

ON A THEOREM OF SCHUR AND ON FRACTIONAL INTEGRALS OF PURELY IMAGINARY ORDER

BY

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1. Let $L_p(a, b)$ be the space of all functions $f(y)$ whose p th power is integrable over (a, b) or which are measurable and essentially bounded over (a, b) for $1 \leq p < \infty$ or $p = \infty$ respectively, with the norm

$$|f|_p = \left\{ \int_a^b |f(y)|^p dy \right\}^{1/p} \quad [1 \leq p < \infty],$$

$$|f|_p = \text{ess. u.b. } |f(y)| \quad [p = \infty];$$

let $p' = p/(p-1)$ and

$$L_p = L_p(0, \infty).$$

The following theorem is in substance due to I. Schur⁽¹⁾:

Let $K(x, y)$ be homogeneous of degree -1 and $K(x, y) \geq 0$ for $0 < x < \infty$, $0 < y < \infty$, let $K(x, y)x^{-1/2} \in L_1$, and let $f(x) \in L_2$; then

$$\left| \int_0^\infty K(x, y)f(x)dx \right|_2 \leq \kappa |f(y)|_2,$$

where

$$\kappa = \int_0^\infty K(x, 1)x^{-1/2}dx = \int_0^\infty K(1, y)y^{-1/2}dy.$$

The constant κ is the best possible.

Of course the inequality is true when $K(x, y)$ takes negative or even complex values also, if we replace κ by

$$\tilde{\kappa} = \int_0^\infty |K(x, 1)| x^{-1/2}dx = \int_0^\infty |K(1, y)| y^{-1/2}dy.$$

However $\tilde{\kappa}$ is not the best possible constant any more. We shall give a better

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⁽¹⁾ Journal für die reine und angewandte Mathematik, vol. 140 (1911), pp. 1-28. The corresponding theorem for $f \in L_p$, $1 < p < \infty$, was proved by G. H. Hardy, J. E. Littlewood, and G. Pólya, vide *Inequalities*, Cambridge, 1934, Theorem 319.

theorem for this case and we shall use it to deal with fractional integrals the order of which is an imaginary number, thus filling a gap in the literature.

Throughout this paper we denote constants depending on the given parameters by the single symbol C ; α and β denote finite numbers such that $\Re(\alpha) > 0$, $\Re(\beta) = 0$.

2. THEOREM I. Let (i) $K(x, y)$ be homogeneous of degree -1 , (ii) $K(x, 1)x^{-1/2} \in L_1$, (iii) $f(x) \in L_2$; then the function

$$Wf = \int_1^\infty K(x, y)f(x)dx$$

exists for almost all values of y in $(0, \infty)$, and

$$\|Wf\|_2 \leq \kappa_0 \|f\|_2 = \max_{-\infty < \tau < \infty} |\omega(\tau)| \cdot \|f\|_2,$$

where

$$\omega(\tau) = \int_0^\infty K(x, 1)x^{-1/2-i\tau}dx = \int_0^\infty K(1, y)y^{-1/2+i\tau}dy, \quad \kappa_0 = \max_{-\infty < \tau < \infty} |\omega(\tau)|.$$

The constant is the best possible.

Obviously $\kappa_0 \leq \tilde{\kappa}$; when $K(x, y) \geq 0$ then $\kappa_0 = \kappa$, as we may see taking $\tau = 0$.

Proof. Without loss of generality we may suppose $K(x, y)$ to be no null-function; then $\kappa_0 > 0$. Let $1 < a < \infty$, $f(x, a) = f(x)$ in (a^{-1}, a) , $f(x, a) = 0$ otherwise, and let

$$M\phi = \lim_{N \rightarrow \infty} \text{sq.} \int_{1/N}^N \phi(x)x^{-1/2+i\tau}dx \quad [\phi \in L_2].$$

In consequence of Schur's theorem W is a bounded linear transformation in L_2 ; the Mellin transform M is a bounded linear transformation from $L_2(0, \infty)$ into $L_2(-\infty, \infty)$. We have

$$\begin{aligned} \int_0^\infty y^{-1/2+i\tau} W\{f(x, a)\} dy &= \int_0^\infty y^{-1/2+i\tau} dy \int_{1/a}^a K(x, y)f(x)dx \\ (2.1) \qquad \qquad \qquad &= \int_{1/a}^a f(x)x^{-1/2+i\tau}dx \int_0^\infty K(1, v)v^{-1/2+i\tau}dv \end{aligned}$$

when we put $y = vx$ and make use of the homogeneousness of $K(x, y)$; the interchanging of the integrations is justified by absolute convergence of the right-hand repeated integral. Since the left-hand integral exists, it must be equal to $MW\{f(x, a)\}$, therefore we have

$$MW\{f(x, a)\} = \omega(\tau)M\{f(x, a)\}.$$

Since $|f(x, a) - f(x)|_2 \rightarrow 0$ [$a \rightarrow \infty$], and $|\omega(\tau)| \leq \kappa_0 < \infty$, by the continuity of the operations M and W we get

$$(2.2) \quad MWf = \omega(\tau)Mf;$$

therefore $|MWf|_2 \leq \kappa_0 |Mf|_2$ in $L_2(-\infty, \infty)$. Now the operator $(2\pi)^{-1/2}M$ is isometric, and so we obtain the first assertion of the theorem.

The function $\omega(\tau)$ is continuous in consequence of (ii) and attains its maximum value at a finite point τ , since, by the Riemann-Lebesgue theorem, $\omega(\tau) \rightarrow 0$ [$\tau \rightarrow \pm \infty$]. Now let λ be any positive number smaller than κ_0 . Then we can easily show the existence of functions $f(x) \in L_2$ such that $|Wf|_2 > \lambda |f|_2$.

Let E be a set of measure $m(E) > 0$ such that $|\omega(\tau)| > \lambda$ in E and E is included in some finite interval. Take $\phi(\tau) = 1$ in E and $\phi(\tau) = 0$ otherwise, and let $f = M^{-1}\phi$. Then from (2.2) we have

$$\begin{aligned} \int_{-\infty}^{\infty} |MWf|^2 d\tau &= \int_E |\omega(\tau)|^2 d\tau > \lambda^2 m(E) \\ &= \lambda^2 \int_{-\infty}^{\infty} |\phi(\tau)|^2 d\tau = 2\pi\lambda^2 \int_0^{\infty} |f(x)|^2 dx, \\ \int_0^{\infty} |Wf|^2 dy &> \lambda^2 \int_0^{\infty} |f(x)|^2 dx. \end{aligned}$$

Hence the theorem is proved.

3. We could give an alternative proof by the theory of "general transforms," without making use of Schur's theorem. Let V be a transformation of the form

$$Vf = \int_0^{\infty} L(x, y)f(x)dx,$$

the infinite integral being defined in some sense. Then it turns out that, roughly speaking, the class of all transformations which are representable in the form $V_1 V_2$ is identical with the class of the transformations

$$Wf = \int_0^{\infty} K(x, y)f(x)dx,$$

where $K(x, y)$ is homogeneous of degree -1 . We leave that proof of I to the reader⁽²⁾.

⁽²⁾ W belongs to the so-called "product-class." We need the lemmas:

A. Let $y^{-1}\chi(y) \in L_2$, let $\omega(\tau) = (\frac{1}{2} - i\tau)M\{y^{-1}\chi(y)\}$ be essentially bounded in $(-\infty, \infty)$, and let $\chi(y)$ have the form $\chi(y) = \int_1^y H(\xi)d\xi + c$, where $c = \chi(1)$ is an arbitrary constant. Then, for any $f \in L_2$, the function

$$g(y) = Wf = \text{l.i.m. sq.}_{N \rightarrow \infty} \int_{1/N}^N H\left(\frac{y}{x}\right)f(x) \frac{dx}{x}$$

4. We replace the customary operators⁽³⁾

$$(4.1) \quad f_{\alpha}^{+}(y) = \frac{1}{\Gamma(\alpha)} \int_0^y (y-x)^{\alpha-1} f(x) dx,$$

$$(4.2) \quad f_{\alpha}^{-}(y) = \frac{1}{\Gamma(\alpha)} \int_y^{\infty} (x-y)^{\alpha-1} f(x) dx$$

by the more general ones

$$(4.3) \quad f_{\eta, \alpha}^{+}(y) = I_{\eta, \alpha}^{+} f = \frac{y^{-\eta-\alpha}}{\Gamma(\alpha)} \int_0^y (y-x)^{\alpha-1} x^{\eta} f(x) dx,$$

$$(4.4) \quad f_{\eta, \alpha}^{-}(y) = J_{\eta, \alpha}^{-} f = \frac{y^{\eta}}{\Gamma(\alpha)} \int_y^{\infty} (x-y)^{\alpha-1} x^{-\eta-\alpha} f(x) dx,$$

where η is a given parameter. Obviously

$$(4.5) \quad f_{\alpha}^{+}(y) = y^{\alpha} I_{0, \alpha}^{+} f, \quad f_{\alpha}^{-}(y) = y^{\alpha} J_{-\alpha, \alpha}^{-} f.$$

In another paper we have proved that $I_{\eta, \alpha}^{+} f$ and $J_{\eta, \alpha}^{-} f$ are bounded linear transformations in L_p for $1 \leq p \leq \infty$ when $\Re(\alpha) > 0$ and when $\Re(\eta) > -1/p'$ or $\Re(\eta) > -1/p$ respectively⁽⁴⁾. Obviously the definitions above have no meaning at all when we replace α by an imaginary number β , but we shall show that the operators $I_{\eta, \beta}^{+} f$ and $J_{\eta, \beta}^{-} f$ exist in some sense for any $f \in L_2$. Those definitions are of importance in the theory of Hankel transforms, as will be shown in a joint paper of A. Erdélyi and myself.

exists, and $Mg = \omega(\tau)Mf$.

Vide H. Kober, Quarterly Journal of Mathematics, Oxford, vol. 8 (1937), pp. 172-185, §6 and Theorem 2A.

B. Let $\phi(x) \in L_1$ and $\psi(y) = y^{-1} \int_0^y \phi(x) x^{1/2} dx$; then $y^{1/2} \psi(y) \rightarrow 0$ for $y \rightarrow 0$ and $y \rightarrow \infty$, and $|\psi|_2 \leq |\phi|_1$.

C. Let $y^{-1} \chi(y) \in L_2$, let $y^{-1/2} \chi(y) \rightarrow 0$ for $y \rightarrow 0$ and $y \rightarrow \infty$, and let $\chi(y)$ have the form as in Lemma A; then

$$(\tfrac{1}{2} - i\tau)M\{y^{-1/2}\chi(y)\} = \lim_{N \rightarrow \infty} \int_{1/N}^N H(y) y^{-1/2+i\tau} dy,$$

if the right-hand limit exists.

Cf. H. Kober, loc. cit., Theorem 3(i).

⁽³⁾ Cf. H. Weyl, Vierteljahrsschrift der Naturforschende Gesellschaft, Zurich, vol. 62 (1917), pp. 296-302; G. H. Hardy and J. E. Littlewood, Mathematische Zeitschrift, vol. 27 (1928), pp. 565-606; E. R. Love and L. C. Young, Proceedings of the London Mathematical Society, (2), vol. 44 (1938), pp. 1-28. The operator $y^{-\alpha} f_{\alpha}^{-}(y)$ exists in L_p with domain L_p and is bounded when $1 \leq p < \{\Re(\alpha)\}^{-1}$. Also cf. J. D. Tamarkin, Annals of Mathematics, (2), vol. 31 (1930), pp. 219-228.

⁽⁴⁾ Cf. *Inequalities*, Theorem 329. H. Kober, Quarterly Journal of Mathematics, Oxford, vol. 11 (1940), pp. 193-211.

Let $\Re(\eta) > -1/2$ and⁽⁴⁾

$$K(x, y) = \begin{cases} \{\Gamma(\alpha)\}^{-1}(y-x)^{\alpha-1}x^{\eta}y^{-\eta-\alpha} & | 0 < x < y|, \\ 0 & [x > y]; \end{cases}$$

then $K(x, y)$ satisfies the hypotheses of Theorem I, and we have

$$\begin{aligned} \omega(\tau) &= \int_0^{\infty} K(x, 1)x^{-1/2-i\tau}dx = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-x)^{\alpha-1}x^{\eta-1/2-i\tau}dx \\ &= \frac{\Gamma(\eta + \frac{1}{2} - i\tau)}{\Gamma(\eta + \alpha + \frac{1}{2} - i\tau)}, \\ I_{\eta, \alpha}^+ f &= \int_0^{\infty} K(x, y)f(x)dx \quad [f \in L_2], \end{aligned}$$

therefore by the theorem

$$|I_{\eta, \alpha}^+ f|_2 \leq \max_{-\infty < \tau < \infty} \left| \frac{\Gamma(\eta + \frac{1}{2} - i\tau)}{\Gamma(\eta + \alpha + \frac{1}{2} - i\tau)} \right| |f|_2 = \kappa_0 |f|_2.$$

When we take $|\alpha| < C$, then, in consequence of a well known property of the gamma function, κ_0 is uniformly bounded for $\Re(\alpha) > 0$; therefore

$$|I_{\eta, \alpha}^+ f|_2 \leq C |f|_2,$$

where C depends on η only. Let β be any fixed imaginary number or zero; then, by a well known theorem on weak convergence, a sequence $\alpha_1, \alpha_2, \alpha_3, \dots$ and a function $\phi(y) \in L_2$ exist such that $I_{\eta, \alpha_n}^+ f$ converges weakly to $\phi(y)$ when α_n tends to β [$n \rightarrow \infty$].

A similar argument applies to $J_{\eta, \alpha}^- f$, and we now define

$$(4.6) \quad I_{\eta, \beta}^+ f = \text{weak limit}_{\alpha_n \rightarrow \beta} I_{\eta, \alpha_n}^+ f \quad [\Re(\eta) > -\frac{1}{2}, f \in L_2]$$

$$(4.7) \quad J_{\eta, \beta}^- f = \text{weak limit}_{\alpha_m \rightarrow \beta} J_{\eta, \alpha_m}^- f$$

for some sequences $\{\alpha_n\}, \{\alpha_m\}$.

5. Strong convergence. Starting from $I_{\eta, \beta}^+ \psi$ and $J_{\eta, \beta}^- \psi$ for step-functions ψ we can show that $I_{\eta, \alpha}^+ f$ or $J_{\eta, \alpha}^- f$ converges to $I_{\eta, \beta}^+ f$ or $J_{\eta, \beta}^- f$ in the strong sense also for any $f \in L_2$ when α tends to β . We can also proceed in a shorter way. By (2.2) we have

$$(5.1) \quad MI_{\eta, \alpha}^+ f = \Gamma(\eta + \frac{1}{2} - i\tau) \{\Gamma(\eta + \alpha + \frac{1}{2} - i\tau)\}^{-1} Mf$$

and, taking

$$K(x, y) = \begin{cases} 0 & [0 < x < y] \\ \{\Gamma(\alpha)\}^{-1}(x-y)^{\alpha-1}x^{-\eta-\alpha}y^{\eta} & [x > y], \end{cases}$$

we get

$$(5.2) \quad MJ_{\eta, \alpha}^{-}f = \Gamma(\eta + \tfrac{1}{2} + i\tau) \{\Gamma(\eta + \alpha + \tfrac{1}{2} + i\tau)\}^{-1}Mf.$$

Let

$$Mf = g(\tau); \quad \omega(\tau; \alpha) = \Gamma(\eta + \tfrac{1}{2} - i\tau)/\Gamma(\eta + \alpha + \tfrac{1}{2} - i\tau).$$

Then

$$\begin{aligned} \int_{-\infty}^{\infty} |MI_{\eta, \alpha}^{+}f - \omega(\tau; \beta)g(\tau)|^2 d\tau &= \int_{-\infty}^{\infty} |g(\tau)|^2 |\omega(\tau; \alpha) - \omega(\tau; \beta)|^2 d\tau \\ &= \int_{-\infty}^{-N} + \int_N^{\infty} + \int_{-N}^N = Z_1 + Z_2 + Z_3. \end{aligned}$$

Since $\omega(\tau; \alpha)$ is bounded in $(-\infty, \infty)$ uniformly when $\Re(\alpha) > 0$ and $|\alpha| < C$, and since $g(\tau) \in L_2(-\infty, \infty)$, we can fix N sufficiently large such that $Z_1 < \epsilon/3$, $Z_2 < \epsilon/3$ uniformly in α for any given $\epsilon > 0$. Now it is easy to show that $Z_3 < \epsilon/3$ when $|\alpha - \beta|$ is sufficiently small. Hence $MI_{\eta, \alpha}^{+}f$ converges strongly to the function $\omega(\tau; \beta)g(\tau)$ and, by the property of the Mellin transformation mentioned above, $I_{\eta, \alpha}^{+}f$ to $M^{-1}\{\omega(\tau; \beta)g(\tau)\}$. By the same argument we get the corresponding result for $J_{\eta, \alpha}^{-}f$, and so we have

THEOREM II. Let $\Re(\eta) > -1/2$, let $\Re(\alpha) > 0$ and $\Re(\beta) = 0$ and $\alpha \rightarrow \beta$, and let $f \in L_2$; then the functions $I_{\eta, \alpha}^{+}f$ and $J_{\eta, \alpha}^{-}f$ converge strongly to $I_{\eta, \beta}^{+}f$ and $J_{\eta, \beta}^{-}f$ respectively, where

$$(5.3) \quad I_{\eta, \beta}^{+}f = M^{-1} \left\{ \frac{\Gamma(\eta + \tfrac{1}{2} - i\tau)}{\Gamma(\eta + \beta + \tfrac{1}{2} - i\tau)} Mf \right\},$$

$$(5.4) \quad J_{\eta, \beta}^{-}f = M^{-1} \left\{ \frac{\Gamma(\eta + \tfrac{1}{2} + i\tau)}{\Gamma(\eta + \beta + \tfrac{1}{2} + i\tau)} Mf \right\}.$$

Evidently $I_{\eta, 0}^{+}f = f$, $J_{\eta, 0}^{-}f = f$.

6. The inversions of the operators $I_{\eta, \beta}^{+}f$, $J_{\eta, \beta}^{-}f$. The operators $(I_{\eta, \beta}^{+})^{-1}$ and $(J_{\eta, \beta}^{-})^{-1}$ are also bounded linear transformations in L_2 . We have

THEOREM III. Let $\Re(\eta) > -1/2$, $\Re(\beta) = 0$, let $f(x) \in L_2$, and let

$$(6.1) \quad I_{\eta, \beta}^{+}f = g(y); \quad J_{\eta, \beta}^{-}f = h(y).$$

Then

$$(6.2) \quad f = I_{\eta+\beta, -\beta}^{+}g; \quad f = J_{\eta+\beta, -\beta}^{-}h.$$

The proof follows from (5.3) and (5.4), immediately; for instance,

$$MI_{\eta+\beta, -\beta}^+ = \frac{\Gamma(\eta + \beta + \frac{1}{2} - i\tau)}{\Gamma(\eta + \beta - \beta + \frac{1}{2} - i\tau)} Mg = Mf.$$

From (5.3) and (5.4) we may also see that both the domain and the range of $(I_{\eta, \beta}^+)^{-1}$ and $(J_{\eta, \beta}^-)^{-1}$ are L_2 , since $\{\omega(\tau; \beta)\}^{-1}$ is bounded in $(-\infty, \infty)$.

Of course the operators $I_{\eta, \alpha}^+$ and $J_{\eta, \alpha}^-$ do not possess this simply property.

7. Application to the customary fractional integrals, to that of Riemann-Liouville and to that of Weyl. Let the operators $X_\alpha f = f_\alpha^+(y)$ and $Y_\alpha f = f_\alpha^-(y)$ be defined by

$$(7.1) \quad X_\beta f = y^\beta I_{0, \beta}^+ f, \quad Y_\beta f = y^\beta J_{-\beta, \beta}^- f \quad [f \in L_2]$$

when they are of imaginary order, in accordance with (4.5). Since $|y^\beta| = 1$, X_β and Y_β are bounded linear transformations in L_2 with domain L_2 , and it is not difficult to show that, for $\alpha \rightarrow \beta$,

$$|y^{\beta-\alpha} f_\alpha^+ - X_\beta|_2 \rightarrow 0, \quad |y^{\beta-\alpha} f_\alpha^- - Y_\beta|_2 \rightarrow 0.$$

The semi-group property of f_α^+ in $L_p(0, a)$ for $1 \leq p \leq \infty$, $0 < a < \infty$ is well known⁽⁵⁾. Here we shall prove

THEOREM IV. *The transformations X_β or Y_β form a group in L_2 .*

Since $X_0 f = I_{0,0}^+ f = f$ and $Y_0 f = J_{0,0}^- f = f$, we have only to prove that, for any imaginary numbers β, γ

$$(7.2) \quad X_\beta X_\gamma = X_{\beta+\gamma}; \quad Y_\beta Y_\gamma = Y_{\beta+\gamma}.$$

We need the following lemmas:

LEMMA 1. *When $f \in L_2$, $\Re(\eta) > -1/2$, $\Re(\lambda) \geq 0$, $\Re(\mu) \geq 0$,*

$$(7.3) \quad I_{\eta+\lambda, \mu}^+ I_{\eta, \lambda}^+ = I_{\eta, \lambda+\mu}^+ = I_{\eta, \lambda}^+ I_{\eta+\lambda, \mu}^+; \quad J_{\eta+\lambda, \mu}^- J_{\eta, \lambda}^- = J_{\eta, \lambda+\mu}^- = J_{\eta, \lambda}^- J_{\eta+\lambda, \mu}^-.$$

LEMMA 2. *When $f \in L_2$, $\Re(\lambda) \geq 0$, $\Re(\nu) = 0$,*

$$(7.4) \quad I_{\eta, \lambda}^+ f = y^\nu I_{\eta+\nu, \lambda}^+ \{x^{-\nu} f(x)\}; \quad J_{\eta, \lambda}^- f = y^{-\nu} J_{\eta+\nu, \lambda}^- \{x^\nu f(x)\}.$$

We can easily prove Lemma 1 by taking the Mellin transforms of both sides and employing (5.1)–(5.4).

The proof of Lemma 2 follows from the definitions (4.3) and (4.4) immediately when $\Re(\lambda) > 0$, since $x^\nu f(x) \in L_2$, $x^{-\nu} f(x) \in L_2$. Taking $\Re(\lambda) > 0$, $\lambda \rightarrow \beta$, we have

⁽⁵⁾ E. Hille, *Annals of Mathematics*, (2), vol. 40 (1939), 4.4. In this paper the theory of semi-groups is developed. Cf. E. Hille, *Proceedings of the National Academy of Sciences*, vol. 24 (1938), pp. 159–161.

$$\begin{aligned}
& |I_{\eta,\lambda}^+ - I_{\eta,\beta}^+|_2 \rightarrow 0; \\
& |y^\gamma I_{\eta+\nu,\lambda}^+ \{x^{-\nu} f(x)\} - y^\gamma I_{\eta+\nu,\beta}^+ \{x^{-\nu} f(x)\}|_2 \\
& \quad = |I_{\eta+\nu,\lambda}^+ \{x^{-\nu} f(x)\} - I_{\eta+\nu,\beta}^+ \{x^{-\nu} f(x)\}|_2 \rightarrow 0,
\end{aligned}$$

and so (7.3) is true for $\lambda = \beta$ also.

Now by (7.1)

$$X_\beta X_\gamma f = y^\beta I_{0,\beta}^+ \{x^\gamma I_{0,\gamma}^+ f\},$$

and $I_{0,\beta}^+ \{x^\gamma \phi\} = y^\gamma I_{\gamma,\beta}^+ \phi$ by Lemma 2. Hence, by Lemma 1,

$$X_\beta X_\gamma f = y^{\beta+\gamma} I_{\gamma,\beta}^+ I_{0,\gamma}^+ f = y^{\beta+\gamma} I_{0,\beta+\gamma}^+ f = X_{\beta+\gamma} f.$$

Similarly we have

$$Y_\beta Y_\gamma f = y^\beta J_{-\beta,\beta}^- \{x^\gamma J_{-\gamma,\gamma}^- f\} = y^{\beta+\gamma} J_{-\beta-\gamma,\beta}^- J_{-\gamma,\gamma}^- f = y^{\beta+\gamma} J_{-\beta-\gamma,\beta+\gamma}^- f = Y_{\beta+\gamma} f.$$

COROLLARY. The transformations $(X_\beta)^{-1}$ and $(Y_\beta)^{-1}$ are linear and bounded in L_2 with domain L_2 , and $(X_\beta)^{-1} = X_{-\beta}$, $(Y_\beta)^{-1} = Y_{-\beta}$.

8. We shall now deal with the corresponding problems in L_p for $p \geq 1$. We do not know if Theorem I can be generalized in some way for $p \neq 2$. Therefore we cannot extend the results of §§4-7 to the general case $f \in L_p$ [$1 \leq p \leq \infty$]. We have to restrict ourselves to certain subspaces of L_p or, as in Theorem VI, to the case when α tends to zero under certain conditions. Moreover we shall discuss the characteristic values (§10).

Let $0 < a < \infty$ and let the step-function $\phi_a(x)$ be defined by $\phi_a(x) = 1$ for $0 \leq x \leq a$, $\phi_a(x) = 0$ otherwise. When $\Re(\xi) > -1$, we easily find

$$(8.11) \quad I_{\xi,\alpha}^+ \phi_a = \frac{\Gamma(\xi + 1)}{\Gamma(\xi + \alpha + 1)} \quad \text{or} \quad \frac{1}{\Gamma(\alpha)} \int_0^{a/y} (1-t)^{\alpha-1} t^\xi dt,$$

$$\begin{aligned}
(8.12) \quad J_{\eta,\alpha}^- \phi_a &= \frac{1}{\Gamma(\alpha + 1)} \left\{ \left(1 - \frac{y}{a}\right)^\alpha \left(\frac{y}{a}\right)^{\eta-1} \right. \\
&\quad \left. + (\eta - 1) \int_{y/a}^1 (1-t)^{\alpha} t^{\eta-2} dt \right\} \quad \text{or} \quad 0
\end{aligned}$$

for $0 < y < a$ or $a < y < \infty$ respectively; therefore, in these open intervals, $I_{\xi,\alpha}^+ \phi_a$ and $J_{\eta,\alpha}^- \phi_a$ certainly exist and are continuous, also when we replace α by a purely imaginary number β , and $I_{\xi,\alpha}^+ \phi_a \rightarrow I_{\xi,\beta}^+ \phi_a$, $J_{\eta,\alpha}^- \phi_a \rightarrow J_{\eta,\beta}^- \phi_a$ as $\alpha \rightarrow \beta$. Hence, for any step-function f , $I_{\xi,\beta}^+ f$ and $J_{\eta,\beta}^- f$ exist and are continuous almost everywhere in $(0, \infty)$, and $I_{\xi,0}^+ f \equiv J_{\eta,0}^- f \equiv f$, as we can easily deduce from (8.11) and (8.12). The following theorem holds:

THEOREM V. Let $1 \leq p < \infty$ and $\Re(\xi) > -1/p' = 1 - 1/p$, $\Re(\eta) > -1/p$, let

$\Re(\lambda) \geq 0$, and let $f(x)$ be any step-function. Then, as $\alpha \rightarrow \lambda$,

$$(8.21) \quad |I_{\zeta, \alpha}^+ f - I_{\zeta, \lambda}^+ f|_p \rightarrow 0,$$

$$(8.22) \quad |J_{\eta, \alpha}^- f - J_{\eta, \lambda}^- f|_p \rightarrow 0.$$

To prove (8.21), we simply take $f = \phi_a$ and $|\alpha - \lambda| < 1$. Then

$$\begin{aligned} \int_0^\infty |I_{\zeta, \alpha}^+ \phi_a - I_{\zeta, \lambda}^+ \phi_a|^p dy &= \int_0^a \left| \frac{\Gamma(\zeta + 1)}{\Gamma(\zeta + \alpha + 1)} - \frac{\Gamma(\zeta + 1)}{\Gamma(\zeta + \lambda + 1)} \right|^p dy \\ &\quad + \int_a^\infty |\psi(\alpha, y) - \psi(\lambda, y)|^p dy = V_1 + V_2, \end{aligned}$$

say, where

$$(8.3) \quad \psi(\alpha, y) = \frac{1}{\Gamma(\alpha)} \int_0^{a/y} (1-t)^{\alpha-1} t^\zeta dt.$$

Obviously $V_1 \rightarrow 0$ as $\alpha \rightarrow \lambda$. When $a < y < \infty$ and y is fixed, then $\psi(\alpha, y) \rightarrow \psi(\lambda, y)$ by the Lebesgue convergence theorem, since $|(1-t)^{\alpha-1} t^\zeta| \leq (1-a/y)^{-1} t^{\Re(\zeta)}$ and $t^{\Re(\zeta)} \in L_1(0, 1)$. To prove $V_2 \rightarrow 0$ we need only show that $|\psi(\alpha, y)| \leq U(y)$, where $U(y)$ does not depend on α and belongs to $L^p(a, \infty)$.

For $2a < y < \infty$, we have $1 - a/y > 1/2$,

$$|\psi(\alpha, y)| \leq \frac{2}{|\Gamma(\alpha)|} \int_0^{a/y} t^{\Re(\zeta)} dt \leq K y^{-\Re(\zeta)-1} = U_0(y) \in L_p(2a, \infty).$$

For $a < y < 2a$, we have

$$\begin{aligned} \psi(\alpha, y) &= \frac{1}{\Gamma(\alpha)} \int_0^{1/2} + \frac{1}{\Gamma(\alpha)} \int_{1/2}^{a/y} = \psi_1(\alpha, y) + \psi_2(\alpha, y), \\ |\psi_1(\alpha, y)| &\leq \frac{2}{|\Gamma(\alpha)|} \int_0^{1/2} t^{\Re(\zeta)} dt = C_1 = U_1(y) \in L_p(a, 2a). \\ |\psi_2(\alpha, y)| &= \frac{1}{|\Gamma(\alpha + 1)|} \left| 2^{-\alpha-\zeta} - \left(1 - \frac{a}{y}\right)^\alpha \left(\frac{a}{y}\right)^\zeta \right. \\ &\quad \left. + \zeta \int_{1/2}^{a/y} (1-t)^\alpha t^{\zeta-1} dt \right| < C_2 = U_2(y) \in L_p(a, 2a). \end{aligned} \quad (8.3.0)$$

Applying Lebesgue's theorem again, we have $V_2 \rightarrow 0$, which completes the proof.

The proof of (8.22) is similar. Let $|\alpha - \lambda| < 1$ again. We have to take into consideration that $J_{\eta, \alpha}^- \phi_a$ is bounded uniformly in α and y for $a/2 < y < a$. Furthermore, for $0 < y < a/2$ we have to replace (8.12) by

$$(8.12.0) \quad \begin{aligned} J_{\eta, \alpha}^- \phi_a &= \frac{2^{1-\eta-\alpha}}{\Gamma(\alpha+1)} + \frac{\eta-1}{\Gamma(\alpha+1)} \int_{1/2}^1 (1-t)^\alpha t^{\eta-2} dt \\ &+ \frac{1}{\Gamma(\alpha)} \int_{y/a}^1 (1-t)^{\alpha-1} t^{\eta-1} dt = W_1 + W_2 + W_3, \end{aligned}$$

where $|W_1| \leq K$, $|W_2| \leq K$, $|W_3| \leq K y^{\eta_1} \subset L_p(0, a/2)$, when $-1/p < \eta_1 < \min(0, \Re(\eta))$. For $2 \leq p < \infty$, the theorem also follows from Theorem IIb (§9). Theorem V holds also when we suppose that ζ and η are not fixed, but that $\zeta \rightarrow \zeta_0$ and $\eta \rightarrow \eta_0$ as $\alpha \rightarrow \lambda$, where $\Re(\zeta_0) > -1/p'$ and $\Re(\eta_0) > -1/p$.

Now for $1 \leq p \leq \infty$ and $\Re(\zeta) > -1/p'$, $\Re(\eta) > -1/p$, we have⁽⁶⁾

$$(8.4) \quad \begin{aligned} |I_{\zeta, \alpha}^+ g|_p &\leq \frac{\Gamma\{\Re(\alpha)\} \Gamma\{\Re(\zeta) + 1/p'\}}{|\Gamma(\alpha)| \cdot \Gamma\{\Re(\zeta + \alpha) + 1/p'\}} |g|_p; \\ |J_{\eta, \alpha}^- g|_p &\leq \frac{\Gamma\{\Re(\alpha)\} \Gamma\{\Re(\eta) + 1/p\}}{|\Gamma(\alpha)| \cdot \Gamma\{\Re(\eta + \alpha) + 1/p\}} |g|_p. \end{aligned}$$

When in (8.21) and (8.22) $\Re(\lambda)$ is greater than zero, then $|I_{\zeta, \alpha}^+ g|_p$ and $|J_{\eta, \alpha}^- g|_p$ are bounded uniformly in α for $|\alpha - \lambda| < \frac{1}{2}\Re(\lambda)$; by approximating to $g(x)$ by a sequence of step-functions we see that, for $\Re(\lambda) > 0$ and $1 \leq p < \infty$, Theorem V is valid for any $g \subset L_p$. It is an open question whether or not it holds for $\Re(\lambda) = 0$ also, when $1 \leq p < 2$ or $2 < p < \infty$ (cf. Theorem II), but it is certainly true in the following sense for $\lambda = 0$:

THEOREM VI. *Let $1 \leq p < \infty$ and $f(x) \subset L_p$, let $\Re(\zeta) > -1/p'$ and $\Re(\eta) > -1/p$; let Θ be any positive number smaller than $\pi/2$, and let $\alpha \rightarrow 0$ with the restriction $|\arg \alpha| \leq \Theta$. Then*

$$|I_{\zeta, \alpha}^+ f - f(y)|_p \rightarrow 0; \quad |J_{\eta, \alpha}^- f - f(y)|_p \rightarrow 0.$$

The proof is an immediate consequence of Theorem V, since in L_p the operations $I_{\zeta, \alpha}^+$ and $J_{\eta, \alpha}^-$ are uniformly bounded for $|\arg \alpha| \leq \Theta$, when $|\alpha| < K$: Let $\alpha = \alpha_1 + i\alpha_2$; then $\alpha_1 > 0$, $|\alpha_2| < \alpha_1 \tan \Theta$, and

$$\frac{\Gamma\{\Re(\alpha)\}}{|\Gamma(\alpha)|} = \frac{\Gamma(\alpha_1 + 1)}{|\Gamma(\alpha + 1)|} \left| 1 + i \frac{\alpha_2}{\alpha_1} \right| \leq \frac{\Gamma(\alpha_1 + 1)}{|\Gamma(\alpha + 1)|} (1 + \tan \Theta) < K.$$

Therefore, by (8.4), the operations have the desired property.

COROLLARY. *Let α be restricted as in Theorem VI and let $f \subset L_p$; then*

$$|y^{-\alpha} f_{\alpha}^+(y) - f(y)|_p \rightarrow 0, \quad |y^{-\alpha} f_{\alpha}^-(y) - f(y)|_p \rightarrow 0$$

for $1 < p < \infty$ and $1 \leq p < \infty$ respectively, as α tends to zero.

⁽⁶⁾ Cf. our paper cited in Footnote 4.

The case $p=1$ is not included for $f_a^+(y)$ by this theorem; some results for $f_a^+(y)$ [$\alpha \rightarrow 0$] under the hypothesis $f \in L_1(0, A)$ were given by Hardy-Littlewood (loc. cit.) and by J. D. Tamarkin⁽⁷⁾.

9. We can state some better theorems for $p > 2$. Let $2 < p \leq \infty$ and let $\mathfrak{M}_p$ be the set of all functions $f \in L_p$ which possess a Mellin transform $F(\tau) = Mf$ in the well defined sense that $f(x)$ is representable in the form

$$(9.1) \quad f(x) = \lim_{a, b \rightarrow \infty} \frac{\text{index } p}{2\pi} \int_{-a}^b F(\tau) x^{-1/p - i\tau} d\tau = M^{-1}F,$$

where $F(\tau) \in L_{p'}(-\infty, \infty)$. In consequence of the well known theory of Fourier transforms, $M^{-1}F$ is a bounded linear transformation from $L_{p'}(-\infty, \infty)$ into $L_p(0, \infty)$ and $\mathfrak{M}_p$ a subspace of L_p and smaller than L_p . When $p=2$ obviously $\mathfrak{M}_p = L_2$ ⁽⁸⁾.

LEMMA 3. Let $2 < p \leq \infty$ and $\Re(\zeta) > -1/p'$ and $\Re(\eta) > -1/p$, and let $f \in \mathfrak{M}_p$. Then $I_{\zeta, \alpha}^+ f$ and $J_{\eta, \alpha}^- f$ belong to $\mathfrak{M}_p$ also, and

$$(9.21) \quad MI_{\zeta, \alpha}^+ f = \omega(\tau, \alpha) Mf;$$

$$(9.22) \quad MJ_{\eta, \alpha}^- f = \chi(\tau, \alpha) Mf,$$

where

$$(9.3) \quad \omega(\tau, \alpha) = \frac{\Gamma(\zeta + 1/p' - i\tau)}{\Gamma(\zeta + \alpha + 1/p' - i\tau)}; \quad \chi(\tau, \alpha) = \frac{\Gamma(\eta + 1/p + i\tau)}{\Gamma(\eta + \alpha + 1/p + i\tau)}.$$

We shall only outline the proof. Let $F_n(\tau) = F(\tau)$ in $(-n, n)$, $F_n(\tau) = 0$ for $|\tau| > n$, and let $f(x, n) = M^{-1}F_n$. Then, for $0 < y < \infty$,

$$\begin{aligned} I_{\zeta, \alpha}^+ \{f(x, n)\} &= \frac{y^{-\zeta - \alpha}}{2\pi\Gamma(\alpha)} \int_0^y (y-x)^{\alpha-1} x^\zeta dx \int_{-n}^n F(\tau) x^{-1/p - i\tau} d\tau \\ &= \frac{1}{2\pi} \int_{-n}^n F(\tau) y^{-1/p - i\tau} d\tau \cdot \frac{1}{\Gamma(\alpha)} \int_0^1 (1-\xi)^{\alpha-1} \xi^{\zeta-1/p - i\tau} d\xi \\ &= M^{-1} \{F_n(\tau) \omega(\tau, \alpha)\}, \end{aligned}$$

and so (9.21) follows by $\int |F - F_n|^{p'} d\tau \rightarrow 0$, $|f(x) - f(x, n)|_p \rightarrow 0$ [$n \rightarrow \infty$], and by (8.4).

⁽⁷⁾ Annals of Mathematics, (2), vol. 31 (1930), pp. 219-228.

⁽⁸⁾ When $2 < p < \infty$, then (9.1) implies

$$F(\tau) = \lim_{a \rightarrow \infty} \frac{\text{index } p'}{\int_{1/a}^a} f(x) x^{-1/p' + i\tau} dx$$

in consequence of the Hille-Tamarkin theorem, vide Bulletin of the American Mathematical Society, vol. 39 (1933), pp. 768-774.

For Lemma 3 see also our paper cited above.

By (9.21) and (9.22) $I_{\xi,\alpha}^+ f$ and $J_{\eta,\alpha}^- f$ are defined also when we replace α by β ; for $\omega(\tau, \beta)$ and $\chi(\tau, \beta)$ are bounded for $-\infty < \tau < \infty$, therefore $\omega(\tau, \beta)Mf = \omega(\tau, \beta)F(\tau)$ and $\chi(\tau, \beta)Mf = \chi(\tau, \beta)F(\tau)$ belong to $L_{p'}(-\infty, \infty)$. Also to every $g \in \mathfrak{M}_p$ corresponds a uniquely determined function $f \in \mathfrak{M}_p$ or $\phi \in \mathfrak{M}_p$ such that $I_{\xi,\beta}^+ f = g$ or $J_{\eta,\beta}^- \phi = g$ respectively (cf. (6.1) and (7.2)). By the same reasoning as in §5 and §7, we have the theorems:

THEOREM IIb. Let $2 \leq p \leq \infty$, let $\Re(\xi) > -1/p'$, $\Re(\eta) > -1/p$, and let $f \in \mathfrak{M}_p$. Then

$$|I_{\xi,\alpha}^+ f - I_{\xi,\lambda}^+ f|_p \rightarrow 0; \quad |J_{\eta,\alpha}^- f - J_{\eta,\lambda}^- f|_p \rightarrow 0$$

as α tends to λ , where $\Re(\lambda) \geq 0$.

LEMMA 4. The operator $X_\beta f$, defined by (7.1), exists in $L_p(0, \infty)$ with both domain and range $\mathfrak{M}_p$ when $2 \leq p \leq \infty$. So does $Y_\beta f$ when $2 \leq p < \infty$.

THEOREM IVb. Under the restrictions of Lemma 4, the transformations X_β or Y_β form a group in L_p , and $|X_\beta - X_{\beta_0}|_p \rightarrow 0$, $|Y_\beta - Y_{\beta_0}|_p \rightarrow 0$ as $\beta \rightarrow \beta_0$.

10. Characteristic values. From (7.1) and (9.2) we easily have

$$(10.11) \quad MX_\beta f = \frac{\Gamma(1/p' - \beta - i\tau)}{\Gamma(1/p' - i\tau)} F(\tau - i\beta) \quad [2 \leq p \leq \infty],$$

$$(10.12) \quad MY_\beta f = \frac{\Gamma(1/p + i\tau)}{\Gamma(1/p + \beta + i\tau)} F(\tau - i\beta) \quad [2 \leq p < \infty],$$

where $f = M^{-1}F \in \mathfrak{M}_p$. We shall now deal with the equations

$$(10.21) \quad X_\beta f = lf,$$

$$(10.22) \quad Y_\beta f = lf,$$

$$(10.23) \quad X_\beta f = l\bar{f}.$$

Let $\beta = i\rho$, where ρ is real. Obviously (10.21) is equivalent to

$$(10.31) \quad h(\tau + \rho) = lh(\tau); \quad F(\tau) = h(\tau)/\Gamma(1/p' - i\tau) \in L_{p'}(-\infty, \infty),$$

and (10.22) or (10.23) to

$$(10.32) \quad k(\tau + \rho) = lk(\tau); \quad F(\tau) = k(\tau)\Gamma(1/p + i\tau) \in L_{p'}(-\infty, \infty)$$

or

$$(10.33) \quad h(\tau + \rho) = l\bar{h}(-\tau); \quad F(\tau) = h(\tau)/\Gamma(1/p' - i\tau) \in L_{p'}(-\infty, \infty).$$

By means of the well known asymptotic expansion of $|\Gamma(\sigma + i\tau)|$ for $t \rightarrow \pm \infty$ we arrive at the following results:

The transformation $X_\beta f$ [$\beta \neq 0$] with domain $\mathfrak{M}_p$ [$2 \leq p \leq \infty$] has no characteristic values at all.

The characteristic values of $Y_\beta f$ [$\beta \neq 0$, domain $\mathfrak{M}_p$, $2 \leq p < \infty$] are the set of points l for which $\exp(-\frac{1}{2}\pi|\beta|) < |l| < \exp(\frac{1}{2}\pi|\beta|)$. To every characteristic value l corresponds an infinity of characteristic functions.

We construct all these functions by (10.32), taking $k(\tau)$ in $(0, |\rho|)$ as an arbitrary function belonging to $L_{p'}(0, |\rho|)$.

Also, from the group property of Y_β , we can deduce the result:

Let $\Re(\kappa) = \Re(\lambda) = 0$, let κ/λ be no rational number, and let $Y_\kappa f = f$ and $Y_\lambda f = f$ and $f \in \mathfrak{M}_p$. Then $f(x) = ce^{-x}$, and $Y_\gamma f = f$ for any γ such that $\Re(\gamma) \geq 0$.

Furthermore, by (10.33), we can prove the result:

The equation (10.23) has a solution $f \in \mathfrak{M}_p$ if, and only if, $|l| = 1$. To every number l of this kind corresponds an infinity of solutions $f \in \mathfrak{M}_p$.

For instance, $f(x) = e^{-1/x} x^{-\beta-1}$ is a solution of (10.23) for $l = 1$.

11. It is not difficult to show that $I_{\xi, \beta}^+ f$ and $J_{\eta, \alpha}^- f$ certainly exist and are integrable for any $f \in L_p$ when $\Re(\xi) > -1/2$ and $\Re(\eta) > -1/p$ and $2 < p < \infty$. We have

THEOREM VII. Let $2 < p < \infty$ and $f(x) \in L_p$; let $\Re(\xi) > -1/2$, $\Re(\eta) > -1/p$. Then there exist functions $I_{\xi, \beta}^+ f$ and $J_{\eta, \alpha}^- f$, defined in $(0, \infty)$ and such that, for any positive finite number A , $I_{\xi, \beta}^+ f$ and $J_{\eta, \alpha}^- f$ belong to $L_2(0, A)$ and that

$$\int_0^A |I_{\xi, \alpha}^+ f - I_{\xi, \beta}^+ f|^2 dy \rightarrow 0; \quad \int_0^A |J_{\eta, \alpha}^- f - J_{\eta, \beta}^- f|^2 dy \rightarrow 0 \quad [\alpha \rightarrow \beta].$$

Proof. Let $f_n(x) = f(x)$ or $f_n(x) = 0$ for $0 \leq x \leq n$ or $x > n$ respectively [$n = 1, 2, \dots$]. Then $f_n(x) \in L_2(0, \infty)$, and by Theorem II, the function $g_\alpha(y, n) = I_{\xi, \alpha}^+ f_n$ belongs to $L_2(0, \infty)$ and converges strongly to a function $g_\beta(y, n)$ as $\alpha \rightarrow \beta$,

$$(11.1) \quad \|g_\alpha(y, n) - g_\beta(y, n)\|_2 \rightarrow 0.$$

We now define $I_{\xi, \beta}^+ f$ in $(0, \infty)$ by putting

$$(11.2) \quad I_{\xi, \beta}^+ f = g_\beta(y, n) \quad \text{for } n-1 < y \leq n \quad [n = 1, 2, \dots].$$

This function then has the desired properties. For $0 < y \leq n$ obviously

$$g_\alpha(y, n) = I_{\xi, \alpha}^+ f,$$

and so

$$(11.3) \quad \int_0^n |I_{\xi, \alpha}^+ f - g_\beta(y, n)|^2 dy \leq \int_0^\infty |g_\alpha(y, n) - g_\beta(y, n)|^2 dy \rightarrow 0$$

as α tends to β . Therefore, for $1 \leq m < n$,

$$\begin{aligned} \left(\int_0^m |g_\beta(y, n) - g_\beta(y, m)|^2 dy \right)^{1/2} &\leq \left(\int_0^m |I_{\tau, \alpha}^+ f - g_\beta(y, m)|^2 dy \right)^{1/2} \\ &+ \left(\int_0^n |g_\beta(y, n) - I_{\tau, \alpha}^+ f|^2 dy \right)^{1/2} \rightarrow 0 \quad [\alpha \rightarrow \beta]. \end{aligned}$$

Hence $g_\beta(y, m) \equiv g_\beta(y, n)$ in $(0, m)$, and

$$(11.4) \quad I_{\tau, \beta}^+ f \equiv g_\beta(y, n)$$

in $(0, n)$ for $n = 1, 2, \dots$.

Therefore $I_{\tau, \beta}^+ f \in L_2(0, n)$ for any n , and, by (11.3) and (11.4),

$$\int_0^n |I_{\tau, \alpha}^+ f - I_{\tau, \beta}^+ f|^2 dy = \int_0^n |I_{\tau, \alpha}^+ f - g_\beta(y, n)|^2 dy \rightarrow 0$$

as $\alpha \rightarrow \beta$; which proves the first part of the theorem.

Let us take (cf. 4.4) $0 < y \leq n$,

$$\begin{aligned} (11.5) \quad J_{\alpha, \eta}^- f &= \frac{y^n}{\Gamma(\alpha)} \left\{ \int_y^n + \int_n^\infty \right\} (x - y)^{\alpha-1} x^{-\eta-\alpha} f(x) dx \\ &= \phi_\alpha(y, n) + \psi_\alpha(y, n). \end{aligned}$$

Plainly $\phi_\alpha(y, n) = J_{\alpha, \eta}^- f_n$ in $(0, n)$. Now $f_n \in L_2(0, \infty)$ and $\Re(\eta) > -1/p > -1/2$, and so, by Theorem II, $J_{\beta, \eta}^- f_n$ exists, belongs to L_2 , and

$$(11.6) \quad \int_0^n |\phi_\alpha(y, n) - J_{\eta, \beta}^- f_n|^2 dy \leq \int_0^\infty |\bar{J}_{\eta, \alpha} f_n - \bar{J}_{\eta, \beta} f_n|^2 dy \rightarrow 0 \quad [\alpha \rightarrow \beta].$$

Let b be any positive number smaller than n . Then there exists a function $\psi_\beta(y, n)$, depending on n but not on b , such that

$$(11.7) \quad \int_0^b |\psi_\alpha(y, n) - \psi_\beta(y, n)|^2 dy \rightarrow 0 \quad [\alpha \rightarrow \beta],$$

as we shall now show. For $0 < y < n$ and $n \leq x < \infty$, the function

$$\chi(x, y, \alpha) = \{\Gamma(\alpha)\}^{-1} y^{-\eta} (x - y)^{\alpha-1} x^{-\eta-\alpha} f(x)$$

tends to $\chi(x, y, \beta)$ as $\alpha \rightarrow \beta$, and

$$|\chi(x, y, \alpha)| < K x^{-\Re(\eta)-1} |f(x)| \in L_1(n, \infty)$$

when y is fixed and $|\alpha| < K$. Hence, by Lebesgue's convergence theorem, $\psi_\beta(y, n)$ exists, and $\psi_\alpha(y, n) - \psi_\beta(y, n) \rightarrow 0$ when $0 < y < n$. For $0 < y \leq b < n$, we have $x - y \geq (1 - b/n)x = cx$,

$$|\psi_\alpha(y, n)| \leq |\Gamma(\alpha)|^{-1} y^{\Re(\eta)} \int_n^\infty c^{\Re(\alpha)-1} x^{-\Re(\eta)-1} |f(x)| dx \leq K y^{\Re(\eta)},$$

where K does not depend on α when $|\alpha - \beta| < 1$. Hence also

$$|\psi_\alpha(y, n) - \psi_\beta(y, n)| \leq Ky^{\Re(\eta)}.$$

Since $y^{\Re(\eta)} \subset L_2(0, b)$, applying Lebesgue's theorem again, we obtain (11.7). In consequence of (11.5)–(11.7), for $h_\beta(y, n) = \phi_\beta(y, n) + \psi_\beta(y, n)$ we have

$$(11.8) \quad \int_0^b |\bar{J}_{\eta, \alpha} f - h_\beta(y, n)|^2 dy \rightarrow 0 \quad [\alpha \rightarrow \beta].$$

When we define $\bar{J}_{\eta, \beta} f$ in $(0, \infty)$ by

$$\bar{J}_{\eta, \beta} f = h_\beta(y, n) \quad \text{for } n-1 \leq y < n \quad [n = 1, 2, \dots],$$

then, by the argument applied in (11.4), it easily follows that $\bar{J}_{\eta, \beta} f \equiv h_\beta(y, n)$ for $0 < y < n$, which completes the proof.

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