ON CIRCAVARIANT MATRICES AND CIRCA-EQUIVALENT NETWORKS

BY

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1. **Introduction.** In recent papers(1) the author has considered various questions concerning the equivalence of quadrics in m-affine n-space and related problems in the theory of (absolutely) equivalent m-terminal pair electrical networks.

The present paper is concerned with the development of certain theorems relating to the theory of congruent matrices which appear to be fundamental to the construction of a somewhat more general theory of (relative) equivalent electrical networks.

Consider the set of matrices B congruent to the matrix A; i.e., $B=P'AP$. In the first section of this paper a theory of circavariant matrices is initiated, general theorems being obtained relating to the restrictions which must be imposed on P in order that one or more of a certain set $A_1, A_1^2, \cdots$ of matrices derived from A each be circavariant. In later sections theorems on the congruence of matrices with P in a modified m-affine space are obtained, together with a set of normal forms.

In the last section, the theory of circavariant matrices is used to initiate a general theory of circa-equivalent networks, the usual theory of equivalent networks appearing as a special case of the general theory.

2. **Congruent and circavariant matrices.** Let $A, B, C, \cdots, P, Q, \bar{A}, \cdots$ be matrices of order n whose elements belong to a field $\mathfrak{F}$. The matrix B is said to be *equivalent*(2) to the matrix A if there exist nonsingular matrices P and Q such that $B=QAP$. The matrix B is said to be *congruent* to A if there exists a nonsingular matrix P such that $B=P'AP$.

Let $C_{r_1\cdots r_t}^{s_1\cdots s_t}$ denote the matrix obtained from C by deleting from C rows $r_1, \cdots, r_t$ and columns $s_1, \cdots, s_t$. Denote $C_{r_1\cdots r_t}^{r_1\cdots r_t}$ by $C_{r_1\cdots r_t}$.

Consider the set $\mathfrak{A}$ of all matrices A of order n whose elements range

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(1) Burington, Richard S., *On the equivalence of quadrics in m-affine n-space and its relation to the equivalence of 2m-pole networks*, these Transactions, vol. 38 (1935), pp. 163–176; hereafter called paper [1].

Burington, Richard S., *Matrices in electric circuit theory*, Journal of Mathematics and Physics, vol. 14 (1935), pp. 235–249; hereafter called paper [2].

Burington, Richard S., *R-matrices and equivalent networks* I, Journal of Mathematics and Physics, vol. 16 (1937), pp. 85–103; hereafter called paper [3].

(2) Throughout this paper it is understood, unless otherwise stated, that all the elements of all matrices used belong to a commutative field $\mathfrak{F}$ whose characteristic is not two.

over $\mathfrak{F}$. With each A associate the set $\mathfrak{B}$ of all matrices congruent to A. If B is any matrix of set $\mathfrak{B}$, there exists a nonsingular matrix P such that

$$B = P'AP. \tag{2.1}$$

If T ranges over the set $\mathfrak{P}$ of all nonsingular matrices of order n, then the matrices $\beta = T'AT$ are all congruent to A, and hence, each β belongs to $\mathfrak{B}$. The matrix P in (2.1) belongs to $\mathfrak{P}$. If (2.1) holds and if there exists a subset $\mathfrak{P}_c$ of $\mathfrak{P}$ such that for all matrices A of $\mathfrak{A}$, and for all matrices P of $\mathfrak{P}_c$,

$$B_{r_1\cdots r_t}^{s_1\cdots s_t} = P'^{r_1\cdots r_t}_{r_1\cdots r_t} A_{r_1\cdots r_t}^{s_1\cdots s_t} P_{s_1\cdots s_t}^{s_1\cdots s_t}, \tag{2.2}$$

then $A_{r_1\cdots r_t}^{s_1\cdots s_t}$ is called a *circavariant matrix* of A under the congruence (2.1). Let $\mathfrak{B}_c$ denote the subset of $\mathfrak{B}$ obtained by letting P range over $\mathfrak{P}_c$.

Thus, if (2.1) holds and if A_1 is a circavariant matrix, then B_1 can be obtained directly from the product $B_1 = P_1' A_1 P_1$, or from B by deleting the first row and column.

The term *circavariant matrix* has been introduced here to avoid confusion with the term *invariant matrix* as used by L. Schur, Littlewood and other writers, as in D. E. Littlewood, *The construction of invariant matrices*, Proceedings of the London Mathematical Society, (2), vol. 43 (1937), pp. 226–240. In paper [1], a *circavariant matrix* was called an *invariant matrix*. In contrast to the definition used in [1], the present definition places greater emphasis on the requirement that (2.2) hold for all matrices of $\mathfrak{A}$. While in [1] P was restricted to (simply) m-affine types, here P is not so constrained.

3. **Conditions that $A_{r_1\cdots r_t}^{s_1\cdots s_t}$ be circavariant.** In paper [1] a system of integer, matric and algebraic invariants of the matrix A of the n-ary quadratic form F was exhibited, under the simply m-affine nonsingular group of linear transformations T, by means of which necessary and sufficient conditions for the simply m-affine congruence with respect to T of two matrices A and B, as well as the equivalence of the two corresponding forms F and G, were given.

Whereas in paper [1] we were concerned with the nature of the matrix A for given simply m-affine matrices P, in the present paper we are concerned as to the content of the subset $\mathfrak{P}_c$, that is, as to the conditions which must be imposed on P in order that $A_{r_1\cdots r_t}^{s_1\cdots s_t}$ be a circavariant matrix of A for the class $\mathfrak{A}$ under (2.1). We shall see that the solution to this question leads to a more general type of matrix P than that used in paper [1].

To begin with, we search for conditions on P in order that A_1 be a circavariant matrix. In other words, with reference to congruence (2.1), under what conditions is the matrix $M = P_1' A_1 P_1$ identically equal to B_1 in the elements of A?

For convenience, we number the first row (and column) of A_1 (P_1, M and B_1) as 2, the second as 3, $\cdots$, the $(n-1)$-th as n. The rows (and columns) of A (P and B) are numbered in the usual way, the first row as 1, the second row as 2, $\cdots$, the nth row as n. Evidently,

$$(3.1)\qquad M \equiv \left(\sum_{r=2}^{n} \sum_{s=2}^{n} p_{rj} a_{rs} p_{sk} \right) \equiv (m_{jk}),$$

where the element in the jth row and kth column of M is m_{jk}, with rows and columns numbered (as agreed above) $j, k=2, \cdots, n$. Also from (2.1)

$$(3.2)\qquad B \equiv \left(\sum_{r=1}^{n} \sum_{s=1}^{n} p_{rj} a_{rs} p_{sk} \right) \equiv (b_{jk}),$$

where b_{jk} is the element in the jth row and kth column, $(j, k=1, 2, \cdots, n)$. In order that B_1 be identical to M in the elements of A, we must have

$$(3.3)\qquad b_{jk} \equiv \sum_{r=1}^{n} \sum_{s=1}^{n} p_{rj} a_{rs} p_{sk} \equiv \sum_{r=2}^{n} \sum_{s=2}^{n} p_{rj} a_{rs} p_{sk} \equiv m_{jk},$$

in the elements of A, for j and k any fixed pair of integers selected from $2, \cdots, n$. From (3.3) with j and k fixed, $(j, k=2, 3, \cdots, n)$, we find that the following identities in the elements of A must hold,

$$(3.4)\qquad \begin{aligned} p_{1j} a_{1s} p_{sk} &\equiv 0 \qquad (s = 1, 2, \cdots, n), \\ p_{rj} a_{r1} p_{1k} &\equiv 0 \qquad (r = 1, 2, \cdots, n). \end{aligned}$$

The cases $j=k$ with $s=1$ give $p_{1j} a_{11} p_{1j} \equiv 0$ in a_{11}, $(j=2, 3, \cdots, n)$, hence p_{1j} must vanish for $j=2, 3, \cdots, n$. With $p_{1j}=0$, $(j=2, \cdots, n)$, all the identities (3.4) are satisfied and A_1 is a circavariant matrix. Since P is nonsingular, $p_{11} \neq 0$. Hence

THEOREM 3.1. *A necessary and sufficient condition that A_1 be a circavariant matrix is that $p_{1j}=0, j=2, \cdots, n$.*

A matrix of the form

$$(3.5)\qquad S \equiv \left(\begin{array}{cccc:cccc} p_{11} & 0 & \cdots & 0 & 0 & & \cdots & 0 \\ 0 & p_{22} & \cdots & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & \cdot & \cdot & & & \cdot \\ 0 & \cdot & \cdots & p_{mm} & 0 & & \cdots & 0 \\ \hdashline p_{m+1,1} & \cdot & \cdots & p_{m+1,m} & p_{m+1,m+1} & & \cdots & p_{m+1,n} \\ \cdot & & & \cdot & \cdot & & & \cdot \\ \cdot & & & \cdot & \cdot & & & \cdot \\ p_{n,1} & \cdot & \cdots & p_{n,m} & p_{n,m+1} & & \cdots & p_{nn} \end{array} \right),$$

with elements in $\mathfrak{F}$ is said to be *m-affine*(3). If $p_{11}=p_{22}= \cdots =p_{mm}=1$, S is said to be *simply m-affine*.

(3) In paper [1], the term *m-affine* means *simply m-affine*.

The matrix B is said to be *m-affine congruent* to A if there exists an m-affine nonsingular matrix S with elements in $\mathfrak{F}$ such that $B=S'AS$.

Theorem 3.1 states that *a necessary and sufficient condition that A_1 be circavariant is that P be 1-affine.* If B is 1-affine congruent to A, then A_1 is circavariant and B_1 is congruent to A_1.

More generally, suppose that we require that A_u be circavariant. Then for j and k any fixed pair of integers selected from $1, 2, \cdots, u-1, u+1, \cdots, n$ the following identity in the elements of A must hold:

$$\sum_{r=1}^{n}\sum_{s=1}^{n} p_{rj}a_{rs}p_{sk} \equiv \sum_{r=1,r\neq u}^{n}\ \sum_{s=1,s\neq u}^{n} p_{rj}a_{rs}p_{sk}. \tag{3.6}$$

This means that the following identities in the elements of A must hold:

$$\begin{aligned} p_{uj}a_{us}p_{sk} &\equiv 0 \qquad (s = 1, 2, \cdots, n), \\ p_{rj}a_{ru}p_{uk} &\equiv 0 \qquad (r = 1, 2, \cdots, n). \end{aligned} \tag{3.7}$$

The cases $j=k\neq u$ with $s=u$ give $p_{uj}a_{uu}p_{uj}=0$, so that each $p_{uj}=0, j\neq u$. We conclude

THEOREM 3.2. *A necessary and sufficient condition that A_u be circavariant is that $p_{uj}=0$ for $j=1, \cdots, n; j\neq u$.*

We note that $d(P)=p_{uu}\cdot d(P_u)$. Since P is nonsingular $p_{uu}\neq 0$, $d(P_u)\neq 0$. From (2.2), we conclude that B_u is congruent to A_u. Hence *if A_u is circavariant, B_u is congruent to A_u.*

Suppose we require A_u^v, $u\neq v$, to be circavariant. Then for j and k any fixed pair of integers selected from $j=1, \cdots, u-1, u+1, \cdots, n$ and $k=1, \cdots, v-1, v+1, \cdots, n$, we must have

$$\sum_{r=1}^{n}\sum_{s=1}^{n} p_{rj}a_{rs}p_{sk} \equiv \sum_{r=1,r\neq u}^{n}\ \sum_{s=1,s\neq v}^{n} p_{rj}a_{rs}p_{sk}$$

identically in the elements of A; i.e.,

$$\begin{aligned} p_{uj}a_{us}p_{sk} &\equiv 0 \qquad (s = 1, \cdots, n), \\ p_{rj}a_{rv}p_{vk} &\equiv 0 \qquad (r = 1, \cdots, n). \end{aligned} \tag{3.8}$$

The cases $j=k=w$, $w\neq u$, $w\neq v$, with $s=u$ and $r=v$ give $p_{uw}a_{uu}p_{uw}=0$ and $p_{vw}a_{vv}p_{vw}=0$, so that $p_{uw}=p_{vw}=0$. The case $j=v$, $k=u$ gives

$$\begin{aligned} p_{uv}a_{us}p_{su} &\equiv 0 \qquad (s = 1, \cdots, n), \\ p_{rv}a_{rv}p_{vu} &\equiv 0 \qquad (r = 1, \cdots, n). \end{aligned} \tag{3.9}$$

Since P is nonsingular at least one $p_{su}\neq 0$. Hence $p_{uv}=0$. Likewise, at least one $p_{rv}\neq 0$, so that $p_{vu}=0$. The case A_v^u leads to the same result. We have

THEOREM 3.3. *A necessary and sufficient condition that A_u^v, $u \neq v$, be a circavariant matrix is that*

$$p_{u\alpha} = 0 \qquad (\alpha = 1, \cdots, n, \alpha \neq u);$$
$$p_{v\beta} = 0 \qquad (\beta = 1, \cdots, n, \beta \neq v).$$

This theorem shows that a necessary and sufficient condition that A_1^2 be circavariant is that P be 2-affine.

Evidently $d(P) = p_{uu}p_{vv} \cdot d(P_{uv})$. Since P is nonsingular, $p_{uu}p_{vv} \neq 0$, $d(P_{uv}) \neq 0$. Hence from (2.2), B_u^v is equivalent to A_u^v. Thus, *B_u^v is equivalent to A_u^v when A_u^v is circavariant.*

If A_{uv}, $u \neq v$, is a circavariant matrix, then for j and k any fixed numbers selected from $w = 1, \cdots, n$, $w \neq u$, $w \neq v$, we must have

$$\sum_{r=1}^{n} \sum_{s=1}^{n} p_{rj} a_{rs} p_{sk} \equiv \sum_{r=1, r \neq u, v}^{n} \quad \sum_{s=1, s \neq u, v}^{n} p_{rj} a_{rs} p_{sk}$$

identically in the elements of A. This means that

$$p_{uj} a_{us} p_{sk} \equiv 0, \qquad p_{vj} a_{vs} p_{sk} \equiv 0 \qquad (s = 1, 2, \cdots, n),$$
$$p_{rj} a_{ru} p_{uk} \equiv 0, \qquad p_{rj} a_{rv} p_{vk} \equiv 0 \qquad (r = 1, 2, \cdots, n) \tag{3.10}$$

identically in the elements of A. The cases $j = k = w = 1, \cdots, n$, $w \neq u$, $w \neq v$, with $s = u$, v give $p_{uw} a_{uu} p_{uw} = 0$, $p_{vw} a_{vv} p_{vw} = 0$. We conclude that $p_{uw} = p_{vw} = 0$. Since P is nonsingular, $p_{uu}p_{vv} - p_{uv}p_{vu} \neq 0$, and $d(P_{uv}) \neq 0$.

THEOREM 3.4. *A necessary and sufficient condition that A_{uv} be a circavariant matrix is that $p_{uw} = p_{vw} = 0$, for $w = 1, \cdots, n$, $w \neq u$, $w \neq v$.*

This theorem shows that a sufficient (but not necessary) condition that A_{12} be circavariant is that P be 2-affine.

Let J be the set $1, \cdots, n$ and let $u_1, u_2, \cdots, u_g$ be any subset U of J, all the elements of U being distinct. Denote by $W = J - U$ the set J with the elements of U removed. If r is not in W, we write $r \not\subset W$.

If $A_{u_1 \cdots u_g}$ is a circavariant matrix, then for j and k any fixed numbers selected from W, we must have

$$\sum_{r=1}^{n} \sum_{s=1}^{n} p_{rj} a_{rs} p_{sk} \equiv \sum_{r=1, r \not\subset W}^{n} \quad \sum_{s=1, s \not\subset W}^{n} p_{rj} a_{rs} p_{sk},$$

identically in the elements of A. This means that

$$p_{u_1 j} a_{u_1 s} p_{sk} \equiv 0, \cdots, p_{u_g j} a_{u_g s} p_{sk} \equiv 0 \qquad (s = 1, 2, \cdots, n),$$
$$p_{rj} a_{r u_1} p_{u_1 k} \equiv 0, \cdots, p_{rj} a_{r u_g} p_{u_g k} \equiv 0 \qquad (r = 1, 2, \cdots, n), \tag{3.11}$$

identically in the elements of A. The cases where $j = k$ and j ranges over W with $s = u_1, u_2, \cdots, u_g$ give

$$p_{u_1 j} a_{u_1 u_1} p_{u_1 j} \equiv 0, \; p_{u_2 j} a_{u_2 u_2} p_{u_2 j} \equiv 0, \cdots, \; p_{u_g j} a_{u_g u_g} p_{u_g j} = 0,$$

from which we conclude that $p_{u_1 j}=p_{u_2 j}= \cdots =p_{u_g j}=0$, for j ranging over W.

THEOREM 3.5. *A necessary and sufficient condition that $A_{u_1 \cdots u_g}$ be circavariant is that $p_{u_1 j}=p_{u_2 j}= \cdots =p_{u_g j}=0$ for j ranging over W.*

From P delete all the rows and columns whose numbers belong to the set W, leaving the matrix P_W. It is easy to see that $d(P)=d(P_W)\cdot d(P_{u_1\cdots u_g})$. Since $d(P)\neq 0$, P_W and $P_{u_1\cdots u_g}$ are both nonsingular, so that $B_{u_1\cdots u_g}$ is congruent to $A_{u_1\cdots u_g}$ when the latter is circavariant.

Let $(u_1, u_2, \cdots, u_t)$ and $(v_1, v_2, \cdots, v_t)$ be two subsets U and V, respectively, of the set I of integers $1, 2, \cdots, n$. Suppose that all the elements of U and V are distinct. Let $L=U-V$ be the set I with the elements of U and V removed. Then

THEOREM 3.6. *A necessary and sufficient condition that $A^{v_1\cdots v_t}_{u_1\cdots u_t}$ (or $A^{u_1\cdots u_t}_{v_1\cdots v_t}$) be circavariant is that P be such that for each u in U, each v in V, and for each λ in L, $p_{u\lambda}=p_{uv}=p_{vu}=0$.*

The proof of this theorem may be obtained from the proof of Theorem 3.3 as follows: equations (3.8) must hold with u ranging over U and v ranging over V. The cases $j=k=\lambda$ with $s=u$, $r=v$, u ranging over U and v over V, lead to the conditions $p_{u\lambda}=0$. The cases $j=v$, $k=u$, with u over U and v over V yield $p_{uv}=0$ and $p_{vu}=0$.

It is easy to see that *$B^{v_1\cdots v_t}_{u_1\cdots u_t}$ is equivalent to $A^{v_1\cdots v_t}_{u_1\cdots u_t}$ in case the latter is circavariant.* For example, if A^{32}_{14} is circavariant, P is such that

$$d(P)=d(E)\cdot d(F)\cdot d(P^{34}_{12})$$

where

$$E\equiv\begin{pmatrix} p_{11} & p_{12}\\ p_{21} & p_{22}\end{pmatrix} \quad\text{and}\quad F\equiv\begin{pmatrix} p_{33} & p_{34}\\ p_{43} & p_{44}\end{pmatrix}.$$

Since P is nonsingular so are E, F, and P^{34}_{12}. From (2.2), B^{34}_{12} is equivalent to A^{34}_{12}.

In a similar manner further theorems concerning the circavariance of $A^{v_1\cdots v_t}_{u_1\cdots u_t}$ for the case when the sets U and V overlap can be stated together with theorems concerning the equivalence of $B^{v_1\cdots v_t}_{u_1\cdots u_t}$ and $A^{v_1\cdots v_t}_{u_1\cdots u_t}$.

4. **Invariants.** From (2.1), $d(B)=[d(A)][d(P)]^2$. Hence, the determinant of A is a *relative invariant* of the set $\mathfrak{B}$ under (2.1) with P ranging over the set $\mathfrak{P}$. Since each P in $\mathfrak{P}$ is nonsingular, the rank of B is equal to the rank of A, so that *the rank of A is an integer invariant for the set $\mathfrak{B}$*. If the field $\mathfrak{F}$ is ordered, the signatures (when defined) of B and A are equal, so that for ordered fields, *the signature of A is an integer invariant for the set $\mathfrak{B}$.*

Suppose in (2.2), $P'_{r_1\cdots r_t}$ and $P_{s_1\cdots s_t}$ are nonsingular and that $A^{s_1\cdots s_t}_{r_1\cdots r_t}$ is a circavariant matrix. Consider any function $G(a_{ij})\equiv G$ of the elements of $A^{s_1\cdots s_t}_{r_1\cdots r_t}$ which is so related to the same function $G(b_{ij})\equiv\overline{G}$ of the elements of

$B^{s_1\cdots s_t}_{r_1\cdots r_t}$ that, in the elements a_{ij},

$$\overline{G} \equiv \alpha G \beta \qquad (\alpha\beta \neq 0), \tag{4.1}$$

where $\alpha \equiv \alpha(P'_{r_1\cdots r_t})$ is a function of the elements of $P'_{r_1\cdots r_t}$ only, and where $\beta = \beta(P_{s_1\cdots s_t})$ is a function of the elements of $P_{s_1\cdots s_t}$ only. Then G is said to be *a circavariant of the set* $\mathfrak{B}$ *with respect to* $A^{s_1\cdots s_t}_{r_1\cdots r_t}$.

If G, G_1^2, G_1, G_2, $\cdots$, $G^{s_1\cdots s_t}_{r_1\cdots r_t}$ are circavariants of the set $\mathfrak{B}$ with respect to the circavariant matrices A, A_1, A_1^2, A_2, $\cdots$, $A^{s_1\cdots s_t}_{r_1\cdots r_t}$, respectively, then any function H of the form

$$H(G) \equiv [G]^{\rho}[G_1]^{\rho_1}[G_1^2]^{\rho_2}[G_2]^{\rho_3} \cdots [G^{s_1\cdots s_t}_{r_1\cdots r_t}]^{\rho_n}, \tag{4.2}$$

where the ρ_i's are real numbers is said to be a *composite circavariant* of the set $\mathfrak{B}$. Let $H(\overline{G}) \equiv \overline{H}$ denote H with G, G_1, $\cdots$, $G^{s_1\cdots s_t}_{r_1\cdots r_t}$ replaced by $\overline{G}$, $\overline{G}_1$, $\overline{G}_1^2$, $\cdots$, $\overline{G}^{s_1\cdots s_t}_{r_1\cdots r_t}$, respectively. Then $\overline{H}$ is of the form

$$\overline{H} = \gamma H \delta \qquad (\gamma\delta \neq 0), \tag{4.3}$$

where

$$\gamma = [\alpha_1(P')][\alpha_2(P_1')] \cdots [\alpha_v(P'_{r_1\cdots r_t})], \quad \delta = [\beta_1(P)][\beta_2(P_1)] \cdots [\beta_v(P_{s_1\cdots s_t})].$$

If $\gamma\delta = 1$, then H is said to be an *absolute circavariant* of $\mathfrak{B}$. If $\gamma = \delta$, then H is said to be a *relative circavariant* of $\mathfrak{B}$.

Consider the set $\mathfrak{B}^{s_1\cdots s_t}_{r_1\cdots r_t}$ of all matrices $B^{s_1\cdots s_t}_{r_1\cdots r_t}$ generated from the circavariant matrix $A^{s_1\cdots s_t}_{r_1\cdots r_t}$ by letting P range over $\mathfrak{P}_\sigma$, with $P'_{r_1\cdots r_t}$ and $P_{s_1\cdots s_t}$ nonsingular. Then the rank $\rho^{s_1\cdots s_t}_{r_1\cdots r_t}$ of each matrix in $\mathfrak{B}^{s_1\cdots s_t}_{r_1\cdots r_t}$ is equal to the rank of $A^{s_1\cdots s_t}_{r_1\cdots r_t}$.

We suppose that $(r_1, \cdots, r_t) \equiv (s_1, \cdots, s_t)$. Then (2.2) is an ordinary congruence. Suppose the field $\mathfrak{F}$ is ordered and that a P exists in $\mathfrak{P}_\sigma$ for which $B_{r_1\cdots r_t}$ is a diagonal matrix, so that $A_{r_1\cdots r_t}$ has a signature $\sigma_{r_1\cdots r_t}$. From (2.2) it follows that the signature of each matrix in the set $\mathfrak{B}_{r_1\cdots r_t}$ is equal to $\sigma_{r_1\cdots r_t}$. Thus,

THEOREM 4.1. *The rank of* $A^{s_1\cdots s_t}_{r_1\cdots r_t}$ *is an integer invariant for the set* $\mathfrak{B}^{s_1\cdots s_t}_{r_1\cdots r_t}$. *If* $\mathfrak{F}$ *is ordered, the signature (when defined) of* $A_{r_1\cdots r_t}$ *is an integer invariant for the set* $\mathfrak{B}_{r_1\cdots r_t}$.

If $A^{s_1\cdots s_t}_{r_1\cdots r_t}$ is a circavariant matrix, then from (2.2)

$$d(B^{s_1\cdots s_t}_{r_1\cdots r_t}) = d(P'_{r_1\cdots r_t}) \cdot d(A^{s_1\cdots s_t}_{r_1\cdots r_t}) \cdot d(P_{s_1\cdots s_t}). \tag{4.4}$$

From (4.4) it is clear that $d(A^{s_1\cdots s_t}_{r_1\cdots r_t})$ is a circavariant for the set $\mathfrak{B}$. If $(r_1, \cdots, r_t) \equiv (s_1, \cdots, s_t)$,

$$d(B^{r_1\cdots r_t}_{r_1\cdots r_t}) = [d(P_{r_1\cdots r_t})]^2 [d(A^{r_1\cdots r_t}_{r_1\cdots r_t})], \tag{4.5}$$

so that

THEOREM 4.2. *The determinants of the matrices of the set* $\mathfrak{B}^{s_1\cdots s_t}_{r_1\cdots r_t}$ *are circavariants for the set* $\mathfrak{B}$ *with respect to* $A^{s_1\cdots s_t}_{r_1\cdots r_t}$. *If* $(r_1, \cdots, r_t) \equiv (s_1, \cdots, s_t)$, *these determinants are relative circavariants of* $\mathfrak{B}$.

It may be remarked that in case $(r_1, \cdots, r_t) \equiv (s_1, \cdots, s_t)$ these determinants are actually ordinary relative invariants of $\mathfrak{B}^{r_1\cdots r_t}_{r_1\cdots r_t}$ under an ordinary congruence of transformation matrix $P_{r_1\cdots r_t}$.

Evidently the ratio of any two circavariants is a composite circavariant for the set $\mathfrak{B}$.

5. **Normal forms for A under P m-affine.** In the theory of electrical networks the cases when $A_1, A_1^2, A_2, \cdots$ are to be circavariant frequently occur, leading to the requirement that P be m-affine. We shall accordingly consider the normal forms of A under P m-affine.

In paper [1] the reduction of A to normal forms was indicated, the case where $m=2$ being considered in detail. In particular, the results obtained indicate that, when P is simply m-affine, every symmetric matrix A with elements in a field $\mathfrak{F}$ (not of characteristic two) with circavariant matrix $A_{1,\cdots,m-1}$ is m-affine congruent in $\mathfrak{F}$ to a matrix B in which the matrix $B_{1,\cdots,m-1}$ is of the form

$$(5.1)\qquad \begin{pmatrix} b_{mm}\cdot & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & \cdot & b_{m+1} & 0 & \cdots & 0 \\ 0 & \cdot & 0 & b_{m+2} & 0\cdots & 0 \\ \cdot & \cdot & \cdot & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & \cdot & 0 & \cdot & \cdot\cdot b_{n-1} & 0 \\ 0 & \cdot & 0 & 0 & \cdot\cdot 0 & b_n \end{pmatrix}, \quad \text{if} \quad \nu = \rho_{1\cdots m-1} - \rho_{1\cdots m} \neq 2,$$

and with a parabolic matrix

$$(5.2)\qquad \begin{pmatrix} 0 & \cdot & 0 & 0 & \cdot\cdot 0 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & \cdot & b_{m+1} & 0 & \cdot\cdot 0 & 0 \\ 0 & \cdot & 0 & b_{m+2} & 0\cdot\cdot & 0 \\ \cdot & \cdot & \cdot & 0 & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & \cdot & 0 & 0 & \cdot\cdot b_{n-1} & 0 \\ 1 & \cdot & 0 & 0 & \cdot\cdot 0 & 0 \end{pmatrix}, \quad \text{if } \nu = 2,$$

the number of nonzero b_i's in $B_{1\cdots m}$ being equal to the rank $\rho_{1\cdots m}$ of $A_{1\cdots m}$.

The parameter b_{mm} is an absolute invariant of A when P is simply m-affine.

We shall let $\sigma_{1\cdots m} \equiv \sigma_{1\cdots m}^{1\cdots m}$ denote the signature of $A_{1\cdots m}$. If the field $\mathfrak{F}$ is real, each positive b_i in $B_{1\cdots m}$ can be reduced (by means of a simply m-affine P) to 1, and each negative b_i to -1. The number of positive b_i's in $B_{1\cdots m}$ is $(\rho_{1\cdots m}+\sigma_{1\cdots m})/2$ and the number of negative b_i's is $(\rho_{1\cdots m}-\sigma_{1\cdots m})/2$, the remaining b_i's each being zero. If $\mathfrak{F}$ is algebraically closed, each nonzero b_i in $B_{1\cdots m}$ can be reduced to 1. No further reduction of $B_{1\cdots m-1}$ is possible when P is simply m-affine. For (5.1) and (5.2), we shall denote the reduced form of $B_{1\cdots m}$ thus obtained by $\delta \equiv [\delta_{m+1}, \cdots, \delta_{r-1}, 0, \cdots, 0]$, a diagonal matrix.

In case $\mathfrak{F}$ is ordered, we shall agree to *regularly arrange*(4) the matrix δ, this always being possible when P is m-affine.

Suppose in (5.1), $b_{mm} \neq 0$. Let P be m-affine with $p_{rs}=\delta_{rs}$, (where $\delta_{rs}=0$ if $r \neq s$, $\delta_{rs}=1$ if $r=s$), except for p_{mm}. If $\mathfrak{F}$ is algebraically closed, select a p_{mm} so that $p_{mm}^2=1/b_{mm}$. If $\mathfrak{F}$ is real, let $p_{mm}=1/(b_{mm})^{1/2}$ if $b_{mm}>0$ and $p_{mm}=1/(-b_{mm})^{1/2}$ if $b_{mm}<0$. Then, in case of (5.1) with $b_{mm} \neq 0$, the matrix B is m-affine congruent to a matrix $C=P'BP$ in which the matrix $C_{1,\cdots,m-1}$ is of the form

$$
\begin{Vmatrix}
b_m & \cdot & 0 & 0\cdot\cdot & \cdots & 0 \\
\cdot & \cdot & \cdot & \cdots & \cdots & \cdot \\
0 & \cdot & \delta_{m+1} & 0\cdot\cdot & \cdots & 0 \\
0 & \cdot & 0 & \cdots & \cdots & 0 \\
\cdot & \cdot & \cdot & \cdots & \cdots & \cdot \\
\cdot & \cdot & \cdot & \cdot\cdot\,\delta_{r-1} & \cdots & \cdot \\
\cdot & \cdot & \cdot & \cdots & 0\cdot\cdot & \cdot \\
\cdot & \cdot & \cdot & \cdots & \cdots & \cdot \\
\cdot & \cdot & \cdot & \cdots & \cdots & \cdot \\
0 & \cdot & 0 & \cdots & \cdots & 0
\end{Vmatrix}, \qquad \text{if } \nu \neq 2, \tag{5.3}
$$

where b_m is 1 if $\mathfrak{F}$ is algebraically closed; and $b_m=1$ or -1 if $\mathfrak{F}$ is real. In the latter case, $b_{mm}=1$ when $\sigma_{1\cdots m-1}=1+\sigma_{1\cdots m}$, $b_{mm}=-1$ when $\sigma_{1\cdots m-1}=-1+\sigma_{1\cdots m}$, and $b_{mm}=0$ when $\sigma_{1\cdots m-1}=\sigma_{1\cdots m}$.

As in paper [1], it is now easy to formulate various theorems. For example, Theorem 3.3 of [1] for P m-affine holds without the requirement on the parameters b_{mm} and b'_{mm}.

THEOREM 5.1. *Let P be m-affine with elements in an algebraically closed field $\mathfrak{F}$, and let $A^{(1)}$ and $A^{(2)}$ be two symmetric matrices of order n in $\mathfrak{F}$ with associated circavariant matrices $A^{(1)}_{1,\cdots,m-1}$ and $A^{(2)}_{1,\cdots,m-1}$. A necessary and sufficient condition that $A^{(1)}_{1,\cdots,m-1}$ and $A^{(2)}_{1,\cdots,m-1}$ be congruent is that they have the*

(4) C. C. MacDuffee, *Theory of Matrices*, Berlin, 1933, pp. 57–58.

same ranks $\rho^{(1)}_{1,\cdots,m-1}$, $\rho^{(1)}_{1,\cdots,m}$, *and* $\rho^{(2)}_{1,\cdots,m-1}$, $\rho^{(2)}_{1,\cdots,m}$, *respectively. If the field* $\mathfrak{F}$ *is real the additional requirement of the equality of the signatures* $\sigma^{(1)}_{1,\cdots,m-1}$, $\sigma^{(1)}_{1,\cdots,m}$ *and* $\sigma^{(2)}_{1,\cdots,m-1}$, $\sigma^{(2)}_{1,\cdots,m}$, *respectively, must be met.*

Case $m=2$. If $m=2$, it was shown in [1] that the symmetric matrix A is simply 2-affine congruent to one of the various normal forms $f_1, f_2, \cdots, f_5$ given below and as indicated in Table I (δ regularly arranged):

$$f_1 \equiv \begin{pmatrix} 0 & 0 & \cdot & 0 & \cdots & 0 & 1 \\ 0 & b_{22} & \cdot & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdot & & & & \\ \vdots & \vdots & \vdots & & \delta & & \\ 0 & 0 & \cdot & & & & \\ 1 & 0 & \cdot & & & & \end{pmatrix}, \qquad f_2 \equiv \begin{pmatrix} b_{11} & b_{12} & \cdot & 0 & \cdots & 0 \\ b_{21} & b_{22} & \cdot & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdot & & & \\ \vdots & \vdots & \vdots & & \delta & \\ 0 & 0 & \cdot & & & \end{pmatrix},$$

$$f_3 \equiv \begin{pmatrix} 0 & 0 & \cdot & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & \cdot & 0 & \cdots & 0 & 0 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & & & & & \\ \vdots & \vdots & \vdots & & & \delta & & \\ 0 & 0 & \cdot & & & & & \\ 1 & 0 & \cdot & & & & & \\ 0 & 1 & \cdot & & & & & \end{pmatrix}, \qquad f_4 \equiv \begin{pmatrix} b_{11} & 0 & \cdot & 0 & \cdots & 0 & b_{1n} \\ 0 & 0 & \cdot & 0 & \cdots & 0 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdot & & & & \\ \vdots & \vdots & \vdots & & \delta & & \\ 0 & 0 & \cdot & & & & \\ b_{1n} & 1 & \cdot & & & & \end{pmatrix},$$

$$f_5 \equiv \begin{pmatrix} b_{11} & 0 & \cdot & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdot & 0 & \cdots & 0 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdot & & & & \\ \vdots & \vdots & \vdots & & \delta & & \\ 0 & 0 & \cdot & & & & \\ 0 & 1 & \cdot & & & & \end{pmatrix}, \qquad \delta \equiv \begin{pmatrix} \delta_3 & 0 \cdot \cdot & \cdots & 0 \\ 0 & \cdots & \cdots & \cdot \\ \cdot & \cdot \cdot \delta_{r-1} & \cdots & \cdot \\ \cdot & \cdots & 0 \cdot \cdot & \cdot \\ 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

If P is 2-affine and nonsingular, it is possible through the proper selection of p_{11} and p_{22} to reduce certain of the matrices $f_1, f_2, \cdots, f_5$ yielding new matrices $g_1, g_2, \cdots, g_5$ having the same general form as $f_1, \cdots, f_5$, respectively, each g_i being 2-affine congruent to f_i $(i=1, \cdots, 5)$. The forms $g_1, \cdots, g_5$ are indicated in Table I. Thus, in the case of f_1, with $\rho_1 = r-2$, g_1 is f_1 with b_{22} replaced by 1 if $\mathfrak{F}$ is algebraically closed, and with b_{22} replaced by 1 or -1 if $\mathfrak{F}$ is real. No further reduction of f_1 is possible by a 2-affine P which preserves the form g_1. The numbers δ_{11}, δ_{22} in Table I denote 1 if $\mathfrak{F}$ is algebraically closed, and denote 1 or -1 if $\mathfrak{F}$ is real. In Case 3, the parameter b_{22} is an absolute invariant. It should be noted that the number of such parameters appearing in the g_i's is just one, whereas in the simply 2-affine case there were several such parameters, b_{ij}, in the forms $f_1, \cdots, f_5$. (See Table I, p. 171, paper [1], which may be constructed from Table I, as here given, by deleting the δ's and by replacing each 1 by the symbol $\neq 0$, and each f_i by g_i.)

TABLE I

Classification of matrix A for the case $m=2$

	$\rho_{12}=r-3,\ r=3, 4, \cdots, n+1$						P 2-affine				
Case	$\rho_1-\rho_{12}$	$\rho_2-\rho_{12}$	ρ	ρ_1	ρ_2	b_1^2	b_{11}	b_{12}	b_{22}	b_{1n}	Form
1	$\neq 2$	$=2$		$r-2$					$\delta_{22}\neq 0$		g_1
2	$\neq 2$	$=2$		$r-3$					0		g_1
3	$\neq 2$	$\neq 2$		$r-2$	$r-2$	$r-2$	$\delta_{11}\neq 0$	1	$b_{22}\neq 0$		g_2
4	$\neq 2$	$\neq 2$		$r-3$	$r-2$	$r-2$	$\delta_{11}\neq 0$	1	0		g_2
5	$\neq 2$	$\neq 2$		$r-3$	$r-2$	$r-3$	$\delta_{11}\neq 0$	0	0		g_2
6	$\neq 2$	$\neq 2$		$r-2$	$r-2$	$r-3$	$\delta_{11}\neq 0$	0	$\delta_{22}\neq 0$		g_2
7	$\neq 2$	$\neq 2$		$r-2$	$r-3$	$r-2$	0	1	$\delta_{22}\neq 0$		g_2
8	$\neq 2$	$\neq 2$		$r-3$	$r-3$	$r-2$	0	1	0		g_2
9	$\neq 2$	$\neq 2$		$r-3$	$r-3$	$r-3$	0	0	0		g_2
10	$\neq 2$	$\neq 2$		$r-2$	$r-3$	$r-3$	0	0	$\delta_{22}\neq 0$		g_2
11	$=2$	$=2$	$r+1$								g_3
12	$=2$	$=2$	r				$\delta_{11}\neq 0$			1	g_4
13	$=2$	$=2$	$r-1$				0			1	g_4
14	$=2$	$\neq 2$	r		$r-2$		$\delta_{11}\neq 0$				g_5
15	$=2$	$\neq 2$	$r-1$		$r-3$		0				g_5

If $\mathfrak{F}$ is real, each form in Table I can be subdivided according to the signatures of A_{12}, A_1, A_2.

The following theorems are now evident:

THEOREM 5.2. *A symmetric matrix A with elements in a field $\mathfrak{F}$ is 2-affine congruent in $\mathfrak{F}$ to one of the forms $g_1, \cdots, g_5$, according to the ranks (and signatures if $\mathfrak{F}$ is real) of the circavariant matrices A, A_1, A_2, A_1^2, A_{12} as indicated in Table* I. *A is simply 2-affine congruent to one of the forms $f_1, \cdots, f_5$ as indicated in Table* I.

THEOREM 5.3. *A necessary and sufficient condition for the 2-affine congruence of two matrices A and C whose elements belong to the real field is that the circavariant matrices A, A_1, A_2, A_1^2, A_{22} and C, C_1, C_2, C_1^2, C_{22} have the same ranks and signatures, respectively, and that in Case 3 with P simply affine (Table* I, *paper* [1]) *the parameters b_{22} and $\bar{b}_{22}$, for A and C respectively, be identically equal. If $\mathfrak{F}$ is algebraically closed, the above holds without the signatures.*

Case $m=m$. The reduction of A to normal forms for P m-affine may be done in a manner similar to that used above for the case when $m=2$.

6. **Applications to the theory of forms and geometry.** It is a simple matter to translate the results of this paper into the language of the theory of bilinear forms under cogredient m-affine transformations.

A geometric study of the locus $F\equiv\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}x_ix_j=0$ in a geometry built upon the group of transformations $x_r=\sum_{s=1}^{n}p_{rs}y_s$, $r=1, \cdots, n$, with (p_{rs}) m-affine can also be made.

7. **Relation to linear networks** [2], [3]. We consider an m-terminal pair bilateral n-mesh linear electrical network $\mathfrak{N}$ containing (lumped) resistances, inductances, and capacitances. Let $E_1, \cdots, E_m$ be the (complex) e.m.f.'s impressed on terminal pairs $1, \cdots, m$, respectively; Q_s and I_s, the (complex) charge and (complex) current, respectively, in mesh s; R_{st}, L_{st}, D_{st} (real numbers), the lumped circuit parameters (resistance, inductance, and elastance, respectively) for mesh s if $s=t$, common to meshes s and t if $s\neq t$. The meshes are so chosen that mesh s, $(s=1, \cdots, m)$, is the only one which passes through the terminal pair s.

Suppose the Kirchhof equations in complex form for the network are

$$B\{Q\} = \{E\}, \tag{7.1}$$

where $B\equiv(b_{rs})$, $b_{rs}=b_{sr}=L_{rs}\lambda^2+R_{rs}\lambda+D_{rs}$, $\lambda=j\omega$, ω being the (real) frequency.

If B is nonsingular,

$$\{Q\} = B^{-1}\{E\}. \tag{7.2}$$

Let $B^{-1}\equiv C\equiv(c_{uv})$. The element c_{uv} is called the *generalized (complex) network admittance*; being a transfer admittance between meshes u and v if $u\neq v$, and a driving-point admittance for mesh u if $u=v$.

Let $Y\equiv(Y_{st})$ be C with all but the first m rows and first m columns deleted. Then

$$\{Q\}_m = Y\{E\}_m, \tag{7.3}$$

where $\{Q\}_m$ and $\{E\}_m$ are the first m rows of $\{Q\}$ and $\{E\}$, respectively. The matrix Y is called a *characteristic (admittance) coefficient matrix* for $\mathfrak{N}$ [2].

Let $\mathfrak{N}_1$ and $\mathfrak{N}_2$ be two m-terminal pair networks of characteristic matrices $Y^{(1)}$ and $Y^{(2)}$, respectively. $\mathfrak{N}_2$ *is said to be circa-equivalent to* $\mathfrak{N}_1$ if there exists a real nonsingular diagonal matrix D such that for all values of λ, $Y^{(1)}=D'Y^{(2)}D$.

It should be noted here that this definition is much more general than the one heretofore used in the theory of equivalent electrical networks. The usual definition, the one given in papers [1], [2], and [3], is a very simple case of the one given in the present paper, being merely the type of circa-equivalence for which D is the identity matrix.

If $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are circa-equivalent, then the admittances $Y_{rs}^{(2)}$, $(r, s=1, \cdots, m)$, are relative circavariants.

Let the diagonal element in the rth row of D be d_r, and let $\Sigma \equiv (\sigma_{rs})$ where $\sigma_{rs} = d_r d_s$. If all of the elements in row k of Σ are equal, $\mathfrak{N}_2$ *is said to be relatively equivalent to* $\mathfrak{N}_1$ *with respect to terminal pair* k. If each element in the kth row of Σ is equal to one, $\mathfrak{N}_2$ *is said to be absolutely equivalent to* $\mathfrak{N}_1$ *with respect to terminal pair* k. If all the elements of Σ are equal to a number σ, $\mathfrak{N}_2$ *is relatively equivalent to* $\mathfrak{N}_1$; and in case $\sigma=1$, $\mathfrak{N}_2$ *is absolutely equivalent to* $\mathfrak{N}_1$.

Consider the set $\mathfrak{M}$ of all m-terminal n-mesh networks. Let $\mathfrak{N}(A)$ be an arbitrary network in $\mathfrak{M}$ having A for a network matrix. For each $\mathfrak{N}$ in $\mathfrak{M}$ select a possible network matrix. Let $\mathfrak{N}$ range over $\mathfrak{M}$. Denote the totality of network matrices so found by $\mathfrak{A}$. With each $\mathfrak{N}(A)$ associate the set $\mathfrak{K}$ of all networks $\mathfrak{N}(B)$ whose matrices B are congruent to A,

$$(7.4) \qquad B = P'AP,$$

where P is restricted to the real field. Let $\mathfrak{P}$ denote the set of all P's which satisfy the above requirements.

Next, with $\mathfrak{N}(A)$ arbitrary, let $\mathfrak{N}(A_k)$ and $\mathfrak{N}(B_k)$ denote the networks, having matrices A_k and B_k, respectively, obtained by opening the mesh k of $\mathfrak{N}(A)$ and $\mathfrak{N}(B)$, respectively. We select a maximal subset $\mathfrak{P}_c$ of $\mathfrak{P}$ such that for all $\mathfrak{N}(A)$ of $\mathfrak{M}$ (that is, for all A of $\mathfrak{A}$), and for all P of $\mathfrak{P}_c$, $B_k = P_k' A_k P_k$, for $k=1, \cdots, m$. In other words, we restrict P to a set $\mathfrak{P}_c$ for which $A_1, A_2, \cdots, A_m$ are circavariant matrices of A. We denote by $\mathfrak{K}_c$ the subset of $\mathfrak{K}$ whose matrices $B, B_1, B_2, \cdots, B_m$ are thus related to $A, A_1, A_2, \cdots, A_m$.

By Theorem 3.2 P must be m-affine. From Theorems 3.3 and 3.4 we know that A_u^v, $(u, v=1, \cdots, m)$, are also circavariant.

The characteristic coefficient matrices for $\mathfrak{N}(B)$ and $\mathfrak{N}(A)$ are

$$(7.5) \qquad Y^{(B)} \equiv (Y_{rs}^{(B)}), \qquad Y^{(A)} \equiv (Y_{rs}^{(A)}),$$

where

$$Y_{rs}^{(B)} = (-1)^{r+s} \frac{d(B_s^r)}{d(B)}, \quad Y_{rs}^{(A)} = (-1)^{r+s} \frac{d(A_s^r)}{d(A)} \qquad (r, s = 1, \cdots, m).$$

If P is m-affine, by Theorem 4.2 we know that the determinants in $Y_{rs}^{(B)}$ are circavariants of the set $\mathfrak{K}$ with respect to A_s^r. In fact, the admittances

$$Y_{rs}^{(B)} = (-1)^{r+s} \frac{d(P_s') \cdot d(A_s^r) \cdot d(P_r)}{d(P) \cdot d(A) \cdot d(P)} = p_{rr}^{-1} y_{rs}^{(A)} p_{ss}^{-1} \qquad (r, s = 1, \cdots, m)$$

are all relative circavariants. Evidently,

$$(7.6) \qquad Y^{(A)} = D'Y^{(B)}D,$$

where

$$D \equiv [p_{11}, \cdots, p_{mm}].$$

This shows that *every network* $\mathfrak{N}(B)$ *of* $\mathfrak{L}_c$ *is circa-equivalent to* $\mathfrak{N}(A)$.

Case $m=2$. Those networks of $\mathfrak{L}_c$ for which $p_{rr}=1$ are absolutely equivalent to $\mathfrak{N}(A)$ with respect to terminal pair r; those for which $p_{11}=p_{22}\neq 1$ are relatively equivalent; and those for which $p_{11}=p_{22}=1$ are absolutely equivalent. By selecting P so that $p_{11}p_{22}=1$, $p_{11}\neq 1$, the transfer admittances of the corresponding $\mathfrak{L}_c$ will all be absolutely circavariant, though the driving-point admittances are only relative circavariants.

If we select a subset $\mathfrak{L}'$ of $\mathfrak{L}$ so that A_1^2 is circavariant, then $Y_{12}^{(A)}$, the transfer admittance between meshes 1 and 2, will be a relative circavariant. The requirement that A_1^2 be circavariant makes P 2-affine so that $\mathfrak{L}'$ is $\mathfrak{L}_c$, and $Y_{11}^{(A)}$ and $Y_{22}^{(A)}$ are also relative circavariants.

If we select a subset $\mathfrak{L}_d$ of $\mathfrak{L}$ for which A_1 is circavariant, then the driving-point admittance $Y_{11}^{(A)}$ is relatively circavariant, but $Y_{12}^{(A)}$ and $Y_{22}^{(A)}$ are not necessarily so.

A subset $\mathfrak{L}_c$ of $\mathfrak{L}$ for which A_1 and A_2 are circavariant makes the admittances $Y_{11}^{(A)}$ and $Y_{22}^{(A)}$ relative circavariants, with P 2-affine; A_1^2 is then circavariant, so that the admittance $Y_{12}^{(A)}$ is also relatively circavariant.

Further results. More generally, in the case of m-terminal pair networks, if $A_{u_1}^{v_1}$, $A_{u_2}^{v_2}$, $\cdots$, A_u, $\cdots$ be circavariant, then the admittances $Y_{u_1v_1}^{(A)}$, $Y_{u_2v_2}^{(A)}$, $\cdots$, $Y_{uu}^{(A)}$, $\cdots$ are relatively circavariant.

The general theory of circa-equivalent networks initiated herein will be developed in greater detail at a later time. It should be noted that the special case when D is the identity matrix yields the usual theory of (absolutely) equivalent networks.

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