t]$ by (ii). Hence $a = [(a, b^s), (a, c^t)]$ by (i). But $(a, b^s) \neq a$. Hence since a is indecomposable, $a = (a, c^t)$ and $a \supset c^t$.

The distributive condition is essential in Theorem 6.5 as is shown by the example in Fig. 3.

Let $\mathfrak{L}_a$ denote the sublattice $\{a', b'', a, c'', b''', z', c''', d', e', z\}$, $\mathfrak{L}_b$ the sublattice $\{d, b', b\}$, and $\mathfrak{L}_c$ the sublattice $\{e, c', c\}$. We define a multiplication over $\mathfrak{L}$ as follows: $u^2=u$, $ux=b$ if $x \in \mathfrak{L}_b$, $ux=c$ if $x \in \mathfrak{L}_c$, $ux=z$ if $x \in \mathfrak{L}_a$. The product of any two elements in $\mathfrak{L}_b$ is b. The product of any two elements in $\mathfrak{L}_c$ is c. The product of any element of $\mathfrak{L}_b$ with an element of $\mathfrak{L}_c$ is z. The product of an element of $\mathfrak{L}$ with an element of $\mathfrak{L}_c$ is z. It is readily verified that the multiplication so defined satisfies $M_1, \cdots, M_{5'}$ and is also commutative. $\mathfrak{L}$ is



$\mathfrak{L}$

FIG. 3

clearly *not* distributive. It can also be verified that $xy \supset [x^s, y^t]$ for suitable s and t. However it is not true that $xy \supset [x, y^s]$ for some s, since $dc \not\supset [d, c^s]$. Furthermore a is indecomposable but *not* semi-primary since $a \supset bc$, but $a \not\supset b^s$ any s and $a \not\supset c^t$ any t.

7. **Ideal lattices with unit.** We turn now to the study of the properties of divisor-free elements in an ideal lattice with unit. We prove first the

LEMMA 7.1. *Let f be a divisor-free element of $\mathfrak{S}$, and let a be any element not divisible by f. Then one and only one of the following formulas holds:*

(1) $fa \supset af$,

(2) $fa = (fa)\cdot f^{-1}$.

We have $(fa\cdot f^{-1})^{-1}\cdot fa \supset f$ by (4.4). Hence either $(fa\cdot f^{-1})^{-1}\cdot fa = u$ or $(fa\cdot f^{-1})^{-1}\cdot fa = f$. In the first case $fa \supset fa\cdot f^{-1}$. But $fa\cdot f^{-1} \supset fa$ by (3.10). Hence $fa = fa\cdot f^{-1}$. If $(fa\cdot f^{-1})^{-1}\cdot fa = f$, then

$$f = f\cdot a^{-1} = ((fa\cdot f^{-1})^{-1}\cdot fa)\cdot a^{-1} = (fa\cdot f^{-1})^{-1}\cdot(fa\cdot a^{-1}) \supset (fa\cdot f^{-1})^{-1}\cdot f.$$

But $(fa\cdot f^{-1})^{-1}\cdot f \supset f$. Hence $(fa\cdot f^{-1})^{-1}\cdot f = f$. But then $f^{-1}\cdot(fa\cdot f^{-1}) = fa\cdot f^{-1}$ or $(f^{-1}\cdot fa)\cdot f^{-1} = fa\cdot f^{-1}$. Then $fa\cdot f^{-1} \supset a\cdot f^{-1} \supset a$. Hence $fa \supset (fa\cdot f^{-1})f \supset af$.

If both (1) and (2) hold, then $fa = fa\cdot f^{-1} \supset af\cdot f^{-1} \supset a$. But then $f = (f, fa) \supset (f, a) = u$, contrary to the assumption that f is a divisor-free element.

We clearly have a similar result for left residuals.

LEMMA 7.2. *Let f be a divisor-free element of a residuated lattice in which* (6.2) *holds. Then f commutes with every element which it does not divide.*

Let $a \in \mathfrak{S}$ such that $f \not\supset a$. Then by Lemma 7.1 either $fa \supset af$ or $fa = fa\cdot f^{-1}$. If $fa = fa\cdot f^{-1}$, then $fa \supset (a^s f^t)\cdot f^{-1} \supset a^s f^{t-1}$ by (6.2). But then $fa \supset fa\cdot f^{-1} \supset (a^s f^{t-1})\cdot f^{-1} \supset a^s f^{t-2}$. Continuing in this manner we finally get $fa \supset a^s$. But then $f = (f, fa) \supset (f, a^s) = u$ since $f \not\supset a^s$. Hence $f = u$ which is contrary to our assumption that f is a divisor-free element. We thus have $fa \supset af$. In a similar manner using left residuals we get $af \supset fa$. Hence $af = fa$.

As a corollary to Lemma 7.2 the divisor-free elements in a residuated lattice for which (6.2) holds always commute. In particular we have from Theorem 6.3

LEMMA 7.3. *If in a residuated lattice every element is expressible as a crosscut of semi-primaries, then the divisor-free elements commute.*

Let $\mathfrak{S}$ be an arbitrary residuated lattice in which the ascending chain condition holds and denote by $\mathfrak{S}'$ the set of all elements x which divide a finite product of divisor-free elements. $\mathfrak{S}'$ is clearly closed under union, crosscut, multiplication and residuation and hence a residuated sublattice of $\mathfrak{S}$. Then

LEMMA 7.4. *Every prime in $\mathfrak{S}'$ is divisor-free.*

Let p be a prime in $\mathfrak{S}'$. Then by the definition of $\mathfrak{S}'$, $p \supset f_1 f_2 \cdots f_r$ where $f_1, f_2, \cdots, f_r$ are divisor-free elements of $\mathfrak{S}$. Hence $p = f_i$ for some i.

LEMMA 7.5. *Every element of $\mathfrak{S}'$ divides a finite product of its divisor-free divisors.*

This lemma follows directly from the following lemma due to Krull [3].

LEMMA 7.6. *Let $\mathfrak{S}$ be a non-commutative residuated lattice in which the ascending chain condition holds. Then each element $a \in \mathfrak{S}$ has only a finite number of minimal prime divisors $p_1, \cdots, p_n$ and a divides a power of $p_1 \cdots p_n$.**

* Krull states this lemma for the more general case where the ascending chain condition is assumed only for prime elements while a residual chain condition holds for all elements. However his proof seems to be in error as he uses the following rule: If $a \supset a_1' a_2'$, then $a \supset a_1 a_2$ where $a_1 = a\cdot a_2'^{-1}$ and $a_2 = a_1'^{-1}\cdot a$. This rule is in general not correct as the following example shows: Let $\mathfrak{S}$ be the lattice defined by the covering relations $u > a > b > c > z$, $b > d > z$. The multiplication is defined by $ux = xu = x$, all $x \in \mathfrak{S}$; $a^2 = a$, and all other products are equal to z. Then $z\cdot c^{-1} = a$, $d^{-1}\cdot z = a$ and $z \supset cd$. However $z \not\supset (z\cdot c^{-1})(d^{-1}\cdot z) = a^2 = a$.

The lemma is readily seen to be correct under the assumption of the ascending chain condition since we may take $a_1 = (a, a_1')$ and $a_2 = (a, a_2')$ and the rule stated above holds.

A further consequence of Lemma 7.6 is the result that $\mathfrak{S}'$ is the maximal residuated sublattice all of whose prime elements are divisor-free.

In certain cases $\mathfrak{S}'$ is simply the Boolean algebra $\mathfrak{B}$ generated by the divisor-free elements. For example we have

THEOREM 7.1. *Let $\mathfrak{S}$ be a residuated lattice with only a finite number of divisor-free elements all of which commute among themselves. If the only elements covered by the divisor-free elements are elements of the Boolean algebra $\mathfrak{B}$ generated by them, then $\mathfrak{S}' = \mathfrak{B}$.*

Proof. Under the hypothesis of the theorem, $f^2 \subset [f, f']$ or $f^2 = f$. But if $[f, f'] \supset f^2$, then $f' \supset f$ which is impossible. Hence $f^2 = f$. But then $[f_1, f_2, \cdots, f_n] = f_1 f_2 \cdots f_n$ and $(f_1 \cdots f_n)^2 = f_1 \cdots f_n$.

If the divisor-free elements do not commute, the theorem does not hold in general. Consider the lattice $\mathfrak{L}$ defined by the covering relations $u > b > c > z$, $u > a > c$. The multiplication is given by $ux = xu = x$, $x \in \mathfrak{L}$, and $ab = c$, $ba = z$, $ac = ca = bc = cb = c^2 = z$, $a^2 = a$, $b^2 = b$, $zx = z$, all $x \in \mathfrak{L}$. Then $\mathfrak{S}' = \mathfrak{L}$ while $\mathfrak{B}$ is the sublattice $\{u, a, b, c\}$.

Applying Theorem 7.1 to hypercomplex systems we obtain

THEOREM 7.2. *A hypercomplex system in which the prime two-sided ideals are commutative is a direct sum of simple two-sided ideals if and only if each irreducible two-sided ideal which is not a prime has at least two prime ideal divisors.*

We conclude this section by giving a variation of a theorem due to Krull.*

THEOREM 7.3. *Each element of $\mathfrak{S}'$ is expressible as a crosscut of a finite number of semi-primaries if and only if the divisor-free elements commute.*

Proof. The second part follows from Lemma 7.3. To prove the first let $a = [a_1, \cdots, a_r]$ be the decomposition of a into coprime indecomposable elements. Then $a_i \supset f_1^{n_1} \cdots f_r^{n_r} = [f_1^{n_1}, \cdots, f_r^{n_r}]$ or $a_i = [(a_i, f_1^{n_1}), \cdots, (a_i, f_r^{n_r})]$ whence $a_i = (a_i, f_j^{n_j})$ for some j. We have then $f_j \supset a_i \supset f_j^{n_j}$. Let $a_i \supset bc$; then $f_j \supset bc$ and hence either $f_j \supset b$ or $f_j \supset c$. Hence either $a_i \supset b^{n_j}$ or $a_i \supset c^{n_j}$. Thus if the divisor-free elements of $\mathfrak{S}$ commute, each element of $\mathfrak{S}'$ may be uniquely represented as a crosscut of coprime semi-primary elements.

IV. ARCHIMEDEAN RESIDUATED LATTICES

8. Throughout this section unless the contrary is explicitly stated it will be assumed that $\mathfrak{S}$ is an ideal lattice in which the ascending and descending chain conditions hold. The unit element of $\mathfrak{S}$ need not be the unit of multiplication.

* Krull proves the theorem for "primary" elements where an element is primary if it has only one divisor-free divisor.

Definition 8.1. *An element a of $\mathfrak{S}$ is said to be nilpotent if $a^s = z$ for some whole number s.*

Lemma 8.1. *The union m of all nilpotent elements of $\mathfrak{S}$ is nilpotent. m is called the radical of $\mathfrak{S}$.*

If $a_1^{t_1} = z$ and $a_2^{t_2} = z$, then $(a_1, a_2)^t = z$ where $t = t_1 + t_2 - 1$. The result follows from the ascending chain condition.

Definition 8.2. *An element s of $\mathfrak{S}$ is said to be simple if $s > z$ where z is the null element of $\mathfrak{S}$.*

Lemma 8.2. *A necessary and sufficient condition that the radical be the null element is that each simple element be idempotent.*

Let $m = z$. If s is a simple element, since $s \supset s^2$, either $s = s^2$ or $s^2 = z$. But if $s^2 = z$, then $z \supset m \supset s$ contrary to Definition 8.2. Suppose now that each simple element is idempotent and let $m \neq z$. Then $m \supset s$ where s is simple, whence $z = m^t \supset s^t = s$, which contradicts the definition of s. Hence $m = z$.

Definition 8.3. *If the radical is the null element, $\mathfrak{S}$ is said to be semisimple.*

Lemma 8.3. *Let $\mathfrak{S}$ be semisimple and s be any simple element of $\mathfrak{S}$. Then $a \supset s \rightleftarrows as = sa = s$, $a \not\supset s \rightleftarrows as = sa = z$.*

Let $a \supset s$. Then $as \supset s^2 = s$ and hence $as = s$. Similarly $sa = s$. If $a \not\supset s$, then $[a, s] = z$ and hence $as = sa = z$.

The position of the radical in the lattice may have important bearing on the arithmetical properties of the lattice. For example, we have the following theorem:

Theorem 8.1. *Let $\mathfrak{S}$ be an archimedean residuated lattice whose divisor-free elements generate a Boolean algebra with null element m. Then the divisor-free elements are the only primes of $\mathfrak{S}$.*

Proof. Since $\mathfrak{S}$ is archimedean there is only a finite number of divisor-free elements. Let p be a prime of $\mathfrak{S}$. Then $p \supset m^t \supset z$ and hence $p \supset m$. But $m = [f_1, \cdots, f_n]$ where $f_1, \cdots, f_n$ are the divisor-free elements of $\mathfrak{S}$. Hence $p \supset [f_1, \cdots, f_n]$ and hence $p = f_i$ for some i.

The conclusion of Theorem 8.1 may be stated in the form $\mathfrak{S} = \mathfrak{S}'$.

Let $\mathfrak{S}_m$ denote the sublattice of all elements x such that $x \supset m$. The study of the structure of $\mathfrak{S}_m$ may be reduced to the study of the structure of semisimple lattices. For since $\mathfrak{S}_m$ is dense in $\mathfrak{S}$ it is closed with respect to residuation and hence has a multiplication (§3). We call this multiplication the multiplication *in* $\mathfrak{S}_m$ and denote it by $a \cdot b$.

Theorem 8.2. *Let $a, b \in \mathfrak{S}_m$. If $ab \in \mathfrak{S}_m$, then $ab = a \cdot b$.*

Proof. $a \cdot b$ is defined by

(i) $(a \cdot b) \cdot b^{-1} \supset a$,

(ii) $x \cdot b^{-1} \supset a,\ x \in \mathfrak{S}_m \rightarrow x \supset a \cdot b$.

Similarly ab is defined by

(i′) $(ab) \cdot b^{-1} \supset a$,

(ii′) $x \cdot b^{-1} \supset a,\ x \in \mathfrak{S} \rightarrow x \supset ab$.

Hence if $ab \in \mathfrak{S}_m$, then $ab \supset a \cdot b$ by (i′), (ii). On the other hand by (i), (ii′), $a \cdot b \supset ab$. Hence $a \cdot b = ab$.

In general we have

LEMMA 8.4. $a \cdot b \supset ab$.

Let now p be a prime element of $\mathfrak{S}$. Then $p \supset m^t = z$ and hence $p \supset m$. Thus $p \in \mathfrak{S}_m$. Now let $p \supset a \cdot b$. Then $p \supset ab$ by Lemma 8.4. Hence either $p \supset a$ or $p \supset b$. We thus have

THEOREM 8.3. *If p is a prime element of $\mathfrak{S}$, then $p \in \mathfrak{S}_m$ and p is a prime in $\mathfrak{S}_m$ with respect to the multiplication in $\mathfrak{S}_m$.*

THEOREM 8.4. *$\mathfrak{S}_m$ is semisimple.*

Proof. Let s be a simple element of $\mathfrak{S}_m$. Then $s > m$. Now $s \supset s \cdot s$. Hence $s = s \cdot s$ or $s \cdot s = m$. But if $s \cdot s = m$, $m \supset s^2$ by Lemma 8.4 and hence $s^{2t} = z$. This contradicts the definition of m. Hence each simple element is idempotent and by Lemma 8.1 $\mathfrak{S}_m$ is semisimple.

The most important application of archimedean residuated lattices is in the theory of hypercomplex systems. More generally, let $\mathfrak{S}$ be the set of two-sided ideals of a non-commutative ring R in which the ascending and descending chain conditions hold for left ideals. Then m is the radical of R. Now the quotient ring R/m is isomorphic to $\mathfrak{S}_m$ and hence is semisimple by Theorem 8.4. However from a well known structure theorem, a semisimple ring is a direct sum of simple two-sided ideals. Its lattice of two-sided ideals is thus a Boolean algebra, and Theorem 8.1 gives

THEOREM 8.5. *The only prime two-sided ideals in a hypercomplex system are the divisor-free ideals.*

9. **Semisimple lattices.** In this section we shall be particularly interested in the sublattices generated by the simple elements of a semisimple lattice $\mathfrak{S}$.

LEMMA 9.1. *There are only a finite number of simple elements in a semisimple lattice $\mathfrak{S}$.*

Let $s_1, s_2, s_3, \cdots$ be an infinite sequence of simple elements. Consider the chain $a_1 \subset a_2 \subset a_3 \subset \cdots$ where $a_i = (s_1, s_2, \cdots, s_i)$. The members of this chain are distinct. For suppose that $a_i = a_{i+1}$; then $(s_1, \cdots, s_i) = (s_1, \cdots, s_{i+1}.)$

Hence we have

$$s_{i+1} = s_{i+1}^{2} = (s_1 s_{i+1}, s_2 s_{i+1}, \cdots, s_{i+1}^{2}) = (s_1, \cdots, s_{i+1}) s_{i+1}$$
$$= (s_1, \cdots, s_i) s_{i+1} = (s_1 s_{i+1}, \cdots, s_i s_{i+1}) = z.$$

This contradicts Definition 8.2. Hence $a_1 \subset a_2 \subset \cdots$ is an infinite ascending chain contradicting the ascending chain condition.

THEOREM 9.1. *Let $\mathfrak{S}$ be a semi-simple lattice. Then if each element of $\mathfrak{S}$ can be expressed as a union of simple elements, $\mathfrak{S}$ is a Boolean algebra.*

Proof. Let $a \in \mathfrak{S}$ have the representation

$$a = (s_1, \cdots, s_n) \tag{9.1}$$

where $s_1, \cdots, s_k$ are distinct simple elements. The representation (9.1) is unique and $s_1, \cdots, s_k$ are the only simple elements which a divides. For let $a = (s_1, \cdots, s_k) = (s_1', \cdots, s_l')$. Multiplying by s_i' we have $s_i' = (s_1 s_i', \cdots, s_k s_i')$. Hence all of the products are null except one, say $s_i s_i'$. Then $s_i s_i' = s_i'$ and hence $s_i \supset s_i'$ by Lemma 8.3. Thus $s_i = s_i'$ and $k = l$. If $a \supset s$, where s is simple and not equal to any of $s_1, \cdots, s_k$, then $(s_1, s_2, \cdots, s_k) = (s_1, \cdots, s_k, s)$ contrary to the result we have just obtained.

We show now that the product of any two elements is equal to their crosscut.

We clearly have $[a, b] \supset ab$. Let $[a, b] = (s_1, \cdots, s_k)$. Then since $a, b \supset [a, b]$, $a = (s_1, s_2, \cdots, s_k, a')$ and $b = (s_1, \cdots, s_k, b')$. Hence

$$ab = (s_1, \cdots, s_k, a')(s_1, \cdots, s_k, b') = (s_1, \cdots, s_k, a'b') \supset [a, b].$$

Thus $[a, b] = ab$.

Since the product is distributive with respect to union, the crosscut must be distributive and hence $\mathfrak{S}$ is distributive. Furthermore $\mathfrak{S}$ is complemented. For let $a = (s_1, \cdots, s_k)$, $u = (s_1, \cdots, s_n)$ and define $a' = (s_{k+1}, \cdots, s_n)$. Then $(a, a') = u$ and $[a, a'] = aa' = (s_1, \cdots, s_n)(s_{k+1}, \cdots, s_n) = z$. Hence $\mathfrak{S}$ is a Boolean algebra.

In an arbitrary semisimple lattice, the set of elements which can be represented as a union of simple elements need not be closed with respect to crosscut as we shall show by an example. However, if we assume the modular* condition we have the following theorem.

THEOREM 9.2. *Let $\mathfrak{S}$ be a modular semisimple lattice. Then the simple elements of $\mathfrak{S}$ generate a Boolean algebra $\mathfrak{S}_B$. Moreover $\mathfrak{S}_B$ is dense in $\mathfrak{S}$.*

Proof. Let U be the set of all elements of $\mathfrak{S}$ which can be expressed as a

* For various statements of the modular axiom see Ore [4].

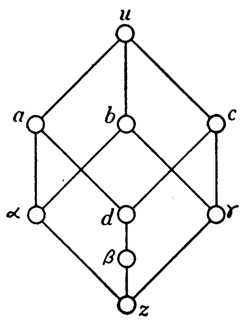
union of simple elements of $\mathfrak{S}$. The set U is obviously closed with respect to union. We shall show that U is dense in $\mathfrak{S}$ and hence closed with respect to crosscut. Let $(s_1, \cdots, s_n) \supset x$, and let $x \supset s_1, \cdots, s_l$, $x \not\supset s_{l+1}, \cdots, s_n$. Then $x = [x, (s_1, \cdots, s_n)] = (s_1, \cdots, s_l, [x, (s_{l+1}, \cdots, s_n)])$ by the modular condition. If $[x, (s_{l+1}, \cdots, s_n)] \neq z$, then there is a simple element s such that $[x, s_{l+1}, \cdots, s_n] \supset s$. But then $x \supset s$ and $(s_{l+1}, \cdots, s_n) \supset s$. Hence

$$s = s(s_{l+1}, \cdots, s_n) = (s_{l+1}s, \cdots, ss_n) = s_is$$

by Lemma 8.3. Thus $s = s_i$ and $x \supset s_i$ contrary to assumption. Hence $[x, (s_{l+1}, \cdots, s_n)] = z$ and $x = (s_1, \cdots, s_l)$.

Since U is dense in $\mathfrak{S}$, it is closed with respect to multiplication and is clearly semisimple. Moreover every element of U can be expressed as a union of simple elements. Hence by Theorem 9.1, $U = \mathfrak{S}_B$ is a Boolean algebra.

To show the significance of the modular condition in the previous theorem we give an example of a non-modular semisimple lattice in Fig. 4.



$\mathfrak{L}$

FIG. 4

If U denotes the set of elements of $\mathfrak{L}$ which can be expressed as a union of simple elements, we define a multiplication over $\mathfrak{L}$ as follows: If $x, y \in U$, $x \neq a$, $y \neq b$, then $xy = [x, y]$, $ac = \beta$, $dx = \beta$ or z according as $x \supset d$ or $x \not\supset d$. It can be readily verified that all of the multiplication postulates are satisfied. Also $\mathfrak{L}$ is non-modular since it contains the non-modular sublattice $\{a, \alpha, d, \beta, z\}$. The simple elements α, β, γ do not generate a Boolean algebra. In fact, U is not closed with respect to crosscut since $d = [(\alpha, \beta), (\beta, \gamma)]$.

THEOREM 9.3. *Let $\mathfrak{S}$ be a modular semisimple lattice. Then if for each simple element s there exists an element $s' \neq u$ such that $(s, s') = u$, $\mathfrak{S}$ is a Boolean algebra.*

Proof. We may take the s''s to be divisor-free elements since if s_i' is not divisor-free, there exists a divisor-free element f_i such that $f_i \supset s_i'$. But then

$(s_i, f_i) \supset (s_i, s_i') = u$. Let $v = (s_1, \cdots, s_n)$. Then the length of chain from v to z is n. But now $[s_1', s_2', \cdots, s_n'] = z$, since if $[s_1', \cdots, s_n'] \neq z$, there exists an s_i such that $[s_1', \cdots, s_n'] \supset s_i$. But then $s_i' \supset s_i$, which is impossible. Since $[s_1', \cdots, s_n'] = z$, the length of chain from u to z is equal to or less than n. But $u \supset v$. Hence $u = v$.

Theorem 9.3 gives immediately

THEOREM 9.4. *A complemented, modular, semisimple lattice is a Boolean algebra.*

We conclude with the statement of Theorem 9.3 in terms of the two-sided ideals of a non-commutative ring.

THEOREM 9.5. *Let R be a ring without radical in which the ascending and descending chain conditions hold for two-sided ideals. Then if for each two-sided ideal $\mathfrak{a}$ there exists an ideal $\mathfrak{a}' \neq R$ such that $(\mathfrak{a}, \mathfrak{a}') = R$, R is a direct sum of two-sided simple ideals.*

Such an ideal $\mathfrak{a}'$ always exists if $\mathfrak{a}$ has a principle unit. For in that case we may take $\mathfrak{a}'$ to be the set of all elements x such that $\mathfrak{a}x = 0$.

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