ON THE NUMBER OF PARTITIONS OF A NUMBER INTO UNEQUAL PARTS[1]

BY

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1. **Introduction.** Let $q(n)$ be the number of partitions of an integer n into unequal parts, or into odd parts[2]. Then

$$
\tag{1.1}
\begin{aligned}
f(x) &= 1 + \sum_{n=1}^{\infty} q(n)x^n = (1+x)(1+x^2)(1+x^3)\cdots \\
&= \frac{1}{(1-x)(1-x^3)(1-x^5)\cdots}.
\end{aligned}
$$

Hardy and Ramanujan[3] indicated that by their fundamental analytic method one can obtain the following result:

$$
\begin{aligned}
q(n) = {} & \frac{1}{2^{1/2}} \frac{d}{dn} J_0\left[i\pi\left\{\tfrac{1}{3}\left(n + \tfrac{1}{24}\right)\right\}^{1/2}\right] \\
& + 2^{1/2} \cos\left(\tfrac{2}{3}\pi n - \tfrac{1}{9}\pi\right) \frac{d}{dn} J_2\left[\tfrac{1}{3} i\pi\left\{\tfrac{1}{3}\left(n + \tfrac{1}{24}\right)\right\}^{1/2}\right] + \cdots \\
& + \text{to } \left[\alpha n^{1/2}\right] \text{ terms} + O(1)
\end{aligned}
$$

where α is an arbitrary constant. This result is less satisfactory than that concerning the number $p(n)$ of partitions (unrestricted) of n, since in the latter case the error term approaches zero with increasing n. Recently Rademacher[4] obtained an equality for $p(n)$. The object of the present paper is to find an equality for $q(n)$. The work of this paper is a straightforward application of Hardy-Ramanujan's method with two modifications. These modifications are Kloosterman's sum and Rademacher's "Farey dissection of infinite order."

The present method may also be applied to find the explicit formula for

$$
\sum_{x=1}^{[n^{1/2}]} p(n-x^2)
$$

where $p(n)$ is the number of unrestricted partitions of n.

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(1) This paper was accepted by Acta Arithmetica before the war.

(2) Cf. MacMahon, *Combinatory Analysis*, vol. 2, 1916, p. 11.

(3) Proceedings of the London Mathematical Society, (2), vol. 17 (1918), pp. 75–115.

(4) Proceedings of the London Mathematical Society, (2), vol. 43 (1937), pp. 241–254.

2. **Statement of the result.** Let

$$\epsilon_{h,k} = \begin{cases} \exp\left(-\pi i\left(\frac{(h'^2-1)}{8}\left(\frac{1-hh'}{k}-1\right)+\frac{h'(1-hh')}{8k}\right.\right. \\ \qquad \left.\left.+\frac{1}{24}\left(k+\frac{1-hh'}{k}\right)(hh'^2-h'-h)\right)\right), & \text{for } 2\mid k, \\ \exp\left(\frac{\pi i}{24}\left(k+\frac{1-hh'}{k}\right)(h+h'-h^2h')\right), & \text{for } 2\nmid k,\ 2\nmid h, \\ \exp\left(-\frac{\pi i}{8}\left(k^2-1-hk+\frac{1}{3}(h+h')\left(hh'k-\frac{hh'-1}{k}\right)\right)\right), & \\ & \text{for } 2\nmid k,\ 2\mid h, \end{cases}$$

and

$$\omega_{h,k} = \begin{cases} \epsilon_{h,k}\exp\left(-\frac{\pi i}{12k}(h+h')\right), & \text{for } 2\mid k, \\ \epsilon_{h,k}\exp\left(-\frac{\pi i}{24k}(2h-h')\right), & \text{for } 2\nmid k, \end{cases}$$

where $hh' \equiv 1 \pmod{k}$, $h \equiv h' \pmod{2}$.

THEOREM. *The number of partitions of an integer n into unequal parts is given by*

$$q(n) = \frac{1}{2^{1/2}} \sum_{k=1,\,k\text{ odd}}^{\infty} \sum_{(h,k)=1,\,0<h\leq k} \omega_{h,k} e^{-2\pi i hn/k} \frac{d}{dn} J_0\left(\frac{i\pi}{k}\left\{\tfrac{2}{3}\left(n+\tfrac{1}{24}\right)\right\}^{1/2}\right),$$

where $J_0(x)$ is the Bessel function of the 0*th order.*

3. **Farey dissection.** By means of Cauchy's integral formula we obtain for (1.1)

$$q(n) = \frac{1}{2\pi i}\int_C \frac{f(x)}{x^{n+1}}\,dx.$$

The path of integration may be the circle defined as $|x| = e^{-2\pi N^{-2}}$ where N is a certain positive integer at our disposal. In the usual way we divide the circle into Farey arcs $\xi_{h,k}$ of order N. The Farey arc $\xi_{h,k}$ is defined by

$$x = \exp(2\pi i h/k - 2\pi N^{-2} + 2\pi i\vartheta), \qquad (h, k) = 1, \tag{3.1}$$

and

$$-\vartheta_1(h, k) = \frac{h+h_1}{k+k_1} - \frac{h}{k} \leq \vartheta \leq \frac{h+h_2}{k+k_2} - \frac{h}{k} = \vartheta_2(h, k) \tag{3.2}$$

where h_1/k_1, h/k, h_2/k_2 are three consecutive fractions in the Farey sequence of order N. It is well known that

$$(3.3)\qquad \frac{1}{k(N+k)} \leqq \vartheta_1(h, k) < \frac{1}{k(N+1)}, \quad \frac{1}{k(N+k)} \leqq \vartheta_2(h, k) < \frac{1}{k(N+1)}.$$

We obtain then

$$(3.4)\qquad q(n) = \frac{1}{2\pi i} \sum_{(h,k)=1, 0<h\leqq k\leqq N} \int_{\xi_{h,k}} \frac{f(x)}{x^{n+1}} dx.$$

Let I_1 and I_2 denote the sums of those terms satisfying $2|k$, and $2\nmid k$, respectively. Then, by (3.4), we have

$$(3.5)\qquad q(n) = I_1 + I_2.$$

4. **Lemmas on Kloosterman's sums.**

Lemma 4.1([5]). *Let*

$$g(N, \vartheta, h, k) = \begin{cases} 1 & \text{for } -\vartheta_1(h, k) \leqq \vartheta \leqq \vartheta_2(h, k), \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$g = \sum_{r=1}^{k} b_r e^{2\pi i r h'/k}$$

where h' iz an integer satisfying

$$hh' \equiv 1 \pmod{k},$$

and b_r is independent of h and

$$\sum_{r=1}^{k} |b_r| < \log 4k.$$

Lemma 4.2. *Let a be an absolute constant. Then*

$$\sum_{0<h\leqq ak, (h,ak)=1, h\equiv l(a)} \exp\left(\frac{2\pi i}{ak}(nk + mh')\right) = O(k^{2/3+\epsilon}(n, k)^{1/3}).$$

Lemma 4.3. *If k is even and $\omega_{h,k}$ as defined in §2, then*

$$S_k = \sum_{1\leqq h\leqq k, (h,k)=1, hh'\equiv 1\,(k)} \omega_{h,k} e^{2\pi i(nh+mh')/k} = O(k^{2/3+\epsilon}(n, k)^{1/3}).$$

Proof. For the sake of simplicity I give here only the proof of the case $24|k$.

([5]) T. Estermann, Abhandlungen aus dem Mathematischen Seminar der Hamburgischen Universität, vol. 7 (1929), pp. 93, 94.

Then

$$S_k = \sum_{1\leq l\leq 24,(l,24)=1} \sum_{1\leq h\leq k,(h,k)=1,hh'\equiv 1,h\equiv l\ (24)} \omega_{h,k}e^{2\pi i(nh+mh')/k}.$$

The inner sum becomes a Kloosterman's sum as in Lemma 4.2. Therefore we have

$$S_k = O(k^{2/3+\epsilon}(n, k)^{1/3}).$$

As to the proof of the other cases, nothing is difficult but a little complicated, and the following fact is used: let

$$F(h, k) = \omega_{h,k}e^{2\pi i(nh+mh')/k};$$

then $F(h+k, k) = F(h, k)$.

LEMMA 4.4. *Let* $2\nmid k$ *and* $\omega_{h,k}$ *be as defined in* §2, *then*

$$S = \sum_{1\leq h\leq k,(h,k)=1,hh'\equiv 1\ (k),h'\ \text{odd}} \omega_{h,k}e^{\pi i(2nh+mh')/k} = O(k^{2/3+\epsilon}(h, k)^{1/3}).$$

The proof is similar to that of Lemma 4.3, only notice that

$$S = \sum_{1\leq h<2k,(h,2k)=1,hh'\equiv 1\ (2k)}$$

5. Lemmas from the theory of the linear transformation of the elliptic modular functions.

LEMMA 5.1. *Suppose that* $2\nmid h$, $2\mid k$; *that* h' *is a positive integer satisfying* $hh'\equiv 1 \pmod k$; *that* $\omega_{h,k}$ *is defined in* §2; *and that*

$$x = \exp\left(-\frac{2\pi z}{k} + \frac{2h\pi i}{k}\right), \qquad x' = \exp\left(-\frac{2\pi}{kz} - \frac{2h'\pi i}{k}\right),$$

where the real part of z *is positive. Then*

$$f(x) = \omega_{h,k}\exp\left(-\frac{\pi}{12kz} + \frac{\pi z}{12k}\right)f(x').$$

Proof. If we take $a=h$, $b=-k$, $c=(1-hh')/k$, $d=h'$, so that $ad-bc=1$, and write

$$x = q^2 = e^{2\pi i\tau}, \qquad x' = Q^2 = e^{2\pi iT},$$
$$\tau = (h + iz)/k, \qquad T = (-h' + i/z)/k,$$

then we can easily verify that

$$T = \frac{c + d\tau}{a + b\tau}.$$

Also, in the notation of Tannery and Molk, we obtain

$$f(x) = \frac{1}{2^{1/3}} q^{-1/12} \frac{\phi(\tau)}{\chi(\tau)}, \qquad f(x') = \frac{1}{2^{1/3}} Q^{-1/12} \frac{\phi(T)}{\chi(T)}.$$

Then

$$\begin{aligned} f(x') &= \frac{1}{2^{1/3}} Q^{-1/12} \frac{\phi(T)}{\chi(T)} = \exp\left(\pi i\left(\tfrac{1}{8}(d^2-1)(c-1) + \frac{cd}{8}\right.\right. \\ &\qquad \left.\left. - \frac{(b-c)(bcd-a)}{24}\right)\right) \frac{1}{2^{1/3}} Q^{-1/12} \frac{\phi(\tau)}{\chi(\tau)} \\ &= \exp\left(\pi i\left(\tfrac{1}{8}(d^2-1)(c-1) + \frac{cd}{8} - \frac{(b-c)(bcd-a)}{24}\right)\right) q^{1/12} Q^{-1/12} f(x) \\ &= \exp\left(\pi i\left(\tfrac{1}{8}(d^2-1)(c-1) + \frac{cd}{8} - \frac{(b-c)(bcd-a)}{24}\right)\right) \\ &\qquad \cdot \exp\left(\frac{\pi}{12k}\left(\frac{1}{z} - z\right)\right) \exp\left(\frac{\pi i}{12k}(h+h')\right) f(x). \end{aligned}$$

LEMMA 5.2. *Suppose that* $2\nmid hk$ *and* $hh' \equiv 1 \pmod{2k}$, *that*

$$f_1(x) = \prod_1^\infty (1 + x^{n-1/2}) = 1 + \sum_{n=1}^\infty q_1(n) x^{n/2}.$$

Then

$$f(x) = \frac{\omega_{h,k}}{2^{1/2}} \exp\left(\frac{\pi}{12k}\left(z + \frac{1}{2z}\right)\right) f_1(x').$$

Proof. As in Lemma 5.1, we have

$$f_1(x) = f_1(q^2) = \prod (1 + q^{2n-1}) = 2^{1/6} q^{1/24} \frac{1}{\chi(\tau)},$$

$$\begin{aligned} f_1(x') &= 2^{1/6} Q^{1/24} \frac{1}{\chi(T)} = 2^{1/6} Q^{1/24} \exp\left(-\frac{(b-c)(abc-d)}{24}\pi i\right) \frac{\phi(\tau)}{\chi(\tau)} \\ &= 2^{1/6} Q^{1/24} \exp\left(-\frac{(b-c)(abc-d)}{24}\pi i\right) 2^{1/3} q^{1/12} f(x) \\ &= \exp\left(-\frac{(b-c)(abc-d)}{24}\pi i\right) 2^{1/2} \\ &\qquad \cdot \exp\left(\frac{\pi i}{24}\left(-\frac{h'}{k} + \frac{i}{kz} + \frac{2h}{k} + \frac{2iz}{k}\right)\right) f(x). \end{aligned}$$

LEMMA 5.3. *Suppose that* $2 | h$, $2 \nmid k$, $hh' \equiv 1 \pmod{k}$, $2 | h'$ *and suppose that*

$$f_2(x) = \prod_1^\infty (1 - x^{n-1/2}) = 1 + \sum q_2(n) x^{n/2}.$$

Then

$$f(x) = \frac{\omega_{h,k}}{2^{1/2}} \exp\left(\frac{\pi}{12k}\left(z + \frac{1}{2z}\right)\right) f_2(x').$$

Proof. We take

$$a = -h, \quad b = k, \quad c = (hh' - 1)/k, \quad d = -h'.$$

Then

$$\begin{aligned} f_2(x') = f_2(Q^2) &= 2^{1/6} Q^{1/24} \frac{\psi(T)}{\chi(T)} \\ &= 2^{1/6} Q^{1/24} \exp\left(\frac{\pi i}{2}\left(\frac{b^2 - 1}{4} + \frac{ab}{4} - \frac{(a+d)(abd - c)}{12}\right)\right) \frac{\phi(\tau)}{\chi(\tau)} \\ &= 2^{1/2} \exp\left(\frac{\pi i}{2}\left(\frac{b^2 - 1}{4} + \frac{ab}{4} - \frac{(a+d)(abd - c)}{12}\right)\right) Q^{1/24} q^{1/12} f(x). \end{aligned}$$

6. **Approximation of the integrand.** Let

$$z = k(N^{-2} - i\vartheta).$$

Then

$$\begin{aligned} I_1 &= \sum_{1 \leqq k \leqq N, 2 | k} \; \sum_{(h,k)=1, 0<h<k} \int_{-k^{-1}(N+1)^{-1}}^{k^{-1}(N+1)^{-1}} g(\vartheta) f(e^{(2\pi i h - 2\pi z)/k}) e^{-2\pi i h n/k + 2\pi z n/k} d\vartheta \\ &= \sum_{1 \leqq k \leqq N, 2 | k} \; \sum_{(h,k)=1, 0<h<k} \int_{-k^{-1}(N+1)^{-1}}^{k^{-1}(N+1)^{-1}} g(\vartheta) \omega_{h,k} e^{(\pi/12k)(z - 1/z)} \\ &\qquad \cdot f(x') e^{-2\pi i h n/k + 2\pi z n/k} d\vartheta \\ &= \sum_{1 \leqq k \leqq N, 2 | k} \; \sum_{(h,k)=1, 0<h<k} \int_{-k^{-1}(N+1)^{-1}}^{k^{-1}(N+1)^{-1}} g(\vartheta) \omega_{h,k} \\ &\qquad \cdot e^{(\pi/12k)(z-1/z) - 2\pi i h n/k + 2\pi z n/k} \sum_{\nu=0}^\infty q(\nu) e^{-(2\pi/kz + 2h'\pi i/k)\nu} d\vartheta \qquad (6.1) \\ &= \sum_{1 \leqq k \leqq N, 2 | k} \; \sum_{(h,k)=1, 0<h<k} \int_{-k^{-1}(N+1)^{-1}}^{k^{-1}(N+1)^{-1}} \sum_{\nu=0}^\infty q(\nu) e^{-(2\pi/kz)(\nu + 1/24) + (2\pi z/k)(n + 1/24)} \\ &\qquad \cdot \sum_{r=1}^{k} b_r e^{2\pi i r h'/k} \omega_{h,k} e^{-2\pi i h n/k - 2\pi i h' \nu/k} d\vartheta. \end{aligned}$$

Since $(1/k)\Re(1/z) \geqq \frac{1}{2}$, we have

$$
\begin{aligned}
|I_1| \leqq & \sum_{1 \leqq k \leqq N, 2|k} \int_{-k^{-1}(N+1)^{-1}}^{k^{-1}(N+1)^{-1}} \sum_{\nu=0}^{\infty} q(\nu) \\
& \cdot \exp\left\{-\frac{2\pi}{k}\left(\nu + \frac{1}{24}\right)\Re\frac{1}{z} + \frac{2\pi}{k}\left(n + \frac{1}{24}\right)\Re z\right\} \\
& \sum_{r=1}^{k} |b_r| \left| \sum_{(h,k)=1} \omega_{h,k} e^{-2\pi i h n/k + 2h'(r-\nu)\pi i/k} \right| d\vartheta \\
= & O\left(\sum_{k=1}^{N} \int_{-k^{-1}(N+1)^{-1}}^{k^{-1}(N+1)^{-1}} \sum_{\nu=0}^{\infty} q(\nu) e^{-\pi(\nu+1/24)} \sum_{r=1}^{k} |b_r| \, k^{2/3} d\vartheta \right) \\
= & O\left(\sum_{k=1}^{N} \log k \cdot k^{2/3} \frac{1}{kN} \right) = O\left(\frac{1}{N} \sum_{k=1}^{N} k^{-1/3+\epsilon} \right) \\
= & O(N^{-1/3+\epsilon}).
\end{aligned}
$$

Let

$$
J = \frac{1}{2^{1/2}} \sum_{k=1, k \text{ odd}}^{N} \sum_{(h,k)=1, 0<h\leqq k} \int_{-k^{-1}(N+1)^{-1}}^{k^{-1}(N+1)^{-1}} g(\vartheta)\omega_{h,k} \cdot e^{(\pi/24k)(2z+1/z) - 2\pi i h n/k + 2\pi z n/k} d\vartheta.
$$

The same method will give us that $|I_2 - J| = O(N^{-1/3+\epsilon})$.

7. **A contour integration.** Let $w = N^{-2} - i\vartheta$. Then

$$
\begin{aligned}
J = & \frac{-i}{2^{1/2}} \sum_{1 \leqq k \leqq N, k \text{ odd}} \sum_{(h,k)=1, 0<h\leqq k} \omega_{h,k} e^{-2\pi i h n/k} \int_{N^{-2} - i\vartheta_2}^{N^{-2} + i\vartheta_1} e^{2\pi w(n+1/24) + \pi/24k^2 w} dw \\
= & \frac{i}{2^{1/2}} \sum_{1 \leqq k \leqq N, k \text{ odd}} \sum_{(h,k)=1, 0<h\leqq k} \omega_{h,k} e^{-2\pi i h n/k} \Bigg(\int_{N^{-2}+i\vartheta_1}^{N^{-2}+ik^{-1}(N+1)^{-1}} \\
& + \int_{N^{-2}+ik^{-1}(N+1)^{-1}}^{-N^{-2}+ik^{-1}(N+1)^{-1}} + \int_{-N^{-2}+ik^{-1}(N+1)^{-1}}^{-N^{-2}-ik^{-1}(N+1)^{-1}} + \int_{-N^{-2}-ik^{-1}(N+1)^{-1}}^{N^{-2}+ik^{-1}(N+1)^{-}} \\
& + \int_{N^{-2}-ik^{-1}(N+1)}^{N^{-2}-i\vartheta_2} - 2\pi i \text{ Residue at } 0 \Bigg) \\
= & K_1 + K_2 + K_3 + K_4 + K_5 + L \text{ (say)}.
\end{aligned}
$$

We have

$$
K_1 = \frac{i}{2^{1/2}} \sum_{1 \leqq k \leqq N, k \text{ odd}} \sum_{(h,k)=1, 0<h\leqq k} \omega_{h,k} e^{-2\pi i h n/k} \cdot \int_{N^{-2}+ik^{-1}(N+k)^{-1}}^{N^{-2}+ik^{-1}(N+1)^{-1}} g(\vartheta) e^{2\pi w(n+1/24) + \pi/24k^2 w} dw.
$$

By Lemma 3.1, we have

$$K_1 = O\left(\sum_{1\leqq k\leqq N,\, k \text{ odd}} k^{2/3+\epsilon}\int_{k^{-1}(N+k)^{-1}}^{k^{-1}(N+1)^{-}} \exp\left\{2\pi\left(n+\frac{1}{24}\right)\Re w + \frac{\pi}{24k^2}\Re\frac{1}{w}\right\} dw\right.$$

$$= O\left(\sum_{k=1}^{N} k^{2/3+\epsilon}e^{-2\pi nN^{-2}}\int_{k^{-1}(N+k)^{-1}}^{k^{-1}(N+1)^{-1}} d\vartheta\right)$$

$$= O(N^{-1/3+\epsilon}).$$

Similar result holds for K_5.

We have

$$\Re\frac{1}{k^2w} = \frac{N^{-2}}{k^2N^{-2}+N^2}, \qquad K_2 = O\left(\sum_{k=1}^{N} N^{-2}k^{2/3+\epsilon}\right) = O(N^{-1/3+\epsilon}).$$

Similar result holds for K_4.

Applying again Kloosterman's argument to K_3, we have also $K_3 = O(N^{-1/3})$.

Finally we find the residue of $\exp(2\pi w(n+1/24)+\pi/24k^2w)$ at $w=0$. We have the expansion

$$e^{2\pi w(n+1/24)} = \sum_{\nu=1}^{\infty}\frac{(2\pi w(n+1/24))^\nu}{\nu!},$$

$$e^{\pi/24k^2w} = \sum_{\mu=1}^{\infty}\frac{1}{\mu!}\left(\frac{\pi}{24k^2w}\right)^\mu.$$

The residue is, therefore,

$$\sum_{\mu=1}^{\infty}\frac{1}{\mu!}\left(\frac{\pi}{24k^2}\right)^\mu\frac{1}{(\mu-1)!}(2\pi(n+\tfrac{1}{24}))^{\mu-1}$$

$$= \frac{1}{2\pi}\frac{d}{dn}\sum_{\mu=1}^{\infty}\frac{1}{(\mu!)^2}\left(\frac{\pi}{24k^2}\right)^\mu(2\pi(n+\tfrac{1}{24}))^\mu$$

$$= \frac{1}{2\pi}\frac{d}{dn}\sum_{\mu=1}^{\infty}\frac{1}{2^{2\mu}(\mu!)^2}\left(\frac{\pi}{k}\left\{\tfrac{1}{3}(n+\tfrac{1}{24})\right\}^{1/2}\right)^{2\mu}$$

$$= \frac{1}{2\pi}\frac{d}{dn}J_0\left(\frac{i\pi}{k}\left\{\tfrac{1}{3}(n+\tfrac{1}{24})\right\}^{1/2}\right).$$

Therefore

$$q(n) = \frac{1}{2^{1/2}}\sum_{k=1,\,k \text{ odd}}^{N}\ \sum_{(h,k)=1,\,0<h\leqq k}\omega_{h,k}e^{-2\pi i hn/k}\frac{d}{dn}J_0\left(\frac{i\pi}{k}\left\{\tfrac{1}{3}(n+\tfrac{1}{24})\right\}^{1/2}\right)$$

$$+ O(N^{-1/3+\epsilon}).$$

Let $N\to\infty$; we obtain the theorem.

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