ON THE STRUCTURE OF DIFFERENTIAL POLYNOMIALS AND ON THEIR THEORY OF IDEALS

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In the first part of this paper a special class of differential ideals(1) is investigated. The results of this section are used in the following one to derive some structural properties of differential polynomials. The last part of the paper is devoted to a special differential ideal.

With the help of some conventions of notation, more precise indications of the scope of our work may be given. Let $\mathfrak{R}$ denote the ring of differential polynomials, with rational numbers for coefficients, in the unknown y. The special class of differential ideals studied in Part I is composed of those generated by y^p, where p is a positive integer. These ideals are among the most simple ideals encountered in the theory of differential equations. Viewed as algebraic entities, however, they are by no means trivial. We denote the ith derivative of y by y_i; $\mathfrak{R}$ thus appears as a polynomial ring with infinitely many indeterminates $y, y_1, y_2, \cdots$. Since the Hilbert basis theorem does not hold on $\mathfrak{R}$, one would expect almost any ideal in $\mathfrak{R}$ to be unruly. By introducing order relations into $\mathfrak{R}$ we have been able to proceed despite the absence of the basis theorem and to obtain fairly comprehensive results concerning these differential ideals. In particular a simple criterion for determining the membership in such an ideal of an element of $\mathfrak{R}$ is obtained which plays a fundamental role in Part II. This second part establishes the abstract counterparts of some results of J. F. Ritt concerning essential manifolds which figure in the decomposition of a manifold into irreducible ones. It has been found possible to present results which cover situations not discussed by him. The differential ideal discussed in Part III is that generated by uv, where u and v are unknowns. Among other properties, it is shown that this ideal has no representation as the intersection or product of two differential ideals, whose manifolds are respectively $u=0$ with v arbitrary, and $v=0$ with u arbitrary. This result owes its interest to the fact that the manifold of the equation $uv=0$ is evidently reducible into the union of the two manifolds just defined.

In a narrow sense, this paper is independent of other literature; the argu-

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(1) For terminology and bibliography, see *Semicentennial Addresses of the American Mathematical Society*, New York, 1938, pp. 35–58. The basic reference for the abstract theory of ideals of differential polynomials is H. W. Raudenbush, *Ideal theory and algebraic differential equations*, these Transactions, vol. 36 (1934), pp. 361–368.

ments make almost no appeal to outside sources. Less strictly, however, the writings of J. F. Ritt and H. W. Raudenbush, Jr., should be cited as furnishing both starting point and direction for this investigation. Indeed, the whole point of Part II lies in its connection with two papers of Ritt (more detailed reference is given in Part II). The congruence notation used in our work is that systematized by E. R. Kolchin(2). Brackets [ ] and braces { } mean respectively the differential ideal and the perfect differential ideal generated by the set of elements they include. The congruences

$$a \equiv b \ [m, n, \cdots]$$
$$a \equiv b \ \{m, n, \cdots\}$$

mean, respectively, $a-b$ is in $[m, n, \cdots]$ and $a-b$ is in $\{m, n, \cdots\}$. We use the term "form" exclusively as an abbreviation for the term "differential polynomial."

PART I. THE DIFFERENTIAL IDEAL GENERATED BY y^p

THE FORM y^p AND ITS DERIVATIVES

1. Let p be any positive integer and let $A = y^p$. We investigate the differential ideal Σ generated by A. Denoting the ith derivative of A by A_i we see that Σ consists of all polynomials $E_0A + E_1A_1 + \cdots + E_rA_r$ where the E_i are any elements of $\mathfrak{R}$. It is sometimes convenient to let $y = y_0$, $A = A_0$.

We shall discuss power products in y and its derivatives, and make a few definitions for this purpose. Let $P = y^{p_0}y_1^{p_1} \cdots y_r^{p_r}$ be such a power product, the p_i being non-negative integers. The degree of P is defined as $\sum p_i$ and its weight is defined as $\sum ip_i$. A power product P is different from a power product $Q = y^{q_0}y_1^{q_1} \cdots y_s^{q_s}$ if some $p_i - q_i$ is different from zero. We understand that if $t > r$ then p_t, the exponent of y_t in P, is zero. If P is different from Q we say that P is *higher* than Q, and Q is *lower* than P, if the first nonzero difference $p_i - q_i$ is positive. If P is higher than Q and R is any power product, then RP is higher than RQ. If P is higher than Q and Q is higher than R, then P is higher than R. A power product P will be called an α *term*, if $p_i + p_{i+1} < p$; $i = 0, 1, 2, \cdots$. Every factor of an α term is an α term. Every power product not an α term will be called a β *term*. Every β term is divisible by an expression $y_i^r y_{i+1}^{p-r}$ with $r \leqq p$.

2. The polynomial A_i, the ith derivative of A, is homogeneous of degree p and isobaric of weight i. A_i is a sum over j of terms $h_{ij}P_j$ where the h_{ij} are positive integers, and the P_j are power products of degree p and weight i. Each power product P_j of this weight and degree is present in A_i with a coefficient h_{ij} different from zero. In particular if $i = rp + s$ (r and s non-negative integers and $s < p$) the term $L_i = y_r^{p-s}y_{r+1}^s$ has the proper weight and degree

(2) *On the exponents of differential ideals*, Annals of Mathematics, (2), vol. 42 (1941), p. 741.

and is present effectively in A_i. It will be called the *leader* of A_i. We show that it is lower than any other term of A_i. It is certainly lower than any term involving a $y_k^{p_k}$ with $k<r$, $p_k>0$. Any term of A_i lower than L_i would thus be of the form $y_r^q y_{r+1}^{q_1} \cdots y_{r+t}^{q_t}$, with $q \leqq p-s$. This would imply that $q_1+q_2+ \cdots +q_t \geqq s$. Thus the weight of such a term would be greater than $qr+(q_1+ \cdots +q_t)(r+1)$ unless $q_2, \cdots, q_t$ were all zero. This last expression exceeds i if $q<p-s$. It follows that any term of A_i distinct from L_i must be higher than L_i.

Reduction of power products

3. We prove the following lemma.

LEMMA 1.1. *For every β term F of $\mathfrak{R}$ there is a congruence*

$$F \equiv \sum_i h_i P_i \ [\Sigma]$$

where the P_i are α terms of the same weight and degree as F and the h_i are rational numbers (they may of course be zero).

F is divisible by the leader L_i of some A_i. Let c_i stand for the coefficient of L_i in A_i. Then $c_iL_i=A_i+(c_iL_i-A_i)$, where the terms in the parenthesis are higher than L_i, or are zero (if i is zero or unity). If $F=c_iL_iF'$, then

$$(1.1) \qquad \begin{aligned} F &= F'A_i + F'(c_iL_i - A_i) \\ &\equiv F'(c_iL_i - A_i) \ [\Sigma]. \end{aligned}$$

All the terms of the right member of this congruence are higher than F and are of the same weight and degree as F. There may be some β terms among them. Each such term is likewise congruent to a sum of higher terms of the same weight and degree. In particular the lowest β term effectively present in (1.1) is congruent to such a sum. This term may be replaced in (1.1) by the appropriate combination of higher terms, yielding a new congruence for F free of this β term and all lower ones. Since there is only a finite number of power products of given weight and degree, this process eventually terminates; what remains in the right member is a linear combination of α terms with rational coefficients.

Canonical representations

4. The above lemma will be complemented by the fact established later that no linear combination of α terms with rational coefficients is in Σ unless the coefficients are all zero. In addition, a canonical representation for the elements of Σ will be obtained, in the following sense. Every element of Σ has a representation $E_0A+ \cdots +E_sA_s$ but the same element may have different representations. A simple example of this is given by the polynomial $2y^2y_1$ which is in the ideal generated by y^2 and may be written $2y_1A$ or yA_1. Our canonical representation will be obtained by choosing the coefficients E_i

from a restricted set of polynomials, with the result that these coefficients are uniquely determined, while still furnishing representations for every element of Σ.

5. In securing the canonical representation for the elements of Σ we shall use forms H of the types

$$H = EA_{i_0}A_{i_1} \cdots A_{i_s}$$

where E is any power product in the y_i and the other factors of H constitute an arbitrary power product in A and its derivatives. It is convenient to write this latter power product as above, without using exponents, in such a way that $i_0 \leqq i_1 \leqq \cdots \leqq i_s$. H is homogeneous and isobaric. Its degree is the degree of E plus $(s+1)p$. Its weight is the weight of E plus $i_0+i_1+\cdots+i_s$. Evidently H is in Σ and conversely every element of Σ is a linear combination of such forms with constant coefficients. We order these forms in the following way: H is *higher* than $H' = E'A_{j_0}A_{j_1} \cdots A_{j_r}$ if either

(a) $A_{i_0}A_{i_1} \cdots A_{i_s}$ is higher than $A_{j_0}A_{j_1} \cdots A_{j_r}$ when both expressions are considered as power products in the A_i and are compared by the method used for power products in the y_i, or

(b) $r=s$, $i_k=j_k$ $(k=0, 1, \cdots, s)$ and E is higher than E' in the sense previously explained.

It should be emphasized that what we order are the symbols used to denote the forms rather than the forms themselves. For instance, for $A=y^2$, criterion (a) implies that $H=A\cdot A$ is higher than $H'=y^2A$ even though H and H' both denote the same form y^4. Thus an expression H is to be considered as different from H' for purposes of ordering, if the set $(i_0, i_1, \cdots, i_s)$ is different from the set $(j_0, j_1, \cdots, j_r)$ or if E is different from E'. We do not insist that the represented forms be different. On the other hand, *equations* connecting $H, H', \cdots$, are to refer in the usual way to the forms denoted by the symbols.

Evidently, of two different expressions H, H', one must be higher than the other. It is clear that our ordering is transitive. Furthermore, if H is higher than H' and G is any power product in the y_i, then GH is higher than GH'.

6. We now introduce the notion of a γ term. An expression $H=EA_{i_0}A_{i_1}\cdots A_{i_s}$ will be called a γ *term* if both (a) and (b) below hold.

(a) $i_s \geqq sp$,

(a_1) if $i_s > sp$ then E is an α term in the letters $y_{s+1}, y_{s+2}, \cdots$,

(a_2) if $i_s = sp$ then E is any power product in the letters $y_s, y_{s+1}, \cdots$.

(b) If $s>0$ then $i_k > kp$, $k=0, 1, \cdots, s-1$.

7. The role of these γ terms is revealed by the following lemma.

LEMMA 1.2. *Every expression H is equal to a sum $\sum r_iR_i$, where the R_i are γ terms of the same weight and degree as H. The r_i are rational numbers.*

This implies that every element of Σ is a linear combination of γ terms with rational coefficients. We shall see later that such a sum is zero only if all the coefficients are zero.

Our proof will consist mainly in showing that if H itself is not a γ term it is a linear combination with constant coefficients of γ terms and of expressions $H', H'', \cdots, H^{(t)}$; the expressions $H^{(i)}$ all being *higher* than H and of the same weight and degree as H. Once this is accomplished the proof can quickly be completed. By replacing the lowest $H^{(i)}$ by its linear combination of γ terms and expressions $H_j^{(i)}$ we obtain for H a new linear combination of γ terms and expressions $H_j^{(i)}$ which is free of that lowest $H^{(i)}$ and all lower ones. A finite number of repetitions of this procedure yields a linear combination of γ terms for H.

8. To devise methods for obtaining this sum of higher terms we consider the obvious equality $yA_1 = py_1A$ and those obtained by differentiating both members of this equation r times, $r = 1, 2, \cdots$. We obtain

$$yA_{r+1} + \sum_{i=1}^{r} C_{r,i}y_iA_{r+1-i} = p\sum_{i=1}^{r+1} C_{r,i-1}y_iA_{r+1-i}. \tag{1.2}$$

The symbols $C_{i,j}$ in equation (1.2) are binomial coefficients. This equation and the original equality together express yA_s with $c>0$ as a sum $\sum c_{si}y_iA_{s-i}$ where i runs through all positive integers not greater than s. An analogous expression may be obtained for y_kA_s, where k and s are positive integers and where $s>kp$. Let r and k be positive integers with the property $r+1-k>kp$. By subtracting the coefficient of y_kA_{r+1-k} in the right member of (1.2) from its coefficient in the left member we obtain

$$\frac{r!}{(k-1)!(r-k)!}\left(\frac{1}{k} - \frac{p}{r+1-k}\right).$$

This number is not zero, since $r+1-k>kp$. Thus under these circumstances y_kA_{r+1-k} is effectively present in (1.2). Equations (1.2) show, then, that y_kA_s with $s>kp$ may be written in the form

$$\sum_{i=1}^{s} d_{ki}y_{k+i}A_{s-i} + \sum_{i=1}^{k} e_{ki}y_{k-i}A_{s+i}$$

where the d_{ki} and e_{ki} are rational numbers which depend on s as well as on i and k. Observe that in the first of these sums the subscript of each A_{s-i} is less than s and in the second the subscript of each y_{k-i} is less than k.

9. We use these equations to derive a useful fact about expressions $y_kA_{i_0}A_{i_1}\cdots A_{i_s}$ when s and k are non-negative integers with $k \leqq s$ and $i_m>mp$, $m=0, \cdots, s$. We show that such an expression is equal to a sum of certain products F_jG_j where the F_j are forms, and the G_j are power products in the A_i. It will be seen that the degree of each G_j in the A_i does not

exceed $s+1$, and that each G_j as a power product in the A_i is higher than $A_{i_0} \cdots A_{i_s}$. For $s=0$, $i_0>0$ we already know that yA_{i_0} is a sum of such products, namely $c_{i_0j}y_jA_{i_0-j}$. We establish the result for $s>0$ by induction. It follows from equations (1.2) that $y_kA_{i_0}A_{i_1} \cdots A_{i_s}$ is equal to

$$\left(\sum_i d_{ki}y_{k+i}A_{i_s-i}\right)A_{i_0} \cdots A_{i_{s-1}} + \left(\sum_i e_{ki}y_{k-i}A_{i_s+i}\right)A_{i_0} \cdots A_{i_{s-1}}.$$

The terms in the first group meet our requirements since each contains a power product in the A_i which is higher than $A_{i_0}A_{i_1} \cdots A_{i_s}$. Each term of the second group contains a factor $y_{k-i}A_{i_0} \cdots A_{i_{s-1}}$. Assume the result true for all integers less than s. Such a factor is then equal to a sum of products F_jG_j where the G_j are power products in the A_i which are higher than $A_{i_0} \cdots A_{i_{s-1}}$ and whose degrees in the A_i do not exceed s. Consequently the G_j are all higher than $A_{i_0}A_{i_1} \cdots A_{i_{s-1}}A_{i_s}$. By letting $F_j' = e_{ki}A_{i_s+i}F_j$ it is seen that the terms in the second group are likewise equal to a sum of the required type.

10. We are now in a position to carry out the proof. An expression H which is not a γ term must fail to satisfy at least one of the conditions (a) and (b). We enumerate the various possibilities and show how for each one the required sum of higher terms may be obtained. It is both permissible and convenient first to discuss those terms $H=EA_{i_0} \cdots A_{i_s}$ which satisfy (b) but not (a), and then to give a complete discussion for those expressions which do not satisfy (b). We follow this plan.

Suppose H satisfies (b), $i_s>sp$ and (a_1) is not satisfied. If E involves only the letters $y_{s+1}, y_{s+2}, \cdots$, it must be a β term. It must then be divisible by the leader L_f of some A_f. As in the proof of Lemma 1.1, we have $E=cE'A_f+F$ where c is a constant and F is a form every one of whose terms is higher than E. Consequently

$$H = cE'A_{i_0} \cdots A_{i_s}A_f + FA_{i_0} \cdots A_{i_s}.$$

The first term in the right member of this equation is the product with c of an expression H' which by criterion (a) above is higher than H. The rest of the right member of this equation consists of a linear combination with rational coefficients of terms $H^{(i)}$ all higher than H by criterion (b). We therefore have the required sum of higher terms.

If $i_s>sp$, if (a_1) is not satisfied and if E contains effectively some letter y_k with $k \leqq s$, we first write $E=y_kE'$. Because H is supposed to satisfy (b), we know that $H'=y_kA_{i_0} \cdots A_{i_s}$ is a sum of products F_jG_j as described earlier. It follows that H is a sum of products $(E'F_j)G_j$. Since the G_j are power products in the A_i whose degrees do not exceed $s+1$ and which are higher than $A_{i_0} \cdots A_{i_s}$, it follows from criterion (a) that all the expressions $H^{(i)}$ in each product $(E'F_j)G_j$ are higher than H. This disposes of expressions H which satisfy (b) but not (a_1). If an H satisfying (b) does not satisfy (a_2) and is such that $i_s=sp$, then its coefficient E must contain effectively some letter y_k with $k<s$. Let $E=y_kE'$ and consider $y_kA_{i_0}\cdots A_{i_{s-1}}$. It equals a sum of products F_jG_j where the G_j are power products in the A_i of degree not more than s and which are higher than $A_{i_0}\cdots A_{i_{s-1}}$. What is important for us, is that the G_j are consequently also higher than $A_{i_0}A_{i_1}\cdots A_{i_{s-1}}A_{i_s}$. It follows that $H=(A_{i_s}E')y_kA_{i_0}\cdots A_{i_{s-1}}$ is a sum of higher expressions of the required sort.

There remains the case of an H which does not satisfy (b). Let r be the smallest integer for which $i_r\leqq rp$ so that if $r>0$ then $i_m>mp$, $m=0,\cdots,r-1$. Our procedure depends on whether $i_r=rp$ or $i_r<rp$. If $i_r=rp$ let $EA_{i_{r+1}}\cdots A_{i_s}$ be expanded into a form $F=\sum h_iE_i$, the h_i being constants and the E_i power products. We have $H=(\sum h_iE_i)A_{i_0}\cdots A_{i_r}$. We consider the expressions $H_i=E_iA_{i_0}\cdots A_{i_r}$, noting that they all satisfy (b). Certain of the E_i may be free of the letters $y, y_1,\cdots,y_{r-1}$. For these E_i the corresponding H_i are γ terms and require no further discussion. If $r=0$, all the E_i have this property and all the H_i are γ terms. On the other hand an E_i which contains effectively some y_k with $k<r$ leads to an H_i which is not a γ term. Such an H_i satisfies (b) but not (a_2). As we have seen, such an H_i is a sum of terms $q_jH_j^{(i)}$ where the q_j are constants and the expressions $H_j^{(i)}$ are all higher than H_i. This of course does not itself imply that the $H_j^{(i)}$ are higher than our original H. But by recalling that the $H_j^{(i)}$ must each contain a power product in the A_i which is higher than $A_{i_0}A_{i_1}\cdots A_{i_r}$ and whose degree does not exceed $r+1$, we see that the $H_j^{(i)}$ are actually higher than H by criterion (a). Our procedure for an H which does not satisfy (b) and for which $i_r<rp$ is the following. We note that r must be greater than zero, since i_r is non-negative. Let the form $EA_{i_r}A_{i_{r+1}}\cdots A_{i_s}$ be expanded as above into the form $F=\sum h_iE_i$. The fact that $i_r<rp$ implies that every term of A_{i_r}, and consequently every term of F, contains effectively some y_k with $k<r$. Then $H=(\sum h_iE_i)A_{i_0}\cdots A_{i_{r-1}}$ is a linear combination of expressions $H_i=E_iA_{i_0}\cdots A_{i_{r-1}}$ which satisfy (b) but do not satisfy the requirement of (a_1) which asks that E be free of $y, y_1,\cdots,y_r$. It is easy to see how the methods of the previous case apply here, and we omit the details of showing that H must be a sum of terms $h_j\ H_j^{(i)}$ where $H_j^{(i)}$ is higher than H.

11. We have now carried out our program of showing that each H not a γ term is a linear combination of expressions H_i, those H_i which are not γ terms being higher than H. The remarks made at the outset of the proof suffice to establish the lemma.

The fundamental lemma

12. We prove the following lemma.

LEMMA 1.3. *Let d and w be positive integers. The number n_γ of γ terms of degree d and weight w does not exceed the number of n_β of β terms which have this weight and degree.*

It will be shown later that we actually have $n_\gamma = n_\beta$. We remind the reader that in computing n_γ one counts the number of distinct symbols which stand for γ terms without considering whether or not the symbols stand for distinct forms.

The proof will consist in associating a unique β term of degree d and weight w with each γ term of this degree and weight. The association will be such that to different γ terms there will correspond different β terms.

13. We require a few more definitions. Let $\mathcal{R}_i$ denote the ring of polynomials with rational coefficients in the letters $y_i, y_{i+1}, \cdots, i=1, 2, 3, \cdots$, and let $\mathcal{R}_0$ denote our original ring $\mathcal{R}$. Let t be any non-negative integer. A form EA_i where $i \geqq tp$ and E is a power product in y and its derivatives will be called an expression K_t if the appropriate one of the following three conditions is satisfied by E.

(i) If $i=tp$, E is any power product of $\mathcal{R}_t$.

(ii) If $tp<i\leqq(t+1)p$, E is any power product of $\mathcal{R}_{t+1}$.

(iii) If $(t+1)p<i$, E is a special power product of $\mathcal{R}_{t+1}$. Let $E=y_{t+1}^{a_1}\cdots y_{t+r}^{a_r}$. We ask that there exist an integer k for which $E'=y_{t+1}^{a_1}\cdots y_{t+k}^{a_k}$ is an α term, and in addition we require for this k that $i\leqq(t+k+1)p-(a_1+\cdots+a_k)$.

Under condition (iii) any α term of $\mathcal{R}_{t+1}$ is acceptable as a coefficient E, for $(t+k+1)p$ increases with k, whereas for large k the exponent a_k is zero, so that $a_1+\cdots+a_k$ remains unchanged. On the other hand under condition (iii) an admissible E need not be an α term. Once a suitable k is found, no restriction whatever is made on the letters $y_{t+k+1}, y_{t+k+2}, \cdots$.

14. We now describe a process by which each such expression K_t determines a β term F of $\mathcal{R}_t$. Let $K=EA_i$ be a definite expression K_t. It comes under one of (i), (ii), (iii).

If K comes under (i) we have $i_t=tp$. Let

$$F = y_t^p E.$$

Clearly F is a β term of $\mathcal{R}_t$ having the same weight and degree as K.

If K comes under (ii) then $tp<i\leqq(t+1)p$. Let $b=(t+1)p-i$. Then b is a non-negative integer and $b<p$. Let

$$F = y_t^b y_{t+1}^{p-b} E.$$

F is a β term of $\mathcal{R}_t$ obtained by replacing A_i in K by the term $y_t^b y_{t+1}^{p-b}$. The degree of this term is p and its weight is $(t+1)p-b=i$. Thus F has the same weight and degree as K. For this case E does not contain the letter y_t so that the exponent of y_t in F is b which is less than p. This distinguishes the F obtained from a K which comes under (ii) from that obtained from a K which comes under (i). However, for both cases the sum of the exponents of y_t and y_{t+1} in F is at least p.

If K comes under (iii) then $(t+1)p<i$. We define

$$s_0 = (t+1)p,$$
$$s_f = (t+f+1)p - (a_1 + \cdots + a_f), \qquad f = 1, \cdots, k.$$

We have $s_f - s_{f-1} = p - a_f$, $f = 1, \cdots, k$. By hypothesis $y_{t+1}^{a_1} \cdots y_{t+k}^{a_k}$ is an α term and in particular each a_f is less than p. Thus each $p-a_f$ is positive, so that $s_0 < s_1 < \cdots < s_k$. Since by hypothesis $i \leqq s_k$ there is an integer m which is such that $1 \leqq m \leqq k$ and for which $s_{m-1} < i \leqq s_m$. Let $b = s_m - i$. Then b is a non-negative integer. Since $b < s_m - s_{m-1}$ and $s_m - s_{m-1} = p - a_m$ we have $b + a_m < p$. Let

$$F = y_t^{a_1} \cdots y_{t+m-1}^{a_m}(y_{t+m}^{b} y_{t+m+1}^{p-b})G \tag{1.3}$$

where $G = y_{t+m+1}^{a_{m+1}} \cdots y_{t+r}^{a_r}$. F is a β term of $\mathfrak{R}_t$. In the transition from K to F the expression A_i is suppressed, $y_{t+j}^{a_j}$ is replaced by $y_{t+j-1}^{a_j}$, $j = 1, \cdots, m$, and the term $y_{t+m}^{b} y_{t+m-1}^{p-b}$ is introduced. G is carried over unchanged. The first operation lowers the weight by i and the degree by p. The second lowers the weight by $a_1 + \cdots + a_m$ and does not change the degree. The introduction of $y_{t+m}^{b} y_{t+m+1}^{p-b}$ augments the degree by p and the weight by $(t+m+1)p-b$. Since

$$(t + m + 1)p - b = s_m + (a_1 + \cdots + a_m) - b = i + (a_1 + \cdots + a_m) \tag{1.4}$$

we see that the net effect of these alterations is to leave the weight and degree unchanged. Note that F contains the factor $y_t^{a_1} \cdots y_{t+m-1}^{a_m} y_{t+m}^{b}$ and that its other letters all have subscripts which exceed $t+m$. This factor is an α term, since $b + a_m < p$ and $y_t^{a_1} \cdots y_{t+m-1}^{a_m}$ is an α term by hypothesis. Since m is positive, it follows that the sum of the exponents of y_t and y_{t+1} in F is less than p. This is a characteristic property of a term F obtained from an expression K which comes under (iii).

15. We have described a procedure for obtaining from any expression K_t a definite β term F of $\mathfrak{R}_t$. We shall investigate this procedure further in order to obtain two useful facts. The first is that by this process different β terms F are assigned to different expressions K. The second is that when $t>0$, then for any integer h such that $(t-1)p < h \leqq i$, the expression FA_h is an expression K_{t-1}. In other words if F is obtained from any K_t in the manner set forth above, then, if $t>0$ and h is as above, FA_h admits one of the three characterizations, (i), (ii), (iii), where the discussion is referred to the integer $t-1$ instead of t. In deriving the first property of the term F we need only show that a β term F cannot be obtained from two different expressions K_t which both come under the same condition of the three listed. This simplification is due to the fact that in describing the procedure it was pointed out how one could infer from a given F which of the three conditions governed the K which determined it. We now list the three possibilities for F and verify the two statements for each one.

16. Let F be determined by an expression $K=EA_i$ which comes under (i). Then $F=y_t^pE$, so that given F one can find E. Since for this case $i=tp$, there is only one EA_i which could lead to F. This proves the first statement. To prove the second, let h be any integer such that $(t-1)p<h\leqq i$. We can easily verify that FA_h is an expression K_{t-1}, coming under (ii). In the case at hand $i=tp$, so that we have $(t-1)p<h\leqq tp$. In addition F is a power product of $\mathfrak{R}_t$. These are precisely the requirements of condition (ii).

17. Now let F be determined by $K=EA_i$, K coming under (ii). We have $F=y_t^by_{t+1}^{p-b}E$ with $0\leqq b<p$ and with E free of $y, y_1, \cdots, y_t$. Again it is obvious that F determines E uniquely and that the subscript i of A_t can also be uniquely determined from the equation $(t+1)p-b=i$. Thus only one expression EA_i can yield F by our procedure. Suppose that $t>0$ and that h is some integer for which $(t-1)p<h\leqq i$. If $h\leqq tp$ then FA is an expression K_{t-1} coming under (ii), since F is in $\mathfrak{R}_t$. If $h>tp$ we show that FA_h is also an expression K_{t-1} but that it then comes under (iii). The inequality $h\leqq i=(t+1)p-b$ enables us to draw this conclusion. The α term required by (iii) is simply y_t^b; the integer k is unity.

18. The case in which $K=EA_i$ comes under (iii) remains. The F which it determines is displayed in (1.3). We noted above that F contains as a factor the α term $y_t^{a_1}\cdots y_{t+m-1}^{a_m}y_{t+m}^b$. In an obvious sense this factor is the "largest" α term which can be split off from F. More precisely, given an $F=y_t^{b_0}y_{t+1}^{b_1}\cdots y_{t+s}^{b_s}$ determined by an expression K_t which comes under (iii), if one chooses the largest g such that $y_t^{b_0}y_{t+1}^{b_1}\cdots y_{t+g}^{b_g}$ is an α term, this last power product will be identical with $y_t^{a_1}\cdots y_{t+m-1}^{a_m}y_{t+m}^b$. We recall that in passing from $K=EA_i$ to F we divided the letters of E into two classes; the letters of one class were replaced by others, and the letters in the other were carried over unaltered. What we have just shown is that given an F determined by a K which comes under (iii) it is possible to determine exactly which letters were in each of the classes. The weight of the A_i involved in K may be computed from

$$i=(t+m+1)p-(a_1+\cdots+a_m)-b.$$

Thus, given such an F it is possible to reconstruct unequivocally the expression K from which it was obtained. This establishes the first property for a K coming under (iii). Assuming now that $t>0$, we proceed to establish the second. Let h be an integer such that $(t-1)p<h\leqq i$. We show that FA_h is an expression K_{t-1}. If $h\leqq tp$, FA_h is clearly an expression K_{t-1} coming under (ii), since F is a power product of $\mathfrak{R}_t$. If $h>tp$, we show that FA_h is an expression K_{t-1} coming under (iii). To do this we must produce an α term and an integer k as described in (iii). Let $F=y_t^{b_0}\cdots y_{t+s}^{b_s}$. Since our calculations are now based on the integer $t-1$, the integer k is required to have the property $(t+k)p-(b_0+\cdots+b_{k-1})\geqq h$. Since $h\leqq i$, we see from (1.4) that the α term $y_t^{a_1}\cdots y_{t+m-1}^{a_m}y_{t+m}^b$ and the integer $m+1$ have the required properties.

19. The proof of the lemma may now be completed. Let $H = EA_{i_0}A_{i_1} \cdots A_{i_s}$ be any γ term of degree d and weight w. H satisfies conditions (a) and (b) defining a γ term. We now show how these conditions make it possible to use the work immediately preceding to carry out our program of assigning a β term to every γ term.

Consider the form $K^{(s)} = EA_{i_s}$. If $i_s = sp$ then $K^{(s)}$ is an expression K_s coming under (i). If $i_s > sp$ it is readily seen that $K^{(s)}$ is likewise an expression K_s, only in this case it comes under (iii). In fact condition (a_1) requires E to be an α term of $\mathfrak{R}_{s+1}$ and it was pointed out above that an integer k of the type required by (iii) can always be found under these circumstances. Thus by splitting off EA_{i_s} from a γ term H we always obtain an expression K_s. Let the weight of E be w_s and its degree be d_s. $K^{(s)}$ determines a β term of $\mathfrak{R}_s$ by the procedure described above. Let it be denoted by $E^{(s)}$. Its weight is $w_s + i_s$ and its degree is $d_s + p$. If $s = 0$ we associate this β term with H. It has the same weight and degree as H because it has the same weight and degree as $K^{(s)}$ and for this case $K^{(s)} = H$.

If $s > 0$ consider the expression $K^{(s-1)} = E^{(s)}A_{i_{s-1}}$. It follows from the definition of γ term that $(s-1)p < i_{i_{s-1}} \leqq i_s$. This inequality permits us to conclude that $K^{(s-1)}$ is actually an expression K_{s-1}. $K^{(s-1)}$ determines a β term $E^{(s-1)}$ of $\mathfrak{R}_{s-1}$ having the same weight $w_s + i_s + i_{s-1}$ and the same degree $d_s + 2p$ as $K^{(s-1)}$. If $s = 1$ we associate this β term with H. It clearly has the same weight and degree as H.

If $s > 1$ we continue in this way. We obtain a sequence $K^{(s)}, K^{(s-1)}, \cdots, K^{(0)}$ and a sequence $E^{(s)}, E^{(s-1)}, \cdots, E^{(0)}$. The sequences are obtained from H by successive applications of the procedure described for obtaining β terms from expressions K_t. Each $K^{(f)} = E^{(f+1)}A_{i_f}$. Each $E^{(f)}$ is the β term of $\mathfrak{R}_f$ determined by $K^{(f)}$. The weight of both $K^{(f)}$ and $E^{(f)}$ is $w_s + i_s + \cdots + i_f$. The degree of both $K^{(f)}$ and $E^{(f)}$ is $d_s + (s-f+1)p$. The sequences are to be continued until $K^{(0)}$ and $E^{(0)}$ are reached. $E^{(0)}$ is a β term of $\mathfrak{R} = \mathfrak{R}_0$ having the same weight and degree as H. We associate $E^{(0)}$ with H.

20. We now prove that if $H_1 = E_1A_{j_0}A_{j_1} \cdots A_{j_r}$ is a γ term different from H, then the β term $E_1^{(0)}$ assigned to it in this way must be different from $E^{(0)}$. H_1 determines the two sequences $K_1^{(r)}, K_1^{(r-1)}, \cdots, K_1^{(0)}$ and $E_1^{(r)}, E_1^{(r-1)}, \cdots, E_1^{(0)}$. We know that each $E^{(f)}$ is determined by at most one $K^{(f)} = E^{(f+1)}A_{i_f}$. We conclude that if $E^{(0)} = E_1^{(0)}$, then for every f for which the symbols are defined, $E^{(f)} = E_1^{(f)}$ and $K^{(f)} = K_1^{(f)}$. If $s = r$ we have immediately that $H = H_1$. Suppose $s \neq r$ and, say, $s < r$. We show that it is impossible to have $E^{(0)} = E_1^{(0)}$ under this assumption. This last equality implies that $K^{(s)} = K_1^{(s)}$, that is, that $EA_{i_s} = E_1^{(s+1)}A_{j_s}$. $E_1^{(s+1)}$ is determined by $K_1^{(s+1)}$ and is consequently a β term. From the definition of γ term E may only be a β term if $i_s = sp$. Thus $j_s = sp$. This is impossible since H_1 is a γ term and for such terms we have $j_f = fp$ only if $f = r$, whereas here we have $j_s = sp$ and $s < r$.

We have shown that every γ term determines a β term of $\mathfrak{R}$ having the same weight and degree, and that distinct γ terms determine distinct β terms. This proves the lemma.

THE STRUCTURE OF THE IDEAL OF y^p

21. We can now prove the following lemma.

LEMMA 1.4. *Let d and w be positive integers. Let* n_α *denote the number of* α *terms* P_i *of degree d and weight w, let* n_β *denote the number of* β *terms* Q_j *of this degree and weight, and let* n_γ *denote the corresponding number of* γ *terms* R_k. *Then* $n_\beta = n_\gamma$, *and a relation*

$$\sum_{i=1}^{n_\alpha} p_i P_i + \sum_{k=1}^{n_\gamma} r_k R_k = 0$$

where the p_i *and* r_k *are rational numbers implies that all the* p_i *and* r_k *are zero.*

Let Q_j be any β term mentioned in the statement of the lemma. By Lemma 1.1 we have

$$Q_j \equiv \sum_i p_{ji} P_i \ [\Sigma], \qquad i = 1, \cdots, n_\beta.$$

Therefore by Lemma 1.2 we have

$$Q_j = \sum_k r_{jk} R_k + \sum_i p_{ji} P_i, \qquad i = 1, \cdots, n_\beta. \tag{1.5}$$

If n_γ were less than n_β some linear combination of the Q_j with rational coefficients would be a similar linear combination of the P_i. This is impossible, so that $n_\gamma \geqq n_\beta$. Applying Lemma 1.3 we see that $n_\gamma = n_\beta$.

22. Every R_k is by definition a homogeneous isobaric polynomial, so that we have

$$R_k = \sum_i a_{ki} P_i + \sum_j b_{kj} Q_j, \qquad k = 1, \cdots, n_\gamma. \tag{1.6}$$

Substituting the right member of (1.6) for R_k in (1.5), we obtain the identities $Q_j = Q_j$, $j = 1, \cdots, n_\beta$. Thus $|r_{jk}| \cdot |b_{kj}| = 1$ and it follows that both $|r_{jk}|$ and $|b_{kj}|$ are different from zero.

From (1.6) and the fact that $|b_{kj}| \neq 0$ we see that any linear combination of the R_k with constant coefficients not all zero must equal a similar linear combination of the P_i and Q_j which involves some Q_j effectively. If such a linear combination of terms R_k were expressible as a linear combination of terms P_i, we would have the absurd result that a linear combination of the Q_j with constant coefficients not all zero was a linear combination of the P_i.

23. THEOREM 1.1. *Let F be any element of* $\mathfrak{R}$. *F is expressible in the form*

$$F = \sum_i p_i P_i + \sum_k r_k R_k$$

where the P_i are α terms, the R_k are γ terms and the p_i, r_k are rational numbers[3]. *For each F there is only one such expression.*

Let F be split up into a sum of homogeneous isobaric polynomials F_h in such a way that any two such polynomials have either different weights or different degrees. Then

$$F = \sum_h F_h$$

and the F_h are uniquely determined by F.

By means of equation (1.5) we have

$$F_h = \sum_i p_{hi}P_i + \sum_k r_{hk}R_k$$

where the d and w of Lemma 1.4 are the degree and weight of F_h. By adding the F_h we obtain

$$F = \sum_h \sum_i p_{hi}P_i + \sum_h \sum_k r_{hk}R_k,$$

and this is the desired equation.

24. To show that F does not have two distinct representations of this type we need only show that if

$$\sum_i p_iP_i + \sum_k r_kR_k = 0 \tag{1.7}$$

then the p_i and r_k are all zero.

Let some p_i or r_k be different from zero. The term which possesses such a coefficient must be cancelled in (1.7) by a sum of other terms of the same weight and degree. This contradiction to Lemma 1.4 establishes our result.

25. Corollary. *No linear combination of α terms with rational*[4] *coefficients is in Σ.*

Let d, w, n_α be as in the statement of Lemma 1.4. The quantity n depends on d and w.

Corollary. *The number of linearly independent* (mod Σ) *elements of $\mathfrak{R}$ which are homogeneous of degree d and isobaric of weight w is n_α.*

26. $\mathfrak{R}$ may be considered as an abelian group with operators, where the group "multiplication" is ordinary addition, and the operators are rational numbers. Theorem 1.1 implies that $\mathfrak{R}$ considered in this way is the direct sum of two groups. One of them is Σ. The other is the additive group generated by

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(3) The same conclusion can be drawn if these symbols stand for any constants, or more generally if they are any elements of a domain of integrity which contains the rational numbers.

(4) See the note to Theorem 1.1.

the totality of α terms. A linearly independent basis for the first group is the totality of γ terms; the totality of α terms forms such a basis for the other.

27. We now determine circumstances under which the n_α of the corollary to Theorem 1.1 is zero. If for a given d and w this number is zero, then every homogeneous isobaric element of $\mathfrak{R}$ having this degree and weight is in Σ. In settling this question we consequently develop a method for establishing the membership in Σ of certain elements of $\mathfrak{R}$ based entirely on an examination of the weights and degrees of their constituent terms.

If d is less than p, every power product of degree d is an α term. To treat the case for which d is not less than p we write[5]

$$\mathfrak{S} = y^{p-1} y_2^{p-1} \cdots y_{2k}^{p-1} y_{2k+2}^{p-1} \cdots .$$

$\mathfrak{S}$ is a formal infinite product whose status in this discussion is that of a visual aid. Let d be a positive integer and write $d=a(p-1)+b$ (a and b non-negative integers with $0<b\leqq p-1$). Let

$$S_d = y^{p-1} y_2^{p-1} \cdots y_{2a-2}^{p-1} y_{2a}^{b}.$$

S_d is an α term of degree d. It is obtained by taking the first d letters of $\mathfrak{S}$ and multiplying them together. We denote the weight of S_d by $w(p, d)$. We have

$$w(p, d) = a(a-1)(p-1) + 2ab.$$

$w(p, d)$ is defined for all integers p greater than unity, and for all positive integers d. Its values are always positive integers or zero. An easy calculation shows that $w(p, d)$ satisfies the difference equation $w(p, d)+2d=w(p, d+(p-1))$, and this fact is used in proving the following result.

28. THEOREM 1.2. *Let d, w and n_α be as in the statement of Lemma* 1.4. *A necessary and sufficient condition that $n_\alpha>0$ is that $w\geqq w(p, d)$.*

In view of our earlier results this is equivalent to asserting that *every power product of degree d and weight $w<w(p, d)$ is in Σ and not every power product of degree d and weight $w\geqq w(p, d)$ is in Σ.*

29. The sufficiency proof is quickly disposed of. S_d is an α term of degree d and weight $w(p, d)$. Let

$$S_d^{(r)} = y^{p-1} y_2^{p-1} \cdots y_{2a-2}^{p-1} y_{2a}^{b-1} y_{2a+r}, \qquad r > 0.$$

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[5] In the remainder of Part I it is assumed that p exceeds unity. The two results enunciated there are seen to be trivially true for p equal to unity, if the weight function introduced at the end of §27 is defined to be plus infinity for p equal to unity and for all positive integral values of d.

$S_d^{(r)}$ is an α term of degree d and weight $w(p, d)+r$. Therefore we see that for any integer d and integer $w \geqq w(p, d)$ there are α terms of degree d and weight w.

30. We begin the necessity proof by observing that when d is less than p, $w(p, d)=0$. Consequently there are no power products of degree $d<p$ and weight $w<w(p, d)$. If our theorem were false there would be an integer $d \geqq p$, and an α term whose degree was d and whose weight was less than $w(p, d)$. We assume this to be the case and force a contradiction.

Let d $(\geqq p)$ be the smallest integer for which there are α terms whose degree is d and whose weight is less than $w(p, d)$. Let P be an α term of degree d and weight w, where w is some integer such that $0 \leqq w<w(p, d)$. Let $P=EP'$, where E is that factor of P of degree $p-1$ which is higher than any other such factor. Then P' is of positive degree and is an α term. Furthermore, since P is an α term the definition of E insures that P' is free of y and y_1. Let the degree of P' be denoted by d' and its weight by w'. Clearly $w' \leqq w$ and $d'<d$. Let P'' be obtained from P' by replacing each letter y_t effectively present in P' by y_{t-2}. P'' is an α term whose degree is d' and whose weight is $w'-2d'$. Our assumption about the minimal character of d, when applied to P'' implies

$$w(p, d') \leqq w' - 2d'.$$

Using the difference equation satisfied by $w(p, d')$ this last inequality yields $w(p, d'+p-1) \leqq w'$. Since $d'+p-1=d$ and $w' \leqq w$ we now have $w(p, d) \leqq w$. This contradiction completes the proof.

THE EXPRESSION FOR A POWER PRODUCT IN THE IDEAL OF y^p

31. Having established the fact that certain power products are in Σ we may naturally inquire as to the number of derivatives of A needed to express them. This question may be precisely formulated in the following way. Let a power product P be in Σ. It is a linear combination of γ terms. Let the lowest of these be the γ term $EA_{i_0} \cdots A_{i_s}$. It is required to determine an upper bound for i_0. This question arises in the following section for a special class of power products. We settle it now for this special class.

COROLLARY. *Let r be a positive integer, let $d=(r+1)p-1$ and let w be a nonnegative integer which does not exceed rd. Then every power product P of degree d and weight w is in Σ and is a linear combination, with forms for coefficients, of A and its derivatives of orders not exceeding rp.*

Let $d'=(r+1)(p-1)+1$. Certainly $d' \leqq d$. We first extract from P a factor P' of degree d' whose weight does not exceed rd'. This is made possible by the fact that the weight of P does not exceed rd. We then show that P', and hence P, is in Σ. Evaluation of $w(p, d')$ yields $r(r+1)(p-1)+2(r+1)$ which exceeds rd'. This shows that P' is in Σ. Actually for large r the weight of P is considerably smaller than $w(p, d)$. This additional restriction makes it possible to estimate relatively easily the number of derivatives of A required to express P.

The proof is by induction. When $r=1$, our assertion is that no more than p derivatives of A are required to obtain P. Any γ term of degree $2p-1$ is of the form A_iE. If i is zero no discussion is required. If i is greater than zero, E must be a power product of degree $p-1$ in the letters $y_1, y_2, \cdots$. The The weight of E is then at least $p-1$. If the weight of the γ term is not to exceed $2p-1$ it must be that $i \leqq p$. Assume now that the result is established for all integers less than some fixed integer r. Let $G=EA_{i_0}A_{i_1}\cdots A_{i_s}$ be a γ term of degree $d=(r+1)p-1$ and weight not greater than rd. It is to be shown that $i_0 \leqq rp$. We need only consider the case in which $i_0>0$. For this case E must be free of y, since G is a γ term. Let E' be obtained from E by diminishing by unity the subscript of each y_t which appears in E. If $s>0$, consider $G'=E'A_{i_1-p}\cdots A_{i_s-p}$. G' is evidently a γ term of degree $d'=d-p$ and weight $w'=w-i_0-d+p$. If i_0 exceeded rp we should then have, using $w \leqq rd$, $w'<rd-rp-d+p$ or $w'<(r-1)d'$. Our induction hypothesis then applies to G' and shows that $i_1-p \leqq (r-1)p$ whence $i_1 \leqq rp$. Since $i_0 \leqq i_1$, the assumption $i_0>rp$ leads to a contradiction. If in G the integer s is zero, so that $G=EA_{i_0}$, a different procedure is required. E must be an α term in the letters $y_1, y_2, \cdots$, of degree $d-p$ and weight $w-i_0$. Consequently E' is an α term of degree $d'=d-p$ and weight $w'=w-i_0-d+p$. Again assume that $i_0>rp$. Using $w \leqq rd$ and $i_0>rp$ we have $w'<(r-1)d'$. Since d' is $rp-1$ it follows as in the outset of this proof that the weight of E' is too small for E' to be an α term. The hypothesis $i_0>rp$ must then be discarded and the induction is carried out.

PART II. SOME THEOREMS ON THE STRUCTURE OF DIFFERENTIAL POLYNOMIALS

THE LOW POWER THEOREM

32. Let $\mathfrak{I}$ be any differential domain of integrity which contains the rational numbers. Throughout this section when we refer to a form in the unknowns $u, v, \cdots, w$, we shall mean a differential polynomial in $u, v, \cdots, w$ whose coefficients are in $\mathfrak{I}$. Indeed the coefficients actually used are for the most part rational numbers. Our work involves auxiliary unknowns which may be specialized with great freedom, and it is with such specialization in mind that the above remarks are made.

33. THEOREM 2.1. *Let*

$$F = \lambda y^p - \sum_{i=1}^{k} u_i B_i \tag{2.1}$$

be a form in $y, \lambda, u_1, \cdots, u_k$ *where* p *is a positive integer and the* B_i *are power products in* y *and its derivatives of degree* $p+1$. *Then there is a positive integer* s *and a form*

$$D = \lambda^s + H$$

where every term of H contains y or one of its derivatives effectively and where D is homogeneous of degree s in λ, $u_1, \cdots, u_k$ *and their derivatives, such that for some positive integer d,*

$$y^d D \equiv 0 \ [F].$$

The questions as to how large d and s need be, and how many derivatives of F are required to obtain $y^d D$ are not answered precisely, but in the proof explicit upper bounds are given for each of these numbers.

34. This theorem is the abstract counterpart of certain results obtained by Ritt. It might be appropriate to discuss this relation before taking up the proof. If y, λ and the u_i are replaced by forms Y, L, U_i in the unknowns $v_1, \cdots, v_n$, then F and D go over into forms F' and D' in the v_i and $Y^d D' \equiv 0$ $[F']$. Let us take Y to be an algebraically irreducible form whose order in v_n is h, L to be a nonzero form not divisible by Y whose order in v_n does not exceed h, and the U_i to be any forms. Then, when $\mathfrak{J}$ is a ring of analytic functions, the relation $Y^d D' \equiv 0$ $[F']$ shows that the general solution of Y is an essential irreducible manifold in the manifold of F'[6]. A somewhat different result is obtained by specializing y as v_1, λ as a form $1+L$ where L vanishes for $v_i = 0$, $i = 1, \cdots, n$, and the u_i as any forms in the v_j. $\mathfrak{J}$ is again a ring of analytic functions. The relation $v_1^d D' \equiv 0$ $[F']$ shows that the solution $v_i = 0$, $i = 1, \cdots, n$, of F' is not contained in any irreducible manifold held by F' but not by v_1[7].

35. We now take up the proof. Let r be the maximum of the weights of the B_i. If r is zero or unity, each B_i is divisible by y^p, and F itself may be factored into a product $y^p(\lambda + H)$ of the required type. Assuming now that $r > 2$, let d be an integer such that the set of all power products in y and its derivatives of degree d whose weight does not exceed $(r-1)d$ is in the differential ideal generated by $A = y^p$. The work of the preceding section proves that there are such integers d. Let these power products be denoted by $P_1, \cdots, P_m$, and let the weight of P_f be denoted by w_f, $f = 1, \cdots, m$. Suppose all the P_f may be expressed as linear combinations of A and its derivatives of order not exceeding t. Then

$$P_f = \sum_{j=0}^{t} C_{fj} A_j, \qquad f = 1, 2, \cdots, m,$$

where the C_{fj} are forms in y. They are homogeneous of degree $d-p$ and isobaric of weight $w_f - j$. Using the fact that $\lambda^{q+1} A_q$ is in the differential ideal

(6) J. F. Ritt, *On the singular solutions of algebraic differential equations*, Annals of Mathematics, (2), vol. 37 (1936), pp. 555–560.

(7) J. F. Ritt, *On certain points in the theory of algebraic differential equations*, American Journal of Mathematics, vol. 60 (1938), p. 9. This paper will be referred to as OCP.

generated by λA and that it is a linear combination of λA and its first q derivatives(8), we have for each P_f

$$\lambda^{t+1}P_f = \sum_{j=0}^{t}\sum_{g=0}^{j} C_{fj}L_g(\lambda A)_{j-g}.$$

Here the symbol $(\lambda A)_{j-g}$ means the $(j-g)$th derivative of λA, and L_g is a homogeneous form in λ of degree t. Referring to (2.1) we have for each P_f

$$\lambda^{t+1}P_f \equiv \sum_{j=0}^{t}\sum_{g=0}^{j} C_{fj}L_g\left(\sum_{i=1}^{k} u_iB_i\right)_{j-g} [F]$$

where $(\sum u_iB_i)_{j-g}$ means the $(j-g)$th derivative of the form inside the parentheses. The right member of this congruence is a linear combination of forms $T=C_{fj}(B_i)_h$, $h\leqq j$, whose coefficients are homogeneous forms in λ and the u_i. These forms T are homogeneous of degree $d+1$, since the B_i are all of degree $p+1$. The weight of each $(B_i)_h$ does not exceed $r+h$, and therefore the weight of each T does not exceed $w_f-j+r+h$. Using $w_f\leqq(r-1)d$ and $h\leqq j$, it follows that the weight of each T does not exceed $(r-1)d+r$. The forms T are thus linear combinations of power products y_cP_f where again the P_f are power products in the y_i of degree d and weight not exceeding $(r-1)d$. We need merely choose $c\geqq r$ if such a y_c appears in a term of the T, while if no such y_c is present effectively, then for any choice of y_c the statement is true.

36. What we have shown is that

$$\lambda^{t+1}P_f \equiv \sum_{j=1}^{m} E_{fj}P_j \; [F], \qquad f=1,\cdots,m,$$

where the E_{fj} are forms in y, and the u_i. E_{fj} is homogeneous of degree t in λ, homogeneous of degree unity in the u_i and homogeneous of degree unity in the y_i. Transposing, we have a system of m linear congruences for the P_f. In the ith congruence the coefficients of P_j with $j\neq i$ is $-E_{ij}$ while the coefficient of P_i is λ^t-E_{ii}. It follows that

$$P_fD \equiv 0 \; [F], \qquad f=1,2,\cdots,m,$$

where D is the determinant of the system of congruences. Clearly D is of the form $\lambda^{m(t+1)}+H$ where H vanishes for $y=0$. Since y^d is one of the P_f we have our result. Observe that H is homogeneous of degree $m(t+1)$ in λ and the u_i.

37. As a supplement to the proof the size of t and d will be examined. Reference to the work of Part I reveals that if $d=rp-1$ then t may be taken as $(r-1)p$. If, on the other hand, $d=r(p-1)+1$, the proof may also be car-

(8) This is obvious for $q=0$. An easy induction establishes it for all positive integral values of q, for the product of λ^q with the qth derivative of A is a form $\lambda^{q+1}A_q$ plus terms all of which contain a factor λ^qA_i with $i<q$.

ried out with possibly a larger value of t. Note that this value of d is in general smaller than the previous one.

Generalization for one unknown

38. This theorem may be extended in the following way. Let

$$F = \lambda y^{p_0} y_1^{p_1} \cdots y_r^{p_r} + \sum u_i B_i \tag{2.2}$$

be a form in y, λ and a finite number of unknowns u_i where $p_0, p_1, \cdots, p_r$ are non-negative integers with $p_r \neq 0$, and where the B_i are power products in the y_i. If the degree of each B_i in the letters $y_r, y_{r+1}, \cdots$, exceeds p_r, then Theorem 2.1 may be applied to F. It yields a relation

$$y_r^d((\lambda y^{p_0} y_1^{p_1} \cdots y_{r-1}^{p_{r-1}})^s + H) \equiv 0 \ [F],$$

where H vanishes for $y_{r+i} = 0$, $i = 0, 1, 2, \cdots$. It is obvious that F admits the solution $y = 0$. This relation shows that every irreducible manifold held by F which contains $y = 0$ and is not held by y_r must be held by the form in the outer parentheses. We are going to show how additional hypotheses on the B_i make it possible to draw a stronger conclusion.

Theorem 2.2. *Let the F be given by* (2.2) *and let the B_i be such that for each integer f, $f = 0, 1, \cdots, r$, the degree of each B_i in the letters $y_f, y_{f+1}, y_{f+2}, \cdots$ exceeds $p_f + p_{f+1} + \cdots + p_r$. Then there is a positive integer s and a form*

$$D = \lambda^s + H$$

where every term of H contains y or one of its derivatives effectively and where D is homogeneous of degree s in λ and the u_i and their derivatives, such that for some positive integer d

$$y_r^d D \equiv 0 \ [F].$$

The stronger conclusion drawn here is that if λ is specialized as any non-zero element of $\mathfrak{J}$, then the solution $y = 0$ of F is contained in no irreducible manifold held by F but not by y_r.

39. The proof is by induction on r. For $r = 0$ this result is practically identical with Theorem 2.1. The only difference is that the B_i may be of degree greater than $p_0 + 1$ in the y_i. However, it is easy to see that by incorporating superfluous factors of the B_i into their coefficients we obtain a form for which Theorem 2.1 may be invoked. We assume the theorem established for integers from zero to $r - 1$ inclusive and prove it for r.

40. We introduce new letters λ', u_i' and a new form F' in y_1, λ', u_i' in the following way. Let Q_i be that factor of B_i of degree p_0 which is higher than any other such factor (if p_0 is zero then Q_i is unity). Let B_i' be defined by $B_i = B_i' Q_i$. Let

$$F' = \lambda' y_1^{p_1} y_2^{p_2} \cdots y_r^{p_r} + \sum u_i' B_i'.$$

F' goes over into F when λ' is replaced in F' by λy^{p_0} and u_i' by $u_i Q_i$. It will be shown that F' satisfies the induction hypothesis for $r-1$. To do this it suffices to show that for all integers f, $f=1, 2, \cdots, r$, the degree of each B_i' in $y_f, y_{f+1}, \cdots$ exceeds $p_f+p_{f+1}+\cdots+p_r$. Suppose there were a B_i' and an f for which this were not so. Because of the way B_i' was defined, our hypothesis would then require that Q_i be of positive degree in some y_t with $t \geqq f$. Since Q_i is the highest factor of B_i of degree p_0, it follows that B_i' is free of $y, y_1, \cdots, y_{f-1}$. Thus the degree of B_i is simply p_0 plus the degree of B_i' in $y_f, y_{f+1}, \cdots$. Clearly the statement that the degree of B_i' in $y_f, y_{f+1}, \cdots$ does not exceed $p_f+p_{f+1}+\cdots+p_r$ conflicts with our assumption that the degree of B_i exceeds $p_0+p_1+\cdots+p_r$.

41. Under our induction hypothesis there is a form $D'=\lambda'^t+H'$ and an integer d_1 such that

$$y_r^{d_1} D' \equiv 0 \ [F']$$

where H' is homogeneous in λ' and the u_i' of degree t and vanishes for $y_1=0$. Let D' become D_1 when λ' and the u_i' are replaced as above. Then $y_r^{d_1}D_1$ is in $[F]$. D_1 is of the form $\lambda^t y^{p_0 t}+H_1$ where H_1 is homogeneous in λ and the u_i of degree t. Every term of H is of degree at least tp_0+1 in the y_i.

It follows from Theorem 2.1 that there is a form $D_2=\lambda^w+H_2$, H_2 homogeneous in λ and the u_i of degree w and vanishing for $y=0$, and an integer k such that $y^k D_2 \equiv 0 \ [D_1]$. By the result stated in the footnote to Theorem 1.1 we find that there are integers h and g such that $y_r^h D_2^g \equiv 0 \ [D_1]$. This same result used again shows that for sufficiently large d, $y_r^d D_2^g$ is in $[y_r^{d_1} D_1]$. This completes the proof, since $y_r^{d_1}D_1$ is in $[F]$, and the form $D=D_2^g$ and the integer d have the required properties.

A SPECIAL CASE

THEOREM 2.3. *Let p be a positive integer, let $A=y^p$, and let A_i be the ith derivative of A. Let*

$$F = A - \sum_{i,j} C_{ij} y_i A_j$$

where i and j have some definite range and where the C_{ij} are forms in y. Then there is a form $D=1+H$, where H is a form in y which vanishes for $y=0$, such that

$$y^p D \equiv 0 \ [F].$$

The distinctive feature of this result is that the integer p is available as the integer d of our previous work. For $p=1$ this theorem is identical with a result obtained in a paper by Ritt and Kolchin(9).

(9) J. F. Ritt and E. R. Kolchin, *On certain ideals of differential polynomials*, Bulletin of the American Mathematical Society, vol. 45 (1939), pp. 895–898.

The device of continued substitution used in the first part of our proof is borrowed from this paper.

42. By hypothesis

$$A \equiv \sum_{i,j} C_{ij} y_i A_j \ [F]. \tag{2.3}$$

If in the right member of (2.3) A_j is replaced by the jth derivative of the whole right member, the result is a congruence

$$A \equiv \sum_{i,j,k} C_{ijk} y_i y_j A_k \ [F]. \tag{2.4}$$

For each C_{ij} of (2.3) which is different from zero, let the sum $i+j$ be computed. Suppose r is the largest of these sums. Then no sum $i+j+k$ for which there is a nonzero C_{ijk} exceeds $2r$. If in the right member of (2.4) the kth derivative of the whole right member of (2.3) is substituted for A_k, a new congruence for A is obtained. After $s-1$ iterations of this substitution, a congruence

$$A \equiv \sum C_{i_1 i_2 \cdots i_{s+1}} y_{i_1} y_{i_2} \cdots y_{i_s} A_{i_{s+1}} \ [F]$$

is obtained, where no sum $i_1+i_2+ \cdots +i_s$ with nonzero $C_{i_1 \cdots i_{s+1}}$ exceeds sr. If $s=(r+1)(p-1)+1$ then by our frequently used criterion, $y_{i_1} y_{i_2} \cdots y_{i_s} \equiv 0[A]$. Thus

$$A \equiv \sum D_{ij} A_i A_j \ [F] \tag{2.5}$$

where D_{ij} is a form in y.

43. We now compute the sum $i+j$ for each D_{ij} effectively present in (2.5). Let s be the largest of these sums. If (2.5) is differentiated s times, we obtain $s+1$ congruences, expressing A_k, $k=0, 1, \cdots, s$, as linear combinations of products $A_i A_j$ whose coefficients are forms in y. Each product so obtained must contain an A_i with $i \leqq s$, for the differentiation introduced an increment of at most s to the sums $i+j$ and they did not exceed s at the outset. We have shown, then, that

$$A_i \equiv \sum_{j=0}^{s} K_{ij} A_j \ [F], \qquad i = 0, 1, \cdots, s,$$

where K_{ij} vanishes for $y=0$. Transposing, we have a system of $s+1$ linear homogeneous congruences for the A_i. In the ith congruence the coefficient of A_j with $j \neq i$ is $-K_{ij}$ while the coefficient of A_i is $(1-K_{ii})$. D, the determinant of the system, is of the form $1+H$ where H vanishes for $y=0$. Since $A_i D \equiv 0 \ [F]$; $i=0, 1, \cdots, s$ we have our result.

Generalizations for several unknowns

44. Our next result concerns systems of forms in the unknowns $y_1, \cdots, y_n, \lambda$ and a finite number of unknowns u_{ij}. As is customary, we shall denote the jth

derivative of y_i with the symbol y_{ij} and the ith derivative of λ by λ_i. The second subscript of the u_{ij} will not mean differentiation, but will simply indicate how these unknowns are displayed in rows and columns.

45. Let s be any positive integer not greater than n. We consider subsets $(i_1, i_2, \cdots, i_s)$ of $(1, 2, \cdots, n)$ where in each subset the numbers i_k are all different. If the binomial coefficient $C_{n,s}$ is denoted by q, there are exactly q such subsets. We suppose a number j, $j=1, \cdots, q$, assigned in any univocal manner to each such subset. We shall consider the system

$$F_j = \lambda y_{i_1}^{p_{j1}} y_{i_2}^{p_{j2}} \cdots y_{i_s}^{p_{js}} + \sum_i u_{ji} B_{ji}, \qquad j = 1, \cdots, q, \tag{2.6}$$

where the p_{ji} are positive integers, and the B_{ji} power products in $y_1, \cdots, y_n$ and their derivatives. We call $\lambda y_{i_1}^{p_{j1}} y_{i_2}^{p_{j2}} \cdots y_{i_s}^{p_{js}}$ the *first term* of F_j. For each form F_j we make the following assumptions concerning the degree of the B_{ji} in the unknowns $y_1, \cdots, y_n$ and their derivatives. These assumptions describe a relation whereby the B_{ji} dominate the first terms of the F_j and it should be understood that each B_{ji} is qualified in this way only by the first term of that form F_j which contains it. Let $(i_a, i_b, \cdots, i_f)$ be any (proper or improper) subset of $(i_1, i_2, \cdots, i_s)$. It is required of B_{ji} that

(1) either

(1a) B_{ji} be divisible by $y_{i_a}^{p_{ja}} y_{i_b}^{p_{jb}} \cdots y_{i_f}^{p_{jf}}$

or

(1b) the degree of B_{ji} in the unknowns $y_{i_a}, y_{i_b}, \cdots, y_{i_f}$, in the unknowns y_k not in the first term of F_j, and in the derivatives of all these unknowns exceeds $p_{ja}+p_{jb}+\cdots+p_{jf}$;

(2) the total degree of each B_{ji} must exceed

$$p_{j1} + p_{j2} + \cdots + p_{js}.$$

THEOREM 2.4. *Let Σ be the differential ideal generated by the F_j. Then there exists a form $D=\lambda^c+H$, where H is a form in the y_i, λ and the u_{ij}, which vanishes for $y_i=0$, $i=1, \cdots, n$, and an integer t such that for every form $V_j=y_{i_1}y_{i_2}\cdots y_{i_s}$*

$$V_j^t D \equiv 0 \ [\Sigma]. \tag{2.7}$$

H is homogeneous of degree c in λ and the u_{ji}.

46. Here, too, our result has considerable contact with Ritt's work in differential equations. Before taking up the proof a few remarks might be made concerning the content of this theorem from the standpoint of differential equations. The unknowns λ and the u_{ij} have been introduced as auxiliaries to facilitate the proof. For purposes of illustration we may suppose λ replaced by unity and the u_{ji} by any forms in the y_i. Equations (2.7) then have the appearance

$$V_j^t(1 + H) \equiv 0 \ [\Sigma], \qquad j = 1, \cdots, q.$$

Obviously Σ admits the solution $y_i = 0$, $i = 1, \cdots, n$. These equations show that any irreducible manifold held by Σ which contains $y_i = 0$, $i = 1, \cdots, n$, must be held by the system V_j, $j = 1, \cdots, q$. Each irreducible manifold in this latter system is found by letting some $n - s + 1$ of the unknowns y_i be zero, the remaining $s - 1$ unknowns being arbitrary. In one extreme case, with $s = 1$, the manifold of the system V_j is precisely $y_i = 0$, $i = 1, \cdots, n$. The essentiality of this solution in Σ was shown by Ritt(10). He also treated the case $s = n$, obtaining(11) the above conclusion as a consequence of an approximation theorem. The intermediate cases, that is, those in which $1 < s < n$, appear here as new results, both from the abstract viewpoint and that of differential equations. The extreme cases owe their novelty to the fact that their proof is abstract. For the extreme case $s = n$, we reverse the procedure followed by Ritt. This result will be established first and then the analogue of Ritt's approximation theorem will be shown to follow from it. For $s = n$ and $\lambda = 1$ we have a single form

$$F = y_1^{p_1} y_2^{p_2} \cdots y_n^{p_n} + \sum u_i B_i.$$

Our hypothesis now reduces to the statements that for each k each B_i is either divisible by $y_k^{p_k}$ or its degree in the y_{kj} exceeds p_k. The total degree of each B_i exceeds $p_1 + \cdots + p_n$. The conclusion is that there is a congruence

$$(y_1 \cdots y_n)^t(1 + H) \equiv 0 \ [F]$$

where H vanishes for $y_i = 0$, $i = 1, \cdots, n$. This special case of Theorem 2.4 will be used in our later work.

47. We prove Theorem 2.4 by induction on s, beginning with $s = 1$ and n arbitrary. For this case our hypothesis states that Σ contains the forms

$$F_j = \lambda y_j^{p_j} + \sum_i u_{ji} B_{ji}, \qquad = 1, \cdots, n,$$

the B_{ji} being power products in the y_i and their derivatives whose total degree exceeds p_j.

Let r be the maximum of the weights of the B_{ji}. We understand that the weight of y_{jk} is k. Let p be the maximum of the p_j. Let $d = n(r+1)(p-1)+1$. Then every power product P_h in the y_{ij} of degree d whose weight w_h does not exceed $r((r+1)(p-1)+1)$ is in some $[y_i^{p_i}]$, for P_h is of degree at least $(r+1)(p-1)+1$ in at least one y_j, and then our earlier result applies. We now follow the procedure used in the proof of Theorem 2.1. We first multiply each P_h by λ^g so that the product is in some $[\lambda y_i^{p_i}]$ and after substitutions and re-

(10) OCP, pp. 5–7.

(11) OCP, p. 14.

arrangements similar to those used in the proof of that theorem we obtain the congruences

$$\lambda^g P_h \equiv \sum_{i=1}^{m} E_{hi} P_i \; [\Sigma], \qquad h = 1, 2, \cdots, m,$$

m being the number of P_h. The only difference is that now the E_{hi} are forms in $y_1, \cdots, y_n$, λ and the u_{ji} which need not be homogeneous in the y_i. It is still true that every term of each E_{hi} involves some y_k effectively. E_{hi} is homogeneous of degree $g-1$ in λ and is homogeneous of degree unity in the u_{ij}. After transposing, we see that

$$P_h D \equiv 0 \; [\Sigma], \qquad h = 1, \cdots, m,$$

where D, the determinant of the transposed system, is of the type described in the statement of this theorem. Since $y_1^d, y_2^d, \cdots, y_n^d$ are all to be found among the P_h we have our result.

48. Continuing with the proof we suppose that the theorem holds for all values of s from unity to some fixed integer, and proceed to show that it holds for the next integer. We shall denote this latter integer by s. In our induction assumptions the only restriction on n is that it be sufficiently large to insure that the statement of the theorem makes sense, that is, n is never less than s. It is otherwise arbitrary.

Referring to (2.6) select all those forms F_j whose first terms contain the letter y_1 effectively. Let p be the maximum of the exponents of y_1 in the first terms of these F_j. By multiplying certain of these F_j by a suitable power of y_1 we obtain a system of forms G_j each of whose first terms contains the letter y_1 exactly to the pth power. The set of forms G_j, being composed of some of these F_j and multiples by a power of y_1 of the others, is certainly in Σ. Furthermore the terms of the G_j which are not first terms satisfy the conditions of our hypothesis relative to the first term of the form which contains them. We now introduce new letters λ', u'_{ji} and new forms G'_j in these letters and $y_2, y_3, \cdots, y_n$. The G'_j will be defined so that their first terms will contain only $s-1$ letters. They will fulfill the conditions of our hypothesis, and will go over into the G_j when appropriate replacements are made for λ' and the u'_{j1}. We begin with

$$G_1 = \lambda y_1^p y_2^{p_{12}} \cdots y_s^{p_{1s}} + \sum u_{1i} \overline{B}_{1i}.$$

The $\overline{B}_{1i}$ are either the original B_{1i} or multiples of them by a power of y_1. As a first step we divide the $\overline{B}_{1i}$ into two classes, those which are divisible by $y_2^{p_{12}} \cdots y_s^{p_{1s}}$ and those which are not. A $\overline{B}_{1i}$ which is divisible by this term may be written as $L_{1i}\, y_2^{p_{12}} \cdots y_s^{p_{1s}}$. Our hypothesis reveals that the degree of L_{1i} in $y_1, \cdots, y_n$ and their derivatives exceeds p. We now have

$$G_1 = \left(\lambda y_1^p + \sum_k u_{1k} L_{1k}\right) y_2^{p_{12}} \cdots y_s^{p_{1s}} + \sum u_{1i} \overline{B}_{1i}$$

where the second summation is performed over all those $\bar{B}_{1i}$ which are in the second class. We further subdivide the $\bar{B}_{1i}$ of the second class into those which contain a factor in y_1 and its derivatives of degree p and those which do not. For a $\bar{B}_{1i}$ of the first kind let Q_{1i} be any such factor and let $\bar{B}_{1i}=Q_{1i}B_{1i}'$. For $\bar{B}_{1i}$ of the second kind the total multiplicity of y_1 and its derivatives in $\bar{B}_{1i}$ is some number $q_{1i}<p$. Let all these letters be split off from $\bar{B}_{1i}$ and multiplied by any factor of $\bar{B}_{1i}$ which contains only the unknowns $y_{s+1}, \cdots, y_n$ and whose degree in these unknowns and their derivatives is $p-q_{1i}$. Our hypothesis permits us to construct such a product for each $\bar{B}_{1i}$ of the second kind. Denoting this product by Q_{1i} we have in Q_{1i} a power product in $y_1, y_{s+1}, \cdots,$ y_n and their derivatives of degree p. Here too we let $\bar{B}_{1i}=Q_{1i}B_{1i}'$. Let

$$G_1' = \lambda_1 y_2^{p_{12}} \cdots y_s^{p_{1s}} + \sum u_{1i}' B_{1i}'.$$

When λ_1 is replaced by $\lambda y_1+\sum u_{1k}L_{1k}$ and u_{1i}' by $u_{1i}Q_{1i}$ then G_1' goes over into G_1.

49. It is readily seen that if $(a, b, \cdots, f)$ is any subset of $(2, 3, \cdots, s)$ then for each B_{1i}' either

(a) B_{1i}' is divisible by $y_a^{p_{1a}}y^{p_{1b}} \cdots y_f^{p_{1f}}$

or

(b) the degree of B_{1i}' in $y_a, y_b, \cdots, y_f, y_{s+1}, \cdots, y_n$ and their derivatives exceeds $p_{1a}+p_{1b}+ \cdots +p_{1f}$.

For a B_{1i}' obtained from a $\bar{B}_{1i}$ of the first kind this is part of our initial hypothesis, since in this case B_{1i}' is the same as $\bar{B}_{1i}$ as far as the unknowns $y_2, y_3, \cdots, y_n$ are concerned. As for a B_{1i}' obtained from a $\bar{B}_{1i}$ of the second kind, recall that $\bar{B}_{1i}=Q_{1i}B_{1i}'$ where Q_{1i} is of degree p in $y_1, y_{s+1}, \cdots, y_n$ and their derivatives and where B_{1i}' is free of y_1 and its derivatives. Since $\bar{B}_{1i}$ is not divisible by $y_1^p y_a^{p_{1a}} y_b^{p_{1b}} \cdots y_f^{p_{1f}}$ (because its degree in y_1 is less than p) our original hypothesis requires that the degree of $\bar{B}_{1i}$ in $y_1, y_a, y_b, \cdots, y_f, y_{s+1}, \cdots, y_n$ and their derivatives exceed $p+p_{1a}+p_{1b}+ \cdots +p_{1f}$. Since Q_{1i} is of degree p it follows that the degree of B_{1i}' exceeds $p_{1a}+p_{1b}+ \cdots +p_{1f}$. Since in addition B_{1i}' is free of y_1 and its derivatives we have our result. To complete the task of showing that G_1' satisfies our hypothesis for the unknowns $y_2, \cdots, y_n$ we must also dispose of the requirement that the degree of each B_{1i}' in $y_2, \cdots, y_n$ exceed $p_{12}+p_{13}+ \cdots +p_{1s}$. Of the original $\bar{B}_{1i}$ the only ones which need not meet this requirement are those divisible by $y_2^{p_{12}} \cdots y_s^{p_{1s}}$. These have been removed from consideration by our choice of λ_1. Thus, referring to the argument just given, if $(a, b, \cdots, f)$ coincides with $(2, 3, \cdots, s)$ the alternative (a) is excluded; alternative (b) is what was to have been established.

50. Proceeding in this way we obtain a form G_j' for each F_j whose first term contains y_1 effectively. We have

$$G_j' = \lambda_j y_{i_2}^{p_{j2}} y_{i_3}^{p_{j3}} \cdots y_{i_s}^{p_{js}} + \sum_i u_{ji}' B_{ji}'$$

where $(i_2, i_3, \cdots, i_s)$ is a subset of $s-1$ distinct numbers of the set $(2, 3, \cdots, n)$ and where each B'_{ji} is a power product in $y_2, y_3, \cdots, y_n$ and their derivatives which dominates the first term of G'_j in the required way. Each G'_j goes over into G_j when λ_j is replaced by $\lambda y_1^p + \sum u_{jk} L_{jk}$ and the u'_{ji} are replaced by $u_{ji}Q_{ji}$. Let the number of these forms be q_1.

51. We now introduce new unknowns λ'' and u''_{ji} and new forms

$$G''_j = \lambda'' y_{i_2}^{p_{j2}} \cdots y_{i_s}^{p_{js}} + \sum u''_{ji} B_{ji}, \qquad j = 1, \cdots, q_1.$$

When in G''_j the unknown λ'' is replaced by $\lambda_1\lambda_2 \cdots \lambda_{q_1}$ and each u''_{ji} by $\lambda_1 \cdots \lambda_{j-1}\lambda_{j+1} \cdots \lambda_{q_1} u'_{ji}$ then G''_j goes over into $\lambda_1 \cdots \lambda_{j-1}\lambda_{j+1} \cdots \lambda_{q_1} G'_j$.

Let Σ'' be the differential ideal generated by the G''_j. Under the terms of the induction hypothesis we conclude that there is a form $D'' = \lambda''^{w_1} + H''_1$ and an integer t_1, such that for every form $y_{i_2} y_{i_3} \cdots y_{i_s}$ we have

$$(2.8) \qquad (y_{i_1} y_{i_2} \cdots y_{i_s})^{t_1} D''_1 \equiv 0 \ [\Sigma''].$$

H''_1 vanishes for $y_i = 0$, $i = 1, \cdots, n$, which is to say that every term of H''_1 contains some y_{ij} effectively. H''_1 is homogeneous of degree w_1 in λ'' and the u''_{ji}. When λ'' and the u''_{ji} are replaced in D''_1 by the forms indicated above, D''_1 goes over into a form

$$D'_1 = (\lambda_1\lambda_2 \cdots \lambda_{q_1})^{w_1} + H'_1.$$

D'_1 is a form in $y_1, \cdots, y_n, \lambda_1, \cdots, \lambda_{q_1}$ and the u'_{ji}. It is homogeneous of degree $q_1 w_1$ in the λ'_i and the u'_{ji}. Now in D'_1 let λ'_j be replaced by $\lambda y_1^p + \sum u_{jk} L_{jk}$ and u'_{ji} by $u_{ji} Q_{ji}$. Let $p_1 = p w_1 q_1$. Then D'_1 goes over into a form

$$D_1 = \lambda^{q_1 w_1} y_1^{p_1} + H_1.$$

D_1 is a form in $y_1, \cdots, y_n, \lambda$ and the u_{ji}. It is homogeneous of degree $q_1 w_1$ in λ and the u_{ji}. We are going to show that the degree in the y_{ij} of each term of H_1 exceeds p_1. To this end we consider two types of terms of H_1, those arising from H'_1 and those from $L = (\lambda_1 \cdots \lambda_{q_1})^{w_1}$. Each λ_j was replaced by $\lambda y_1^p + \sum u_{jk} L_{jk}$ where the degree of each L_{jk} in $y_1, \cdots, y_n$ and their derivatives exceeded p. Therefore the degree in the y_{ij} of every term of L except $\lambda^{q_1 w_1} y_1^{p_1}$ exceeds p_1. This accounts for terms of H_1 arising from L. As for those arising from H'_1 recall that H'_1 is homogeneous of degree $q_1 w_1$ in the λ_i and the u'_{ji}. Since each λ_i contributes at least p to the degree of H_1 in the y_{ij} and each u'_{ji} contributes exactly p to this degree, the λ_i and u'_{ji} contribute at least $p_1 = p w_1 = q_1$. Because each term of H'_1 was of positive degree in the y_{ij} it follows that the terms of H_1 arising from H'_1 also have a degree in the y_{ij} which exceeds p_1. This verifies our assertion about D_1.

After these replacements are made, equation (2.8) becomes

$$(y_{i_2} y_{i_3} \cdots y_{i_s})^{t_1} D_1 \equiv 0 \ [\Sigma].$$

Consequently, for every $V_j = y_{i_1} y_{i_2} \cdots y_{i_s}$

$$V_j^{t_1} D_1 \equiv 0 \ [\Sigma].$$

By singling out all of the original F_j whose first terms contain effectively y_k, $k = 2, 3, \cdots, n$, and repeating the above procedure for each k, we obtain finally n forms

$$D_k = \lambda^{q_k w_k} y_k^{p_k} + H_k, \qquad k = 1, \cdots, n,$$

where H_k is homogeneous of degree $q_k w_k$ in λ and the u_{ji} and every term of H_k contains a power product in the y_{ij} whose degree exceeds p_k. There is an integer t such that

$$(2.9) \qquad V_j^t D_k \equiv 0 \ [\Sigma], \qquad k = 1, 2, \cdots, n; j = 1, 2, \cdots, q.$$

Let w be the maximum of the numbers $w_k q_k$ and let each D_k be multiplied by $\lambda^{w - w_k}$. Using the same symbols to denote the modified forms, we see that equations (2.9) still hold, and that now the D_k are homogeneous of degree w in λ and the u_{ji}. What we have accomplished by this alteration is to obtain a set of forms $D_k = \lambda^w y_k^{p_k} + H_k$, $k = 1, \cdots, n$, to which we may apply our result for the case $s = 1$. It follows that there is a form $D = \lambda^e + H$ and an integer a such that

$$(2.10) \qquad y_i^a D \equiv 0 \ [D_1, D_2, \cdots, D_n], \qquad i = 1, 2, \cdots, n.$$

Every term of H contains some y_{ij} effectively, and H is homogeneous of degree e in λ and the u_{ji}. (Actually the conclusion that the case $s = 1$ entitles us to draw is that $D = (\lambda^w)^b + H$ where H is homogeneous of degree b in λ^w and the coefficients of the H_k. Since these coefficients are themselves homogeneous of degree w in λ and the u_{ji}, the above conclusion is justified.)

52. We now show that there is an integer t such that

$$V_j^t D \equiv 0 \ [\Sigma], \qquad j = 1, 2, \cdots, q.$$

This will complete the proof, since D meets all our other requirements.

We know that for each D_k there is a power of V_j such that its product with D_k is in Σ. This is likewise true of any derivative of the D_k. We chose the integer h sufficiently large so that the product of V_j with any D_k or with any derivative of a D_k which appears effectively in the right member of some congruence (2.10) shall be in Σ. A single h serves for all j. Clearly

$$V_j^h y_i^a D \equiv 0 \ [\Sigma], \qquad j = 1, \cdots, q; \ i = 1, \cdots, n.$$

The integer $h + a$ thus has the required property; for every j, $V_j^{h+a} D$ is in Σ.

An approximation theorem

53. Our next theorem is an application of Theorem 2.4. Its statement and proof presuppose Raudenbush's theory of perfect differential ideals. The coefficient domain is an arbitrary differential field of characteristic zero.

THEOREM 2.5. *Let Σ be a prime ideal of forms in $y_1, \cdots, y_n$ which does not contain $V=y_1y_2\cdots y_n$ and which is such that every form of Σ vanishes for $y_i=0$, $i=1, \cdots, n$. Let r be a positive integer and let $u_1, \cdots, u_n$ be unknowns. If in each form of Σ, y_{ij} is replaced by $(u_i^r)_j$, $i=1, \cdots, n$, Σ goes over into a system σ of forms in $u_1, \cdots, u_n$. Let $\{\sigma\}$ be the perfect ideal generated by σ and let it be the intersection of prime ideals $\Omega_1, \cdots, \Omega_e$. Then there is an Ω_i which does not contain $W=u_1u_2\cdots u_n$ and every form of Ω_i vanishes for $u_i=0$, $i=1, \cdots, n$.*

What this amounts to in the theory of differential equations is that the approximation theorem which holds for $r=1$ holds for any positive integral value of r. In this form our theorem has been established by Ritt([12]). What we shall prove is thus the abstract counterpart of Ritt's approximation theorem relative to the rth roots of the functions constituting a solution of an irreducible system of differential equations. Our proof is indirect. We assume the theorem false and force a contradiction.

54. If the theorem were false, every Ω_i which did not contain W would contain a form $1+B_i$ where B_i vanishes for $u_i=0, i=1, \cdots, n$. Clearly W is not in each Ω_i for then V would be in Σ. Our assumption that the theorem is false implies that there actually are such forms $1+B_i$. Let their product be $1+B$. B vanishes for $u_i=0$, $i=1, \cdots, n$, and $W(1+B)$ is in $\{\sigma\}$. Let $F=W(1+B)$. There is an integer s such that $F^s\equiv 0$ $[\sigma]$. We work back from F^s to a form of Σ.

The forms of σ were obtained from those of Σ by the transformation $y_{ij}=(u_i^r)_j$. Thus while σ is not a different ideal it is closed with respect to differentiation. The inverse of the above transformation may be obtained from the formulas $u_{ij}=((u_iy_{i1})/(ry_i))_{j-1}$ where the subscript outside the parentheses denotes differentiation. These formulas show that u_{ij}, $j=1, 2, 3, \cdots$, may be expressed as the product of u_i with a polynomial (rational coefficients) in y_{i1}/y_i and its first $j-1$ derivatives. Each term of these expressions for the u_{ij} is the quotient of a polynomial in u_i and the y_{ij} by a power of y_i. The total degree of the numerator in u_i and y_{ij} exceeds the degree of the denominator.

55. We examine the effect of making the above replacements for the derivatives of the u_i in a form in the u_i no term of which is free of all the u_{ij}. We obtain a rational function of the u_i and the y_{ij} whose least common denominator is a power product $y_1^{p_1}y_2^{p_2}\cdots y_n^{p_n}$. When the rational function is written in the form $P/(y_1^{p_1}y_2^{p_2}\cdots y_n^{p_n})$ with P a polynomial in the u_i and the

([12]) OCP, pp. 7–14.

y_{ij}, then for each i for which $p_i \neq 0$ each term of P not divisible by $y_i^{p_i}$ is of degree greater than p_i in u_i, y_i and its derivatives. In addition, the total degree of every term of P in the u_i, y_{ij} exceeds $p_1+p_2+\cdots+p_n$. If, for some k, $p_k=0$, then P may be free of the letters u_k, y_{kj}.

Let us suppose these replacements made in F^s. An expression $W^s(1+T)$ is obtained which involves the u_i and the y_{ij}. The expression T is a rational function whose numerator is a polynomial L in the u_i and the y_{ij} and whose denominator is a power product $y_1^{a_1}y_2^{a_2}\cdots y_n^{a_n}$. The remarks made above about the degree of P hold also for L relative to the exponents $a_1, a_2, \cdots, a_n$. F^s belongs to $[\sigma]$ and is a linear combination of forms of σ whose coefficients are forms in $u_1, u_2, \cdots, u_n$. When the above replacements are made in a form of σ, what is obtained is a form in u_i^r and the y_{ij} which, when u_i^r is replaced by y_i, becomes a form of Σ. The fractions with which we deal are produced only by the coefficients which figure in the linear combination. Let $W^s(1+T)$ be multiplied by the least common denominator $y_1^{b_1}y_2^{b_2}\cdots y_n^{b_n}$ of these coefficients. A form

$$G = W^s(y_1^{b_1}y_2^{b_2}\cdots y_n^{b_n} + K)$$

is obtained. G is a polynomial in the u_i and the y_{ij}. It is a linear combination, with polynomials in these letters for coefficients, of polynomials in u_i^r and the y_{ij}, these latter polynomials having the property that when u_i^r is replaced by y_i in them, forms of Σ are obtained. Each term of K dominates $y_1^{b_1}y_2^{b_2}\cdots y_n^{b_n}$ in the same way that each term of L dominates $y_1^{a_1}y_2^{a_2}\cdots y_n^{a_n}$.

56. Let ω be a primitive rth root of unity. Let $G', G'', \cdots, G^{(m)}$ denote the forms obtained by replacing u_i by $\omega^k u_i$ in G; $k=1, \cdots, r-1$; $i=1, \cdots, n$, the replacements being made independently on all the u_i in all possible ways. The product $GG'\cdots G^{(m)}$ is a form in u_i^r and the y_{ij}. Our final substitution is to replace u_i^r by y_i, $i=1, \cdots, n$, in this product. We denote this form in the y_i by P. We have

$$P = V^q(y_1^{q_1}y_2^{q_2}\cdots y_n^{q_n} + H)$$

where H is a form in the y_{ij} such that the total degree of every one of its terms exceeds $q_1+q_2+\cdots+q_n$. In addition, every term of H not divisible by y_i^q has a total degree in y_i and its derivatives which exceeds q_i provided that $q_i \neq 0$. Clearly P is in Σ. We are going to show that its presence in Σ contradicts our hypothesis.

Since Σ is prime and does not contain V we must have

$$y_1^{q_1}y_2^{q_2}\cdots y_n^{q_n} + H \equiv 0 \ [\Sigma].$$

Let N denote the form constituting the left member of this congruence. We recall that every form of Σ vanishes for $y_i=0$, $i=1, \cdots, n$, and that H also has this property. Consequently not all the exponents q_i of the first term of N are zero. Renumbering the letters if necessary, suppose that $q_1, q_2, \cdots, q_t$ are not zero, while $q_{t+1}=q_{t+2}=\cdots=q_n=0$. We have $1\leqq t\leqq n$. We consider N as a form in $y_1, \cdots, y_t$. It follows from Theorem 2.4 that there is a form $1+R$, R vanishing for $y_1=0$, $i=1, \cdots, t$, and an integer a such that

$$(y_1y_2\cdots y_t)^a(1+R)\equiv 0\ [N].$$

Since N and, consequently, $(y_1y_2\cdots y_t)^a(1+R)$ are in Σ we have the following contradiction. Σ is prime, so it must contain either $y_1y_2\cdots y_t$ or $1+R$. By hypothesis Σ does not contain $y_1y_2\cdots y_t$. By hypothesis every form of Σ vanishes for $y_i=0$, $i=1, \cdots, n$, and since $1+R$ does not have this property it cannot be in Σ.

PART III. THE DIFFERENTIAL IDEAL GENERATED BY uv

THE FORM uv AND ITS DERIVATIVES

57. Let u and v be unknowns. We investigate the differential ideal Ω generated by the form $X=uv$. For most of the discussion, the underlying ring will be that of forms in the unknowns u and v whose coefficients are rational numbers. Results obtained under these circumstances carry over readily to more general ones.

Our arguments follow the pattern of those used in the discussion of the ideal generated by y^p. We begin with some conventions concerning power products in u and v and their derivatives. We retain for power products in the u_i alone the definitions concerning weight, degree and order already made for the unknown y; and likewise for power products in the v_i alone. A power product P in both the u_i and the v_i may be written in the form UV where U involves only the u_i and V only the v_i. The *signature* of P is defined as an ordered pair of numbers (d_1, d_2) where d_1 and d_2 are the respective degrees of U and V.

We can without fear of confusion describe as *homogeneous* a form all of whose terms have the same signature. The weight of P is defined as the sum of the weights of U and V. A power product P is defined as *higher* than $P'=U'V'$, and P' as *lower* than P, if (a) U is higher than U' or (b) $U=U'$ and V is higher than V'. Evidently if P is different from P' it must be either higher than P or lower than P'. It is evident that our ordering is transitive. Furthermore, if P is higher than P' and G is any power product, then GP is higher than GP'. A *β term* is defined in the following way. Let $P=UV$ be of signature (d_1, d_2). P is a β term if V effectively contains some v_k with $k<d_1$. This implies of course that $d_1>0$. All other power products are called *α terms*. In particular unity, any power product in the u_i alone and any power product in the v_i alone are α terms. It will be noted that these definitions do not respect the symmetry of Ω.

58. We prove the following lemma.

LEMMA 3.1. *Let d_1, d_2, w be any non-negative integers. Every β term $P=UV$ of signature (d_1, d_2) and weight w is congruent* mod Ω *to a linear combination with rational coefficients of α terms of this weight and signature.*

We need only show that P is either in Ω or is a congruent mod Ω to a linear combination with rational coefficients of power products of signature (d_1, d_2) and weight w, all of which are higher than P. Arguments identical with those used in the proof of Lemma 1.1 show how the proof may then be completed.

59. The proof is by induction on d_1, with d_2 and w arbitrary, starting with $d_1=1$. For this case $P=u_kV$ where $k\geqq 0$ and V involves only the v_i. The fact that P is a β term implies that v is effectively present in P and we have $P=u_kvV'$. If $k=0$ then P is in Ω and requires no further discussion. Assume now that $k>0$ and consider X_k, the kth derivative of $X=uv$. It is a form

$$u_kv+\sum_{i=1}^{k}c_iu_{k-i}v_i$$

where the c_i are positive integers. Thus

$$\begin{aligned} P &= \left(X_k-\sum c_iu_{k-i}v_i\right)V' \\ &\equiv -\left(\sum c_iu_{k-i}v_i\right)V' \ [\Omega]. \end{aligned}$$

Evidently each term $u_{k-i}v_iV'$ is higher than P and has the same weight and signature as P, so that the statement is verified for $d_1=1$. Observe that in comparing the terms in the right member of this congruence with P, an examination of their factors in the u_i alone reveals that they are higher than P. We attach this fact to the induction hypothesis and assume that the lemma, with this additional restriction, is valid for all power products of signature (d_1, d_2) and weight w, and d_2 and w arbitrary, provided that d_1 is less than some integer $d>0$. It will be shown that it likewise holds for power products of signature (d, d_2) and weight w.

60. Let P be such a β term. Then P effectively contains some v_k with $k<d$. Let u_r be that derivative of u of highest order which is effectively present in P. If $r=0$, then P is in Ω and needs no further discussion. Assuming $r>0$, write

$$P = u_rv_kU'V'.$$

Now the $(r+k)$th derivative of X is a form

$$X_{r+k} = cu_rv_k+\sum_{i=1}^{k}c_iu_{r+i}v_{k-i}+\sum_{i=1}^{r}c_i'\,u_{r-i}v_{k+i}$$

where c, the c_i and the c_i' are non-negative integers. Certainly c and the c_i' are different from zero and the c_i are zero only if $k=0$. Thus

$$P \equiv \sum_{i=1}^{k} d_i u_{r+i} v_{k-i} U'V' + \sum_{i=1}^{r} d_i' u_{r-i} v_{k+i} U'V' \quad [\Omega] \tag{3.1}$$

where the d_i', d_i are rational numbers. The terms $u_{r-i}v_{k+i}U'V'$ are all of the same weight and signature as P and are all higher than P in the proper way. Consequently only the terms $u_{r+i}v_{k-i}U'V'$ need be considered. Ignoring u_{r+i} for the moment, consider the terms $v_{k-i}U'V'$. Since U' is of degree $d-1$ and $d>k$ these are all β terms. The induction hypothesis applied to them shows that each is congruent to a linear combination of terms $U''V''$ of the same weight and signature, and with U'' higher than U'. Since U'' has the same degree as U', and U' involves only the letters $u, u_1, \cdots, u_r$, it follows that for some $t<r$ (t is a non-negative integer) the exponent of u_t in U'' exceeds that of u_t in U', while for all non-negative integers $s<t$ the exponents of u_s in U'' and in U' are the same. We see then that each $u_{r+i}U''$ is higher than U. Since in the congruence (3.1) each $u_{r+i}v_{k-i}U'V'$ may be replaced by a linear combination of terms $u_{r+i}U''V''$ of the proper weight and signature, the result follows.

Canonical representations

61. A γ *term* is defined to be a form

$$G = EX_{i_1}X_{i_2} \cdots X_{i_s}$$

where E is an α term and $X_{i_1}X_{i_2} \cdots X_{i_s}$ is any power product in the X_i of positive degree. Let E be of signature (d_1, d_2) and weight w. The signature of G is defined to be (d_1+s, d_2+s) and its weight to be $w+i_1+ \cdots +i_s$. G is actually a homogeneous isobaric form of this signature and weight. The following lemma shows that the forms defined here as γ terms are entirely analogous to those so defined relative to the ideal $[y^p]$.

Lemma 3.2. *Let H be any homogeneous isobaric element of Ω. Then H may be expressed as a linear combination with rational coefficients of γ terms all of which we have the same weight and signature as H.*

H is a linear combination with rational coefficients of terms $KX_{i_1} \cdots X_{i_r}$, K being some power product in the u_i and v_i. If K is not an α term Lemma 3.1 asserts the existence of a congruence

$$K \equiv \sum c_i K_i \quad [\Omega]$$

where the K_i are α terms and the c_i are constants. This congruence may be written as an equality

$$K = \sum_i c_i K_i + \sum_{i,j} c_{ij} K_{ij} X_j$$

where the c_{ij} are constants and the K_{ij} power products. If the right member of this equation is substituted for K in the term $KX_{i_1} \cdots X_{i_r}$ in question, a linear combination of γ terms plus a linear combination of terms $K_{ij}X_jX_{i_1}\cdots X_{i_r}$ is obtained. These terms all have a degree in the X_i which exceeds that of the original one. Repetition of this process a finite number of times in the finitely many terms of H must yield an expression for H of the required nature.

THE FUNDAMENTAL LEMMA

62. The following lemma is the counterpart of Lemma 1.3.

LEMMA 3.3. *Let d_1, d_2, w be non-negative integers. The number of γ terms of signature (d_1, d_2) and weight w does not exceed the number of β terms of this weight and signature.*

The plan of the proof is the same as that of Lemma 1.3. A definite β term of this weight and signature will be assigned to each γ term of this weight and signature in such a way that different β terms are assigned to different γ terms.

63. We consider expressions EX_h of the following description. If E is an α term it may be completely arbitrary. If E is a β term it is restricted by h, the exact statement of the restriction requiring that E be written out explicitly. In this case let

$$E = u_{i_1}^{a_1}\cdots u_{i_r}^{a_r}v_{j_1}^{b_1}\cdots v_{j_s}^{b_s} \tag{3.2}$$

where the a_i and b_i are positive integers. We might state explicitly that the subscripts satisfy the relations $i_1 < \cdots < i_r$ and $j_1 < \cdots < j_s$. Let t be the smallest integer for which $a_1+a_2+\cdots+a_t > j_1$. Our restriction on E is that $i_t+j_1 \geqq h$.

A procedure for associating a β term with such an expression will now be described, the β term to have the same weight and signature as EX_h. The β term will have the general appearance u_av_bE where $a+b=h$. Thus requirements of weight and signature will evidently be met. Some preliminary calculations must be made before the β term can actually be produced.

64. Let E be given by equation (3.2). Let $e_i = a_1+a_2+\cdots+a_i$, $i=1,\cdots,r$. Let $f_1=i_1$ and $f_{2n+1}=i_{n+1}+e_n$, $n=1,\cdots,r-1$. Let $f_{2n}=i_n+e_n$, $n=1,\cdots,r$. Finally let f_{2r+1} be an integer which exceeds both h and f_{2r}. Obviously $f_1<f_2<\cdots<f_{2r+1}$. Then either $h\leqq f_1$ or there is a positive integer c such that $f_c<h\leqq f_{c+1}$. Three cases are treated, depending on the relation h bears to the f_i. The cases are

(i) $h\leqq f_1$. E' is defined by

$$E' = u_hvE.$$

(ii) The integer c mentioned above is odd, say $2d-1$, so that $f_{2d-1}<h\leqq f_{2d}$. Then E' is defined by

$$E' = u_{i_d}v_{h-i_d}E.$$

(iii) The integer c is even, say $2d$, so that $f_{2d} < h < f_{2d+1}$. Then E' is defined by

$$E' = u_{h-e_d} v_{e_d} E.$$

NOTE. We admit the possibility that E contains no u_i effectively so that the quantities e_i cannot be computed. For this case the quantity f_{2r+1} is simply f_1, and the case is covered by (i).

The problem now is to show that these assignments always lead to β terms and that distinct expressions are assigned to distinct β terms. It will first be shown that this procedure always leads to a β term, and then the following characterization of the u_a and v_b used as factors with the E will be obtained. The v_b will be shown to be such that no v_k is effectively present in E' with $k < b$. The u_a will be shown to be such that for no $k < a$ does the degree of E' in $u, u_1, \cdots, u_k$ exceed b. The three cases will be treated separately and for each one the validity of these remarks will be shown.

65. For case (i) the fact that E' contains the factor $u_h v$ proves that E' is a β term. It is evident that E' contains no v_k with $k < b$ because b in this case is zero. Finally the fact that $h \leqq i_1$ and $a = h$ shows that E' contains no u_j effectively with $j < a$.

66. For case (ii) it is desirable to write out the inequality $f_{2d-1} < h \leqq f_{2d}$ in full. It states

$$(3.3) \qquad i_d + a_1 + \cdots + a_{d-1} < h \leqq i_d + a_1 + \cdots + a_d.$$

(If d is unity this is to mean $i_1 < h \leqq i_1 + a_1$.) In this case E is multiplied by v_b with $b = h - i_d$. The degree of E' in the u_i exceeds that of E in the u_i and the latter degree is certainly not less than $a_1 + \cdots + a_d$. It is a consequence of (3.3) that $a_1 + \cdots + a_d$ is not less than $h - i_d$ so that the degree of E' in the u_i exceeds $h - i_d$. Then E' is a β term since it contains v_b effectively with b less than the degree of E' in the u_i. To show that E' contains no v_k with $k < b$ observe that then E would also contain this v_k. Such an integer would be less than $a_1 + \cdots + a_d$ and this fact with the supposition $k + i_d < b + i_d = h$ would mean that E did not obey the restriction imposed on it. To show that the factor u_a has the property described, note that it is a consequence of (3.3) that $a_1 + \cdots + a_{d-1} < h - i_d = b$. Since E' is identical with E as far as the letters $u, u_1, \cdots, u_{i_d - 1}$ are concerned and a is i_d, this inequality shows that for no $k < a$ does the degree of E' in $u, u_1, \cdots, u_k$ exceed b.

67. We now turn to case (iii). The inequality $f_{2d} < h < f_{2d+1}$ written out in full becomes

$$(3.4) \qquad i_d + a_1 + \cdots + a_d < h \leqq i_{d+1} + a_1 + \cdots + a_d$$

(for $d = r$ this is to mean $i_r + a_1 + \cdots + a_r < h$).

The factor v_b used with E is in this case defined by $b = a_1 + \cdots + a_d$. Since the degree of E' in the u_i exceeds that of E, and since this latter degree is at least b, it follows that E' is a β term because it contains v_b effectively. To show that E' contains no v_k effectively with $k<b$ observe that then E would also contain this v_k. It follows from (3.4) that $i_d+k<h$ and since $b=a_1+\cdots+a_d>k$ the restriction imposed on E could not be satisfied. We now show that u_a is such that for all $k<a$ the degree of E' in $u, u_1, \cdots, u_k$ does not exceed b. This degree is the same as that of E in these letters. The integer a is defined as $h-e_d$. It follows from (3.4) that $i_d<h-e_d\leqq i_{d+1}$. The degree of E' in the letters u_k with $k<a$ is thus $a_1+\cdots+a_d$. This number is precisely b, which verifies the statement.

68. It can now be shown that this procedure assigns distinct β terms E' to distinct expressions EX_h. Let E' be any β term. It must contain some of the v_i effectively. Let b be the smallest subscript for which v_b is effectively present in E'. Since E' is a β term its degree in the u_i exceeds b. Let a be the smallest integer such that the degree of E' in $u, u_1, \cdots, u_a$ exceeds b. Then if EX_h led to E' by the method described above, it must have been that $E'=u_av_bE$, $h=a+b$. Thus given E' there is only one possibility for EX_h.

We need the additional fact that if EX_h determines E' as above, and if g is any non-negative integer such that $g\leqq h$, then $E'X_g$ is also an admissible expression. This means that if k is such that the degree of E' in $u, u_1, \cdots, u_k$ exceeds t and v_t actually appears in E', then $k+t\geqq g$. It has already been shown that a and b are the smallest relevant integers and since $a+b=h$ it follows that $a+b\geqq g$.

69. We are now in a position to describe the way in which β terms may be associated with γ terms. Let $G=E_rX_{i_1}X_{i_2}\cdots X_{i_r}$ be a γ term (it is assumed that $i_1\leqq i_2\leqq\cdots\leqq i_r$ and that $r\geqq 1$). E_r is an α term, so that EX_{i_r} is an expression of the type considered and determines a β term E_{r-1} of the same weight and signature. If $r=1$ this β term is associated with G. If $r>1$ it has been shown that $E_{r-1}X_{i_{r-1}}$ is also an admissible expression and determines a β term E_{r-2}. E_{r-2} has the same weight and signature as $E_rX_{i_r}X_{i_{r-1}}$. If $r=2$ then E_{r-2} is associated with G. If $r>2$ we continue. In this way a sequence of β terms $E_{r-1}, E_{r-2}, \cdots, E_0$ is obtained where each E_{j-1} is the β term determined by $E_jX_{i_j}$. E_0 has the same weight and signature as G. We associate it with G.

It will be shown that if G and G' are γ terms associated with E_0 and E_0', and if $E_0=E_0'$, then G and G' are the same. Let $G'=E_s'X_{j_1}\cdots X_{j_s}$. G' and G are to be considered as identical if and only if $E_r=E_s'$, $r=s$ and $(i_1, \cdots, i_r)=(j_1, \cdots, j_s)$. Suppose $E_0=E_0'$. E_0 was determined by $E_1X_{i_1}$ and E_0' by $E_1'X_{j_1}$. Because of the uniqueness property of the procedure used, it follows that $E_1=E_1'$ and $i_1=j_1$. Suppose $r\leqq s$. Then by reconstructing $E_2X_{i_2}, \cdots, E_rX_{i_r}$ it follows that $E_k=E_k'$ and $i_k=j_k$ for $k=1, \cdots, r$. Now E_r is an α term, so that E_r' is also an α term. The only power product in the sequence $E_0', E_1', \cdots, E_s'$ which is an α term is E_s'. It follows that $s=r$ and thus $G=G'$.

It has been shown that every γ term determines a β term having the same weight and signature and that distinct γ terms determine distinct β terms. This completes the proof.

THE STRUCTURE OF THE IDEAL OF uv

70. We now can prove the following lemma.

LEMMA 3.4. *Let d_1, d_2, w be non-negative integers. Let n_α denote the number of α terms A_i of signature (d_1, d_2) and weight w, let n_β denote the number of β terms of this signature and weight, and let n_γ denote the corresponding number of γ terms G_j. Then $n_\beta = n_\gamma$ and a relation*

$$\sum_{i=1}^{n_\alpha} a_i A_i + \sum_{j=1}^{n_\gamma} g_j G_j = 0$$

where the a_i and g_j are rational numbers implies that all the a_i and g_j are zero.

The proof of this lemma is identical with that of Lemma 1.4, so that no further argument will be given. It will be noted that if the a_i and g_j are elements of any differential domain of integrity which contains the rational numbers and over which u and v are unknowns, the same conclusion can be drawn.

71. The above lemmas combine to yield the following theorem.

THEOREM 3.1. *Let $\mathfrak{J}$ be any differential domain of integrity which contains the rational numbers. Let F be any differential polynomial in the unknowns u and v. Then F is expressible in the form*

$$F = \sum a_i A_i + \sum g_j G_j, \qquad a_i, g_i \in \mathfrak{J},$$

where the A_i are α terms and the G are γ terms. For each F there is only one such expression.

For the proof of this theorem the reader is referred to the proof of Theorem 1.1.

COROLLARY. *Let d_1, d_2, w be as in the statement of Lemma 3.4. Let the $\mathfrak{J}$ considered above be a field. Then the number of linearly independent* (mod Ω) *forms with coefficients in $\mathfrak{J}$ which are homogeneous and isobaric of this signature and weight is n_α.*

COROLLARY. *No linear combination of α terms with coefficients in $\mathfrak{J}$ is in Ω.*

INDECOMPOSABILITY OF THE IDEAL OF uv

72. The preceding work enables us to indicate a striking difference between ordinary (algebraic) ideals of polynomials in a finite number of unknowns and differential ideals of such polynomials. This difference is manifested by the differential ideal of uv, as we proceed to show. The mani-

fold of this ideal is reducible into the union of two irreducible manifolds, namely $u=0$ with v arbitrary and $v=0$ with u arbitrary. Nonetheless we are going to show that the differential ideal $[uv]$ has no representation as the intersection or product of two differential ideals whose manifolds are respectively the first and the second just described. Let Σ_1 be any differential ideal of differential polynomials in the unknowns u and v whose manifold is $u=0$ with v arbitrary. Then[13] Σ_1 contains some power of u, say u^r. Again if Σ_2 is a similar differential ideal whose manifold is $v=0$ with u arbitrary, then some power v^t belongs to Σ_2. Suppose that the ideal $[uv]$ had a representation as the intersection or the product of Σ_1 and Σ_2. Since Σ_2 contains v^t, it contains some power v_r^s, and the form $u^r v_r^s$ is in the product and intersection of Σ_1 and Σ_2. This form is an α term and is thus not in $[uv]$. This proves our contention.

The power products in the ideal of uv

73. Theorem 3.2. *Let d_1, d_2, w, n_α be as in Lemma 3.4. A necessary and sufficient condition that $n_\alpha > 0$ is that $w \geqq d_1 d_2$.*

This is equivalent to the assertion that *every power product of signature (d_1, d_2) and weight $w < d_1 d_2$ is in Ω and not every power product of this signature and weight $w \geqq d_1 d_2$ is in Ω.*

We need only investigate the circumstances under which α terms exist. Let $w = d_1 d_2 + h$ with $h \geqq 0$. Then $u^{d_1} v_{d_1}^{d_2 - 1} v_{d_1+h}$ is an α term of signature (d_1, d_2) and weight w. This disposes of the sufficiency condition. We now show that there are no α terms of signature (d_1, d_2) and weight less than $d_1 d_2$. If $d_1 d_2 = 0$, there are no power products with this property and certainly no α terms. If $d_1 d_2 > 0$, such an α term would have to be such that every v_k effectively present in it would have a subscript not less than d_1. Since it must have d_2 such letters v_k its weight is at least $d_1 d_2$.

74. It might be pointed out that this discussion of Ω applies to more general ideals, in that the X_i need not be the ith derivative of uv. If each X_i is a homogeneous isobaric polynomial in the u_i and v_i of signature (1, 1) and weight i, and is such that every term of this signature and weight is present in X_i with a nonzero coefficient, then the whole discussion applies verbatim. The coefficients of the X_i must be confined to some field and may otherwise be arbitrary.

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(13) Raudenbush, loc. cit.

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