THE CONFORMAL THEORY OF CURVES

BY

AARON FIALKOW

TABLE OF CONTENTS

1. **Introduction.** Classical differential geometry is the metric theory of euclidean 3-space R_3. Its generalization, Riemannian geometry, is the metric theory of an n-dimensional Riemannian manifold[1] V_n. On the whole, the development of these geometries has proceeded in two main directions. Naturally these two directions are not mutually exclusive; occasionally they overlap in the common development of some subject.

One approach is the study of the metric transformations of the manifolds as a whole upon each other. This is the intrinsic theory of the space. In classical geometry, this point of view yields rather meager results since the intrinsic theory of R_3 is almost synonymous with the discovery of the complete group of motions in R_3. In Riemannian geometry, the intrinsic theory has considerably greater significance. The discovery of the process of covariant differentiation with respect to the first fundamental form of V_n and of the Riemann curvature tensor of V_n are important milestones in the development of this theory. This approach reaches its culmination in the fundamental theorem which states the conditions under which two Riemann spaces are isometric.

The other approach is the study of curves, surfaces and other subspaces and configurations in the enveloping R_3 or V_n and their behavior when the

---

[1] We denote an n-dimensional Riemann space, Einstein space, euclidean space and a space of constant curvature by V_n, E_n, R_n and S_n respectively.

enveloping manifold undergoes any metric transformation. In its most common development today, this study is based upon the process of covariant differentiation. As is well known, by repeated use of this type of differentiation, a system of Frenet equations of the subspace is obtained. These equations involve a number of metric geometric objects[2]: the $(n-1)$ curvatures and arc length for curves, the coefficients of the first and second fundamental forms for hypersurfaces, and so on. These geometric objects constitute the foundation upon which the detailed geometry of curves, surfaces and subspaces is built.

Classical differential geometry concerns itself almost exclusively with this second approach and a very considerable portion of Riemannian geometry has also evolved in this direction. The development of conformal Riemannian geometry, however, presents a different picture. Here the main emphasis has been upon the intrinsic conformal theory of the manifolds; that is, the investigation of the conformal transformations of Riemann spaces as a whole upon each other. This point of view is maintained in the early papers of Weyl[3] and Schouten[4] on conformal Riemannian geometry which mark the modern beginning of that subject. The fundamental conformal curvature tensor is discovered in these papers and is used in order to obtain a complete characterization of conformally euclidean Riemann spaces. These results are a continuation of classical theorems such as the theorem of Liouville on the conformal transformations of R_3 on itself.

The central problem of the intrinsic theory is the question of the conformal equivalence of Riemann spaces V_n. In order to effect a solution of this problem, T. Y. Thomas has considered the conformal tensor $g_{ij}/g^{1/n}$ where g_{ij} is the metric tensor of V_n and g is the determinant $|g_{ij}|$. This tensor remains invariant under conformal transformations of the metric tensor of V_n. The Christoffel symbols[5] formed with respect to this tensor (called the conformal parameters) have a complicated law of transformation under coordinate transformations and one cannot define a simple covariant derivative of tensors by means of these parameters. However, by formal methods based upon

(2) By a geometric object we mean an abstract object having a unique set of components, depending on the coordinates and their differentials to a specified order, in any coordinate neighborhood of the manifold. Hence the law of transformation of the components under coordinate changes must be transitive.

(3) H. Weyl, *Reine Infinitesimalgeometrie*, Mathematische Zeitschrift, vol. 2 (1918), pp. 384–411.

(4) J. A. Schouten, *Über die konforme Abbildung n-dimensionaler Mannigfaltigkeiten mit quadratischer Massbestimmung auf eine Mannigfaltigkeit mit euklidischer Massbestimmung*, Mathematische Zeitschrift, vol. 11 (1921), pp. 58–88.

(5) These Christoffel symbols were first defined by J. M. Thomas in another way. Cf. J. M. Thomas, *Conformal invariants*, Proceedings of the National Academy of Sciences, vol. 12 (1926), pp. 389–393.

the conformal parameters, it is possible to obtain a solution to the conformal equivalence problem for Riemann spaces([6]).

The investigations of conformal Riemannian geometry by Cartan([7]) and Schouten([8]) affords another method for the development of this subject. This method depends upon the introduction of $(n+2)$ homogeneous coordinates (the generalization of tetracyclic and pentaspherical coordinates) into the local euclidean space R_n of the V_n. Still another path for the study of intrinsic conformal geometry is indicated by the recent results of Schouten and Haantjes([9]) which suggest a projective treatment of conformal geometry.

Thus there exists a variety of general methods for the development of the intrinsic conformal theory of Riemann spaces. While this formal intrinsic theory is complete, the conformal theory of configurations in a Riemann space has been largely neglected. This fact is all the more remarkable when one considers that such a theory would always have real significance whereas this is rarely the case for the corresponding metric theory of configurations in a general V_n. To illustrate this point, we note that while a curve in a general V_n has $(n-1)$ curvatures which are metric invariants, these invariants are not very meaningful if the V_n does not admit any metric transformations other than the identity (as is usually the case). This state of affairs is never encountered in the conformal theory of configurations since every V_n always admits an infinite number of conformal mappings on conformally equivalent Riemann spaces.

One of the earliest results belonging to the conformal theory of configurations in V_n is the theorem which states that the lines of curvature of a hypersurface of V_n remain invariant under conformal transformations of V_n, first proved for a general V_n by Schouten and Struik([10]). They also proved a considerable number of similar results, some of which are not purely conformal theorems since they depend upon metric properties of the configuration and upon the particular conformal transformation to which the V_n is subjected.

([6]) T. Y. Thomas, *The Differential Invariants of Generalized Spaces*, 1934, chap. 4. O. Veblen, *Formalism for conformal geometry*, Proceedings of the National Academy of Sciences, vol. 21 (1935), pp. 168–173.

([7]) E. Cartan, *Les espaces à connexion conforme*, Annales de la Société Polonaise de Mathématique, vol. 2 (1923), pp. 171–221.

([8]) J. A. Schouten, *On the place of conformal and projective geometry in the theory of linear displacements*, Proceedings, K. Akademie van Wetenschappen, Amsterdam, vol. 27 (1924), pp. 407–424.

([9]) J. A. Schouten and J. Haantjes, *Beiträge zur allgemeinen (gekrümmten) konformen Differentialgeometrie*. I, II, Mathematische Annalen, vol. 112 (1936), pp. 594–629; vol. 113 (1936), pp. 568–583.

([10]) J. A. Schouten and D. J. Struik, *Un théorème sur la transformation conforme dans la géométrie différentielle à n dimensions*, Comptes Rendus de l'Académie des Sciences, vol. 176 (1923), pp. 1597–1600. Also cf. J. A. Schouten and D. J. Struik, *Einführung in die neueren Methoden der Differentialgeometrie*, vol. 2, 1938, pp. 199–215 for this topic as well as a general discussion of conformal Riemannian geometry.

The recent investigations of Sasaki[11], Modesitt[12], and the author[13] are of this general character. The papers of Kasner[14], Lipke[15], Schouten[16], and the writer[17] on conformal geodesics (natural families of curves) also belong in this category.

A number of investigators have used the formal methods which were devised to obtain a solution of the equivalence problem in the conformal intrinsic theory, in order to develop a conformal theory of curves and other subspaces. Among these developments are the results[18] of Sasaki[19] and Yano[20]. The formal apparatus used in these papers is necessarily quite complicated because their methods follow those used in the intrinsic theory. The various derivatives which have been devised for the development of the intrinsic theory have a strongly formal character and their structure is more complicated than that of ordinary covariant differentiation.

But while this formal apparatus may be inevitable in the case of the intrinsic theory, it is not essential for the development of the conformal theory of a subspace. For the subspace introduces additional structure into the Riemann space V_n by means of which we find a relative conformal scalar[21] at points of the subspace. By means of this relative conformal scalar, it is possible to define a new simple type of differentiation (with respect to the subspace) which plays a role analogous to ordinary covariant differentiation in metric Riemannian geometry. This differentiation process enjoys all the usual properties of covariant differentiation as well as a number of others which give it its distinctive conformal character.

---

[11] S. Sasaki, *Some theorems on conformal transformations of Riemannian spaces*, Proceedings of the Physico-Mathematical Society of Japan, (3), vol. 18 (1936), pp. 572–578.

[12] V. Modesitt, *Some singular properties of conformal transformations between Riemann spaces*, American Journal of Mathematics, vol. 60 (1938), pp. 325–336.

[13] A. Fialkow, *Conformal transformations and the subspaces of a Riemann space*, Bulletin of the American Mathematical Society, abstract 43-9-328.

[14] E. Kasner, *Natural families of trajectories: conservative fields of force,* these Transactions, vol. 10 (1909), pp. 201–219.

[15] J. Lipke, *Natural families of curves in a general curved space of n dimensions*, these Transactions, vol. 13 (1912), pp. 77–95.

[16] J. A. Schouten, *Über die Umkehrung eines Satzes von Lipschitz*, Nieuw Archief voor Wiskunde, vol. 15 (1928), pp. 97–102.

[17] A. Fialkow, *Conformal geodesics*, these Transactions, vol. 45 (1939), pp. 443–473.

[18] Possibly the work of Hlavatý also belongs in this category. These papers are not accessible to the writer. Cf. V. Hlavatý, *Zur Konformgeometrie* III, Proceedings, K. Akademie van Wetenschappen, Amsterdam, vol. 38 (1935), pp. 1006–1011.

[19] S. Sasaki, *On the theory of curves in a curved conformal space*, Science Reports of the Imperial University of Tokyo (1), vol. 27 (1939), pp. 392–409; *On the theory of surfaces in a curved conformal space*, ibid., vol. 28 (1940), pp. 261–285; *Geometry of the conformal connexion*, ibid., vol. 29 (1940), pp. 219–267.

[20] K. Yano, *Sur la théorie des espaces à connexion conforme*, Journal of the Faculty of Science, Imperial University of Tokyo, vol. 4 (1939), pp. 40–57.

[21] This term is defined in §2.

By "conformal differentiation," we arrive at a sequence of normal vector spaces and fundamental forms for the subspace which are unchanged by conformal transformations of V_n. These "conformal fundamental forms" constitute the foundation upon which a detailed conformal geometry of subspaces may be built. Formally, this entire theory is considerably simpler than the previous investigations of conformal Riemannian geometry, its technical aspects being no more involved than those of the ordinary metric geometry of Riemann spaces. While we are concerned with the same general subject as that dealt with by Sasaki and Yano, there is no actual overlapping either of results or of methods. We note, however, as is shown in §15, that our results may be used to develop a conformal theory of curves based upon the conformal tensor $g_{ij}/g^{1/n}$ which is formally analogous to the investigations mentioned above.

In the present paper, we develop the foundations of the conformal theory of curves, reserving the treatment of other subspaces for later publication([22]). We note that this separate treatment is not prompted by pedagogic reasons alone, but is a natural separation. For in our development of the conformal geometry of a subspace of V_n, two mutually exclusive cases arise which must be treated separately: (1) curves and (2) subspaces whose dimensionality exceeds one.

It is well known that there is a metric (congruence) theory of curves in the plane but no conformal theory. That an analytic curve can have no conformal properties follows from the theorem: *Every analytic curve in the plane is conformally equivalent to a straight line.* It is the object of this paper to show that a conformal theory of curves *does* exist in any Riemann space whose dimensionality exceeds 2 and to develop this theory. Accordingly, we study those properties of a curve which remain unchanged when the enveloping Riemann space V_n of dimensionality $n > 2$ undergoes any conformal mapping, not necessarily on itself.

The principal tool is a new kind of tensor differentiation which has conformal meaning—"the conformal derivative." By systematic use of the conformal derivative we derive the conformal analogues of the ordinary (metric) Frenet equations. We find $n-1$ differential "conformal curvatures" $J_1, J_2, \cdots, J_{n-1}$ and an integral "conformal arc length" S which are unchanged by any conformal transformation of the Riemann space. This means that *if $V_n \leftrightarrow \overline{V}_n$, $C \leftrightarrow \overline{C}$ by a conformal map, then the J's are the same functions of S for C and $\overline{C}$.*

The converse holds in spaces which are conformal to a euclidean space.

([22]) Some of the principal results in the curve theory are stated without proof in a previous note having the same title as the present paper which appeared in the Proceedings of the National Academy of Sciences, vol. 26 (1940), pp. 437–439. Corresponding results in the conformal theory of a subspace appear in two abstracts in the Bulletin of the American Mathematical Society, abstracts 46-11-487 and 47-3-156.

In this case, we have the fundamental conformal equivalence theorem: *If V_n and $\overline{V}_n$ are conformal to a euclidean space and the J's for both C and $\overline{C}$ are the same functions of S, then a conformal mapping exists for which $V_n \leftrightarrow \overline{V}_n$, $C \leftrightarrow \overline{C}$.* This is the conformal analogue of the metric congruence theorem which holds in a euclidean space and in a space of constant curvature.

We also prove the existence theorem: *In any V_n, a curve exists for which the J's are preassigned continuous functions of S. This curve is uniquely determined by a set of initial conditions which is found explicitly.*

The conformal curvatures of a curve C in V_n have rather simple geometric properties if V_n is conformal to an Einstein space or, more particularly, to a euclidean space. Thus *if V_n is conformally euclidean then $J_\alpha \equiv 0$ $(1 \leq \alpha \leq n-2)$ if and only if C is conformally equivalent to a curve in a euclidean n-space whose $(\alpha+1)st$ metric curvature vanishes. Another example: If V_n is conformal to a euclidean space then the $n-1$ conformal curvatures of a curve and their derivatives with respect to the conformal length constitute a complete set of conformal differential invariants of the curve.*

If $n = 2$, the results of this paper apply if the conformal transformations are restricted to mappings applied to spaces of constant curvature which are similar to and include the inversive transformations of the plane.

While the results of this paper bear a close analogy to those which hold in the metric theory, in some cases the proofs are markedly different. Thus, the first of the "conformal Frenet equations" is not obtained, as in the classic case, by differentiating the unit tangent vector. For it will be seen later that the conformal derivative of the unit tangent always vanishes identically. As another important point of difference, we note that only $n-2$ of the conformal curvatures occur as coefficients in the conformal Frenet equations. The $(n-1)st$ conformal curvature, while as essential as the other curvatures, is found in an entirely different way and does not have the same properties as the others.

These essentially novel features which distinguish the conformal geometry of curves from the metric geometry are also present in an analogous form in the corresponding theory for any subspace. For example, a hypersurface has three "conformal fundamental forms" instead of the anticipated two forms and four sets of integrability conditions instead of the classic Gauss-Codazzi equations. Furthermore, the conformal behavior of subspaces whose dimensionality is at least 4 is typical, while the cases of dimension number 3, 2, and 1 respectively are increasingly degenerate. There is no corresponding analogue in the metric theory.

As an important special case, this theory obviously includes the "natural geometry" of curves in euclidean n-space under the continuous group of conformal mappings of the euclidean space upon itself. The transformations of this group are the products of inversions with respect to a hypersphere, motions and transformations of similitude (Liouville's theorem). This means

that our results constitute the inversive theory of curves when applied to this continuous group of transformations of a euclidean space. In particular, *the curves along which all the conformal curvatures are constants are the paths of the inversive group.*

A detailed inversive geometry of plane curves and of curves and surfaces in R_3 has been developed by Thomsen, Blaschke and Takasu in their books on conformal differential geometry[23]. Their investigations constitute a theory of curves and surfaces in R_2 and R_3 which is complete in its essential parts and anticipates many of our results for this important but special case. However, their methods depend upon the systematic use of tetracyclic and pentaspherical coordinates and therefore differ completely from the methods which are employed here. The inversive theory of plane curves has also been developed by a number of other writers using still different methods.

We note that the subject of this paper is also somewhat connected with the "natural geometry" of a curve associated with any group of transformations of the plane into itself. This theory was originated by Pick[24] and has subsequently been developed by Kowalewski[25] and his students.

2. **Riemann spaces conformal to V_n, conformal tensors.** Let V_n be a real Riemann space whose coordinate manifold is of class[26] C^m and whose real metric tensor, defined over the manifold, is positive definite[27] and of class C^{m-1} with $m \geq 1$. Briefly, we say V_n is a Riemann space of class[28] C^m.

---

[23] W. Blaschke and G. Thomsen, *Vorlesungen über Differentialgeometrie*, vol. 3, 1929; T. Takasu, *Differentialgeometrien in den Kugelräumen*, vol. 1, 1938.

We are obliged to a referee for these references. Due to our unfamiliarity with tetracyclic and pentaspherical coordinates, it is difficult for us to determine precisely the extent to which duplication of results occurs. In general, these books would appear to contain most of our theorems for curves in R_2 and for curves and surfaces in R_3 under the inversive group. These books also contain other results on the detailed inversive geometry of R_2 and R_3 which lie beyond the scope of our present investigations. These references have been incorporated into the revision of the introduction and we have also included references to a number of papers which have appeared since this paper was first written.

[24] G. Pick, *Natürliche Geometrie ebener Transformationsgruppen*, Sitzungsberichte der Academie der Wissenschaften, Vienna, vol. 115 (1906), p. 139.

[25] G. Kowalewski, *Vorlesungen über allgemeine natürliche Geometrie und Liesche Transformationsgruppen*, 1931, chap. 3.

[26] The definitions of the class of a coordinate manifold and of a Riemann space are based upon the discussion which appears in the paper by T. Y. Thomas, *Recent trends in geometry*, American Mathematical Society Semicentennial Publications, vol. 2 (1938), pp. 98–99, 104. In particular, if the coordinate manifold is of class C^m, then the admissible coordinate systems are related to each other by transformations of class C^m.

[27] The greater part of the following discussion and of the results of the paper will hold even if the metric tensor is indefinite provided that it is not singular. The only real novelty arises when a vector is a null vector. We shall not consider the indefinite case.

[28] We shall assume the reality, existence and continuity of whatever functions occur in the proofs. At the outset of the proof of an important theorem we shall simply indicate sufficient conditions for the satisfaction of this assumption in order to avoid frequent interruptions of the discussion for essentially non-geometric matters.

Suppose([29]) $\{x^i\}$ are admissible real local coordinates in a coordinate neighborhood of any point of V_n. In each coordinate neighborhood, we write the first fundamental form of V_n as

$$(2.1) \qquad\qquad ds^2 = g_{ij}dx^idx^j.$$

Since the results of this paper are local theorems which hold for a sufficiently small neighborhood of a point we shall restrict ourselves to a portion of V_n which is a neighborhood $U(P)$ of a point P coverable by a single coordinate system $\{x^i\}$. We shall refer to $U(P)$ as the Riemann space V_n and use similar language in connection with other Riemann spaces which appear in the paper.

Let([30]) $\overline{V}_n$ be a real Riemann space of class C^m whose first fundamental form may be written as

$$(2.2) \qquad\qquad d\bar{s}^2 = \bar{g}_{ij}d\bar{x}^id\bar{x}^j$$

where $\{\bar{x}^i\}$ are allowable local coordinates. *Then*([31]) $\overline{V}_n$ *is conformal to* V_n *by means of a transformation of class* C^m *(briefly:* $\overline{V}_n$ *is conformal to* V_n*)* if a one-to-one point transformation T exists between the points P of V_n and the points $\overline{P}$ of $\overline{V}_n$ which may be written (locally) as

$$(2.3) \qquad \bar{x}^i = \bar{x}^i(x^1, x^2, \cdots, x^n), \qquad x^i = x^i(\bar{x}^1, \bar{x}^2, \cdots, \bar{x}^n)$$

so that the real functions

$$\bar{x}^i(x^1, x^2, \cdots, x^n), \qquad x^i(\bar{x}^1, \bar{x}^2, \cdots, \bar{x}^n)$$

are of class C^m and

$$(2.4) \qquad\qquad d\bar{s} = e^\sigma ds$$

at corresponding points. It follows that $\sigma(x^i)$ is a real function of class C^{m-1} and that the form (2.2) is positive definite. We refer to $\sigma(x^i)$ as *the conformal mapping function of* V_n *on* $\overline{V}_n$, or briefly, as the *mapping function*. Whenever we say that $\overline{V}_n$ is conformal to V_n, it is to be understood that the conformal transformation is of class C^m.

The transformation T may be written in the simple form

$$(2.5) \qquad\qquad \bar{x}^i = x^i$$

after a suitable change of coordinates. For if we transform the coordinate

---

([29]) Throughout this paper the indices h, i, j, k have the range $1, 2, \cdots, n$. It is to be understood that a tensor equation in which an index is not summed is valid for each value of the index within its range. A covariant or contravariant index which appears twice in an expression is to be summed over the appropriate range.

([30]) We denote a Riemann space conformal to V_n by $\overline{V}_n$. Thus $\overline{E}_n$ and $\overline{R}_n$ signify spaces which are conformally equivalent to an Einstein space and a euclidean space respectively. A geometric object in $\overline{V}_n$ corresponding to the geometric object F in V_n is denoted by $\overline{F}$.

([31]) This clause may obviously be replaced by "V_n is conformal to $\overline{V}_n$ by means of a transformation of class C^m."

neighborhoods of $\overline{V}_n$ according to (2.3) considered as an admissible coordinate transformation, points in $\overline{V}_n$ and V_n with the same coordinates correspond and the conformal transformation becomes (2.5). Throughout this paper, unless a contrary assumption is explicitly made, we shall always assume that coordinate systems have been chosen so that (2.5) holds. In these coordinate systems,

$$(2.6) \qquad\qquad \bar{g}_{ij} = e^{2\sigma}g_{ij}, \qquad \bar{g}^{ij} = e^{-2\sigma}g^{ij}$$

where g^{ij} and $\bar{g}^{ij}$ are the contravariant components of the metric tensors. Conversely, if (2.5) is a point transformation of the points of V_n and $\overline{V}_n$ and (2.6) holds at corresponding points where the mapping function $\sigma(x^i)$ is a real function of class C^{m-1}, it follows that $\overline{V}_n$ is conformal to V_n.

The problem of the conformal equivalence of Riemann spaces leads quite naturally to the study of the conformal Riemann space V_n. The conformal Riemann space V_n of class C^m is a space whose coordinate manifold is of class C^m and whose fundamental geometric object, defined over the manifold, is the set of all second order, symmetric, positive definite tensors of class C^{m-1},

$$(2.7) \qquad\qquad \{e^{2\sigma}g_{ij}\},$$

any two of which are equal except for a positive multiplicative scalar factor of class C^{m-1}. The conformal tensor $g_{ij}/g^{1/n}$ constructed from any tensor g_{ij} belonging to (2.7) is independent of the particular tensor which is chosen. For this reason, T. Y. Thomas[32] has defined the conformal Riemann space V_n by using this tensor instead of the set (2.7) as the fundamental geometric object of V_n.

It is natural to associate the set of all conformally equivalent Riemann spaces

$$(2.8) \qquad\qquad \{\overline{V}_n\}$$

of class C^m whose metric tensors (in some allowable coordinate system) belong to (2.7) with the conformal Riemann space V_n. Indeed, as is easy to see, the geometric properties of V_n (which are independent of the factor $e^{2\sigma}$) are conformal properties of the set of Riemann spaces (2.8). Throughout this paper, whenever we refer to the conformally equivalent Riemann spaces V_n, $\overline{V}_n$, it will be understood that these spaces are *any* two spaces of the set of conformally equivalent Riemann spaces (2.8).

The above discussion shows that, formally, our conformal theory of curves is the theory, under the identity transformation, of a curve and an enveloping coordinate manifold on which is defined a second order, symmetric, positive definite tensor up to a positive scalar multiplicative factor. It is shown in §15, that our results may be used to develop a conformal theory of curves which is based directly on the conformal tensor $g_{ij}/g^{1/n}$.

---

[32] T. Y. Thomas, loc. cit., p. 119.

Let $T_{j_1\cdots j_v}^{i_1\cdots i_w}$ be components of a tensor at a point P of V_n whose values depend upon geometric objects of V_n and of its subspaces ([33]). Let $\overline{V}_n$ be any Riemann space conformal to V_n and let $\overline{T}_{j_1\cdots j_v}^{i_1\cdots i_w}$ be the components of the tensor at $\overline{P}$ whose values depend in the same way upon the corresponding geometric objects of $\overline{V}_n$ and its corresponding subspaces. Then if (2.6) holds and

(2.9)
$$\overline{T}_{j_1\cdots j_v}^{i_1\cdots i_w} = (e^{\sigma})^u\, T_{j_1\cdots j_v}^{i_1\cdots i_w}$$

we call $T_{j_1\cdots j_v}^{i_1\cdots i_w}$ a *relative conformal tensor of weight* u. The law of transformation of $T_{j_1\cdots j_v}^{i_1\cdots i_w}$ between any two $\overline{V}_n$ as well as between any two coordinate systems is consistent. If $u=0$, the tensor has the same components in V_n and $\overline{V}_n$. In this case, we call $T_{j_1\cdots j_v}^{i_1\cdots i_w}$ a *conformal tensor*. If $u=v-w$, we say that $T_{j_1\cdots j_v}^{i_1\cdots i_w}$ is a *conformetric tensor*. As will be seen below, these latter tensors have both metric and conformal properties.

Under the assumption that a relative conformal scalar Q exists in V_n, one may construct a conformetric tensor or a conformal tensor corresponding to every relative conformal tensor. More generally, under the same assumption, if $T_{j_1\cdots j_v}^{i_1\cdots i_w}$ obeys (2.9), one may construct a corresponding relative conformal tensor which satisfies an equation like (2.9) with u replaced by an arbitrary u'. For suppose that the transformation law([34]) of Q is $\overline{Q}=e^{\sigma}Q$. Then $Q^{u'-u}\cdot T_{j_1\cdots j_v}^{i_1\cdots i_w}$ is a relative conformal tensor which satisfies (2.9) with u replaced by u'. We note that every relative conformal tensor (including conformetric tensors) is the product of a conformal tensor by a relative conformal scalar.

As a consequence of our definitions it follows that if the components of a relative conformal tensor are zero in V_n, they are zero in any $\overline{V}_n$. This fact permits us to write conformal tensor equations which retain their meaning under conformal transformations. The sum, difference, inner and outer product of conformetric tensors (conformal tensors) is also a conformetric tensor (conformal tensor).

If λ^i is a conformetric contravariant vector, the condition (2.9) becomes $\overline{\lambda}^i=e^{-\sigma}\lambda^i$. It follows that the direction of λ^i in V_n coincides with the direction of $\overline{\lambda}^i$ in $\overline{V}_n$. Since $\bar{g}_{ij}\overline{\lambda}^i\overline{\lambda}^j=g_{ij}\lambda^i\lambda^j$, the length of λ^i remains unchanged under any conformal mapping. Thus λ^i has a conformally invariant direction and (metric) length. Conversely, any vector for which this is true must be a conformetric vector. If the length of a conformetric vector is unity, then the vector is called a *unit conformetric vector*. Any conformetric scalar is a conformal scalar or invariant. It is easy to show that any conformetric tensor

---

([33]) Examples of such geometric objects which will be used in this paper are: the metric tensor g_{ij}, the Christoffel symbols of the second kind, the unit tangent and principal normal of a curve.

([34]) There is no loss of generality in this assumption, for if $\overline{Q}=(e^{\sigma})^t\cdot Q$, the relative conformal scalar $|Q|^{1/t}$ has the desired transformation law.

which is not a scalar (and only these tensors) may be represented in the usual way[35] by means of adjoint n-beins of conformetric vectors. As follows from (2.6), g_{ij} and g^{ij} are conformetric tensors. Hence the indices of any conformetric tensor (but not of a conformal tensor) may be raised or lowered using g_{ij}, g^{ij} in the usual way and the result will be a conformetric tensor.

In this paper, we consider the conformal geometry of a curve in V_n. The curve introduces additional structure into V_n, by means of which a relative conformal scalar is found at points of the curve. In view of the existence of this relative conformal scalar, one may find a conformal vector corresponding to any conformetric vector, and conversely. Thus it is chiefly a matter of convenience whether we use conformetric vectors or conformal vectors. Our work is based upon unit conformetric vectors. The "conformal derivative" of such vectors is somewhat simpler than the "conformal derivative" of conformal vectors. However, we note that the analogous conformal theory of any subspace whose dimensionality exceeds one is developed by the use of conformal tensors.

3. **The conformal derivative.** We suppose that the class, defined in §2, of any two Riemann spaces V_n and $\overline{V}_n$ belonging to (2.8), is at least 2, that is, $m \geq 2$. Then it follows from (2.6) that [36]

$$(3.1) \qquad \left\{ \overline{i \atop jk} \right\} = \left\{ i \atop jk \right\} + \delta^i_j \sigma_{,k} + \delta^i_k \sigma_{,j} - g_{jk} g^{ih} \sigma_{,h}$$

where

$$\left\{ i \atop jk \right\}, \qquad \left\{ \overline{i \atop jk} \right\}$$

are the Christoffel symbols of the second kind for V_n and $\overline{V}_n$ respectively. Let

$$x^i = x^i(z)$$

represent a real curve C in V_n. Let $dx^i/dz \neq 0$ at each point of this z-interval for at least one value of i. We also suppose that the functions $x^i(z)$ are of class C^p where p is a fixed integer subject to the inequalities

$$(3.2) \qquad m \geq p \geq 2.$$

Then it is easy to show that $s = s(z)$ is an allowable change of parameter where s is an arc length parameter determined up to an additive constant and a choice of sign. Hence the equation of C may be written as[37]

---

[35] A. Duschek and W. Mayer, *Lehrbuch der Differentialgeometrie*, vol. 2 (1930), pp. 14–15.

[36] The comma denotes covariant differentiation with respect to the x's and the form (2.1) and the δ^i_j are the Kronecker deltas.

[37] Note that $x^i(z)$ and $x^i(s)$ are different functions of their respective variables. This remark also applies to the functions $\bar{x}^i(z)$ and $\bar{x}^i(\bar{s})$ which are defined below.

$$x^i = x^i(s), \qquad\qquad a_1 < s < b_1,$$

where $x^i(s)$ are of class C^p. Then the unit tangent ν^i and the principal normal μ^i are given by

$$(3.3) \qquad\qquad \nu^i = \frac{dx^i}{ds},$$

$$(3.4) \qquad\qquad \mu^i = \frac{d^2x^i}{ds^2} + \left\{\begin{matrix} i \\ jk \end{matrix}\right\} \frac{dx^j}{ds} \frac{dx^k}{ds}.$$

The curve $\overline{C}$ in $\overline{V}_n$ which corresponds to C under the conformal transformation (2.5) is

$$\bar{x}^i = \bar{x}^i(z)$$

where $\bar{x}^i(z)$ and $x^i(z)$ are the same functions of z and corresponding points have the same value of z. The conditions (3.2) also apply to $\overline{C}$. As shown above in the case of C, the curve $\overline{C}$ may also be referred to an arc length parameter $\bar{s}$ and written as

$$\bar{x}^i = \bar{x}^i(\bar{s}), \qquad\qquad \bar{a}_1 < \bar{s} < \bar{b}_1.$$

Naturally the points for which $s = \bar{s}$ do not correspond since metric arc length is not a conformal parameter. The unit tangent $\bar{\nu}^i$ and the principal normal $\bar{\mu}^i$ of $\overline{C}$ are given by equations similar to (3.3) and (3.4). From these equations and (2.4), (2.6), (3.1), (3.3) and (3.4), we find that

$$(3.5) \qquad\qquad \bar{\nu}^i = e^{-\sigma}\nu^i,$$

$$(3.6) \qquad\qquad \bar{\mu}^i = e^{-2\sigma}\left[\mu^i - \sigma_{,h}(g^{ih} - \nu^i\nu^h)\right].$$

The tensor $g^{ih} - \nu^i\nu^h$ is the projection tensor[38] of the vector space orthogonal to ν^i. If we write μ_i and $\bar{\mu}_i$ for the covariant components of the principal normals, $\mu_i = g_{ij}\mu^j$, $\bar{\mu}_i = \bar{g}_{ij}\bar{\mu}^j$ and it follows from (2.6) and (3.6) that

$$(3.7) \qquad\qquad \bar{\mu}_i = \mu_i - \sigma_{,i} + \sigma_{,k}\nu^k\nu_i$$

where ν_i is the covariant tangent defined by $\nu_i = g_{ij}\nu^j$.

Let $\lambda^i(t)$ be the components of a conformetric contravariant vector of class C^1 defined along C where t is any (not necessarily allowable) conformal parameter along C related to z (or s) by a parameter transformation of class C^1. Then dx^i/dt exists and is continuous. Since λ^i is a conformetric vector,

$$(3.8) \qquad\qquad \bar{\lambda}^i = e^{-\sigma}\lambda^i.$$

We write the absolute derivative with respect to t and the form (2.1) of this vector as $D\lambda^i/Dt$ so that

---

[38] Duschek-Mayer, loc. cit., pp. 44–45.

$$(3.9) \qquad \frac{D\lambda^i}{Dt} = \frac{d\lambda^i}{dt} + \begin{Bmatrix} i \\ hk \end{Bmatrix} \lambda^h \frac{dx^k}{dt} .$$

If we write the analogous equation for $\overline{D}\overline{\lambda}{}^i/\overline{D}t$ and simplify by the use of (3.1), (3.8) and (3.9), we obtain

$$(3.10) \qquad \frac{\overline{D}\overline{\lambda}{}^i}{\overline{D}t} = e^{-\sigma} \left[\frac{D\lambda^i}{Dt} + \sigma_{,h}\lambda^h \frac{dx^i}{dt} - g_{hk}\lambda^h \frac{dx^k}{dt} g^{ij}\sigma_{,j} \right].$$

We substitute the values for $\sigma_{,h}$ and $\sigma_{,j}g^{ij}$ which are obtained from (3.6) and (3.7) in (3.10) and simplify the resulting equation by the use of (2.4), (2.6), (3.3) and (3.8). This gives

$$\frac{\overline{D}\overline{\lambda}{}^i}{\overline{D}t} + \bar{\mu}_h\bar{\lambda}^h \frac{dx^i}{dt} - \bar{g}_{hk}\bar{\lambda}^h \frac{dx^k}{dt} \bar{\mu}^i = e^{-\sigma} \left[\frac{D\lambda^i}{Dt} + \mu_h\lambda^h \frac{dx^i}{dt} - g_{hk}\lambda^h \frac{dx^k}{dt} \mu^i \right].$$

It follows that $\delta\lambda^i/\delta t_C$ given by

$$(3.11) \qquad \frac{\delta\lambda^i}{\delta t_C} = \frac{D\lambda^i}{Dt} + \mu_h\lambda^h \frac{dx^i}{dt} - g_{hk}\lambda^h \frac{dx^k}{dt} \mu^i$$

is a conformetric vector, that is,

$$\frac{\delta\bar{\lambda}^i}{\delta t_{\overline{C}}} = e^{-\sigma} \frac{\delta\lambda^i}{\delta t_C} .$$

The subscript C is used in the symbol $\delta/\delta t_C$ to indicate that the definition of this symbol depends upon the curve C as well as the process of differentiation. Since $\delta/\delta t_C$ will always be evaluated with respect to the same curve C in V_n, we shall usually write $\delta/\delta t$ for $\delta/\delta t_C$ without danger of ambiguity.

In order to arrive at a meaning for the operator $\delta/\delta t$ when applied to any tensor, we assume that $\delta/\delta t$ satisfies the following requirements[39]:

$$(\alpha) \qquad \frac{\delta\phi}{\delta t} = \frac{D\phi}{Dt}$$

if ϕ is any scalar;

$$(\beta) \qquad \frac{\delta\lambda^i}{\delta t} = \frac{D\lambda^i}{Dt} + \Omega_j^i\lambda^j \frac{ds}{dt}$$

where λ^i is any contravariant vector (not necessarily a conformetric vector) and $\Omega_j^i = \nu^i\mu_j - \mu^i\nu_j$;

----

[39] This discussion is analogous to a similar one for ordinary covariant differentiation by Mayer. Cf. Duschek-Mayer, loc. cit., vol. 2, pp. 31–33.

$$(\gamma) \qquad \frac{\mathfrak{d}}{\mathfrak{d}t}(S \cdot T) = \frac{\mathfrak{d}S}{\mathfrak{d}t} \cdot T + S \cdot \frac{\mathfrak{d}T}{\mathfrak{d}t}$$

where S and T are any tensors

$$(\delta) \qquad \sum_{i=j}\left(\frac{\mathfrak{d}}{\mathfrak{d}t} T_j^i\right) = \frac{\mathfrak{d}}{\mathfrak{d}t}\left(\sum_{i=j} T_j^i\right)$$

where i and j are any two indices of the tensor T, one contravariant and the other covariant. (That is, the contraction operation for tensors $\sum_{i=j}$ and the $\mathfrak{d}/\mathfrak{d}t$ operation are commutative.)

On the basis of these properties, we shall find a unique expression for $\mathfrak{d}T/\mathfrak{d}t$. We first consider a covariant vector ξ_i. Then, according to (α),

$$\frac{\mathfrak{d}}{\mathfrak{d}t}(\lambda^i \xi_i) = \frac{D}{Dt}(\lambda^i \xi_i).$$

Because of (γ) and (δ), this may be written as

$$\frac{\mathfrak{d}\lambda^i}{\mathfrak{d}t} \cdot \xi_i + \lambda^i \frac{\mathfrak{d}\xi_i}{\mathfrak{d}t} = \frac{D\lambda^i}{Dt}\xi_i + \lambda^i \frac{D\xi_i}{Dt}.$$

It follows that $\lambda^i(\mathfrak{d}\xi_i/\mathfrak{d}t)$ is an invariant for all λ^i and hence $\mathfrak{d}\xi_i/\mathfrak{d}t$ is a covariant vector (if it exists). In this last equation, we substitute the value for $\mathfrak{d}\lambda^i/\mathfrak{d}t$ given by (β) and simplify. This gives

$$\lambda^i\left(\frac{\mathfrak{d}\xi_i}{\mathfrak{d}t} - \frac{D\xi_i}{Dt} + \Omega_i^j \xi_j \frac{ds}{dt}\right) = 0.$$

Since λ^i is an arbitrary vector, it follows that

$$\frac{\mathfrak{d}\xi_i}{\mathfrak{d}t} = \frac{D\xi_i}{Dt} - \Omega_i^j \xi_j \frac{ds}{dt}$$

or

$$(3.12) \qquad \frac{\mathfrak{d}\xi_i}{\mathfrak{d}t_C} = \frac{D\xi_i}{Dt} - \xi_h \frac{dx^h}{dt}\mu_i + \xi_h \mu^h g_{ij}\frac{dx^j}{dt}.$$

It is clear that $\mathfrak{d}/\mathfrak{d}t$ as applied to covariant vectors ξ_i satisfies those conditions (α) to (δ) which have meaning in this case.

To extend this definition to a tensor of any kind $T_{j_1\cdots j_v}^{i_1\cdots i_w}$, we form the invariant

$$(3.13) \qquad T_{j_1\cdots j_v}^{i_1\cdots i_w}{}_{(1)}\lambda^{j_1}\cdots{}_{(v)}\lambda^{j_v}{}_{(1)}\xi_{i_1}\cdots{}_{(w)}\xi_{i_w}$$

where the λ^j and ξ_i are arbitrary vectors. If we apply the $\mathfrak{d}/\mathfrak{d}t$ operator to

(3.13) and proceed as in the derivation of (3.12), using (α) to (δ) and (3.12), we find that

$$(3.14) \quad \frac{\delta}{\delta t_C} T^{i_1 \cdots i_w}_{j_1 \cdots j_v} = \frac{D}{Dt} T^{i_1 \cdots i_w}_{j_1 \cdots j_v} + \sum_{\alpha=1}^{w} T^{i_1 \cdots i_{\alpha-1} h i_{\alpha+1} \cdots i_w}_{j_1 \cdots j_v} \Omega^{i_\alpha}_h \frac{ds}{dt}$$

$$- \sum_{\beta=1}^{v} T^{i_1 \cdots i_w}_{j_1 \cdots j_{\beta-1} h j_{\beta+1} \cdots j_v} \Omega^{h}_{j_\beta} \frac{ds}{dt}.$$

It follows that $\delta T^{i_1 \cdots i_w}_{j_1 \cdots j_v}/\delta t$ is a tensor of the same kind as $T^{i_1 \cdots i_w}_{j_1 \cdots j_v}$ and that the definition of $\delta T/\delta t$ stated in (3.14) satisfies conditions (α) to (δ). Equations (3.11) and (3.12) are special cases of (3.14).

The definition of $\delta T/\delta t$ stated in (3.14) may be based upon symmetric coefficients of connection Γ^i_{jk} just as ordinary covariant differentiation is based upon the Christoffel symbols $\{^i_{jk}\}$. We define the Γ^i_{jk} by[40]

$$\Gamma^i_{jk} = \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} + \mu_j \delta^i_k + \mu_k \delta^i_j - g_{jk}\mu^i$$

and note that (3.14) is equivalent to

$$\frac{\delta}{\delta t} T^{i_1 \cdots i_w}_{j_1 \cdots j_v} = \frac{d}{dt} T^{i_1 \cdots i_w}_{j_1 \cdots j_v} + \sum_{\alpha=1}^{w} T^{i_1 \cdots i_{\alpha-1} h i_{\alpha+1} \cdots i_w}_{j_1 \cdots j_v} \Gamma^{i_\alpha}_{hk} \frac{dx^k}{dt}$$

$$- \sum_{\beta=1}^{v} T^{i_1 \cdots i_w}_{j_1 \cdots j_{\beta-1} h j_{\beta+1} \cdots j_v} \Gamma^{h}_{j_\beta k} \frac{dx^k}{dt}.$$

We now find the law of transformation of $\delta T^{i_1 \cdots i_w}_{j_1 \cdots j_v}/\delta t$ when $T^{i_1 \cdots i_w}_{j_1 \cdots j_v}$ is a relative conformal tensor of weight u; that is, when $T^{i_1 \cdots i_w}_{j_1 \cdots j_v}$ obeys (2.9). As follows from the definition of Γ^i_{jk} and (2.6), (3.1), (3.6) and (3.7),

$$\overline{\Gamma}^i_{jk} = \Gamma^i_{jk} + \frac{d\sigma}{ds} (\nu_j \delta^i_k + \nu_k \delta^i_j - g_{jk}\nu^i).$$

By means of this relation and the definition for $\delta T^{i_1 \cdots i_w}_{j_1 \cdots j_v}/\delta t$ by means of the Γ^i_{jk}, we find upon applying the $\delta/\delta t$ operator to (2.9) that

$$\frac{\overline{\delta}}{\delta t} \overline{T}^{i_1 \cdots i_w}_{j_1 \cdots j_v} = e^{\sigma u} \left\{ \frac{\delta}{\delta t} T^{i_1 \cdots i_w}_{j_1 \cdots j_v} + (u + w - v) \frac{d\sigma}{dt} \cdot T^{i_1 \cdots i_w}_{j_1 \cdots j_v} \right\}.$$

---

[40] Since the Γ^i_{jk} differ from the Christoffel symbols

$$\left\{ \begin{matrix} i \\ jk \end{matrix} \right\}$$

by a tensor, it is immediate that they must transform like coefficients of connection under coordinate transformations. For example, cf. L. P. Eisenhart, *Non-Riemannian Geometry*, American Mathematical Society Colloquium Publications, vol. 8, 1927, p. 48.

If Q is a relative conformal scalar of weight -1 so that $\overline{Q} = e^{-\sigma}Q$, the last equations may be written in invariant form by noting that they are equivalent to the equations

$$S^{i_1\cdots i_w}_{j_1\cdots j_v} = \frac{\mathfrak{d}}{\mathfrak{d}t}\, T^{i_1\cdots i_w}_{j_1\cdots j_v} + (u + w - v)\left(\frac{d\log Q}{dt}\right) T^{i_1\cdots i_w}_{j_1\cdots j_v},$$

$$\overline{S}^{i_1\cdots i_w}_{j_1\cdots j_v} = e^{\sigma u} S^{i_1\cdots i_w}_{j_1\cdots j_v}.$$

Hence $S^{i\cdots i}_{j\cdots j}$ is a relative conformal tensor of weight u. In particular, if $T^{i\cdots i}_{j\cdots j}$ is a conformetric tensor so that $u = v - w$, it follows that $\mathfrak{d}T^{i\cdots i}_{j\cdots j}/\mathfrak{d}t$ is also a conformetric tensor. This fact exhibits the conformal property of the operator $\mathfrak{d}/\mathfrak{d}t$ and thus justifies the definition: *The tensor $\mathfrak{d}T^{i\cdots i}_{j\cdots j}/\mathfrak{d}t$ defined by* (3.14) *is called the conformal derivative (with respect to the curve C) of $T^{i\cdots i}_{j\cdots j}$ with respect to t.*

The conformal derivative at a point is thus dependent not only on the metric of V_n but also on the curve C (or rather, on the second order element of C). This dependence of differentiation on a curve as well as the space is analogous to the similar dependence of parallel displacement of vectors in a general Riemann space. In this respect the conformal derivative also resembles the derivative (with respect to a curve) which has been defined for any Finsler space by Synge[41] and Taylor[42]. Indeed, it is very likely that the results of this paper may be generalized to apply to any Finsler space.

The geometry which is based upon the conformal derivative will appear in a separate paper. We note here a number of fundamental properties of the conformal derivative which are elementary consequences of the preceding remarks and the definition:

(A) *The conformal derivative with respect to a conformal parameter of a conformetric tensor is a conformetric tensor.*

(B) *The conformal derivative of any tensor is a tensor.*

(C) *Conformal differentiation of the sum, difference, inner and outer product of tensors obeys the same rules as ordinary differentiation.*

(D) *The conformal derivative of g_{ij}, g^{ij}, δ^i_j is zero; that is,*

$$(3.15) \qquad \frac{\mathfrak{d}g_{ij}}{\mathfrak{d}t} = \frac{\mathfrak{d}g^{ij}}{\mathfrak{d}t} = \frac{\mathfrak{d}\delta^i_j}{\mathfrak{d}t} = 0.$$

(E) *The conformal derivative (with respect to a curve C) of the unit tangent vector of C is zero, that is,*

$$(3.16) \qquad \frac{\mathfrak{d}\nu^i}{\mathfrak{d}t} = 0.$$

[41] J. L. Synge, *A generalization of the Riemannian line element*, these Transactions, vol. 27 (1925), p. 64.

[42] J. H. Taylor, *A generalization of Levi-Civita's parallelism and the Frenet formulas*, these Transactions, vol. 27 (1925), pp. 255–257.

The conformal derivative (3.14) is with respect to t which is a conformal parameter, that is, corresponding points of C and $\overline{C}$ have the same value of t. The equations of C and $\overline{C}$ are frequently given in terms of metric arc length parameters s and $\bar{s}$. In this case, the conformal derivative of $T^{i_1 \cdots i_w}_{j_1 \cdots j_v}$ with respect to s is given by (3.14) with t replaced by s. It follows that

$$(3.17) \qquad \frac{\mathfrak{d}}{\mathfrak{d}s} T^{i_1 \cdots i_w}_{j_1 \cdots j} = \frac{dt}{ds} \cdot \frac{\mathfrak{d}}{\mathfrak{d}t} T^{i_1 \cdots i_w}_{j_1 \cdots j_v}.$$

Since

$$(3.18) \qquad \frac{dt}{d\bar{s}} = e^{-\sigma} \frac{dt}{ds},$$

dt/ds is a relative conformal scalar and it follows from (3.17) that if $T^{i_1 \cdots i_w}_{j_1 \cdots j_v}$ is a conformetric tensor then $\mathfrak{d}T^{i_1 \cdots i_w}_{j_1 \cdots j_v}/\mathfrak{d}s$ is the product of a relative conformal scalar by a conformetric tensor. The conformal derivative with respect to s has the properties (B), (C), (D), and (E) mentioned above. We note that the existence of a conformal parameter t implies the existence of a relative conformal scalar. For according to (3.18), dt/ds is a relative conformal scalar. Conversely, let Q be a relative conformal scalar which we may assume transforms so that $\overline{Q} = e^{-\sigma}Q$. Then the solution t of the differential equation $dt/ds = Q(s)$ is a conformal parameter.

A simple geometric interpretation of $\mathfrak{d}\lambda^i/\mathfrak{d}t$ is possible in terms of the ideas of projection and ordinary (metric) differentiation. We denote by $N_\phi \lambda^i$ the projection of λ^i in the vector space normal to an arbitrary vector ϕ^i. Then if ψ^i, ψ_i are the unit contravariant and covariant vectors which span the vector space determined by ϕ^i

$$N_\phi \lambda^i = \lambda^i - \lambda^h \psi_h \cdot \psi^i.$$

Let

$$(3.19) \qquad \omega^i = N_\nu \lambda^i$$

where ν^i is the unit tangent to the curve C. It follows that

$$\lambda^i = \omega^i + \alpha \nu^i$$

where

$$(3.20) \qquad \omega^i \nu_i = 0$$

and α is a scalar. Then

$$\frac{\mathfrak{d}\lambda^i}{\mathfrak{d}t} = \frac{\mathfrak{d}\omega^i}{\mathfrak{d}t} + \frac{d\alpha}{dt} \nu^i + \alpha \frac{\mathfrak{d}\nu^i}{\mathfrak{d}t}$$

according to (C). According to (3.11), (3.16), (3.20) and the last equation,

$$\frac{\delta\lambda^i}{\delta t} = \frac{D\omega^i}{Dt} + \mu_h\omega^h\frac{dx^i}{dt} + \frac{d\alpha}{dt}\nu^i.$$

By absolute differentiation of (3.20) with respect to t and the form (2.1) and use of (3.3) and (3.4), we obtain

$$\frac{D\omega^h}{Dt}\nu_h + \omega^h\mu_h\frac{ds}{dt} = 0.$$

Now

$$N_\nu\frac{D\omega^i}{Dt} = \frac{D\omega^i}{Dt} - \frac{D\omega^h}{Dt}\nu_h\cdot\nu^i.$$

It follows from (3.19) and the last three equations that

(3.21) $$\frac{\delta\lambda^i}{\delta t} = N_\nu\frac{D}{Dt}N_\nu\lambda^i + \frac{d\alpha}{dt}\nu^i$$

which is the desired interpretation. In particular, if λ^i is orthogonal to the curve C then $\alpha = 0$ and $N_\nu\lambda^i = \lambda^i$ so that

(3.22) $$\frac{\delta\lambda^i}{\delta t} = N_\nu\frac{D\lambda^i}{Dt}.$$

It follows that if λ^i is orthogonal to C then $\delta\lambda^i/\delta t$ is also orthogonal to C.

It is also possible to define a derivative which has the property that when applied to a conformal vector it yields a conformal vector. If we suppose that λ^i is a conformal vector of class C^1 then

(3.23) $$\bar{\lambda}^i = \lambda^i.$$

We proceed with this equation precisely as with (3.8) and find that

$$\frac{\bar{\delta}\bar{\lambda}^i}{\bar{\delta}t} = \frac{\delta\lambda^i}{\delta t} + \alpha\lambda^i$$

where $\delta\lambda^i/\delta t$ is defined by (3.11) and α is a scalar. It follows from this equation that

(3.24) $$\frac{\mathcal{D}\lambda^i}{\mathcal{D}t_C} = N_\lambda\frac{\delta\lambda^i}{\delta t_C}$$

is a conformal vector. In view of the more complicated structure of $\mathcal{D}\lambda^i/\mathcal{D}t_C$, our work shall be in terms of conformetric vectors and the conformal derivative.

Another type of differentiation for which the derivative of a conformal vector λ^i of class C^1 is a conformal vector may be defined if a nonzero rela-

tive conformal scalar Q of class C^1 exists along the curve. We may assume that the law of transformation of Q is

$$(3.25) \qquad \overline{Q} = e^{-\sigma}Q.$$

Then in virtue of (3.23) and (3.25), $Q\lambda^i$ is a conformetric vector and, in accordance with (A), $\mathfrak{d}(Q\lambda^i)/\mathfrak{d}t$ is also a conformetric vector. This fact and (3.25) show that $\delta\lambda^i/\delta t_{(C,Q)}$ defined by

$$(3.26) \qquad \frac{\delta\lambda^i}{\delta t_{(C,Q)}} = Q^{-1}\frac{\mathfrak{d}Q\lambda^i}{\mathfrak{d}t_C} = \frac{d\log Q}{dt}\cdot\lambda^i + \frac{\mathfrak{d}\lambda^i}{\mathfrak{d}t_C}$$

is a conformal vector. The type of differentiation defined for contravariant vectors by (3.26) may be extended to conformal tensors $T^{i_1\cdots i_w}_{j_1\cdots j_v}$. One readily finds by reasoning similar to that used in the derivation of (3.26) that $\delta T^{i_1\cdots i_w}_{j_1\cdots j_v}/\delta t_{(C,Q)}$ defined by

$$(3.27) \qquad \begin{aligned} \frac{\delta}{\delta t_{(C,Q)}} T^{i_1\cdots i_w}_{j_1\cdots j_v} &= Q^{v-w}\frac{\mathfrak{d}}{\mathfrak{d}t_C}(Q^{w-v}\cdot T^{i_1\cdots i_w}_{j_1\cdots j_v}) \\ &= (w-v)\frac{d\log Q}{dt}\cdot T^{i_1\cdots i_w}_{j_1\cdots j_v} + \frac{\mathfrak{d}}{\mathfrak{d}t_C} T^{i_1\cdots i_w}_{j_1\cdots j_v} \end{aligned}$$

is a conformal tensor. As a consequence of (3.24) and (3.26), we note that

$$(3.28) \qquad \frac{\mathcal{D}\lambda^i}{\mathcal{D}t_C} = N_\lambda\frac{\delta\lambda^i}{\delta t_{(C,Q)}}.$$

While the results of this paper are based upon the $\mathfrak{d}/\mathfrak{d}t$ process, they may be derived equally well using the $\delta/\delta t$ type of differentiation defined by (3.27).

If $T^{i_1\cdots i_w}_{j_1\cdots j_v}$ is any tensor (not necessarily a conformal tensor), we define $\delta T^{i_1\cdots i_w}_{j_1\cdots j_v}/\delta t_{(C,Q)}$ by means of (3.27). Since $\delta T^{i_1\cdots i_w}_{j_1\cdots j_v}/\delta t_{(C,Q)}$ will always be evaluated with respect to the same curve C, we may write $\delta/\delta t_Q$ for $\delta/\delta t_{(C,Q)}$. We easily prove that the properties (B) and (C) hold for the $\delta/\delta t_Q$ type of conformal differentiation and that (A), (D), and (E) are replaced by the analogous statements:

(A') *The $\delta/\delta t_Q$ derivative of a conformal tensor is a conformal tensor.*

(D') *The $\delta/\delta t_Q$ derivative of $Q^2 g_{ij}$, $Q^{-2}g^{ij}$, δ^i_j is zero.*

(E') *The $\delta/\delta t_{(C,Q)}$ derivative of the conformal vector $Q^{-1}v^i$ which corresponds to the unit tangent of C is zero.*

Just as $\mathfrak{d}/\mathfrak{d}t_C$ differentiation may be defined by means of the Γ^i_{jk}, so the definition of $\delta/\delta t_{(C,Q)}$ differentiation may be based upon symmetric coefficients of connection Γ'^i_{jk}. These Γ'^i_{jk} are defined along points of the curve by means of the equations([43])

([43]) The footnote concerning the law of transformation of the Γ^i_{jk} also applies here so that the Γ'^i_{jk} transform like coefficients of connection under coordinate transformations.

$$\Gamma_{jk}^{\prime i} = \left\{ {i \atop jk} \right\} + \left(\mu_j + \nu_j \frac{d \log Q}{ds} \right) \delta_k^i + \left(\mu_k + \nu_k \frac{d \log Q}{ds} \right) \delta_j^i$$

$$- g_{jk} \left(\mu^i + \nu^i \frac{d \log Q}{ds} \right).$$

Then a simple calculation shows that (3.27) is equivalent to

$$\frac{\delta}{\delta t_{(C,Q)}} T_{j_1 \cdots j_.}^{i_1 \cdots i_w} = \frac{d T_{j_1 \cdots j_.}^{i_1 \cdots i_w}}{dt} + \sum_{\alpha=1}^{w} T_{j_1 \cdots j_.}^{i_1 \cdots i_{\alpha-1} h i_{\alpha+1} \cdots i_w} \Gamma_{hk}^{\prime i_\alpha} \frac{dx^k}{dt}$$

$$- \sum_{\beta=1}^{v} T_{j_1 \cdots j_{\beta-1} h j_{\beta+1} \cdots j_.}^{i_1 \cdots i_w} \Gamma_{j_\beta k}^{\prime h} \frac{dx^k}{dt}.$$

The law of transformation of the $\Gamma_{jk}^{\prime i}$ under a conformal transformation of V_n is found by using the definition of $\Gamma_{jk}^{\prime i}$ and equations (2.4), (2.6), (3.1), (3.5), (3.6), (3.7) and (3.25). It is

$$\boldsymbol{\Gamma}_{jk}^{\prime i} = \boldsymbol{\Gamma}_{jk}^{\prime i}.$$

The conformal invariance of the $\Gamma_{jk}^{\prime i}$ provides another proof of (A'). In the same way, it may be shown that if $T_{j_1 \cdots j_.}^{i_1 \cdots i_w}$ is a relative conformal tensor of weight u then

$$\frac{\delta}{\delta t_Q} T_{j_1 \cdots j_.}^{i_1 \cdots i_w} + u \frac{d \log Q}{dt} \cdot T_{j_1 \cdots j_.}^{i_1 \cdots i_w}$$

is also a relative conformal tensor of weight u. We remark that the conformal theory of a general subspace is based upon conformal tensors and a type of "conformal differentiation" analogous to the $\delta/\delta t_{(C,Q)}$ differentiation.

4. **The conformal Frenet equations.** We suppose that the inequalities (3.2) hold throughout this section unless it is stated otherwise. Let $_{(1)}\theta^i(t)$ where t is the conformal parameter defined in the previous section be an arbitrary unit conformetric vector of class C^r ($m - 1 \geq r \geq 1$) defined along C. Then the corresponding vector defined along $\overline{C}$ in $\overline{V}_n$ is given by $_{(1)}\bar{\theta}^i = e^{-\sigma} \, _{(1)}\theta^i$ and must also be a vector of class C^r. We shall derive conformal analogues of the ordinary Frenet equations for the vector $_{(1)}\theta^i$ subject to the assumption that the "normals" in these equations exist and are of class C^1. For the satisfaction of this last assumption it is sufficient but not necessary that[44]

(4.1) $m \geq p \geq n + 1,$ $r \geq n.$

By conformal differentiation of $_{(1)}\theta^i$ with respect to t, we obtain $\delta_{(1)}\theta^i/\delta t$. If $\delta_{(1)}\theta^i/\delta t$ is not identically zero, at points where at least one component does

---

[44] We may weaken these assumptions by replacing n by the integer τ which is defined below.

not vanish we may write

(4.2)
$$\frac{\eth_{(1)}\theta^i}{\eth t} = H_1 \cdot {}_{(2)}\theta^i$$

where $H_1 \neq 0$ and ${}_{(2)}\theta^i$ is a unit vector, that is,

(4.3)
$$g_{ij}\, {}_{(2)}\theta^i\, {}_{(2)}\theta^j = 1.$$

Both H_1 and ${}_{(2)}\theta^i$ are determined except for an initial choice of sign.

By conformal differentiation with respect to t of

(4.4)
$$g_{ij}\, {}_{(1)}\theta^i\, {}_{(1)}\theta^j = 1$$

and use of (3.15) and (4.2), we obtain

(4.5)
$$g_{ij}\, {}_{(1)}\theta^i\, {}_{(2)}\theta^j = 0.$$

We suppose that the class of ${}_{(2)}\theta^i$ is C^1. Then, since ${}_{(2)}\theta^i$ is a unit vector, $\eth\, {}_{(2)}\theta^i/\eth t$ is normal to ${}_{(2)}\theta^i$. If $\eth\, {}_{(2)}\theta^i/\eth t$ is not contained in the linear vector space determined by ${}_{(1)}\theta^i$, it may be written as

(4.6)
$$\frac{\eth_{(2)}\theta^i}{\eth t} = A_1\, {}_{(1)}\theta^i + H_2\, {}_{(3)}\theta^i$$

where ${}_{(3)}\theta^i$ is a unit vector normal to ${}_{(1)}\theta^i$ and ${}_{(2)}\theta^i$ and $H_2 \neq 0$. From (4.4) and (4.6) we find that $A_1 = g_{ij}(\eth\, {}_{(2)}\theta^i/\eth t)\, {}_{(1)}\theta^j$. Now, if we differentiate (4.5) conformally with respect to t and use (3.15), (4.2) and (4.3) we obtain

$$H_1 + g_{ij}\, {}_{(1)}\theta^i \frac{\eth_{(2)}\theta^j}{\eth t} = 0.$$

Hence (4.6) becomes

(4.7)
$$\frac{\eth_{(2)}\theta^i}{\eth t} = -H_1\, {}_{(1)}\theta^i + H_2\, {}_{(3)}\theta^i.$$

Equations (4.2) and (4.7) are the first two analogues of the Frenet equations[45]. We shall prove the general formula by means of the customary proof by mathematical induction. Suppose

(4.8)
$$\frac{\eth_{(\alpha)}\theta^i}{\eth t} = -H_{\alpha-1}\, {}_{(\alpha-1)}\theta^i + H_\alpha\, {}_{(\alpha+1)}\theta^i$$

with the convention[46] $H_0 = 0$ where $\alpha = 1, 2, \cdots, l-1$ and where the vectors ${}_{(\beta)}\theta^i$ satisfy the relations

[45] For the general metric Frenet equations cf. L. P. Eisenhart, *Riemannian Geometry*, 1926, pp. 103–107. Also cf. Duschek-Mayer, loc. cit., pp. 59–62.

[46] If $l = n$, we also make the convention that $H_n = 0$.

(4.9)
$$g_{ij\ (\beta)}\theta^i\ _{(\gamma)}\theta^j = \delta_\gamma^\beta$$

where β, $\gamma = 1, 2, \cdots, l$. If the class of $_{(l)}\theta^i$ is C^1 we shall show that the equations (4.8) and (4.9) hold if the ranges of α, β, γ are each increased by one.

The conformal derivative $\mathfrak{d}\ _{(l)}\theta^i/\mathfrak{d}t$ has a representation of the form

(4.10)
$$\frac{\mathfrak{d}\ _{(l)}\theta^i}{\mathfrak{d}t} = A_1\ _{(1)}\theta^i + \cdots + A_l\ _{(l)}\theta^i + \omega^i$$

where $g_{ij\ (\beta)}\theta^i\omega^j = 0$ $(\beta = 1, 2, \cdots, l)$. As above, we find

$$A_\beta = g_{ij\ (\beta)}\theta^i\frac{\mathfrak{d}\ _{(l)}\theta^i}{\mathfrak{d}t}, \qquad\qquad \beta = 1, 2, \cdots, l.$$

As a result of (4.9), we find that

$$g_{ij}\frac{\mathfrak{d}\ _{(l)}\theta^i}{\mathfrak{d}t}\ _{(\beta)}\theta^j = -g_{ij}\frac{\mathfrak{d}\ _{(\beta)}\theta^i}{\mathfrak{d}t}\ _{(l)}\theta^j.$$

For $\beta = l$, this shows that $A_l = 0$. If we substitute for $\mathfrak{d}\ _{(\beta)}\theta^i/\mathfrak{d}t$ $(\beta = 1, 2, \cdots, l-2)$ from (4.8) and use (4.9), we find that $A_1 = A_2 = \cdots = A_{l-2} = 0$. In the same way, the work for $\beta = l-1$ shows that $A_{l-1} = -H_{l-1}$. Furthermore if ω^i is not identically a zero vector, then at points where at least one of its components does not vanish we may write $\omega^i = H_l\ _{(l+1)}\theta^i$ where $H_l \neq 0$ and $g_{ij\ (l+1)}\theta^i\ _{(l+1)}\theta^j = 1$. Then (4.10) becomes (4.8) with $\alpha = l$ and the lth formula holds. It is clear that (4.9) is also true for the greater range of β, γ.

This process of constructing successive θ^i, H may be continued until we arrive at a vector $_{(\tau)}\theta^i$ (whose class is assumed to be C^1) such that $\mathfrak{d}\ _{(\tau)}\theta^i/\mathfrak{d}t$ is contained in the linear vector space determined by $_{(1)}\theta^i$, $_{(2)}\theta^i$, $\cdots$, $_{(\tau)}\theta^i$. Then $H_\tau = 0$ by definition and (4.8) and (4.9) hold for α, β, $\gamma = 1, 2, \cdots, \tau$. In this case, by definition, $H_{\tau+1} = \cdots = H_n = 0$. We shall sometimes write $_{(\tau+1)}\theta^i, \cdots, _{(n)}\theta^i$ for any vectors which obey (4.9). Of course, $\tau \leq n$. We call (4.8) where the $_{(\alpha)}\theta^i$ obey (4.9) for $\alpha = 1, 2, \cdots, \tau$ the *conformal Frenet equations in V_n for the vector $_{(1)}\theta^i$ and the parameter t.*

If V_n is mapped conformally on $\overline{V}_n$, as already shown, $_{(1)}\bar{\theta}^i$ must exist and have at least one continuous derivative. Hence an equation corresponding to (4.2) in $\overline{V}_n$ exists and may be written as

(4.11)
$$\frac{\overline{\mathfrak{d}}\ _{(1)}\bar{\theta}^i}{\overline{\mathfrak{d}}t} = \overline{H}_1\ _{(2)}\bar{\theta}^i$$

where $_{(2)}\bar{\theta}^i$ is a unit vector. According to §3 (B), $\overline{\mathfrak{d}}\ _{(1)}\bar{\theta}^i/\overline{\mathfrak{d}}t = e^{-\sigma}(\mathfrak{d}\ _{(1)}\theta^i/\mathfrak{d}t)$. This last equation and (4.2) and (4.11) show that $_{(2)}\theta^i$ is a unit conformetric vector, that is, $_{(2)}\bar{\theta}^i = e^{-\sigma}\ _{(2)}\theta^i$ and that H_1 is a conformal scalar, that is, $\overline{H}_1 = H$. It follows that the classes of $_{(2)}\bar{\theta}^i$ and $_{(2)}\theta^i$ are equal.

Since the class of $_{(2)}\theta^i$ was assumed to be C^1, this must also be true of $_{(2)}\bar{\theta}^i$

so that the second conformal Frenet equation in $\overline{V}_n$ is valid. A comparison of this equation with (4.7) using the properties of the conformal derivative readily shows that $_{(3)}\theta^i$ is a unit conformetric vector and H_2 is a conformal scalar and that $_{(3)}\overline{\theta}^i$ and $_{(3)}\theta^i$ are of class C^1.

This method of reasoning may be continued and leads to the following conclusion: *If the conformal Frenet equations in V_n exist then the conformal Frenet equations in $\overline{V}_n$ also exist, that is,*

$$\frac{\overline{\delta}\ _{(\alpha)}\overline{\theta}^i}{\overline{\delta}t} = -\ \overline{H}_{\alpha-1}\ _{(\alpha-1)}\overline{\theta}^i + \overline{H}_\alpha\ _{(\alpha+1)}\overline{\theta}^i, \qquad \overline{H}_0 = \overline{H}_\tau = 0,$$

$$\overline{g}_{ij}\ _{(3)}\overline{\theta}^i\ _{(\gamma)}\overline{\theta}^j = \delta_\gamma^\beta$$

where $\alpha, \beta, \gamma = 1, 2, \cdots, \tau$. In addition the $_{(\alpha)}\theta^i$ and H_α satisfy the equations

(4.12) $$_{(\alpha)}\overline{\theta}^i = e^{-\sigma}\ _{(\alpha)}\theta^i, \qquad \overline{H}_\alpha = H_\alpha.$$

If t is replaced by another conformal parameter t' and we denote the quantities in the corresponding conformal Frenet equations[47] by primes, then $_{(\alpha)}\theta'^i = {}_{(\alpha)}\theta^i$, $H'_\alpha = (dt/dt')\cdot H_\alpha$. Hence the $_{(\alpha)}\theta^i$ are independent of the parametrization and the H_α are multiplied by the same conformal scalar dt/dt'. In the same way if the metric arc length s is used instead of t, one finds that the $_{(\alpha)}\theta^i$ are unchanged and H_α are multiplied by the same relative conformal scalar dt/ds. We call $H_1, H_2, \cdots, H_{\tau-1}$ the *associate conformal curvatures of the vector $_{(1)}\theta^i$ and the parameter t of orders* $1, 2, \cdots, \tau-1$ and say that $_{(2)}\theta^i, {}_{(3)}\theta^i, \cdots, {}_{(\tau)}\theta^i$ are the *associate conformal directions of the vectors $_{(1)}\theta^i$ of orders* $1, 2, \cdots, \tau-1$. We also say that $_{(2)}\theta^i, {}_{(3)}\theta^i, \cdots, {}_{(\tau)}\theta^i$ are obtained from $_{(1)}\theta^i$ by the *Frenet process.*

When the vector $_{(1)}\theta^i$ is normal to C, $H_{n-1}\equiv 0$. For according to the discussion following (3.21) and the conformal Frenet equations, if $H_1H_2 \cdots H_{n-2}\not\equiv 0$ then $_{(1)}\theta^i, {}_{(2)}\theta^i, \cdots, {}_{(n-1)}\theta^i$ are all normal to ν^i. Now, if $H_{n-1}\not\equiv 0$ then $_{(n)}\theta^i$ would also be orthogonal to ν^i. Since this is impossible in virtue of (4.9) it follows that $H_{n-1}\equiv 0$, $_{(n)}\theta^i = \nu^i$. If $H_1H_2 \cdots H_{n-2}\equiv 0$ then $H_{n-1}\equiv 0$ by definition.

The familiar and most useful metric Frenet equations are those obtained when the first vector is the unit tangent ν^i and the parameter is a metric arc length parameter s. In this case, the equations are

(4.13) $$\frac{D\ _{(\alpha)}\nu^i}{Ds} = -\ k_{\alpha-1}\ _{(\alpha-1)}\nu^i + k_\alpha\ _{(\alpha+1)}\nu^i, \qquad \alpha = 1, 2, \cdots, n,$$

where we make the convention that $k_0 = k_n = 0$. The metric invariants $k_1, k_2, \cdots, k_{n-1}$ are the successive (metric) curvatures and $_{(1)}\nu^i\ (=\nu^i)$, $_{(2)}\nu^i, \cdots, {}_{(n)}\nu^i$ are the unit tangent and successive unit (metric) normals which obey the relations

---

[47] These equations exist if the conformal Frenet equations for the parameter t exist.

(4.14) $$g_{ij\ (\alpha)}\nu^i\ {}_{(\beta)}\nu^j = \delta^{\beta}_{\alpha}.$$

The conformal Frenet equations given by (4.8) hold for any first vector ${}_{(1)}\theta^i$ and any parameter t. In order to derive an exact analogue of (4.13), we must indicate a "natural" vector and a "natural" conformal parameter which will play roles analogous to the roles of ν^i and s respectively in the Frenet equations (4.13). We note that the obvious selection for ${}_{(1)}\theta^i$, namely ν^i, is an unfruitful one. For in view of (3.16), the first associate conformal curvature of ν^i vanishes identically.

In order to obtain a solution of this problem, for the remainder of this section we shall replace the assumption stated in (3.2) by the inequalities

(4.15) $$m \geqq p \geqq 3.$$

Then the components of the vectors μ^i and $\bar{\mu}^i$ have continuous first derivatives. If we consider a conformetric vector whose components in $\overline{V}_n$ are $\bar{\mu}^i$ then, according to (3.6), the corresponding components in V_n are $e^{-\sigma}[\mu^i - \sigma_{,h}(g^{ih} - \nu^i\nu^h)]$. It follows from §3 (A) and (3.17) that

$$\frac{\overline{\mathfrak{b}}\bar{\mu}^i}{\overline{\mathfrak{b}}\bar{s}} = e^{-2\sigma}\left[\frac{\mathfrak{b}}{\mathfrak{b}s}\left\{e^{-\sigma}[\mu^i - \sigma_{,h}(g^{ih} - \nu^i\nu^h)]\right\}\right].$$

Since $\bar{\mu}^i$ and μ^i are normal to ν^i, (3.22) applies so that the last equation becomes

(4.16) $$\overline{N}_\nu\frac{\overline{D}\bar{\mu}^i}{\overline{D}\bar{s}} = e^{-3\sigma}\left[N_\nu\frac{D\mu^i}{Ds} - \sigma_{hk}\nu^k(g^{ih} - \nu^i\nu^h)\right]$$

where

(4.17) $$\sigma_{hk} = \sigma_{,hk} - \sigma_{,h}\sigma_{,k}.$$

Now according to the Frenet equations (4.13),

$$\mu^i = k_1\ {}_{(2)}\nu^i, \qquad \frac{D\ {}_{(2)}\nu^i}{Ds} = -k_1\nu^i + k_2\ {}_{(3)}\nu^i.$$

Hence $N_\nu(D\mu^i/Ds) = (dk_1/ds)\ {}_{(2)}\nu^i + k_1k_2\ {}_{(3)}\nu^i$ and an analogous equation obtains for $\overline{N}_\nu(\overline{D}\bar{\mu}^i/\overline{D}\bar{s})$. In virtue of these equations, (4.16) becomes

(4.18) $$\frac{d\bar{k}_1}{d\bar{s}}\ {}_{(2)}\bar{\nu}^i + \bar{k}_1\bar{k}_2\ {}_{(3)}\bar{\nu}^i = e^{-3\sigma}\left[\frac{dk_1}{ds}\ {}_{(2)}\nu^i + k_1k_2\ {}_{(3)}\nu^i - \sigma_{hk}\nu^k(g^{ih} - \nu^i\nu^h)\right].$$

We now find an equivalent expression for σ_{hk} which will permit us to write (4.18) in an invariant form. Under the assumption (4.15), the Riemann curvature tensors R_{hijk} of V_n and $\overline{R}_{hijk}$ of $\overline{V}_n$ exist and are continuous. The Ricci tensor R_{ij} and the invariant curvature R of V_n are defined by $R_{ij} = g^{hk}R_{hijk}$,

$R = g^{ij}R_{ij}$ with analogous definitions for the corresponding tensors in $\overline{V}_n$. A straightforward calculation, using (2.6) and (3.1) as well as the definitions of $\dot{R}_{hijk}$, R_{ij} and R gives[48]

$$e^{-2\sigma}\overline{R}_{hijk} = R_{hijk} + g_{hk}\sigma_{ij} + g_{ij}\sigma_{hk} - g_{hj}\sigma_{ik} - g_{ik}\sigma_{hj}$$
$$+ (g_{hk}g_{ij} - g_{hj}g_{ik})\Delta_1\sigma,$$

(4.19)

and

$$(n-1)(n-2)\sigma_{hk} = (n-1)[\overline{R}_{hk} - R_{hk}] - \tfrac{1}{2}[\bar{g}_{hk}\overline{R} - g_{hk}R]$$
$$- \tfrac{1}{2}(n-1)(n-2)\Delta_1\sigma g_{hk}$$

(4.20)

where $\Delta_1\sigma$ is the differential parameter of the first order defined by $\Delta_1\sigma = g^{ij}\sigma_{,i}\sigma_{,j}$. If[49] $n > 2$, it follows from (2.6), (3.5) and (4.20) that (4.18) may be written as

$$\frac{d\bar{k}_1}{d\bar{s}}\,_{(2)}\bar{v}^i + \bar{k}_1\bar{k}_2\,_{(3)}\bar{v}^i + \frac{1}{n-2}\overline{R}_{hk}\bar{v}^k(\bar{g}^{ih} - \bar{v}^i\bar{v}^h)$$

$$= e^{-3\sigma}\left[\frac{dk_1}{ds}\,_{(2)}\nu^i + k_1k_2\,_{(3)}\nu^i + \frac{1}{n-2}R_{hk}\nu^k(g^{ih} - \nu^i\nu^h)\right].$$

This is the invariant form[50] of the law of transformation of $D\mu^i/Ds$.

We suppose that the members of this equation are not zero and write

(4.21) $$J^2\eta^i = \frac{dk_1}{ds}\,_{(2)}\nu^i + k_1k_2\,_{(3)}\nu^i + \frac{1}{n-2}R_{hk}\nu^k(g^{ih} - \nu^i\nu^h)$$

---

[48] For example, cf. L. P. Eisenhart, loc. cit., pp. 89–90, especially equations (28.5) and (28.9).

[49] The assumption $n > 2$ is to hold throughout §§5–12 inclusive. The case $n = 2$ is considered in §14 and it is shown that the theorems which we obtain in §§5–12 apply in a modified sense.

[50] It is also possible to write the law of transformation of the principal normal given by (3.7) in an invariant form. For, as a consequence of (3.1),

$$n\sigma_{,i} = \left\{\overline{\begin{matrix}i\\ij\end{matrix}}\right\} - \left\{\begin{matrix}i\\ij\end{matrix}\right\}.$$

But

$$\left\{\begin{matrix}i\\ij\end{matrix}\right\} = \frac{\partial}{\partial x^j}\log g^{1/2}$$

where $g = |g_{ik}|$ is the determinant whose elements are the components of the metric tensor. (For a proof of this statement, cf. L. P. Eisenhart, loc. cit., p. 18.) Hence

$$\sigma_{,j} = \frac{\partial}{\partial x^j}\log \bar{g}^{1/2n} - \frac{\partial}{\partial x^j}\log g^{1/2n}.$$

If we substitute this expression for $\sigma_{,j}$ in (3.7), it becomes

$$\bar{\mu}_i + \frac{\partial}{\partial x^k}\log \bar{g}^{1/2n}(\delta_i^k - \bar{v}^k\bar{v}_i) = \mu_i + \frac{\partial}{\partial x^k}\log g^{1/2n}(\delta_i^k - \nu^k\nu_i).$$

Thus a conformal geometric object is defined. As it is not a tensor, a formal theory based upon it would be quite complicated and we shall therefore not consider it further in this paper.

where $g_{ij}\eta^i\eta^j = 1$ and write an analogous equation for $\bar{J}^2\bar{\eta}^i$. It follows that η^i is a unit conformetric vector, that is, the direction of η^i remains unchanged if V_n is subjected to a conformal transformation. We call η^i the *first conformal normal* of the curve C and shall also write it as $_{(1)}\eta^i$. The quantity J which is the unique positive root of J^2 is a relative conformal scalar having the transformation law

$$(4.22) \qquad\qquad\qquad \bar{J} = e^{-\sigma}J.$$

We call J the *relative conformal curvature* of C. As a result of (2.4) and (4.22), S defined (up to an additive constant and choice of sign) by

$$(4.23) \qquad\qquad\qquad S = \int J\,ds$$

remains invariant under conformal transformations and is a conformal scalar. We call S a *conformal arc length parameter* of C. If the values of the same arc length parameter are S_1 and S_2 at two points P_1 and P_2 of C, we call $|S_1 - S_2|$ the *conformal arc length* or *conformal length* of the arc P_1P_2 (or P_2P_1) of C. It is clear that the conformal length is independent of the choice of the conformal arc length parameter. Since S is a conformal scalar, *the conformal length of an arc of a curve is unchanged by any conformal transformation of V_n.* The vector η^i and the parameter S have roles in the conformal theory analogous to those of ν^i and s in the metric theory.

As a consequence of (4.15) and (4.21), the parameter transformation (4.23) is of class C^1 so that we may apply the preceding results of this section with the conformal parameter S replacing t. We note that the inequalities

$$(4.24) \qquad\qquad\qquad m \geqq p \geqq n + 2$$

which are analogous to and here replace (4.1) are sufficient to insure the existence of conformal Frenet equations for the vector η^i and the parameter S. The conditions (4.24) include (4.15). We summarize some of these remarks in the theorem

THEOREM 4.1. *Let C be a curve of class C^p ($p \geqq 3$) in a V_n of class C^m ($m \geqq p$) and dimensionality $n > 2$ and let the relative conformal curvature of C be different from zero. Let the classes*[51] *of the first conformal normal $_{(1)}\eta^i$ and of each successive unit normal $_{(2)}\eta^i$, $_{(3)}\eta^i$, $\cdots$, $_{(\tau)}\eta^i$ obtained from $_{(1)}\eta^i$ by the Frenet process be C^1. Then there exists a set of scalars J_1, J_2, $\cdots$, $J_{\tau-1}$ ($\tau \leqq n - 1$) such that*

$$(4.25) \qquad \frac{\delta\,_{(\alpha)}\eta^i}{\delta S} = -J_{\alpha-1}\,_{(\alpha-1)}\eta^i + J_\alpha\,_{(\alpha+1)}\eta^i, \qquad J_0 = J_\tau = 0;\ \alpha = 1, 2, \cdots, \tau,$$

where S is a conformal arc length parameter. The $_{(\alpha)}\eta^i$ form a normalized τ-bein

---

[51] It is unnecessary to assume the existence of the $_{(\alpha)}\eta^i$. For $_{(1)}\eta^i$ exists and the existence of $_{(\omega)}\eta^i$ if $J_{\omega-1} \neq 0$ follows from the fact that $_{(\omega-1)}\eta^i$ has a derivative.

orthogonal to the tangent vector ν^i

$$g_{ij}\,{}_{(\alpha)}\eta^i\,{}_{(\beta)}\eta^j = \delta_\beta^\alpha, \qquad\qquad \alpha, \beta = 1, 2, \cdots, \tau,$$

$$g_{ij}\,{}_{(\alpha)}\eta^i\,\nu^j = 0.$$

If $C \leftrightarrow \overline{C}$, $V_n \leftrightarrow \overline{V}_n$ by a conformal transformation, then equations analogous to (4.25) hold in $\overline{V}_n$ and at corresponding points of C and $\overline{C}$, the directions of the ${}_{(\alpha)}\eta^i$ correspond

(4.26) $$\qquad\qquad {}_{(\alpha)}\bar\eta^i = e^{-\sigma}\,{}_{(\alpha)}\eta^i \qquad\qquad \alpha = 1, 2, \cdots, \tau,$$

and the J's are equal

$$\overline{J}_\gamma = J_\gamma, \qquad\qquad \gamma = 1, 2, \cdots, \tau - 1.$$

We call (4.25) the *conformal Frenet equations*. In writing these equations we have replaced ${}_{(1)}\theta^i$, ${}_{(2)}\theta^i$, $\cdots$, ${}_{(\tau)}\theta^i$; H_1, H_2, $\cdots$, $H_{\tau-1}$ in (4.8), (4.9) and (4.12) by ${}_{(1)}\eta^i\,(=\eta^i)$, ${}_{(2)}\eta^i$, $\cdots$, ${}_{(\tau)}\eta^i$; J_1, J_2, $\cdots$, $J_{\tau-1}$ which we call the *first, second, $\cdots$, τth conformal normals; first, second, $\cdots$, $(\tau-1)$st conformal curvatures.*

The vector ${}_{(1)}\lambda^i$ defined by

(4.27) $$\qquad\qquad {}_{(1)}\lambda^i = J^{-1}\,{}_{(1)}\eta^i$$

is a conformal vector as follows from (4.22) and (4.26). Systems of conformal Frenet equations which involve only conformal vectors and conformal scalars may be derived corresponding to the initial conformal vector ${}_{(1)}\lambda^i$ by use of the derivatives defined in (3.24) and (3.26). We suppose that the relative conformal scalar Q defined in (3.25) is J. Then one easily finds as a consequence of (3.26), (3.28), (4.25) and (4.27) that

(4.28) $$\frac{\delta\,{}_{(\alpha)}\lambda^i}{\delta S_J} = -J_{\alpha-1}\,{}_{(\alpha-1)}\lambda^i + J_\alpha\,{}_{(\alpha+1)}\lambda^i, \quad J_0 = J_\tau = 0; \alpha = 1, 2, \cdots, \tau,$$

(4.29) $$\frac{\mathcal{D}\,{}_{(\alpha)}\lambda^i}{\mathcal{D}S} = -J_{\alpha-1}\,{}_{(\alpha-1)}\lambda^i + J_\alpha\,{}_{(\alpha+1)}\lambda^i, \quad J_0 = J_\tau = 0; \alpha = 1, 2, \cdots, \tau$$

where the conformal vectors ${}_{(\alpha)}\lambda^i$ correspond to the ${}_{(\alpha)}\eta^i$; that is,

$$ {}_{(\alpha)}\lambda^i = J^{-1}\,{}_{(\alpha)}\eta^i, \qquad {}_{(\alpha)}\overline{\lambda}^i = {}_{(\alpha)}\lambda^i.$$

5. **The $(n-1)$st conformal curvature J_{n-1}.** As a consequence of the remarks in the paragraph preceding (4.13), since η^i is orthogonal to ν^i, a curve can have at most $n-2$ nonzero conformal curvatures arising in the conformal Frenet equations. We shall now construct still another conformal invariant, unrelated to J_1, J_2, $\cdots$, $J_{\tau-1}$ and the conformal Frenet equations, which plays the role of the $(n-1)$st conformal curvature J_{n-1}. This curvature is as essential as the others in the development of the theory. However, the defini-

tion of J_{n-1} differs completely from that of the other conformal curvatures. Indeed, as we shall show later, there are important qualitative differences in the properties of the first $(n-2)$ conformal curvatures and the $(n-1)$st conformal curvature. This is natural because $J_1, J_2, \cdots, J_{n-2}$ are measures of the variation of the conformal normals whereas J_{n-1} is defined in connection with the variation of the relative conformal curvature.

It is assumed that (4.15) holds throughout this section except where it is explicitly replaced by another assumption. If we multiply the respective members of (3.6) and (3.7) by each other and sum,

$$(5.1) \qquad \bar{k}_1^2 = \psi^2 k_1^2 + 2\psi\psi_{,i}\mu^i + \Delta_1\psi - \left(\frac{d\psi}{ds}\right)^2$$

where $\Delta_1\psi = g^{ij}\psi_{,i}\psi_{,j}$ and

$$(5.2) \qquad \psi = e^{-\sigma}$$

and $d\psi/ds = \psi_{,h}\nu^h$. We differentiate the last equation with respect to s and the form (2.1) and use (3.4). This gives

$$\frac{d^2\psi}{ds^2} = \psi_{,hk}\nu^h\nu^k + \psi_{,i}\mu^i.$$

When the value of $\psi_{,i}\mu^i$ from this equation is substituted in (5.1), it becomes

$$(5.3) \qquad \bar{k}_1^2 = \psi^2 k_1^2 + 2\psi\frac{d^2\psi}{ds^2} - \left(\frac{d\psi}{ds}\right)^2 - 2\psi\psi_{,hk}\nu^h\nu^k + \Delta_1\psi.$$

Since, in virtue of (4.17) and (5.2),

$$\sigma_{hk} = -\psi_{,hk}/\psi,$$

(4.20) becomes

$$-2\psi\psi_{,hk} + \Delta_1\psi \cdot g_{hk}$$

$$= 2\psi^2\left[\frac{1}{n-2}(\bar{R}_{hk} - R_{hk}) - \frac{1}{2(n-1)(n-2)}(\bar{g}_{hk}\bar{R} - g_{hk}R)\right].$$

This equation and the fact that $\bar{\nu}^i = \psi\nu^i$ show that (5.3) is equivalent to

$$(5.4) \qquad \bar{k}_1^2 + \bar{K} = \psi^2(k_1^2 + K) + 2\psi\frac{d^2\psi}{ds^2} - \left(\frac{d\psi}{ds}\right)^2,$$

where

$$(5.5) \qquad K = \frac{1}{(n-1)(n-2)}[R - 2(n-1)R_{hk}\nu^h\nu^k],$$

and $\bar{K}$ is defined by an expression constructed from the analogous quantities

in $\overline{V}_n$. If V_n is an S_n of constant curvature K, then $\overline{K}$ is found to be equal to K. If V_n is an Einstein space E_n of constant mean curvature ρ, then $\overline{K} = \rho/(n-1)$.

Now let $Q(s)$ be any nonzero relative conformal scalar of class C^2 which transforms according to the law

$$(5.6) \qquad \overline{Q}(\bar{s}) = \psi Q(s).$$

According to (2.4) and (5.2), $d\bar{s}/ds = \psi^{-1}$. If we use this fact and differentiate (5.6) with respect to s, we obtain

$$\frac{d\psi}{ds} = \left[\frac{d\overline{Q}}{d\bar{s}} \cdot Q^2 - \frac{dQ}{ds} \cdot \overline{Q}^2\right] / Q^2 \overline{Q},$$

$$\psi \frac{d^2\psi}{ds^2} = \left[\frac{d^2\overline{Q}}{d\bar{s}^2} \overline{Q} Q^4 - \frac{d^2Q}{ds^2} Q\overline{Q}^4 - \frac{d\overline{Q}}{d\bar{s}} \frac{dQ}{ds} \overline{Q}^2 Q^2 \right.$$

$$\left. + 2\left(\frac{dQ}{ds}\right)^2 \overline{Q}^4 - \left(\frac{d\overline{Q}}{d\bar{s}}\right)^2 Q^4 \right] / Q^4 \overline{Q}^2.$$

Substitution of these results in (5.4) shows that

$$\left[2\overline{Q} \frac{d^2\overline{Q}}{d\bar{s}^2} - 3\left(\frac{d\overline{Q}}{d\bar{s}}\right)^2 - (\bar{k}_1^2 + \overline{K})\overline{Q}^2\right] / \overline{Q}^4$$

$$= \left[2Q \frac{d^2Q}{ds^2} - 3\left(\frac{dQ}{ds}\right)^2 - (k_1^2 + K)Q^2\right] / Q^4.$$

This expression is equivalent to (5.1) if $Q(s)$ obeys (5.6). It is therefore the invariant form of the law of change of the first curvature of a curve when V_n is subjected to a conformal transformation. We summarize these results in the theorem

THEOREM 5.1. *Let $Q(s)$ be a nonzero relative conformal scalar defined along a curve in a V_n $(n > 2)$ whose law of transformation is $\overline{Q}(\bar{s}) = e^{-\sigma}Q(s)$. Then*

$$(5.7) \qquad \left[2Q \frac{d^2Q}{ds^2} - 3\left(\frac{dQ}{ds}\right)^2 - (k_1^2 + K)Q^2\right] / Q^4$$

is a conformal scalar.

If we replace the assumption (4.15) by the stronger inequalities

$$(5.8) \qquad m \geq p \geq 5,$$

then these conditions are sufficient that $J(s)$ defined by (4.21) and having the transformation law (4.22) be a relative conformal scalar of class C^2. Theorem 5.1 applied to $J(s)$ yields the conformal invariant

$$(5.9) \qquad J_{n-1} = \left[2J \frac{d^2J}{ds^2} - 3\left(\frac{dJ}{ds}\right)^2 - (k_1^2 + K)J^2\right] / J^4.$$

We call J_{n-1} the $(n-1)$st *conformal curvature* of the curve C. If J is referred to a conformal arc length parameter S, we find by use of (4.23) that (5.9) becomes

$$(5.10) \qquad J_{n-1} = \left[2J \frac{d^2 J}{dS^2} - 3 \left(\frac{dJ}{dS} \right)^2 - (k_1^2 + K) \right] / J^2.$$

In a similar way, the conformal scalar (5.7) may be written in terms of Q and its derivatives with respect to S.

   We note that if J_{n-1} exists in a V_n of class C^3 then it follows from the definition of J_{n-1} and (4.22) that $\overline{J}_{n-1}$ must exist in $\overline{V}_n$. This observation plus a similar one implicit in the italicized statement containing (4.12) shows that *if the conformal curvatures $J_1, J_2, \cdots, J_{\tau-1}, J_{n-1}$ exist for a curve C in a V_n of class C^3, then the conformal curvatures also exist for the conformal image curve $\overline{C}$ in $\overline{V}_n$.* Since the conformal arc length of a curve remains invariant under conformal transformations, it is always possible to pick conformal arc length parameters S and $\overline{S}$ on C and $\overline{C}$ so that corresponding points are given by $S = \overline{S}$. Then, according to Theorem 4.1 and the preceding paragraphs, all the conformal curvatures (if they exist) and S are the same at corresponding points of C and $\overline{C}$. As an immediate consequence of this remark, we have the theorem

   THEOREM 5.2. *If $n > 2$ and $V_n \leftrightarrow \overline{V}_n$, $C \leftrightarrow \overline{C}$ by a conformal map, then a conformal arc length parameter S may be chosen so that corresponding points of C and $\overline{C}$ have the same value of S and the conformal curvatures $J_1, J_2, \cdots, J_{n-1}$ are the same functions of S for C and $\overline{C}$.*

   The analogous theorem in the metric theory of a curve is well known.

   6. The existence theorem. In §§4 and 5, it was shown that any curve in V_n which satisfies certain general conditions determines (except for sign) τ ($\leqq n - 1$) continuous conformal curvatures. We now prove the following converse:

   THEOREM 6.1. *Let V_n be any Riemann space of class C^4 and dimensionality $n > 2$. Suppose that*

$$(6.1) \qquad J_1(S), J_2(S), \cdots, J_{\tau-1}(S), J_{n-1}(S), \qquad \tau \leqq n - 1; a < S < b,$$

are any continuous functions no one of which, except possibly J_{n-1} vanishes identically in any subinterval of $a < S < b$. Let x_0^i be the coordinates of any point P of V_n; J_0, L_0 any two real numbers of which the first is positive; $v_0^i, {}_{(1)}\eta_0^i, {}_{(2)}\eta_0^i, \cdots, {}_{(\tau)}\eta_0^i$ any normalized $(\tau+1)$-bein at P; μ_0^i any vector at P orthogonal to v_0^i; and S_0 any real number so that $a < S_0 < b$. Then there exists a curve

$$(6.2) \qquad x^i = x^i(S)$$

of class C^3 defined in some subinterval of $a < S < b$ about S_0 which has S as a

conformal arc length parameter and the functions (6.1) *as conformal curvatures. For* $S = S_0$, *the curve passes through* P *so that its moving conformal* $(\tau+1)$*-bein and principal normal take the positions* v_0^i, $_{(1)}\eta_0^i$, $_{(2)}\eta_0^i$, $\cdots$, $_{(\tau)}\eta_0^i$ *and* μ_0^i *respectively and the values of the relative conformal curvature and its derivative with respect to* S *at* P *are* J_0 *and* L_0 *respectively. Any other curve with these properties will coincide with* (6.2) *in the common interval of definition.*

Briefly, this theorem states that curves exist in V_n whose conformal curvatures are any arbitrary continuous functions of the conformal arc length and that any such curve is uniquely determined by a number of initial conditions. For the proof, we consider the system of $[2+n(\tau+3)]$ differential equations

$$\frac{dx^i}{dS} = J^{-1}v^i,$$

$$\frac{dv^i}{dS} = J^{-1}\left(-\begin{Bmatrix} i \\ jk \end{Bmatrix} v^j v^k + \mu^i\right),$$

$$\frac{d\mu^i}{dS} = J^{-1}\left[-\begin{Bmatrix} i \\ jk \end{Bmatrix} \mu^j v^k - g_{jk}\mu^j\mu^k v^i + J^2\,_{(1)}\eta^i \right.$$
$$\left. - \frac{1}{n-2} R_{jk}v^j(g^{ki} - v^k v^i)\right],$$

$$\frac{d\,_{(1)}\eta^i}{dS} = -J^{-1}\begin{Bmatrix} i \\ jk \end{Bmatrix} \,_{(1)}\eta^j v^k - J^{-1}g_{jk}\,_{(1)}\eta^j\mu^k v^i + J_1(S)\cdot\,_{(2)}\eta^i,$$

(6.3)　$$\frac{d\,_{(2)}\eta^i}{dS} = -J^{-1}\begin{Bmatrix} i \\ jk \end{Bmatrix} \,_{(2)}\eta^j v^k - J^{-1}g_{jk}\,_{(2)}\eta^j\mu^k v^i - J_1(S)\,_{(1)}\eta^i + J_2(S)\,_{(3)}\eta^i,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$\frac{d\,_{(\tau)}\eta^i}{dS} = -J^{-1}\begin{Bmatrix} i \\ jk \end{Bmatrix} \,_{(\tau)}\eta^j v^k - J^{-1}g_{jk}\,_{(\tau)}\eta^j\mu^k v^i - J_{\tau-1}(S)\,_{(\tau-1)}\eta^i,$$

$$\frac{dJ}{dS} = L,$$

$$\frac{dL}{dS} = \frac{J^{-1}}{2}\left[J^2 J_{n-1}(S) + 3L^2 + g_{jk}\mu^j\mu^k \right.$$
$$\left. + \frac{1}{(n-1)(n-2)}\left\{R - 2(n-1)R_{jk}v^j v^k\right\}\right)\right]$$

in the $[2+n(\tau+3)]$ unknowns J, L, x^i, v^i, μ^i $_{(1)}\eta^i$, $_{(2)}\eta^i$, $\cdots$, $_{(\tau)}\eta^i$. This system of equations has been obtained by rewriting equations (3.3), (3.4), (4.21), (4.25) and (5.10) in the normal form for ordinary differential equations using the definitions of the conformal derivative, S and K which are given by (3.11), (4.22) and (5.5) respectively. The right members are continuous for

any x^i in the coordinate system; any ν^i, μ^i, $_{(\alpha)}\eta^i$ $(\alpha = 1, 2, \cdots, \tau)$; any J different from zero; any L; and any S in the interval $a < S < b$. They are of class C^1 in all the dependent variables since the g_{ij} are of class C^3. Therefore we can apply the fundamental existence theorem for such a system of differential equations.

According to this theorem, there exists a set of functions $J(S)$, $L(S)$, $x^i(S)$, $\nu^i(S)$, $\mu^i(S)$, $_{(\alpha)}\eta^i(S)$ of class C^1 satisfying (6.3) which are defined in a sufficiently small interval I of $a < S < b$ about S_0 and which assume the values J_0, L_0, ν_0^i, μ_0^i, $_{(\alpha)}\eta_0^i$ for $S = S_0$. Any two solutions of (6.3) which have the same initial conditions are identical in their common interval of definition. It is readily seen that $J(S)$ is of class C^2. Hence, as a consequence of the special form of the first two equations, $x^i(S)$ are actually of class C^3. The existence theorem applies to a fixed coordinate system. Nevertheless the solutions in one coordinate system will transform under a change of coordinates into the solutions in any other since the differential equations (6.3) may be written in the invariant form, tensor = tensor.

The solutions $x^i(S)$ determine a curve C whose parametric equations are (6.2). We now show that the dependent variables in (6.3) as well as the J's actually have the geometric significance for C that is stated in the theorem. Let

$$A_{\alpha\beta}(S) = g_{ik}\,_{(\alpha)}\eta^i(S)\,_{(\beta)}\eta^k(S),$$

$$A_{00}(S) = g_{ik}\,\nu^i(S)\nu^k(S),$$

$$A_{0\alpha}(S) = g_{ik}\nu^i(S)\,_{(\alpha)}\eta^k(S),$$

$$B(S) = g_{ik}\nu^i(S)\mu^k(S),$$

$$C_{\alpha}(S) = g_{ik}\mu^i(S)\cdot\,_{(\alpha)}\eta^k(S).$$

We prove that these quantities keep their initial values, that is,

(6.4)
$$A_{\alpha\beta}(S) = \delta_\beta^\alpha, \qquad A_{00}(S) = 1,$$
$$A_{0\alpha}(S) = 0, \qquad B(S) = 0$$

all along C. A straightforward calculation using (6.3) shows that

(6.5)
$$\frac{dA_{\alpha\beta}}{dS} = -J^{-1}C_\alpha A_{0\beta} - J_{\alpha-1}A_{(\alpha-1)\beta} + J_\alpha A_{(\alpha+1)\beta}$$
$$\qquad - J^{-1}C_\beta A_{0\alpha} - J_{\beta-1}A_{(\beta-1)\alpha} + J_\beta A_{(\beta+1)\alpha},$$

$$\frac{dA_{00}}{dS} = 2BJ^{-1},$$

$$\frac{dA_{0\alpha}}{dS} = J^{-1}C_\alpha(1 - A_{00}) - J_{\alpha-1}A_{0(\alpha-1)} + J_\alpha A_{0(\alpha+1)};$$

$$\frac{dB}{dS} = J^{-1}g_{ik}\mu^i\mu^k(1 - A_{00}) + JA_{01}.$$

The derivation of these equations is somewhat simplified by noting that $dA_{\alpha\beta}/dS = DA_{\alpha\beta}/DS$, and so on.

In these equations, we think of $A_{\alpha\beta}$, A_{00}, $A_{0\alpha}$, B as independent variables and the other quantities as known functions of S given by the solution of (6.3). Then (6.5) is a system of ordinary differential equations in the normal form, and the right members satisfy the conditions for the existence of a unique solution determined by arbitrary initial values of the A's and B and for S in the interval I on which the solution of (6.3) is defined. It is easy to verify that (6.4) is a solution of (6.5) for S in I. Since the ν_0^i, $_{(\alpha)}\eta_0^i$, μ_0^i were chosen so that (6.4) holds for $S = S_0$, it follows that (6.4) is true for any S belonging to I. Consequently ν^i is orthogonal to μ^i and ν^i, $_{(\alpha)}\eta^i$ form a normalized $(\tau+1)$-bein all along C. Since $A_{00} = 1$, $\nu^i = dx^i/ds$ where s is a metric arc length parameter and $dS = Jds$. It is then readily seen from the second equation in (6.3) that μ^i is the principal (metric) normal, and from the third equation that J and $_{(1)}\eta^i$ are the relative conformal curvature and the first conformal normal of C. After this result, it is immediate that S is a conformal arc length parameter, the J's are the conformal curvatures and the $_{(\alpha)}\eta^i$ are the conformal normals of C. This completes the proof of the theorem.

The set of geometric objects $M = \{x^i, \nu^i, _{(\alpha)}\eta^i, \mu^i, J, L\}$ which together with S comprise the set of initial conditions of the theorem may be defined independently of any curve in V_n. The ν^i, $_{(\alpha)}\eta^i$ form an arbitrary normalized $(\tau+1)$-bein and μ^i is any vector normal to ν^i and J and L are any scalars of which the first is positive. Under any conformal transformation of V_n, the respective objects of M transform according to the laws (2.5), (3.5), (4.26), (3.7), (4.22) and

$$(6.6) \qquad\qquad \bar{L} = e^{-\sigma}(L - \sigma_{,i}\nu^i)$$

respectively. The set $\{x^i, \nu^i, _{(\alpha)}\eta^i, J, L\}$ is called an *M-set*. The *order* of the M-set is the integer $\tau+1$ whose maximum value is n. The role of an M-set of order $\tau+1$ in the conformal theory of a curve is analogous to that of a normalized $(\tau+1)$-bein in the metric theory. If the geometric objects of an M-set have the geometric significance described in Theorem 6.1 for some curve C, then the M-set is said to be *associated* with C at the point whose coordinates belong to M.

One may refer the curve C to a metric arc length parameter s. For (4.23) defines S as a function of s which may be substituted in the equations (6.2) of C. Since J has two continuous derivatives and $x^i(S)$ are of class C^3, C is also of class C^3 when s is the parameter of the curve.

If V_n is an Einstein space E_n of constant mean curvature ρ then since

$$(6.7) \qquad\qquad R_{ij} = -\rho g_{ij}$$

the term $R_{jk}\nu^k(g^{ij} - \nu^i\nu^j)/(n-2)$ is identically zero in the third equation of (6.3). As a result, it is readily seen that in this case the hypothesis of Theo-

rem 6.1 may be weakened so that E_n is a space of class C^3 instead of class C^4 and the remainder of the theorem will hold as stated.

We now show that if V_n is an $\overline{E}_n$, the hypothesis of Theorem 6.1 may be weakened in the same manner as for an E_n. Suppose that $\overline{E}_n$ is of class C^3 and that $\sigma(x^i)$ is a mapping function (necessarily of class C^2) which maps $\overline{E}_n$ conformally on E_n. Let $J_1(S)$, $J_2(S)$, $\cdots$, $J_{r-1}(S)$, $J_{n-1}(S)$ be the preassigned conformal curvatures and let an M-set $\overline{M} \equiv \left\{ \bar{x}_0^i, \bar{\nu}_0^i, {}_{(\alpha)}\bar{\eta}_0^i, \bar{\mu}_0^i, \overline{J}_0, \overline{L}_0 \right\}$ and S_0 be a set of initial conditions of the sort enumerated in Theorem 6.1 and suppose that M, S_0 is the set of quantities in E_n corresponding to these initial conditions under the mapping determined by $\sigma(x^i)$. In accordance with the preceding paragraph the set M, S_0 and the J's determine a curve C of class C^3 in E_n. Geometric objects like those of the set M exist at all points of C and have continuous first derivatives.

Let $\overline{C}$ be the image of C in $\overline{E}_n$. Since $\overline{C}$ is of class C^3, $\bar{\nu}^i$, $\bar{\mu}^i$, ${}_{(1)}\bar{\eta}^i$, $\overline{J}$ all exist. As a result of the existence of $\overline{J}$, (4.22) holds. From (4.22) and the fact that J and σ are each of class C^2, it follows that $\overline{J}$ is also so that $\overline{L}$ exists and is of class C^1. It follows that S is a conformal arc length parameter for $\overline{C}$ as well as for C. Both curves have the same parametric equations in terms of the parameter S.

Since ${}_{(1)}\bar{\eta}^i$ exists, ${}_{(1)}\bar{\eta}^i = e^{-\sigma}{}_{(1)}\eta^i$. From the classes of σ and ${}_{(1)}\eta^i$, we infer that ${}_{(1)}\bar{\eta}^i$ has a continuous derivative so that the first conformal Frenet equation

$$\frac{\overline{\mathfrak{d}}\,{}_{(1)}\bar{\eta}^i}{\overline{\mathfrak{d}}S} = \overline{J}_1\,{}_{(2)}\bar{\eta}^i$$

holds. It follows from Theorem 4.1 that $\overline{J}_1 = J_1$ and ${}_{(2)}\bar{\eta}^i = e^{-\sigma}{}_{(2)}\eta^i$. Hence ${}_{(2)}\bar{\eta}^i$ has a continuous first derivative and we may proceed as before. In this way one shows that all the conformal normals exist and have continuous first derivatives and that the (existent) conformal curvatures of $\overline{C}$ are the preassigned functions J_1, J_2, $\cdots$, J_{r-1}, J_{n-1}. The initial conditions given by $\overline{M}$, S_0 must be satisfied because of the manner in which C was constructed. The unique determination of $\overline{C}$ follows readily from the known uniqueness of C. Thus the conclusions of Theorem 6.1 hold in the case of an $\overline{E}_n$ even if the class of $\overline{E}_n$ is only C^3. In particular, the existence theorem applies when the space is an $\overline{R}_n$ of class C^3.

7. **The conformal equivalence theorem.** The fundamental theorem in the metric theory is the congruence theorem for a curve in a euclidean space or a space of constant curvature: All curves in S_n with equal curvatures $k_\alpha(s)$ are congruent, that is, they may be made to coincide by a motion in S_n. We now develop some minor results leading to the analogous theorem in the conformal theory of a curve in any conformally euclidean space $\overline{R}_n$. The present proof is based upon the existence theorem. Another proof which does not use the results of Theorem 6.1 exists and depends upon the following ideas: Suppose

C_1 and C_2 are curves in two conformally euclidean spaces whose conformal curvatures are the same functions of their conformal arc lengths. Then a mapping is established between C_1 and C_2 so that points with equal values of the conformal arc length parameters correspond. It may then be shown that this mapping may be imbedded in a conformal transformation between the two spaces. This conformal transformation necessarily maps the spaces on each other in such a manner that C_1 and C_2 correspond. The proof just outlined will not be given in this paper.

If an S_n of constant curvature K is mapped conformally on an S_n' of constant curvature K' where S_n and S_n' are spaces of class C^3 and $n > 2$, then one readily finds from (4.19) and (4.20) that the mapping function $\sigma(x^i)$ satisfies the differential equations

$$(7.1) \qquad \sigma_{ij} = -\tfrac{1}{2}\left[e^{2\sigma}K' - K + \Delta_1\sigma\right]g_{ij}.$$

Conversely, if a transformation whose mapping function is a solution of (7.1) is applied to S_n, the image space must be S_n'. Suppose that both S_n and S_n' are euclidean spaces. Then $K = K' = 0$. In this case (7.1) may be written as

$$(7.2) \qquad \psi_{,ij} = \frac{\Delta_1\psi}{2\psi} \cdot g_{ij}$$

where ψ is defined by (5.2). Let the x^i be cartesian rectangular coordinates. Then $g_{ij} = \delta_j^i$ and (7.2) becomes

$$\frac{\partial^2\psi}{\partial x^i\partial x^j} = \frac{\Delta_1\psi}{2\psi}\delta_j^i.$$

The solution of these equations is easily found to be

$$(7.3) \qquad\qquad \psi = a, \qquad\qquad a > 0,$$

or

$$(7.4) \qquad\qquad \psi = \sum_{i=1}^{n} b(x^i - d^i)^2, \qquad\qquad b > 0,$$

where a, b, d^i are real constants. The point given by $x^i = d^i$ is a singular point of any conformal transformation associated with (7.4). It follows that *every mapping function that maps R_n ($n > 2$) on itself conformally must satisfy (7.3) or (7.4) and conversely.* We consider the group of conformal transformations of R_n into itself more fully in the next section. At present, we only prove the following theorem:

THEOREM 7.1. *A conformal transformation of R_n ($n > 2$) into itself exists which transforms any given M-set M into any other given M-set $\overline{M}$ of the same order.*

It is shown in §8 that if the order of M is n then the transformation is uniquely determined. The proof of the theorem follows. Let the sets M and $\overline{M}$ be $\{x^i, \nu^i, {}_{(\alpha)}\eta^i, \mu_i, J, L\}$ and $\{\bar{x}^i, \bar{\nu}^i, {}_{(\alpha)}\bar{\eta}^i, \bar{\mu}_i, \overline{J}, \overline{L}\}$ respectively. We consider two cases.

(1) Suppose $\bar{\mu}_i = \mu_i$ and $\overline{L}/\overline{J} = L/J$ at the point P whose coordinates x^i belong to M. Then a unique positive[52] constant a exists so that $\overline{J} = aJ$, $\overline{L} = aL$. If we subject R_n to the magnification of similitude with center at P determined by $\psi = e^{-\sigma} = a$ it follows from (3.7), (4.22) and (6.6) that $\{\mu_i, J, L\}$ transform into $\{\bar{\mu}_i, \overline{J}, \overline{L}\}$. Suppose that the magnification transforms ν^i, ${}_{(\alpha)}\eta^i$ into ν'^i, ${}_{(\alpha)}\eta'^i$. Then the normalized bein ν'^i, ${}_{(\alpha)}\eta'^i$ at P may be mapped on $\bar{\nu}^i$, ${}_{(\alpha)}\bar{\eta}^i$ at $\overline{P}$ by a motion in R_n where $\overline{P}$ is the point whose coordinates $\bar{x}^i$ belong to $\overline{M}$. Since $\bar{\mu}_i, \overline{J}, \overline{L}$ are metric geometric objects, this motion leaves them unchanged. Hence M is transformed into $\overline{M}$ by means of a magnification of similitude (7.3) followed by a motion.

(2) Suppose at least one of the equations in the hypothesis of case (1) is untrue. We choose cartesian rectangular coordinates in R_n so that P is the origin of coordinates and $\nu^1 = \nu_1 = 1$, $\nu^\gamma = \nu_\gamma = 0$, $(\gamma = 2, 3, \cdots, n)$. It follows that $\mu_1 = \bar{\mu}_1 = 0$. We now determine unique constants b, d^i so that a conformal transformation associated with (7.4) transforms M into $\overline{M}$. At the origin, we find from (7.4) that

$$e^{-\sigma} = b\sum_{j=1}^{n} d^{j^2} \qquad \sigma_{,i} = 2d^i / \sum_j d^{j^2}.$$

If we substitute these values in (3.7), (4.22) and (6.6) and write

$$A_i = (\mu_i - \bar{\mu}_i)/2, \qquad B = \overline{J}/J, \qquad D = (\overline{J}L - J\overline{L})/2J$$

we find

$$A_1 = 0, \qquad d^\gamma = A_\gamma \sum_j d^{j^2}, \qquad \gamma = 2, 3, \cdots, n,$$

$$B = b\sum_j d^{j^2}, \qquad D = bd^1.$$

These equations lead to

$$bB = B^2 \sum_{\gamma=2}^{n} A_\gamma^2 + D^2.$$

Since $J > 0$ and $\overline{J} > 0$, $B > 0$ and the above equation gives a unique solution for b. The value of b obviously cannot be negative and also cannot be zero since in this latter case $A_1 = A_2 = \cdots = A_n = D = 0$ which is impossible according to the hypothesis. Then $d^1 = D/b$ and $d^\gamma = BA_\gamma/b$. The unique numbers $d^1, d^2, \cdots, d^n$ cannot all be zero for the reason just given. Hence the origin is

(52) The positive sign of a and, in case (2) below, of b is due to our definition of J as non-negative. If both positive and negative J's were permitted, the present discussion and the conformal equivalence theorem would be needlessly complicated.

not a singular point of any transformation associated with (7.4). Any such transformation T would transform $\{\mu_i, J, L\}$ into $\{\bar{\mu}_i, \bar{J}, \bar{L}\}$. If it also changes ν^i, $_{(\alpha)}\eta^i$ at P into ν'^i, $_{(\alpha)}\eta'^i$ at P' then the normalized bein ν'^i, $_{(\alpha)}\eta'^i$ at P' may be transformed into $\bar{\nu}^i$, $_{(\alpha)}\bar{\eta}^i$ at $\bar{P}$ by a motion T_1 of R_n. Then $T_1 T$ transforms M into $\bar{M}$. The proof of the theorem is thus completed. A similar theorem exists for any $\bar{R}_n$ and may be proved by mapping $\bar{R}_n$ conformally on R_n and using Theorem 7.1. We may now easily prove the fundamental conformal equivalence theorem:

THEOREM 7.2. *Let $_{(1)}\bar{R}_n$ and $_{(2)}\bar{R}_n$ be conformally euclidean spaces of class C^3 and dimensionality $n > 2$ and let C_1 and C_2 be curves in $_{(1)}R_n$ and $_{(2)}R_n$ respectively whose conformal curvatures are the same functions of the conformal arc length. Then a conformal transformation exists so that $_{(1)}\bar{R}_n \leftrightarrow _{(2)}\bar{R}_n$ and $C_1 \leftrightarrow C_2$.*

In short, this theorem states that curves in $\bar{R}_n$'s whose $J_1, J_2, \cdots, J_{r-1}, J_{n-1}$ are the same functions of S are conformally equivalent. To prove the theorem, we note that conformal transformations T_1 and T_2 exist which map $_{(1)}\bar{R}_n$ and $_{(2)}\bar{R}_n$ respectively on R_n. For these transformations we also have $C_1 \to \bar{C}_1$ and $C_2 \to \bar{C}_2$ respectively where $\bar{C}_1$ and $\bar{C}_2$ are curves in R_n. Since $J_1, J_2, \cdots, J_{r-1}, J_{n-1}$ exist for C_1 and C_2, in accordance with the italicized statement preceding Theorem 5.2, they exist for $\bar{C}_1$ and $\bar{C}_2$ also. As a consequence of Theorem 5.2, the J's for $\bar{C}_1$ and $\bar{C}_2$ are the same functions of their respective conformal arc lengths S.

Let P_1 and P_2 be two points which belong to $\bar{C}_1$ and $\bar{C}_2$ respectively such that the conformal arc length parameters for $\bar{C}_1$ and $\bar{C}_2$ at P_1 and P_2 have the same value S_0. Let M_1 and M_2 be the M-sets associated with $\bar{C}_1$ and $\bar{C}_2$ respectively at P_1 and P_2. As a consequence of Theorem 7.1, a conformal transformation T of R_n into itself exists which transforms M_1 into M_2. This same transformation transforms $\bar{C}_1$ into some curve C' passing through P_2 for $S = S_0$ and having the associated M-set M_2 at P_2. For reasons like those given in connection with $\bar{C}_1$ and $\bar{C}_2$, the J's for C' and $\bar{C}_2$ are the same functions of S. It follows from Theorem 6.1 that C' coincides with $\bar{C}_2$ in a sufficiently small neighborhood of P_2. Hence the conformal transformation $T_2^{-1}TT_1$ transforms $_{(1)}\bar{R}_n$ into $_{(2)}\bar{R}_n$ mapping sufficiently small arcs of C_1 and C_2 on each other. This proves Theorem 7.2. As a result of Theorem 6.1 and Theorem 7.2, the equations

$$J_1 = J_1(S), J_2 = J_2(S), \cdots, J_{r-1} = J_{r-1}(S), J_{n-1} = J_{n-1}(S)$$

may be regarded as conformal intrinsic equations of a curve in $\bar{R}_n$ determining the curve up to a conformal transformation of the space. A detailed conformal geometry of curves could be developed by a study of important particular conformal intrinsic equations.

8. **Groups of conformal transformations in euclidean space R_n and in a conformally euclidean space $\bar{R}_n$.** A euclidean space R_n admits a group G of

conformal transformations of the space. According to the italicized statement preceding Theorem 7.1, the mapping function which is associated with any of these transformations must satisfy (7.3) or (7.4). Suppose that T_1 and T_2 are two transformations of G which are associated with the same ψ. Then T_1 and T_2 induce changes in the metric element of arc ds of R_n which may be written as

$$ds_1 = \psi^{-1}ds, \qquad ds_2 = \psi^{-1}ds$$

respectively. If we write the point transformations as

$$T_1(P) = P_1, \qquad T_2(P) = P_2$$

then the point transformation T_3 defined by $T_3(P_1) = P_2$ is a conformal transformation belonging to G for which the induced change in metric is $ds_1 = ds_2$. Hence T_3 is a euclidean motion and T_2 may be written as $T_2 = T_3T_1$.

If ψ is given by (7.3), one transformation associated with ψ is a magnification of similitude having any point of R_n as center. If ψ is defined by (7.4), one transformation associated with ψ is readily found to be the inversion with respect to the hypersphere whose center is given by $x^i = d^i$ and whose radius r is $b^{-1/2}$. For the equations of this inversion are

$$x'^i = \frac{r^2(x^i - d^i)}{\sum_j (x^j - d^j)^2} + d^i.$$

From these equations, we find that

$$\sum_i dx'^{i^2} = \frac{r^4}{[\sum_j(x^j - d^j)^2]^2} \cdot \sum_i dx^{i^2}$$

which is a conformal transformation of R_n with ψ given by (7.4). These results and the remarks of the first paragraph of this section prove the theorem of Liouville([53]).

THEOREM 8.1. *The most general conformal map of R_n $(n > 2)$ on itself is the product of an inversion with respect to a hypersphere by a motion or the product of a magnification of similitude by a motion.*

Now in the proof of Theorem 7.1, it was noted that the geometric objects $\{\mu_i, J, L\}$ belonging to the two M-sets M and $\overline{M}$ uniquely determine the constants a or b and d^i which define a mapping function associated with a transformation of G. Hence if the orders of M and $\overline{M}$ are each n so that the ν^i, $_{(\alpha)}\eta^i$ form normalized n-biens, it is readily seen in consequence of Theorem 8.1 that the two M-sets determine a *unique* transformation of G.

([53]) For $n > 3$, this theorem was proved by S. Lie, *Über Complexe, insbesondere Linien- und Kugelcomplexe, mit Anwendung auf die Theorie partieller Differentialgleichungen*, Mathematische Annalen, vol. 5 (1872), pp. 145–246. Also cf. L. Bianchi, *Lezioni di Geometria Differenziale*, 2d edition, vol. 1, 1902, pp. 375, 376.

Suppose that the geometric objects in $\overline{M}$ are fixed while those which belong to M range over all admissible values. Then the corresponding conformal transformations range over the totality of transformations belonging to G. Hence the geometric objects of M determine the parameters of G. The normalized n-bein at any point provides $n(n+1)/2$ independent constants, the vector μ^i orthogonal to ν^i contributes $n-1$ additional constants and J, L are two more parameters. As a result the group G has exactly $(n+1)(n+2)/2$ essential parameters.

Let $\overline{R}_n$ $(n>2)$ be a conformally euclidean space and suppose that T_1 is a conformal transformation which transforms $\overline{R}_n$ into R_n. If T is any transformation belonging to G, then the conformal transformation $\overline{T} = T_1^{-1}TT_1$ maps $\overline{R}_n$ on itself. The totality of these transformations $\{\overline{T}\}$ form the complete group $\overline{G}$ of conformal transformations of $\overline{R}_n$ upon itself. The group $\overline{G}$ is the conformal image in $\overline{R}_n$ of the group G in R_n and is obviously independent of the particular mapping T_1. In consequence of the preceding discussion we have the theorem.

THEOREM 8.2. *Every conformally euclidean space $\overline{R}_n$ $(n>2)$ admits a continuous group of conformal transformations on itself having $(n+1)(n+2)/2$ essential parameters.*

Any path in R_n of the group G is a curve C which is described by a point as the latter undergoes the transformations of a one parameter subgroup of G. If P_1 and P_2 are any two points of C, a conformal transformation belonging to this subgroup exists which maps C on itself so that P_1 coincides with P_2. Now it may be shown that if([54]) the relative conformal curvature $J \neq 0$ then the conformal curvatures $J_1, J_2, \cdots, J_{r-1}, J_{n-1}$ of C exist. Hence it follows from Theorem 5.2 that the J's have the same values at P_1 and P_2 and therefore each conformal curvature is constant along the curve.

Conversely, let C be a curve of R_n each of whose conformal curvatures is equal to a constant. In consequence of the conformal equivalence theorem, a transformation of G exists which maps C on itself so that any two given points P_1 and P_2 coincide. These transformations may be chosen (in those cases where they are not already determined) so that they belong to a one parameter subgroup of G. Hence C is a path of the group G.

The paths in $\overline{R}_n$ of the group $\overline{G}$ are readily seen to be the conformal images of the paths in R_n of G. If $\overline{R}_n$ is of class C^3, the italicized statement preceding Theorem 5.2 shows that the conformal curvatures of each path of $\overline{G}$ exist. If the line of reasoning employed in the discussion of the paths of G is now followed, we arrive at the following theorem:

THEOREM 8.3. *The paths whose relative conformal curvatures do not vanish*

([54]) It is easy to prove, using the results of §10, that every curve in $\overline{R}_n$ whose relative conformal curvature vanishes is a path of $\overline{G}$.

of the group of conformal transformations of an $\overline{R}_n$ *($n > 2$) of class* C^3 *upon itself coincide with the curves in* $\overline{R}_n$ *each of whose conformal curvatures is equal to a constant.*

The results of this and the preceding sections obviously apply to a curve in R_n which is subjected to any transformation of the continuous group G. Consequently, at least for curves in R_n, the results of this paper might have been obtained by using the classical methods of Lie and his followers which depend upon the elimination of the parameters of G. Our results would thus constitute the inversive theory of curves when applied to the inversive group G. The present methods based upon the conformal derivative yield considerably simpler proofs than would be obtained by elimination of the parameters of G and the results have greater applicability, being valid for a curve in any V_n.

From the standpoint of Lie theory, this inversive geometry is simply the "natural geometry" of curves in an R_n under the transformations of G. It is analogous to the well known natural geometry of curves in an R_n associated with the metric group. It has been shown by Pick[55] that it is possible to develop a "natural geometry" of curves (at least when the enveloping space is an R_2) for an arbitrary continuous group and this theory has been carried further by Kowalewski[56] and his students using the methods of the classical Lie theory.

9. Conformal differential invariants. A conformal differential invariant H of a curve C in V_n whose equations are $x^i = x^i(t)$ is a scalar function of the variables:

$$(9.1) \qquad x^i(t), \frac{dx^i}{dt}, \frac{d^2x^i}{dt^2}, \cdots, g_{ij}[x^k(t)], \frac{\partial g_{ij}}{\partial x^h}, \frac{\partial^2 g_{ij}}{\partial x^h \partial x^k}, \cdots,$$

defined along C which, at any point P of C, has the same value for any admissible change of coordinates x^i or of the parameter t and whose value also remains unchanged if V_n is mapped conformally on $\overline{V}_n$. This last condition, which gives H its conformal character, is equivalent to the assumption that H is invariant if the g_{ij} and their derivatives are replaced respectively by $e^{2\sigma}g_{ij}$ and their derivatives while the other variables in (9.1) are unchanged.

The simplest conformal differential invariants of C are its conformal curvatures. The most important problem concerning conformal invariants is the discovery of all such invariants. Before answering this question (at least for curves in an $\overline{R}_n$) we discuss the relationship between two apparently distinct processes for constructing new conformal differential invariants from known invariants.

----

[55] G. Pick, loc. cit., p. 139.

[56] G. Kowalewski, loc. cit., chap. 3.

If H is a conformal scalar, then dH/dS, where S is a conformal arc length parameter, is also a conformal invariant. This classical method of constructing new invariants is a direct consequence of the properties of S. In particular, any function of the conformal curvatures and their derivatives with respect to S is a conformal invariant. A different method is apparently provided by Theorem 5.1. For the relative conformal scalar Q defined by

$$(9.2) \qquad Q = J \cdot H$$

obeys (5.6) and it follows from Theorem 5.1 that the scalar

$$(9.3) \qquad q = \left[2Q \frac{d^2Q}{ds^2} - 3\left(\frac{dQ}{ds}\right)^2 - (k_1^2 + K)Q^2 \right]/Q^4$$

with Q defined by (9.2) is a conformal invariant if the derivatives of Q in (9.3) exist. We now show that q may be expressed as a function of J_{n-1} and conformal differential invariants obtained from H by the classical method.

A straightforward calculation using (4.23) shows that

$$(9.4) \qquad \begin{aligned} \frac{dQ}{ds} &= H \cdot \frac{dJ}{ds} + J^2 \frac{dH}{ds}, \\ \frac{d^2Q}{ds^2} &= H \cdot \frac{d^2J}{ds^2} + 3J \frac{dJ}{ds} \frac{dH}{dS} + J^3 \frac{d^2H}{dS^2}. \end{aligned}$$

If we substitute the values given by (9.2) and (9.4) in (9.3) and use (5.9), we find

$$(9.5) \qquad q = \left[H^2 \cdot J_{n-1} - 3\left(\frac{dH}{ds}\right)^2 + 2H \frac{d^2H}{dS^2} \right]/H^4.$$

This discussion shows that the only conformal scalar that one may construct by the method implicit in Theorem 5.1 and which is not obtainable by the classical method is J_{n-1} itself. To obtain J_{n-1}, we simply set $H=1$ in (9.5). Every other q is a function of J_{n-1} and conformal invariants obtained from H by the classical method.

We return to the problem of finding all conformal differential invariants of a curve. Since the value of any conformal scalar H is independent of the parametrization of the curve, we may replace t in (9.1) by a conformal arc length parameter S. In virtue of equations (6.3), the successive derivatives of x^i with respect to S may be written as functions of the conformal curvatures $J_1, J_2, \cdots, J_{\tau-1}, J_{n-1}$ and their derivatives with respect to S and of the g_{ij} and their derivatives with respect to the x^k and of x^i, ν^i, $_{(\alpha)}\eta^i$, μ^i, J, L. Since the principal normal μ^i is orthogonal to ν^i, it may be written as[57]

---

[57] If $\tau < n-1$, we choose $_{(\tau+1)}\eta^i$, $_{(\tau+2)}\eta^i$, $\cdots$, $_{(n-1)}\eta^i$ as any normalized $(n-\tau-1)$-bein orthogonal to ν^i, $_{(1)}\eta^i$, $_{(2)}\eta^i$, $\cdots$, $_{(\tau)}\eta^i$.

$\mu^i = \sum_\alpha a_{\alpha\ (\alpha)} \eta^i$. Hence H is a function $f(\omega_1, \omega_2, \cdots, \omega_r)$ where the ω's are to be replaced by the variables:

$$x^i, \nu^i, {}_{(\alpha)}\eta^i; \ a_\alpha, J, L; \ g_{ij}, \frac{\partial g_{ij}}{\partial x^h}, \frac{\partial^2 g_{ij}}{\partial x^h \partial x^k}, \cdots;$$

$$(9.6) \quad J_1, \frac{dJ_1}{dS}, \frac{d^2J_1}{dS^2}, \cdots; J_2, \frac{dJ_2}{dS}, \frac{d^2J_2}{dS^2}, \cdots; J_{r-1}, \frac{dJ_{r-1}}{dS}, \frac{d^2J_{r-1}}{dS^2}, \cdots;$$

$$J_{n-1}, \frac{dJ_{n-1}}{dS}, \frac{d^2J_{n-1}}{dS^2}, \cdots.$$

Of these variables, the conformal curvatures and their derivatives with respect to S as well as ν^i, ${}_{(\alpha)}\eta^i$ are conformal geometric objects while g_{ij} and their derivatives with respect to x^k and a_α, J, L are metric geometric objects. When the enveloping space of the curve is subjected to a conformal transformation, $\overline{H}$ is the same function $f(\omega_1, \omega_2, \cdots, \omega_r)$ with the ω's replaced by the variables $\bar{x}^i$, $\bar{\nu}^i$, ${}_{(\alpha)}\bar{\eta}^i$, $\bar{a}_\alpha$, and so on, which correspond to (9.6) under the conformal mapping and $\overline{H} = H$ at corresponding points. We now prove the following theorem of which the converse statement has already been demonstrated:

THEOREM 9.1. *The most general conformal differential invariant of a curve in an $\overline{R}_n$ $(n > 2)$ is a function of the conformal curvatures and their derivatives with respect to a conformal arc length parameter. Conversely, every such function is a conformal differential invariant.*

In the proof of the theorem, we consider a conformal differential invariant H defined at any point of a curve in an $\overline{R}_n$. By means of a conformal mapping of $\overline{R}_n$ on R_n, H becomes a conformal scalar defined at any point P of a curve C in R_n. We therefore first discuss invariants of C in R_n. In R_n, the coordinates x^i may be chosen so that they belong to a rectangular cartesian coordinate system U such that $g_{ij} = \delta^t_j$, $\partial g_{ij}/\partial x^h = 0$, $\partial^2 g_{ij}/\partial x^h \partial x^k = 0$, $\cdots$ throughout a region of R_n containing P and at P, $x^i = 0$, $\nu^i = \delta^t_1$, ${}_{(\alpha)}\eta^i = \delta^t_{(\alpha+1)}$. If R_n is subjected to a conformal transformation[58]

$$x^i = x^i(\bar{x}^1, \bar{x}^2, \cdots, \bar{x}^n), \qquad \bar{x}^i = \bar{x}^i(x^1, x^2, \cdots, x^n)$$

belonging to G, C is mapped on another curve $\overline{C}$ in R_n and P corresponds to $\overline{P}$. In this transformation, the $\bar{x}^i$ have been chosen so that they belong to a rectangular cartesian coordinate system $\overline{U}$ such that $\bar{g}_{ij} = \delta^t_j$, $\partial \bar{g}_{ij}/\partial \bar{x}^h = 0$, $\partial^2 \bar{g}_{ij}/\partial \bar{x}^h \partial \bar{x}^k = 0$, $\cdots$ throughout a region of R_n containing $\overline{P}$ and at $\overline{P}$, $\bar{x}^i = 0$, $\bar{\nu}^i = \delta^t_1$, ${}_{(\alpha)}\bar{\eta}^i = \delta^t_{(\alpha+1)}$. It is clear that in the coordinate systems U and $\overline{U}$, the g_{ij} and their derivatives as well as x^i, ν^i, ${}_{(\alpha)}\eta^i$ remain unchanged by con-

---

[58] In this proof, we no longer assume that points with the same coordinates correspond as was done previously.

formal transformations of G. Of course the conformal curvatures and their derivatives with respect to S also remain constant. Hence in these coordinate systems, the only possible variables of H are a_α, J, L.

As a consequence of Theorem 7.1, a conformal transformation of G exists which transforms the set a_α, J, L into any other set $\bar{a}_\alpha$, $\bar{J}$, $\bar{L}$ with $J>0$, $\bar{J}>0$. Hence a_α, J, L behave like independent variables with respect to conformal transformations of G so that if any of the a_α, J, L are effective variables, H cannot remain invariant. This contradiction shows that H is independent of a_α, J and L. Hence, in the coordinate system U, H may be written as

(9.7)
$$H = H\left(J_1, \frac{dJ_1}{dS}, \frac{d^2J_1}{dS^2}, \cdots ; J_2, \frac{dJ_2}{dS}, \cdots ; \right.$$
$$\left. J_{\tau-1}, \frac{dJ_{\tau-1}}{dS}, \cdots ; J_{n-1}, \frac{dJ_{n-1}}{dS}, \cdots \right).$$

Since the values of H as well as of the J's and their derivatives with respect to S do not depend on the coordinate system, (9.7) is valid in every coordinate system. (Or otherwise: The values of the g_{ij} and their derivatives and of x^i, ν^i, $_{(\alpha)}\eta^i$ at P behave like independent variables with respect to coordinate transformations, subject to normality conditions for ν^i, $_{(\alpha)}\eta^i$ while H and the conformal curvatures and their derivatives with respect to S remain constant. Therefore H cannot involve g_{ij}, x^i, ν^i, $_{(\alpha)}\eta^i$ as effective variables.) This observation proves the theorem for curves in R_n and conformal transformations belonging to G. But since H and the J's and their derivatives with respect to S are unchanged by a conformal mapping of R_n on an $\bar{R}_n$, (9.7) is also valid for curves in $\bar{R}_n$. The proof of Theorem 9.1 is thus complete([59]).

It is clear that a similar proof in the metric theory would show that *every metric differential invariant of a curve in an R_n is a function of the metric curvatures and their derivatives with respect to a metric arc length parameter, and conversely*. We note that Theorem 9.1 is not true in general Riemann spaces. For in a space V_n whose dimensionality n exceeds 3, the Weyl conformal curvature tensor C^h_{ijk} is different from zero([60]) if V_n is not an $\bar{R}_n$. Then unit vectors $_{(1)}\theta^i$, $_{(2)}\theta^i$, $_{(3)}\theta^i$, $_{(4)}\theta^i$ any two of which are either identical or mutually orthogonal exist at a point P of V_n such that $C^h_{ijk}\,_{(1)}\theta_h\,_{(2)}\theta^i\,_{(3)}\theta^j\,_{(4)}\theta^k$ is different from zero. Now, in accordance with Theorem 6.1, curves C_1 and C_2 exist in V_n whose conformal curvatures are the same functions of the conformal arc length and which pass through P so that the θ^i's are vectors of their moving conformal $(\tau+1)$-beins and whose relative conformal curvatures J at P are

([59]) A very short non-constructive proof for the analytic case may be based on Theorem 7.2.

([60]) H. Weyl, loc. cit., p. 404, and J. A. Schouten, loc. cit., p. 80.

any two different positive constants a and b. Since C_{ijk}^h is a conformal tensor, it follows readily from (4.22) and (4.26) that H defined by

$$H = J^{-2} C_{ijk}^h {}_{(1)}\theta_h {}_{(2)}\theta^i {}_{(3)}\theta^j {}_{(4)}\theta^k$$

is a conformal differential invariant of C_1 and C_2. Since $a \neq b$, H has different values at P for C_1 and C_2. Since the values of corresponding conformal curvatures and their derivatives with respect to conformal arc length parameters are the same for C_1 and C_2, H cannot be a function of the conformal curvatures and their derivatives with respect to a conformal arc length parameter. A similar example in the metric theory of a curve in a space which is not a space of constant curvature could be constructed by means of the Riemann curvature tensor R_{hijk}.

10. **Conformal null curves.** The results of the preceding sections obviously cannot be applied to a curve C whose relative conformal curvature vanishes identically. In this case the conformal arc length between any two points of C is zero, and conversely. Consequently, we call any curve along which $J \equiv 0$, a *conformal null curve*[61]. As a consequence of (4.21) with $J = 0$, it follows that in any V_n of class C^4 (class C^3 if V_n is conformal to an E_n), a unique conformal null curve is determined by a set of initial values for x^i, ν^i, μ^i. Hence, in contrast to the metric case, real conformal null curves exist in V_n even though its first fundamental form is positive definite. In virtue of (4.22), the conformal image of a conformal null curve is also a conformal null curve. We now derive a number of simple properties of these curves. The first of these is the theorem:

THEOREM 10.1. *A curve C in V_n ($n > 2$) is a conformal null curve if and only only if a Riemann space $\overline{V}_n$ and a conformal transformation of V_n on $\overline{V}_n$ exist such that C is mapped by this transformation on a geodesic of $\overline{V}_n$ which is also a line of principal Ricci curvature of $\overline{V}_n$.*

In the proof, we begin with the fact that a conformal transformation of V_n on some $\overline{V}_n$ exists which maps any curve (not necessarily a conformal null curve) on a geodesic of $\overline{V}_n$. In other words, every curve is conformally geodesic in some $\overline{V}_n$. The proof of this result is simple and will appear in another paper. As a consequence of this proposition, the given conformal null curve C is conformally equivalent to a geodesic $\overline{C}$ of some $\overline{V}_n$. For $\overline{C}$, $\bar{k}_1 = \bar{k}_2 = 0$. Of course the relative conformal curvature $\overline{J}$ of $\overline{C}$ is also zero. It follows from the definition of $\overline{J}^2\bar{\eta}^i$ in the equation analogous to (4.21) that

$$(10.1) \qquad \overline{R}_{hk}\bar{\nu}^k(\bar{g}^{ih} - \bar{\nu}^i\bar{\nu}^h) = 0$$

at each point of C. If we multiply this equation by $\bar{g}_{ij}$ and sum for i,

----

[61] In the following discussion we do not consider the non-real conformal null curves which are solutions of $ds^2 = g_{ij}dx^i dx^j = 0$.

$$(10.2) \qquad [\overline{R}_{jk} - A\overline{g}_{jk}]\overline{\nu}^k = 0$$

where $A = \overline{R}_{jk}\overline{\nu}^j\overline{\nu}^k$. Hence $\overline{\nu}^k$ is a Ricci principal direction[62] of $\overline{V}_n$ so that $\overline{C}$ is a geodesic line of principal Ricci curvature. Conversely, if $\overline{C}$ is a curve of this kind, $\overline{k}_1 = \overline{k}_2 = 0$ and (10.2) holds. In virtue of these equations, (10.1) is true and $\overline{J}^2\overline{\eta}^i = 0$. From this equation it follows that $\overline{J} = 0$. In accordance with (4.22), J must vanish identically on any conformal image C of $\overline{C}$. This completes the proof of the theorem.

If V_n is an E_n, (6.7) holds and $R_{hk}\nu^k(g^{ih} - \nu^i\nu^h) = 0$ for every curve in E_n. Hence, as a consequence of (4.21), $J = 0$ if and only if k_1 is constant and k_2 is zero. In this case, the curve is a geodesic circle of E_n. We state these facts in this theorem:

THEOREM 10.2. *Every conformal null curve of an E_n $(n > 2)$ is a geodesic circle of E_n, and conversely.*

As a result of this theorem, a curve in an $\overline{E}_n$ is a conformal null curve if and only if it is conformal to a geodesic circle in E_n. If the E_n is an R_n, any geodesic circle may be mapped on a straight line of R_n by a conformal transformation belonging to G. This proves the theorem:

THEOREM 10.3. *The necessary and sufficient condition that a curve C in an $\overline{R}_n$ $(n > 2)$ be a conformal null curve is that C be the conformal image of a straight line in R_n.*

As an immediate consequence of the theorem, we note that if C_1 and C_2 are conformal null curves of two conformally euclidean spaces $_{(1)}\overline{R}_n$ and $_{(2)}\overline{R}_n$ respectively then a conformal transformation exists so that $_{(1)}\overline{R}_n \leftrightarrow _{(2)}\overline{R}_n$ and $C_1 \leftrightarrow C_2$. In other words, *all conformal null curves in $\overline{R}_n$'s are conformally equivalent.*

11. Curves in an Einstein space E_n. The defining equation (4.21) for the first conformal normal and the relative conformal curvature of a curve C in V_n $(n > 2)$ becomes

$$(11.1) \qquad J^2\eta^i = \frac{dk_1}{ds}\,_{(2)}\nu^i + k_1k_2\,_{(3)}\nu^i$$

at a point of C whenever the additional equation

$$(11.2) \qquad R_{hk}\nu^k(g^{ih} - \nu^i\nu^h) = 0$$

is satisfied. A line of reasoning similar to that employed in connection with (10.1) and (10.2) shows that (11.2) is the necessary and sufficient condition that ν^i be a Ricci principal direction of V_n. Now the second of the Frenet equations (4.13) is equivalent to

---

[62] L. P. Eisenhart, loc. cit., pp. 113–114.

$$\frac{D\mu^i}{Ds} = -k_1^2 \, {}_{(1)}\nu^i + \frac{dk_1}{ds} \, {}_{(2)}\nu^i + k_1 k_2 \, {}_{(3)}\nu^i$$

where $\mu^i = k_1 \, {}_{(2)}\nu^i$ so that (3.22) shows that

(11.3)
$$\frac{\mathfrak{d}\mu^i}{\mathfrak{d}s} = \frac{dk_1}{ds} \, {}_{(2)}\nu^i + k_1 k_2 \, {}_{(3)}\nu^i.$$

A comparison of (11.1) and (11.3) and the preceding remarks prove the theorem:

THEOREM 11.1. *The necessary and sufficient condition that*

(11.4)
$$J^2\eta^i = \frac{\mathfrak{d}\mu^i}{\mathfrak{d}s}$$

be true at a point of a curve of V_n ($n>2$) is that the tangent to the curve at the point be a Ricci principal direction of V_n.

Of course, if (11.4) is true at all points of C, then C must be a line of principal Ricci curvature of V_n. It follows that (11.4) *is true for all curves of V_n if and only if V_n is an E_n.* We suppose this to be the case in the remainder of the section.

Let C be a curve in E_n. If we find the successive conformal derivatives of (11.1) with respect to s and make use of (3.22), (4.23) the Frenet equations (4.13) and the conformal Frenet equations (4.25) we obtain a series of equations of the form

$$B_1^1 \, {}_{(1)}\eta^i = A_2^1 \, {}_{(2)}\nu^i + A_3^1 \, {}_{(3)}\nu^i,$$

$$B_1^2 \, {}_{(1)}\eta^i + B_2^2 \, {}_{(2)}\eta^i = A_2^2 \, {}_{(2)}\nu^i + A_3^2 \, {}_{(3)}\nu^i + A_4^2 \, {}_{(4)}\nu^i,$$

(11.5)
$$\cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots ,$$

$$B_1^\omega \, {}_{(1)}\eta^i + B_2^\omega \, {}_{(2)}\eta^i + \cdots + B_\omega^\omega \, {}_{(\omega)}\eta^i = A_2^\omega \, {}_{(2)}\nu^i + A_3^\omega \, {}_{(3)}\nu^i + \cdots$$
$$+ A_{\omega+2}^\omega \, {}_{(\omega+2)}\nu^i.$$

The B's are functions of the relative conformal curvature, the conformal curvatures (except J_{n-1}) and their derivatives while the A's are functions of the metric curvatures and their derivatives. Thus

$$B_1^1 = J^2, \qquad B_1^2 = 2J \frac{dJ}{ds}, \qquad B_2^2 = J^3 J_1,$$

and

$$A_2^1 = \frac{dk_1}{ds}, \qquad A_3^1 = k_1 k_2, \qquad A_2^2 = \frac{d^2 k_1}{ds^2} - k_1 k_2^2,$$

$$A_3^2 = 2\frac{dk_1}{ds}k_2 + k_1\frac{dk_2}{ds}, \qquad A_4^2 = k_1 k_2 k_3.$$

By means of equations (11.5), we may write J and the conformal curvatures as functions of the metric curvatures and their derivatives with respect to s. Thus, from the first equation, it follows that

$$J^4 = \left(\frac{dk_1}{ds}\right)^2 + (k_1 k_2)^2.$$

One readily finds, if $\omega \leq n-1$, that

(11.6)
$$B_\omega^\omega = J^{\omega+1} \cdot J_1 J_2 \cdots J_{\omega-1},$$

$$B_\omega^{\omega+1} = \frac{d}{ds} B_\omega^\omega + J J_{\omega-1} B_{\omega-1}^\omega,$$

$$A_{\omega+2}^\omega = k_1 k_2 \cdots k_{\omega+1},$$

$$A_{\omega+1}^\omega = \frac{d}{ds} A_{\omega+1}^{\omega-1} + k_\omega A_\omega^{\omega-1}.$$

We now derive a number of results which interrelate the zeros of the metric and conformal curvatures. The first of these is the theorem[63] which follows

THEOREM 11.2. *If $k_{\tau+1}=0$ and either $k_w=0$ ($w<\tau+1$) or $dk_{\tau+1}/ds=0$ ($0 \leq \tau \leq n-2$) at a point of a curve in E_n ($n>2$) then $J J_1 J_2 \cdots J_\tau = 0$ at this point.*

The proof follows. If $J J_1 J_2 \cdots J_{\tau-1} = 0$ at a point P, the theorem is proved. We consider the case where $J J_1 J_2 \cdots J_{\tau-1} \neq 0$ at P. Suppose that the hypothesis of the theorem is satisfied at P. Then (11.6) with $\omega = \tau, \tau+1$ shows that $A_{\tau+2}^\tau = A_{\tau+2}^{\tau+1} = A_{\tau+3}^{\tau+1} = 0$ at P. Now the assumption $J J_1 J_2 \cdots J_{\tau-1} \neq 0$ at P makes it possible to solve the first τ equations of (11.5) for ${}_{(1)}\eta^i, {}_{(2)}\eta^i, \cdots, {}_{(\tau)}\eta^i$ as vectors in the linear vector space V determined by ${}_{(2)}\nu^i, {}_{(3)}\nu^i, \cdots, {}_{(\tau+1)}\nu^i$. Since ${}_{(1)}\eta^i, {}_{(2)}\eta^i, \cdots, {}_{(\tau)}\eta^i$ are independent vectors, they may serve as a basis for V so that ${}_{(2)}\nu^i, {}_{(3)}\nu^i, \cdots, {}_{(\tau+1)}\nu^i$ may be written as linear combinations of ${}_{(1)}\eta^i, {}_{(2)}\eta^i, \cdots, {}_{(\tau)}\eta^i$. If these solutions for the ν^i's are substituted in the right member of the $(\tau+1)$st equation of (11.5), it is possible to solve for $B_{\tau+1}^{\tau+1} {}_{(\tau+1)}\eta^i$ as a vector in the linear vector space V. Since all the conformal normals are independent, it follows that $B_{\tau+1}^{\tau+1} = J^{\tau+2} J_1 J_2 \cdots J_{\tau-1} J_\tau = 0$ which completes the proof of the theorem.

We now derive a result which is in the nature of a converse of the above theorem.

---

[63] If $\tau=0$, we write $J_0 = J$. In this case, some of the statements in the proofs of §§11 and 12 are either immediate or vacuous. The slight amendments which this necessitates can easily be supplied by the reader.

THEOREM 11.3. *If $J_\tau = 0$ $(0 \leq \tau \leq n-2)$ at a point of a curve in E_n $(n > 2)$ then $k_1 k_2 \cdots k_{\tau+2} = 0$ at this point.*

We first consider the case where $J J_1 J_2 \cdots J_{\tau-1} \neq 0$ at the point P. Then, as above, it is possible to solve the first τ equations of (11.5) for $_{(1)}\eta^i$, $_{(2)}\eta^i$, $\cdots$, $_{(\tau)}\eta^i$ as linear combinations of $_{(2)}\nu^i$, $_{(3)}\nu^i$, $\cdots$, $_{(\tau+1)}\nu^i$, $_{(\tau+2)}\nu^i$. Since $J_\tau = 0$, $B_{\tau+1}^{\tau+1} = 0$ and $_{(\tau+1)}\eta^i$ is absent from the $(\tau+1)$st equation of (11.5). Hence, if we substitute the above solutions for $_{(1)}\eta^i$, $_{(2)}\eta^i$, $\cdots$, $_{(\tau)}\eta^i$ in the left member of the $(\tau+1)$st equation, $A_{\tau+3}^{\tau+1}\,_{(\tau+3)}\nu^i$ is found to lie in the linear vector space determined by $_{(2)}\nu^i$, $_{(3)}\nu^i$, $\cdots$, $_{(\tau+2)}\nu^i$. As a consequence, $A_{\tau+3}^{\tau+1} = k_1 k_2 \cdots k_{\tau+2} = 0$ at P.

We now proceed to the case where $J J_1 J_2 \cdots J_{\tau-1} = 0$ at P. In this case, an $\omega \geq 0$ exists so that $J_\omega = 0$ and $J J_1 J_2 \cdots J_{\omega-1} \neq 0$ at P. The discussion of the preceding paragraph then applies so that $k_1 k_2 \cdots k_{\omega+2} = 0$ at P. The theorem is thus proved in all cases.

As a consequence of the two preceding theorems, we note this theorem:

THEOREM 11.4. *If $k_{\tau+1} \equiv 0$ $(0 \leq \tau \leq n-2)$ along a curve in E_n $(n > 2)$, then $J_\tau \equiv 0$. If $J_\tau \equiv 0$ along a curve in E_n, then $k_{\tau+2} \equiv 0$.*

The proof is immediate. For if $k_{\tau+1} \equiv 0$, then Theorem 11.1 shows that $J J_1 J_2 \cdots J_\tau \equiv 0$. If the J's are continuous, it follows that a J_ω $(\omega \leq \tau)$ exists which vanishes identically along the curve. Then $J_{\omega+1} = J_{\omega+2} = \cdots = J_\tau \equiv 0$. The second statement in the theorem is demonstrated in a similar manner.

We shall not prove but simply note that a theorem analogous to Theorem 12.3 exists in the case of a curve which is contained in a hypersurface E_{n-1} with indeterminate lines of curvature of an enveloping E_n.

12. **Curves in a conformally euclidean space $\overline{R}_n$.** In the metric theory of a curve in an R_n $(n > 2)$, $k_{\tau+1} \equiv 0$ $(\tau \geq 0)$ implies that the curve lies in a $(\tau+1)$-dimensional totally geodesic subspace of R_n. (The same result holds in an S_n.) We denote such a subspace which is called a $(\tau+1)$-plane by $P_{\tau+1}$. Hence $P_{\tau+1}$ is an $R_{\tau+1}$ having zero normal curvature in an enveloping R_n. If R_n is mapped conformally on itself or any other $\overline{R}_n$, then the image of $P_{\tau+1}$ in $\overline{R}_n$ is denoted by $\overline{P}_{\tau+1}$ and is called *a conformal $(\tau+1)$-plane.* We assume that the class of $\overline{R}_n$ is at least C^3. Now it can be shown that a subspace having umbilical points maps into a subspace with umbilical points under any conformal transformation of the enveloping space[64]. If R_n is mapped conformally on itself, then it follows that the $\overline{P}_{\tau+1}$ of R_n must be umbilical and hence are the $(\tau+1)$-dimensional spheres $S_{\tau+1}$ of R_n. This also is a consequence of the fact that every $(\tau+1)$-sphere is equivalent to a $(\tau+1)$-plane by means of a suitable inversion in R_n. Similarly, any $(\tau+1)$-dimensional subspace $V_{\tau+1}$ of an $\overline{R}_n$ having only umbilical points must be a $\overline{P}_{\tau+1}$, and conversely. For if $\overline{R}_n$ is

[64] For example, cf. J. A. Schouten and D. J. Struik, *Einführung in die neueren Methoden der Differentialgeometrie*, vol. 2, 1938, p. 211.

mapped conformally on R_n, V_{r+1} is transformed into an umbilical subspace of R_n, that is, an S_{r+1}. Since S_{r+1} is a $\overline{P}_{r+1}$, V_{r+1} is one also. The converse is established by reversing this argument. Thus the $\overline{P}_{r+1}$ of an $\overline{R}_n$ are in one-to-one correspondence with the $(r+1)$-spheres of R_n. In this section, some relations between the $\overline{P}_{r+1}$ of an $\overline{R}_n$ and the conformal curvature J_r are derived. In particular, we obtain a conformal analogue of the theorem stated in the first sentence of this section.

A $\overline{P}_{r+1}$ in R_n is simply a $(r+1)$-sphere and is determined by $r+3$ points which do not lie in the same r-sphere. By a conformal transformation of R_n on an $\overline{R}_n$, it is readily seen that, in general, a $\overline{P}_{r+1}$ of $\overline{R}_n$ is determined by $r+3$ points of $\overline{R}_n$. The *osculating conformal* $(r+1)$-*plane* at a point P of a curve C in $\overline{R}_n$ is a $\overline{P}_{r+1}$ whose order of contact with C at P is not exceeded by any other $\overline{P}_{r+1}$. Since $\overline{P}_{r+1}$ is determined by $r+3$ points, the order of contact of the osculating $\overline{P}_{r+1}$ with C is not less than $r+2$. If this order of contact exceeds $r+2$, we say that the osculating P_{r+1} *hyperosculates* the curve C.

The discussion of the osculating $\overline{P}_{r+1}$ is considerably simplified by first converting this $\overline{P}_{r+1}$ into a P_{r+1} in R_n. We now proceed to this simplification. Let $\overline{P}_{r+1}$ be the osculating conformal $(r+1)$-plane at a point P of a curve C in $\overline{R}_n$. Then a conformal transformation T exists so that $\overline{R}_n \leftrightarrow R_n$, $\overline{P}_{r+1} \leftrightarrow P_{r+1}$. Suppose $C \leftrightarrow C_1$, $P \leftrightarrow P_1$. Since T is continuous, the order of contact is preserved and P_{r+1} is a $(r+1)$-plane in R_n whose order of contact with C_1 at P_1 is equal to that of $\overline{P}_{r+1}$ with C at P. We choose rectangular cartesian coordinates x^i in R_n so that P_1 is the origin of coordinates and the moving n-bein ${}_{(1)}\nu^i, {}_{(2)}\nu^i, \cdots, {}_{(n)}\nu^i$ of C_1 takes the position

$$(12.1) \qquad {}_{(1)}\nu^i = \delta^i_1,\ {}_{(2)}\nu^i = \delta^i_2, \cdots,\ {}_{(n)}\nu^i = \delta^i_n$$

at P. Since the order of contact of P_{r+1} with C_1 is at least $r+2$, P_{r+1} osculates C_1 (actually P_{r+1} hyperosculates C_1 since the "normal" order of contact of an osculating $(r+1)$-plane with a curve is $r+1$) and hence must contain ${}_{(1)}\nu^i, {}_{(2)}\nu^i, \cdots, {}_{(r+1)}\nu^i$ at P_1[65]. As a consequence of (12.1), the equations of P_{r+1} are

$$(12.2) \qquad \begin{aligned} x^1 &= x^1, \quad x^2 = x^2, \cdots, x^{r+1} = x^{r+1}, \\ x^{r+2} &= 0, \quad x^{r+3} = 0, \cdots, \quad x^n = 0. \end{aligned}$$

If the equations of C are $x^i = x^i(s)$ where s is a metric arc length parameter, then these equations may be written as[66]

---

[65] The well known fact that an osculating P_{r+1} contains the tangent vector and the first r normals at the point of contact need not be assumed as is done here but may readily be demonstrated using equations (12.1), (12.3) and (12.4).

[66] The analyticity of the $x^i(s)$ is not necessary as we may replace (12.3) by a finite series using the extended theorem of the mean.

$$(12.3) \qquad x^i = (x^i)_0 + \left(\frac{dx^i}{ds}\right)_0 \cdot s + \left(\frac{d^2x^i}{ds^2}\right)_0 \frac{s^2}{2!} + \cdots$$

where the subscript zero signifies that the corresponding expression is to be evaluated for $x^i = 0$. Since $dx^i/ds = {}_{(1)}\nu^i$, we find by successive differentiation with respect to s and use of (4.13) that

$$\frac{dx^i}{ds} = \overset{1}{D_1}\,{}_{(1)}\nu^i,$$

$$\frac{d^2x^i}{ds^2} = \overset{2}{D_1}\,{}_{(1)}\nu^i + \overset{2}{D_2}\,{}_{(2)}\nu^i,$$

$$(12.4) \qquad \frac{d^3x^i}{ds^3} = \overset{3}{D_1}\,{}_{(1)}\nu^i + \overset{3}{D_2}\,{}_{(2)}\nu^i + \overset{3}{D_3}\,{}_{(3)}\nu^i,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$\frac{d^\omega x^i}{ds^\omega} = \overset{\omega}{D_1}\,{}_{(1)}\nu^i + \overset{\omega}{D_2}\,{}_{(2)}\nu^i + \cdots + \overset{\omega}{D_\omega}\,{}_{(\omega)}\nu^i.$$

The D's are functions of the metric curvatures and their derivatives. Thus $D_1^1 = 1$, $D_1^2 = 0$, $D_2^2 = k_1$, $D_1^3 = -k_1^2$, $D_2^3 = dk_1/ds$, $D_3^3 = k_1k_2$. It is a simple consequence of (4.13), that if $0 \leq \omega \leq n-2$,

$$D_{\omega+2}^{\omega+2} = k_1k_2 \cdots k_{\omega+1},$$

$$(12.5)$$

$$D_{\omega+1}^{\omega+2} = \frac{d}{ds}\,D_{\omega+1}^{\omega+1} + k_\omega \cdot D_\omega^{\omega+1}.$$

Now by definition, C_1 has contact of order w with $P_{\tau+1}$ at P_1 if the perpendicular distance from a nearby point P_2 on C_1 to $P_{\tau+1}$ is an infinitesimal of order $w+1$ with respect to the infinitesimal arc length P_1P_2. By a comparison of (12.2) and (12.3), it follows that the order of contact of C_1 with $P_{\tau+1}$ is one less than the lowest power of s which occurs in the expansions of $x^{\tau+2}(s)$, $x^{\tau+3}(s)$, $\cdots$, $x^n(s)$ in (12.3). In virtue of (12.1) and (12.4), the equations (12.3) for $i = \tau+2, \tau+3$ are

$$x^{\tau+2} = D_{\tau+2}^{\tau+2} \cdot \frac{s^{\tau+2}}{(\tau+2)!} + D_{\tau+2}^{\tau+3} \cdot \frac{s^{\tau+3}}{(\tau+3)!} + \cdots,$$

$$(12.6)$$

$$x^{\tau+3} = D_{\tau+3}^{\tau+3} \cdot \frac{s^{\tau+3}}{(\tau+3)!} + \cdots,$$

where the D's are to be evaluated at the origin and it is seen that $x^{\tau+4}(s)$, $\cdots$, $x^n(s)$ involve powers of s greater than $\tau+3$. Since the order of contact between C_1 and $P_{\tau+1}$ is at least $\tau+2$,

(12.7)
$$D_{\tau+2}^{\tau+2} = 0$$

is a necessary condition. If $\overline{P}_{\tau+1}$ hyperosculates C, then the order of contact between C_1 and $P_{\tau+1}$ exceeds $\tau+2$, and conversely. Hence

(12.8)
$$D_{\tau+2}^{\tau+3} = 0, \qquad D_{\tau+3}^{\tau+3} = 0$$

are necessary and sufficient conditions[67] that the osculating $\overline{P}_{\tau+1}$ hyperosculate C. The conditions (12.7) and (12.8) are stated in terms of the metric curvatures of C_1. We now find an equivalent statement in terms of the conformal curvatures of C_1 at P_1 and hence indirectly, in terms of the conformal curvatures of C at P.

By comparison of (11.6) and (12.5),

(12.9)
$$D_{\omega+2}^{\omega+2} = A_{\omega+2}^{\omega}, \qquad D_{\omega+1}^{\omega+2} = A_{\omega+1}^{\omega}.$$

According to (11.5) and (12.1), at the origin,

(12.10)
$$_{(\omega)}\eta^1 = 0,$$
$$_{(\omega)}\eta^{\omega+3} = {}_{(\omega)}\eta^{\omega+4} = \cdots = {}_{(\omega)}\eta^n = 0.$$

It follows from (11.5), (12.1), (12.9) and (12.10) that

(12.11)
$$D_{\omega+2}^{\omega+2} = B_\omega^{\omega} {}_{(\omega)}\eta^{\omega+2},$$
$$D_{\omega+2}^{\omega+3} = B_\omega^{\omega+1} {}_{(\omega)}\eta^{\omega+2} + B_{\omega+1}^{\omega+1} {}_{(\omega+1)}\eta^{\omega+2}.$$

As a consequence of (11.6) and (12.11), the necessary condition (12.7) is equivalent to

(12.12)
$$J^{\tau+1}J_1J_2 \cdots J_{\tau-1} {}_{(\tau)}\eta^{\tau+2} = 0,$$

and the necessary and sufficient conditions (12.8) are equivalent to

(12.13)
$$B_\tau^{\tau+1} {}_{(\tau)}\eta^{\tau+2} + J^{\tau+2}J_1J_2 \cdots J_{\tau-1}J_\tau \cdot {}_{(\tau+1)}\eta^{\tau+2} = 0,$$
$$J^{\tau+2}J_1J_2 \cdots J_{\tau-1}J_\tau \cdot {}_{(\tau+1)}\eta^{\tau+3} = 0.$$

Suppose that $JJ_1J_2 \cdots J_{\tau-1} \neq 0$ at P_1. Then (12.12) shows that $_{(\tau)}\eta^{\tau+2} = 0$ at P_1. Then $J_\tau = 0$ is the only solution of (12.13). For, if possible, let J_τ be different from zero at P_1. Then, from (12.13), $_{(\tau+1)}\eta^{\tau+2} = {}_{(\tau+1)}\eta^{\tau+3} = 0$. As a consequence of (12.10), the rank of the vectors $_{(1)}\eta^i, {}_{(2)}\eta^i, \cdots, {}_{(\tau+1)}\eta^i$ is less than $\tau+1$ which is impossible. Hence, if $JJ_1J_2 \cdots J_{\tau-1} \neq 0$ at P_1, (12.13) is equiva-

---

[67] The discussion leading to equations (12.6) did not use the fact that $P_{\tau+1}$ hyperosculates C_1 and hence applies to the osculating $P_{\tau+1}$ of a curve C_1 in all cases. Hence (12.7) is the necessary and sufficient condition that the osculating $P_{\tau+1}$ hyperosculate C_1. If (12.8) holds as well as (12.7), then the order of contact between C_1 and $P_{\tau+1}$ is at least $\tau+3$.

lent to $J_\tau = 0$ at P_1. Since the zeros of $J, J_1, J_2, \cdots, J_\tau$ for C and C_1 coincide, this completes the proof of the theorem:

THEOREM 12.1. *A conformal $(\tau+1)$-plane $(0 \leqq \tau \leqq n-2)$ which osculates a curve in an $\overline{R}_n$ $(n>2)$ at a point where $JJ_1J_2 \cdots J_{\tau-1} \neq 0$ hyperosculates the curve if and only if $J_\tau = 0$ at the point of contact.*

Incidentally, we have shown that if $JJ_1J_2 \cdots J_{\tau-1} \neq 0$ and $J_\tau = 0$ at P then C in $\overline{R}_n$ is conformal to a C_1 in R_n so that (12.7) and (12.8) hold at the corresponding point P_1 of C_1. In virtue of (12.5), the solution of (12.7) and (12.8) is either

$$(12.14) \qquad\qquad k_u = 0, \qquad \frac{dk_u}{ds} = 0, \qquad\qquad u \leqq \tau + 1,$$

or

$$(12.15) \qquad\qquad k_w = 0, \qquad k_u = 0, \qquad\qquad w < u \leqq \tau + 1.$$

The solution (12.14) with $u < \tau + 1$ is impossible under the assumption that $JJ_1J_2 \cdots J_{\tau-1} \neq 0$ at P. For if $u < \tau + 1$, in accordance with Theorem 11.2, $JJ_1J_2 \cdots J_{u-1} = 0$ at P_1 which contradicts the hypothesis. Similarly Theorem 11.2 shows that the solution (12.15) is possible only if $u = \tau + 1$. These remarks prove the following converse of Theorem 11.2 for a curve in a conformally euclidean space:

THEOREM 12.2. *If $JJ_1J_2 \cdots J_{\tau-1} \neq 0$ and $J_\tau = 0$ $(0 \leqq \tau \leqq n-2)$ at a point of a curve C in an $\overline{R}_n$ $(n>2)$, then C is conformally equivalent to a curve in R_n at whose corresponding point $k_{\tau+1} = 0$ and either $dk_{\tau+1}/ds = 0$ or $k_w = 0$ $(w < \tau + 1)$.*

If C is in a $\overline{P}_{\tau+1}$ of $\overline{R}_n$, then this $\overline{P}_{\tau+1}$ hyperosculates C at each of its points so that, in accordance with Theorem 12.2, $JJ_1J_2 \cdots J_{\tau-1} J_\tau = 0$ at each point of C. A proof similar to that of Theorem 11.4 shows that $J_\tau \equiv 0$ along C. Conversely, if $J_\tau \equiv 0$ along C, the osculating $\overline{P}_{\tau+1}$ of C hyperosculates C at the point of contact. However this fact need not imply that C is contained in a $\overline{P}_{\tau+1}$ since the osculating conformal $(\tau+1)$-plane may differ from point to point. We shall show that this conjecture is never actually realized and that $J_\tau \equiv 0$ implies that C lies in a $\overline{P}_{\tau+1}$ of $\overline{R}_n$.

We consider the conformal geometry of a curve C which is in a $P_{\tau+1}$ of R_n. Suppose that the equations of the $P_{\tau+1}$ in R_n which contains the curve C are

$$x^\alpha = y^\alpha, \qquad \alpha = 1, 2, \cdots, \tau + 1, \qquad x^{\tau+2} = 0, \cdots, x^n = 0,$$

where the x^i are rectangular cartesian coordinates for R_n and the y^α are the rectangular cartesian coordinates of a point of $P_{\tau+1}$. Let ϕ^i be the components in the x^i of any vector in the tangent vector space of $P_{\tau+1}$ and ψ^α the corresponding components in the y^α. Then

(12.16) $$\phi^i = \psi^\alpha \cdot x^i_{,\alpha}$$

where the comma denotes covariant differentiation with respect to the y^α and the first fundamental form of P_{r+1}. In particular, the components ν^i in the x's and θ^α in the y's of the unit tangent of C are connected by the equation $\nu^i = \theta^\alpha \cdot x^i_{,\alpha}$.

Let $D\phi^i/Ds$ and $D\psi^\alpha/Ds^*$ denote the absolute derivatives of ϕ^i and ψ^α with respect to the arc length of C and the first fundamental forms of R_n and P_{r+1} respectively. If we find the absolute derivative of both members of (12.16), we obtain

(12.17) $$\frac{D\phi^i}{Ds} = \frac{D\psi^\alpha}{Ds^*} \cdot x^i_{,\alpha} + x^i_{,\alpha\beta}\psi^\alpha \theta^\beta$$

where $x^i_{,\alpha\beta} = \partial^2 x^i/\partial y^\alpha \partial y^\beta$. It is clear from the defining equations of P_{r+1} that the tensor $x^i_{,\alpha\beta}$ satisfies the equation $x^i_{,\alpha\beta} = 0$. Therefore (12.17) becomes

(12.18) $$\frac{D\phi^i}{Ds} = \frac{D\psi^\alpha}{Ds^*} \cdot x^i_{,\alpha}.$$

If we equate the projections in the normal vector space of C of each member of (12.18) and make use of (3.22), we obtain

(12.19) $$\frac{\flat\phi^i}{\flat s} = \frac{\flat\psi^\alpha}{\flat s^*} \cdot x^i_{,\alpha}.$$

Let $\phi^i = \nu^i$, $\psi^\alpha = \theta^\alpha$ in (12.18). Then

$$\mu^i = \lambda^\alpha \cdot x^i_{,\alpha}$$

where μ^i and λ^α are the principal normals of C_1 in R_n and P_{r+1} respectively. As a result of this last equation, we may apply (12.19) with $\phi^i = \mu^i$, $\psi^\alpha = \lambda^\alpha$. If $\tau > 1$, after account is taken of Theorem 11.1, we obtain

(12.20) $$J^2 \eta^i = J^{*2} \eta^{*\alpha} \cdot x^i_{,\alpha}$$

where J^*, $\eta^{*\alpha}$ are geometric objects of P_{r+1} analogous to J, η^i in R_n. A similar notation is used for other conformal geometric objects in P_{r+1}. We refer to geometric objects of P_{r+1} and R_n as *surface* and *space* geometric objects respectively. (We do not pause to prove that it is unnecessary to assume the existence of both the set of surface conformal objects and the set of space conformal objects. The existence of either set implies the existence of the other.)

From (12.20)

(12.21) $$J = J^*,$$

(12.22)
$$\eta^{i} = \eta^{*\alpha} \cdot x^{i}_{,\alpha}.$$

Since the metric arc length parameters s, s^* may be chosen along C so that $s = s^*$, equations (4.23) and (12.21) lead to the conclusion that conformal arc length parameters may be chosen so that

(12.23)
$$S = S^*.$$

Then, from (12.19)

(12.24)
$$\frac{\delta\phi^{i}}{\delta S} = \frac{\delta\psi^{*}}{\delta S^*} \cdot x^{i}_{,\alpha}.$$

As an immediate consequence of the definition (5.10), and (12.21) and (12.23), the last surface conformal curvature J_τ^* and the last space conformal curvature J_{n-1} are equal.

In virtue of (12.22), we may let $\phi^{i} = \eta^{i}$, $\psi^{\alpha} = \eta^{*\alpha}$ in (12.24). If we make use of the conformal Frenet equations, the resulting equation is

(12.25)
$$J_1 \, _{(2)}\eta^{i} = J_1^{*} \, _{(2)}\eta^{*\alpha} \cdot x^{i}_{,\alpha}$$

from which

(12.26)
$$J_1 = J_1^{*}, \qquad _{(2)}\eta^{i} = {}_{(2)}\eta^{*\alpha} \cdot x^{i}_{,\alpha}.$$

We now let $\phi^{i} = {}_{(2)}\eta^{i}$, $\psi^{\alpha} = {}_{(2)}\eta^{*\alpha}$ in (12.24) and simplify the resulting equation by means of (4.25) and (12.25). This gives

$$J_2 = J_2^{*}, \qquad _{(3)}\eta^{i} = {}_{(3)}\eta^{*\alpha} \cdot x^{i}_{,\alpha}.$$

Proceeding in this manner, we find that the successive surface conformal normals and conformal curvatures and the corresponding space conformal normals and conformal curvatures are equal. Since only $\tau - 1$ of the conformal curvatures which occur in the conformal Frenet equations of C as a curve in $P_{\tau+1}$ can be different from zero, it follows that the space conformal curvature J_τ vanishes identically.

It remains to consider the cases $\tau = 0$, 1 which were excluded in the above discussion. If $\tau = 0$, C is a geodesic and $J = 0$. If $\tau = 1$, we write $\lambda^{\alpha} = k_1^{*}\xi^{\alpha}$ where k_1^{*} is the surface first curvature and ξ^{α} is the surface first (metric) normal of C. If we proceed in the same way as above, we obtain

(12.27)
$$J^2\eta^{i} = \frac{dk_1^{*}}{ds^*} \xi^{\alpha} \cdot x^{i}_{,\alpha},$$

$$J_1 = 0$$

instead of (12.20) and (12.25).

Conformal-geometric properties similar to those derived above are enjoyed by any curve C contained in a $\overline{P}_{r+1}$ of an $\overline{R}_n$. For a suitable conformal transformation exists so that $\overline{R}_n \leftrightarrow R_n$, $\overline{P}_{r+1} \leftrightarrow P_{r+1}$, $C \leftrightarrow C_1$. Accordingly, the above discussion applies to C_1 in a P_{r+1} of R_n. However, since the vectors and scalars mentioned in this discussion are the conformal normals, conformal curvatures and conformal arc length, in virtue of Theorem 4.1 and Theorem 5.2, the results apply equally well to the curve C in $\overline{P}_{r+1}$ in $\overline{R}_n$. These remarks complete the proof of the theorem.

THEOREM 12.3. *If a curve C in an $\overline{R}_n$ $(n>2)$ is contained in a conformal $(\tau+1)$-plane $\overline{P}_{r+1}$ $(0 \leq \tau \leq n-2)$ then the τth space conformal curvature J_τ vanishes along C. If $\tau>1$, the conformal arc length, conformal normals and conformal curvatures of C as a curve in the space $\overline{R}_n$ and as a curve in the surface $\overline{P}_{r+1}$ are related by the equations*:

$$S = S^*,$$
$$_{(u)}\eta^i = {}_{(u)}\eta^{*\alpha} \cdot x^i_{,\alpha}, \qquad\qquad u = 1, 2, \cdots, \tau,$$
$$J = J^*$$
$$J_w = J^*_w, \qquad\qquad w = 1, 2, \cdots, \tau-1,$$
$$J_{n-1} = J^*_\tau.$$

It is now a simple matter to prove the converse theorem.

THEOREM 12.4. *If the τth conformal curvature J_τ $(0 \leq \tau \leq n-2)$ of a curve C in an $\overline{R}_n$ $(n>2)$ vanishes identically, then C lies in a conformal $(\tau+1)$-plane of $\overline{R}_n$.*

The proof follows. We first consider the case $\tau>1$. Let

(12.28) $\qquad\qquad J_1(S), J_2(S), \cdots, J_{\tau-1}(S), J_{n-1}(S),$

be the conformal curvatures of C in $\overline{R}_n$. According to Theorem 6.1, a curve C_1 exists in R_{r+1} whose conformal curvatures are the functions (12.28). If the R_{r+1} is imbedded as a P_{r+1} of R_n, C_1 considered as a curve in R_n will have the same functions (12.28) for its conformal curvatures as a result of Theorem 12.3. Then the conformal equivalence theorem shows that a conformal mapping exists so that $\overline{R}_n \leftrightarrow R_n$, $C \leftrightarrow C_1$. It follows that C lies in the conformal image of P_{r+1} which is a $\overline{P}_{r+1}$ of $\overline{R}_n$. This completes the proof for this case.

We now consider the remaining cases, $\tau=0$ and $\tau=1$. If $\tau=0$, C is a conformal null curve and, by Theorem 10.3, C is conformal to a straight line in R_n. Hence C lies in a conformal 1-plane.

If $\tau=1$, the only conformal curvature which is not zero is $J_{n-1}(S)$. The results of §14 make it possible to give a proof for this case in precisely the same manner as was done for $\tau>1$. However, we now indicate an alternative

proof which is independent of §14. Known existence theorems for differential equations permit us to establish the existence of a solution $k_1^*(s^*)$ of the system of equations

$$(12.29) \qquad 2q\frac{d^2q}{ds^{*2}} - 3\left(\frac{dq}{ds^*}\right)^2 - k_1^{*2}q^2 = q^4 \cdot J_{n-1}(Q),$$

$$(12.30) \qquad q^2 = \left|\frac{dk_1^*}{ds^*}\right|,$$

$$(12.31) \qquad Q = \int qds^*.$$

Then a curve C_1 exists in R_2 whose (metric) first curvature and (metric) arc length are k_1^* and s^* respectively. Let R_2 be imbedded as a 2-plane in R_n. According to Theorem 12.3, the only nonzero conformal curvature of C_1 as a curve in R_n is the last conformal curvature. The relative conformal curvature J of C_1 is related to k_1^* by (12.27). Then (12.30) leads to the conclusion that $J = q$. A comparison of (12.31) with (4.23) shows that $Q = S$. Now the space first curvature k_1 equals the surface curvature k_1^*. It follows from (5.9) and (12.29) that the last conformal curvature of C_1 in R_n is $J_{n-1}(S)$. According to Theorem 7.2, a conformal transformation exists so that $\overline{R}_n \leftrightarrow R_n$ $C \leftrightarrow C_1$. Hence C must lie in the conformal image of the 2-plane which contains C_1. This completes the proof of Theorem 12.4.

As an immediate consequence of Theorem 12.4, if $J_\tau = 0$, C is conformal to a curve C_1 which is contained in a $(\tau+1)$-plane of R_n. Hence the $(\tau+1)$st metric curvature of C_1 vanishes. We state this result, which is similar to that of Theorem 11.4, in the theorem:

THEOREM 12.5. *If $J_\tau \equiv 0$ $(0 \leq \tau \leq n-2)$ along a curve C in an $\overline{R}_n$ $(n > 2)$, then C is conformally equivalent to a curve in R_n whose $(\tau+1)$st metric curvature $k_{\tau+1}$ is identically zero.*

Since $\overline{P}_{\tau+1}$ in R_n is a $(\tau+1)$-sphere, we have the following corollary of Theorem 12.3 and Theorem 12.4: *The necessary and sufficient condition that a curve of R_n $(n > 2)$ lie in a $(\tau+1)$-sphere of R_n is that J_τ be identically zero.* For the case $n = 3$, $\tau = 1$, the condition becomes $J_1 = 0$. This condition, when J_1 is evaluated in terms of the metric curvatures of the curve, is known[68], but the classic derivation differs completely from that of the present paper.

13. **Circular conformal transformations.** The characteristic property of the inversive group G defined in R_n is that every circle (including straight lines) of R_n is mapped on a circle under any transformation belonging to G. A generalization to any V_n of the circle in R_n is the geodesic circle defined as the curve whose first (metric) curvature is constant and whose second curva-

---

[68] L. P. Eisenhart, *Differential Geometry*, 1909, p. 36.

ture vanishes identically. It is therefore defined by the equations

$$(13.1) \qquad \frac{dk_1}{ds} = 0, \qquad k_2 = 0.$$

The ordinary (metric) existence theorem for curves proves that a unique geodesic circle is determined by the values of $_{(1)}\nu^i$, $_{(2)}\nu^i$, k_1 at an arbitrary point of V_n. A conformal transformation of V_n on $\overline{V}_n$ which maps the geodesic circles of V_n and $\overline{V}_n$ on each other is called a *circular conformal transformation*. It is a natural generalization of the inversive transformations belonging to G. In a manner similar to that of §2, we may define tensors which have an invariant character with respect to circular conformal transformations. Analogous to the definitions in §2 we may thus define relative circular conformal tensor, circular conformetric tensor and circular conformal tensor.

According to the definition of circular conformal transformations, if (13.1) holds for a curve in V_n then $d\bar{k}_1/d\bar{s} = 0$, $\bar{k}_2 = 0$ is true for the conformal image in $\overline{V}_n$. If these results are substituted in (4.18), we find that the mapping functions σ of the circular conformal transformations coincide with the solutions of the differential equations

$$(13.2) \qquad \sigma_{hk}\nu^k(g^{ih} - \nu^i\nu^h) = 0,$$

where ν^i is an arbitrary unit vector of V_n. Equation (13.2) is analogous to (10.1). A line of reasoning similar to that employed in the discussion following (10.1) shows that ν^i in (13.2) is a principal direction determined by σ_{ij}. Since ν^i is arbitrary, it follows that *the necessary and sufficient condition that σ be the mapping function of a circular conformal transformation of V_n is that it satisfy the equations*

$$(13.3) \qquad \sigma_{ij} = \phi g_{ij}.$$

Thus a V_n admits circular conformal transformations if and only if (13.3) has solutions. In a previous paper([69]), we have shown that a very large class of V_n's actually exist which admit such transformations. In particular, as follows from (4.20), (6.6) and (13.3), *any conformal transformation between Einstein spaces of dimensionality $n > 2$ is circular. Conversely, the conformal image of an Einstein space under a circular conformal transformation is also an Einstein space.* A detailed study of the existence questions concerning conformal transformations between E_n's has been made by Brinkmann([70]). We also note, as was shown at the beginning of §7, *any conformal transformation between spaces of constant curvature of dimensionality $n > 2$ is circular.* The converse of this statement is true even if $n = 2$, that is, *if a circular conformal*

([69]) A. Fialkow, *Conformal geodesics*, loc. cit., §12.

([70]) H. W. Brinkmann, *Einstein spaces which are mapped conformally on each other*, Mathematische Annalen, vol. 94 (1925), pp. 119–145.

*transformation is applied to a space of constant curvature S_n, the conformal image
of S_n is also a space of constant curvature S_n'.* A proof of this statement has already been given in §7 if $n > 2$. The following proof also applies to the case
$n = 2$.

Since $S_n \leftrightarrow R_n$ by a circular conformal transformation, the conformal image
spaces of S_n and R_n under all circular conformal transformations coincide. It
therefore suffices to prove the above italicized statement for an R_n only. Let
the x^i be rectangular cartesian coordinates. Then, if ψ is defined by (5.2), the
first fundamental form of any V_n equivalent to the R_n by a circular conformal
transformation is

$$(13.4) \qquad d\bar{s}^2 = \psi^{-2}\left[dx^{1^2} + dx^{2^2} + \cdots + dx^{n^2}\right]$$

where, as a consequence of (13.3), ψ is a solution of

$$\frac{\partial^2 \psi}{\partial x^i \partial x^j} = - \phi \psi \delta_j^i.$$

The solution of these equations exists only if ϕ may be written in the form

$$\phi = - 2a / \sum_{i=1}^{n} (ax^{i^2} + b^i x^i + c^i)$$

where a, b^i, c^i are real constants and the solution ψ is given by

$$(13.5) \qquad \psi = \sum_{i=1}^{n} (ax^{i^2} + b^i x^i + c^i).$$

A comparison of (13.4) and (13.5) shows that $d\bar{s}^2$ is the first fundamental
form[71] of an S_n whose Riemannian curvature is $\sum_{i=1}^{n}(4ac^i - b^{i^2})$. The italicized statement is thus proved.

Analogous to the present conformal theory of curves, we may develop a
theory based upon circular conformal transformations. In this theory, we
would restrict the conformal mapping functions to solutions of (13.3) and
would consider as the enveloping space of the curve only those Riemann
spaces which admit circular conformal transformations. All the results of the
present theory would also hold with reference to circular conformal transformations. However, a number of new features also present themselves some
of which are now indicated.

As a consequence of (4.18) and (13.3), the vector ξ^i defined by

$$(13.6) \qquad \xi^i = \frac{dk_1}{ds} {}_{(2)}\nu^i + k_1 k_2 {}_{(3)}\nu^i$$

is a relative circular conformal vector defined along the curve C which has the

[71] We are here using a result which is stated in L. P. Eisenhart, *Riemannian Geometry*,
1926, p. 85.

transformation law $\bar{\xi}^i = e^{-3\sigma}\xi^i$. This vector is defined even if $n = 2$. In a manner analogous to that indicated by (4.21), ξ^i determines a relative circular conformal scalar and a unit circular conformetric vector. The conformal Frenet equations for this vector and the circular conformal arc length defined with the aid of the above relative scalar yields a new sequence of "circular conformal normals" and "circular conformal curvatures." If $n > 2$, the vector λ^i defined by

$$\lambda^i = J^2\eta^i - \xi^i = \frac{1}{n-2} R_{hk}\nu^k(g^{ih} - \nu^i\nu^h)$$

is also a relative circular conformal vector with the transformation law $\bar{\lambda}^i = e^{-3\sigma}\lambda^i$. Also, as follows from (4.20) and (13.3), if([72]) $n > 2$ any two orthogonal vectors ω^i and θ^i which are relative circular conformal vectors determine a relative circular conformal scalar Ω defined by

$$\Omega = R_{hk}\omega^h\theta^k.$$

Of course, in an E_n, $J^2\eta^i = \xi^i$ and $\lambda^i = \Omega = 0$. Similar relative circular conformal scalars and tensors could be defined using (4.19) and (13.3).

14. **The two-dimensional case.** The principal interest of circular conformal transformations lies in the possibility of utilizing them in order to develop a conformal geometry of curves in a two-dimensional Riemann space. This possibility is realized in the present section and, except where restrictive assumptions are explicitly stated, the results apply to any V_2 which admits a circular conformal transformation. Obviously the S_2's (including R_2) are such surfaces. According to an italicized statement of the preceding section, circular conformal transformations applied to surfaces of constant curvature map them on surfaces of constant curvature. Hence the theorems of this section include the circular conformal geometry of curves in surfaces of constant curvature under any circular conformal correspondence between these surfaces. In this case, as will be seen, a complete theory including the equivalence theorem is obtained. This circular conformal geometry includes the inversive theory of curves in R_2 as a special case since the transformations considered in the inversive theory are the circular conformal transformations of R_2 on itself.

But there are many V_2's besides the obvious S_2's which admit circular conformal transformations. For it may be shown, using the results of a previous paper([73]), that every V_2 applicable to a surface of revolution and only these V_2's admit circular conformal transformations. The conformal image space of any of these V_2's under a circular conformal map is a surface which is also applicable to a surface of revolution. These remarks indicate the existence of a fairly large class of V_2's other than S_2's to which the present discussion applies.

([72]) If $n = 2$, $\Omega = 0$.

([73]) A. Fialkow, loc. cit., p. 471, equations (12.7) to (12.10) inclusive.

Any vector λ^i defined along a curve C in one of these V_2's may be resolved into tangential and normal components and written as

$$\lambda^i = \alpha\,_{(1)}\nu^i + \beta\,_{(2)}\nu^i.$$

As a result of (3.21) and the Frenet equations (4.13) for C, the conformal derivative of λ^i is defined by

$$(14.1) \qquad \frac{\delta\lambda^i}{\delta t} = \frac{d\alpha}{dt}\,_{(1)}\nu^i + \frac{d\beta}{dt}\,_{(2)}\nu^i.$$

The circular conformal vector ξ^i in V_2 defined by (13.6) becomes

$$J^2\eta^i = \frac{dk}{ds}\,_{(2)}\nu^i$$

where, since no ambiguity is involved, we have written k for the geodesic curvature of C in V_2 instead of the usual k_1 and where similarly to (4.21), we have written $\xi^i = J^2\eta^i$. Hence the *unit circular conformetric normal* is given by

$$(14.2) \qquad \eta^i = \pm\,_{(2)}\nu^i$$

where the algebraic sign agrees with the sign of dk/ds and the *relative circular conformal curvature J* is defined by

$$(14.3) \qquad J = +\left|\frac{dk}{ds}\right|^{1/2}.$$

It follows that J is identically zero if and only if C is a geodesic circle of V_2. We exclude these curves from the present discussion. The quantity J has the transformation law (4.22) and the normal η^i or $_{(2)}\nu^i$ transforms according to

$$(14.4) \qquad _{(2)}\bar\nu^i = e^{-\sigma}_{(2)}\nu^i.$$

Indeed, since the vector space normal to $_{(1)}\nu^i$ is one-dimensional and is a conformal geometric object, $_{(2)}\nu^i$ must be a conformetric vector whose direction remains unchanged under all conformal transformations (even including the non-circular ones). As in the conformal theory, the *circular conformal arc length parameters S* are defined by (4.23) with J determined by (14.3). As a consequence of (4.23) and (14.3), $dS^2 = \pm dk\,ds$ and $JdS = \pm dk$. In virtue of (14.1) and (14.2), the circular conformal Frenet equations become the trivial equation $\delta\eta^i/\delta S = 0$ so that no "curvatures" arise in connection with the Frenet process.

We now define the circular conformal invariant of the curve C in V_2 which is analogous to J_{n-1} in the conformal theory. If we multiply (4.19) by $_{(1)}\nu^h\,_{(2)}\nu^i\,_{(1)}\nu^j\,_{(2)}\nu^k$ and sum using (3.5), (13.3), (14.4) and $g_{ij}\,_{(1)}\nu^i\,_{(2)}\nu^j = 0$, we obtain

$$(14.5) \qquad e^{2\sigma}\overline{K} = K - 2\phi - \Delta_1\sigma$$

where K and $\overline{K}$ are the Gaussian (or Riemann) curvatures of V_2 and of its circular conformal image $\overline{V}_2$. Differentiation of (5.2) shows that $\Delta_1\psi = \psi^2\Delta_1\sigma$. This fact and equations (5.2), (5.3) and (14.5) lead to the conclusion that

$$\bar{k}^2 + \overline{K} = \psi^2(k^2 + K) + 2\psi\frac{d^2\psi}{ds^2} - \left(\frac{d\psi}{ds}\right)^2.$$

If we proceed with this equation as with (5.4), an analogue of Theorem 5.1 is obtained and the *circular conformal curvature* J_1 defined by equation (5.9) with $n = 2$, $\overline{K} = K$ is a circular conformal invariant. According to (14.3),

$$(14.6) \qquad J_1 = \left[4\frac{dk}{ds}\frac{d^3k}{ds^3} - 5\left(\frac{d^2k}{ds^2}\right)^2 - 4(k^2 + K)\left(\frac{dk}{ds}\right)^2\right] \Big/ 4\left|\frac{dk}{ds}\right|^3.$$

The formally simpler quantity H defined by

$$H = \left[4\frac{dk}{ds}\frac{d^3k}{ds^3} - 5\left(\frac{d^2k}{ds^2}\right)^2 - 4(k^2 + K)\left(\frac{dk}{ds}\right)^2\right] \Big/ \left(\frac{dk}{ds}\right)^3$$

is obviously a circular conformal invariant, being equal to $\pm 4J_1$.

The invariant H was first obtained in the special case where $K = \overline{K} = 0$ by Mullins[74] as a differential invariant of a plane curve under the group of inversions of R_2 into itself. He found H in a different form than that given above using the methods of the Lie theory. This invariant was also found by Liebmann[75], Kubota[76], Morley[77], and Patterson[78] in connection with the inversion geometry of the plane. Their methods differed from that of Mullins and the present paper, depending in most cases upon the use of the Schwarzian derivative. Some of these writers referred to S and H as the "inversive length" and "inversive curvature" respectively of a plane curve. The books of Blaschke-Thomsen[79] and Takasu[80] develop the inversive geometry of plane and space curves based upon the use of tetracyclic and pentaspherical coordinates. Recently, Maeda[81] obtained a number of new

[74] G. W. Mullins, *Differential Invariants under the Inversion Group*, Columbia University dissertation, 1917.

[75] Liebmann, *Beiträge zur Inversionsgeometrie der Kurven*, Sitzungberichte der Bayerischen Academie der Wissenschaften Munich, vol. 1 (1923), p. 79. This paper is not accessible to the author.

[76] T. Kubota, *Beiträge zur Inversionsgeometrie*, Tokyo Imperial University Science Reports, vol. 13 (1924–1925), p. 243.

[77] F. Morley, *On differential inversive geometry*, American Journal of Mathematics, vol. 48 (1926), p. 144 and *Inversive Geometry*, 1933, pp. 137–142.

[78] B. C. Patterson, *The differential invariants of inversive geometry*, American Journal of Mathematics, vol. 50 (1928), p. 556.

[79] Blaschke-Thomsen, loc. cit.

[80] T. Takasu, loc. cit.

[81] J. Maeda, *Geometric meanings of the inversion curvature of a plane curve*, Japanese Journal of Mathematics, vol. 16 (1940), pp. 177–232.

geometric interpretations of the inversive curvature of a plane curve by methods which depended upon the use of functions of a complex variable and the Schwarzian derivative.

We now enumerate those theorems of the previous sections for which analogues exist in the circular conformal geometry of V_2. A simpler phrasing of many of these theorems is possible if $n = 2$. In all cases the proof in the two-dimensional case, after trivial modifications, parallels the proof already given with the understanding that $n = 2$ and that S, J, J_1, η^i have the significance indicated in this section and that the word "conformal" is to be replaced by the words "circular conformal." In particular, we note that the expression $R_{jk}\nu^k(g^{ij} - \nu^i\nu^j)/(n-2)$ is to be omitted wherever it appears in a proof. After these conventions, we list the following theorems (besides those already mentioned) as having two-dimensional analogues for all surfaces applicable to a surface of revolution: Theorems 5.2, 6.1, 10.2. In Theorem 6.1, the class of V_2 may be C^3 instead of C^4. In Theorem 10.2, the condition that V_2 be an E_2 is universally true and should be omitted. The following theorems have analogues only if V_2 is an S_2: Theorems 7.1, 7.2, 8.1, 8.2, 8.3, 9.1, 10.3. The theorems of §11 are trivially true for curves in V_2 and the theorems of §12 are trivially true for curves in S_2. We note however as a consequence of (12.27), (14.3) and (5.9) with $n = 2$ (or the equivalent equation (14.6)) that Theorem 12.3 has the following analogue if $\tau = 1$ (using the notation of the theorem): *If a curve C in an R_n (or S_n) is contained in a 2-sphere $\overline{P}_2$ of R_n (or S_n) then the first space conformal curvature J_1 vanishes along C and the following equations hold*:

$$S = S^*, \qquad J = J^*, \qquad J_{n-1} = J_1^*$$

where S^*, J^*, J_1^* are the circular conformal arc length, relative curvature and curvature respectively of C in $\overline{P}_2$. This result may now be used to prove Theorem 12.4 for the case $\tau = 1$.

The equivalence theorem for curves in the inversion geometry of the plane was proved in a different form by Kubota and stated in the present form by Morley and Patterson. The plane curves along which H is a constant were studied by Mullins who showed that they are the inversive images of the logarithmic spirals. According to the two-dimensional analogue of Theorem 8.3, they are the paths of the inversive group.

In this section, we have developed the circular conformal theory of curves in a V_2. While a curve has circular conformal invariants, it cannot have any conformal invariants since every analytic curve in a V_2 is conformally equivalent to a straight line in R_2. However the horn angle formed by two tangent curves C_1 and C_2 in a V_2 does have a conformal invariant which we now derive. Since $n = 2$, $k_2 = 0$ so that, as before, we may write k for k_1. Also

(14.7) $$g^{ij} = {}_{(1)}\nu^i \, {}_{(1)}\nu^j + {}_{(2)}\nu^i \, {}_{(2)}\nu^j.$$

If we use (4.13), (14.4) and (14.7), then (3.6) is equivalent to

(14.8) $\bar{k} = e^{-\sigma}(k - \sigma_{,h}\nu^h)$.

Similarly (4.18) becomes, after account is taken of (14.4) and (14.7)

(14.9) $\dfrac{d\bar{k}}{d\bar{s}} = e^{-2\sigma}\left(\dfrac{dk}{ds} - \sigma_{hk\ (2)}\nu^h\ _{(1)}\nu^k\right)$.

It follows from (14.8) and (14.9) that

$$\left[\frac{dk_{(1)}}{ds_{(1)}} - \frac{dk_{(2)}}{ds_{(2)}}\right]\Big/ [k_{(1)} - k_{(2)}]^2,$$

where the subscripts (1) and (2) refer to the geometric objects of C_1 and C_2 respectively, is a conformal invariant of the horn angle. It is the measure of the horn angle discovered by Kasner[82] for the plane case who also proved that it is sufficient for a conformal characterization of the horn angle (except possibly for invariants of infinite order). Kasner's results have been extended to any two-dimensional Riemann space by Comenetz[83]. A very detailed geometry of horn angles based upon the above measure has been developed by Kasner[84]. We note that the method of this paper may be utilized to obtain conformal invariants of a horn angle in any V_n $(n > 2)$. However the results are not as significant as in the two-dimensional case since each of the constituent curves of the horn angle has conformal properties of $n > 2$.

15. **Curves in a conformal Riemann space V_n.** In this section, we show how our previous results may be used to develop a theory of curves in a V_n which is based upon the tensor $g_{ij}/g^{1/n}$. Following T. Y. Thomas, we define the conformal Riemann space V_n of class C^m as the space whose coordinate manifold is of class C^m and whose fundamental geometric object G_{ij}, defined over the manifold, is of class C^{m-1}. The tensor G_{ij} is a symmetric, positive definite relative tensor of weight $-2/n$ with respect to coordinate transformations, that is,

(15.1) $G'_{ij} = \Delta^{-2/n}G_{hk}\dfrac{\partial x^h}{\partial x'^i}\dfrac{\partial x^k}{\partial x'^i}$

where Δ is the Jacobian $\left|\partial x^i/\partial x'^i\right|$ of the transformation. It follows from (15.1) that $G = \left|G_{ij}\right|$ is a scalar, that is,

[82] E. Kasner, *Conformal geometry*, Proceedings of the Fifth International Congress of Mathematicians, vol. 2 (1912), p. 81.

[83] G. Comenetz, *Conformal geometry on a surface*, Annals of Mathematics, (2), vol. 39 (1938), pp. 863–871.

[84] E. Kasner, *Trihornometry: A new chapter of conformal geometry*, Proceedings of the National Academy of Sciences, vol. 23 (1937), pp. 337–341 and *Fundamental theorems of trihornometry*, Science, vol. 85 (1937), pp. 480–482.

(15.2) $$G' = G.$$

For reasons stated in §2, we restrict ourselves to a portion of V_n which is a neighborhood U of a point coverable by a single coordinate system $\{x^i\}$. We shall continue to refer to U as the conformal Riemann space V_n. Let F denote the set of all positive functions Ω of class C^{m-1} defined over V_n which are relative scalars with respect to coordinate transformations having the transformation law

(15.3) $$\Omega' = |\Delta|^{1/n} \cdot \Omega.$$

If Ω belongs to F, then the geometric object g_{ij} defined by

(15.4) $$g_{ij} = \Omega^2 G_{ij}$$

is a tensor, that is,

$$g'_{ij} = g_{hk} \frac{\partial x^h}{\partial x'^i} \frac{\partial x^k}{\partial x'^j}.$$

We note as a consequence of (15.4) that

$$\Omega = g^{1/2n}/G^{1/2n}.$$

If $\bar{\Omega}$ is any other scalar in F then

(15.5) $$\bar{g}_{ij} = \bar{\Omega}^2 G_{ij}$$

is also a tensor. Also, as follows from (15.3), e^{σ} defined by

(15.6) $$e^{\sigma} = \frac{\bar{\Omega}}{\Omega}$$

is an absolute scalar with respect to coordinate transformations. As a consequence of (15.4), (15.5) and (15.6),

(15.7) $$\bar{g}_{ij} = e^{2\sigma} g_{ij}.$$

Let U and $\bar{U}$ be any two coordinate neighborhoods of class C^m whose points correspond to those of U by means of point transformations which are of class C^m in the local coordinates of the neighborhoods. Then, as was shown in §2, allowable coordinate systems $\{x^i\}$ may be chosen in U and $\bar{U}$ so that corresponding points of U and U and of $\bar{U}$ and U have the same coordinates. Throughout this discussion, we assume that the coordinates $\{x^i\}$ are chosen so that points with the same coordinates correspond. In these coordinate systems, the tensors g_{ij} and $\bar{g}_{ij}$ defined over U and $\bar{U}$ respectively determine two Riemann spaces V_n and $\bar{V}_n$ of class C^m whose respective metric tensors they are. According to (15.7), the induced transformation which maps points of U

and $\overline{U}$ with the same coordinates is a conformal correspondence of the points of V_n and $\overline{V}_n$. As Ω ranges over all values of the set F, the corresponding σ's range over all functions of class C^{m-1} over U and the $\bar{g}_{ij}$ are any tensors related to one of them by (15.7). Associated with V_n is the set of conformal correspondences of class C^m whose domains are the $\overline{V}_n$. We denote this set by([85]) Ψ.

Corresponding to any geometric object in V_n, there exists a set of geometric objects, one in each Riemann space $\overline{V}_n$, which are in conformal correspondence by means of the transformations of Ψ. For example, corresponding to a tensor $T^{t_1 \cdots t_w}_{j_1 \cdots j_v}$ in V_n are the set of conformal tensors in the $\overline{V}_n$ whose components coincide with those of $T^{t_1 \cdots t_w}_{j_1 \cdots j_v}$. Conversely, every conformal tensor in the $\overline{V}_n$ determines a unique tensor in V_n. Indeed relative conformal tensors in the $\overline{V}_n$ also define tensors in V_n. For, in accordance with the remarks of §2, every relative conformal tensor in $\overline{V}_n$ corresponds to a unique conformal tensor if a relative conformal scalar exists in the space. As a consequence of these remarks, any theorem concerning conformal geometric objects in the $\overline{V}_n$ which is independent of the particular mapping function $\sigma(x^i)$, and hence depends only on the set F and not on the particular function Ω belonging to F, is also a theorem about geometric objects of V_n. These observations apply to the conformal theory of curves in conformally equivalent Riemann spaces which is developed in the previous sections of the paper. Consequently the previous results also constitute a theory of curves in V_n. If the Weyl conformal curvature tensor C^h_{ijk} of V_n vanishes, V_n is a flat conformal space. In this case, the $\overline{V}_n$ are the conformally euclidean spaces $\overline{R}_n$ which are related by the correspondences of Ψ and the previous theorems lead to a complete theory of curves in V_n.

In what follows, we give the outline of the theory of curves which is based (formally) upon the tensor G_{ij}. If we write $e^\phi = \Omega$, then (15.4) becomes $g_{ij} = e^{2\phi}G_{ij}$ which is analogous to (2.6) with σ, g_{ij}, $\bar{g}_{ij}$ replaced by ϕ, G_{ij}, g_{ij} respectively. Of course the analogy is not complete since ϕ is not a scalar and G_{ij} is not a simple tensor with respect to coordinate transformations. However, this complication does not affect the argument which follows since we remain in the same coordinate system. However, the geometric objects which we define (arc length, curvatures, normals) are absolute scalars and vectors with respect to coordinate transformations.

Let

$$x^i = x^i(s)$$

([85]) We may assume that G is constant. If G were a nonconstant function $A(x^i)$, the equivalence theory for V_n would be reducible to the corresponding theory for a Riemann space. For, as a consequence of (15.2) and (15.7), the tensor f_{ij} defined by

$$f_{ij} = [g^{hk}A_{,h}A_{,k}] \cdot g_{ij}$$

has the same components for every Ω belonging to F. The equivalence theory of V_n is therefore the equivalence theory of the Riemann space whose metric tensor is f_{ij}.

be the equations of a curve C in V_n. We define

$$\begin{Bmatrix} i \\ jk \end{Bmatrix}$$

as the Christoffel symbols of the second kind formed from G_{ij} and θ^i, ξ^i by the equations

$$\theta^i = \frac{dx^i}{ds}$$

$$\xi^i = \frac{d^2x^i}{ds^2} + \begin{Bmatrix} i \\ jk \end{Bmatrix} \frac{dx^j}{ds} \frac{dx^k}{ds}$$

where s is determined by

$$G_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds} = 1.$$

Hence θ^i and ξ^i are the formal analogues of the unit tangent and principal normal of the corresponding curve C in V_n. The "relative curvature" J and the vector $_{(1)}n^i$ of C are defined by an equation analogous to (4.21) so that, as in (4.22),

(15.8) $$J = e^{-\phi}J, \qquad _{(1)}\eta^i = e^{-\phi} \, _{(1)}n^i$$

where the geometric objects J and $_{(1)}\eta^i$ refer to the curve C in V_n. Therefore S, defined by

$$S = \int J ds$$

is equal to the integral invariant S given by (4.23). Since S remains unchanged under transformations of coordinates, this is also true of S so that S is a scalar. It plays the role of an "arc length parameter" for the curve C in V_n.

We define $_{(1)}\lambda^i$ by $_{(1)}\lambda^i = J^{-1} \, _{(1)}n^i$. It follows from (15.8) that $_{(1)}\lambda^i = _{(1)}\lambda^i$ where $_{(1)}\lambda^i$ is defined by (4.27). Since $_{(1)}\lambda^i$ is a conformal vector, $_{(1)}\lambda^i$ must transform like a vector under coordinate changes. It is the "first normal" of C.

Let Γ^i_{jk} be defined by an equation similar to (3.29). Then

$$\Gamma^i_{jk} = \begin{Bmatrix} i \\ jk \end{Bmatrix} + \left(\xi_j + \theta_j \frac{d \log J}{ds} \right) \delta^i_k + \left(\xi_k + \theta_k \frac{d \log J}{ds} \right) \delta^i_j$$
$$- G_{jk} \left(\xi^i + \theta^i \frac{d \log J}{ds} \right).$$

Then, as in (3.31), it follows that $\Gamma^i_{jk} = \Gamma^i_{jk}$. Since the Γ^i_{jk} transform like coefficients of connection under coordinate transformations, this is also true for

the Γ^i_{jk}. We use these Γ^i_{jk} to define the "derivative" of any tensor $T^{t_1 \cdots t_w}_{j_1 \cdots j_v}$ in a manner analogous to the definition given by (3.30). Consequently, this derivative has properties similar to those described in (A'), (B), (C), (D') and (E'). In particular, the analogue of (A') states that the Γ^i_{jk} derivative of any tensor $T^{t_1 \cdots t_w}_{j_1 \cdots j_v}$ transforms like a tensor of the same kind under coordinate transformations.

If this derivative is applied successively to $_{(1)}\lambda^i$ a sequence of equations analogous to (4.28) is obtained. They are

$$\frac{\delta_{(\alpha)}\lambda^i}{\delta S} = - J_{\alpha-1}{}_{(\alpha-1)}\lambda^i + J_\alpha{}_{(\alpha+1)}\lambda^i \quad J_0 = J_\tau = 0, \qquad \alpha = 1, 2, \cdots, \tau,$$

and are the Frenet equations of the curve C. The proofs of §4 show that the J_α are scalars and that the $_{(\alpha)}\lambda^i$ are vectors with respect to coordinate transformations in V_n. They are the "curvatures" and "normals" of C. The "$(n-1)$st curvature" J_{n-1} may be obtained as in §5 and the other results of this paper also have application here.

We note that the $\delta/\delta S$ process of differentiation defined by means of the Γ^i_{jk} is with respect to and depends upon the curve C in V_n and is therefore not appropriate for the purpose of obtaining a characterization of the entire space V_n (unless one could define a congruence of curves intrinsically in V_n). While the applicability of the derivative appears limited in this sense, its simple structure and conformal properties noted in §3 make it a suitable tool in the theory of curves.

BROOKLYN COLLEGE,
BROOKLYN, N. Y.