OPERATIONS IN BANACH SPACES

BY

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The starting point for this investigation was an attempt to generalize the well known theorem of Silverman and Toeplitz [25](1) on regularity of a sequence-to-sequence transformation. This theorem may be stated as follows: *If a transformation of sequences* $\{t_n\}$ *of real numbers to sequences* $\{s_m\}$ *is defined from a matrix* $\{a_{mn}\}$, $m, n = 1, 2, \cdots$, *by the equations* $s_m = \sum_n a_{mn}t_n$, *the transformation is regular—that is, is defined everywhere and takes every convergent sequence* $\{t_n\}$ *into another convergent sequence with the same limit—if and only if the matrix* $\{a_{mn}\}$ *satisfies the conditions* (a) $\lim_m \sum_n a_{mn} = 1$, (b) $\lim_m a_{mn} = 0$ *for each* n, *and* (c) *there is a* K *such that* $\sum_n |a_{mn}| \leq K$ *for every* m. In the special case under consideration, the fact that regularity implies condition (c) (the non-trivial part of the proof) can be derived from a theorem of Banach [3, p. 80, Theorem 5]: If A and B are Banach spaces, and if U_n, $n = 1, 2, \cdots$, are linear operators on A to B, such that $\lim \sup_n \|U_n(a)\| < \infty$ for each a in A, then $\lim \sup_n \|U_n\| < \infty$.

If the sequence of integers is replaced by a directed set X, it is known that A, B, X, and U_x can be chosen for which the similar statement relating $\lim \sup_x \|U_x(a)\|$ and $\lim \sup_x \|U_x\|$ is false; sections 1–3 of this paper consider these cases in an attempt to solve the problem of boundedness: Characterize those Banach spaces A and B, and directed sets X such that choosing the linear operators U_x on A to B so that $\lim \sup_x \|U_x(a)\| < \infty$ for each a in A implies that $\lim \sup_x \|U_x\| < \infty$. Section 1 is a review of pertinent facts about directed sets and convergence (mostly due to Moore and Smith [19], G. Birkhoff [5], and Tukey [27]). Section 2 studies the relations among three topologies in the space of operators on A to B. In §3 the problem of boundedness is studied but not completely solved.

The second part of the paper is concerned with certain special operators on some function spaces. In §4 the space is that of the totally measurable functions on a set Y to a Banach space B; a class of operators on this space is defined in terms of additive, real-valued set-functions and the relations among various topologies in this set of operators is given; this is used in §6 to give a general form to a theorem of Vulich [28]. In §5 the functions studied are the measurable functions on Y to B; the operations on this space are defined in terms of completely additive, limited, set-functions whose values are transfor-

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(1) Numbers in brackets refer to the bibliography at the end of the paper.

mations instead of real numbers. A corollary of the results obtained there is this: *The real-valued, finitely additive set-function* Ψ *has the property that* $\sum_i \Psi(E_i)$ *converges for every sequence* $\{E_i\}$ *of disjoint measurable sets if and only if the total variation of* Ψ *is finite.* This has as a corollary the well known result that a completely additive, real-valued set-function is of bounded variation.

Section 6 contains the applications of these theorems on Banach spaces to the problems which originally started the investigation beginning with two general theorems on regularity conditions. The first includes the Silverman-Toeplitz theorem and many others of the same type; the second generalizes a result of Vulich which says that a transformation defined by a matrix $\{a_{mn}\}$ takes every convergent sequence of points of a Banach space B into another such sequence with the same limit, if and only if the transformation defined by $\{a_{mn}\}$ is regular for real sequences. The section closes with some sample corollaries of these results.

1. **Directed sets and convergence.** This section contains a short discussion of properties of directed sets which will be useful throughout this paper. A non-empty set Y of elements y is called *directed* by a relation $>$ (read "follows" or "succeeds") if the pairs of points y_1, y_2 for which the relation $y_1 > y_2$ holds are subject to the conditions (a) if $y_1 > y_2$ and $y_2 > y_3$ then $y_1 > y_3$ (transitivity) and (b) each pair of points y_1, y_2 in Y has a common successor in Y; that is, there is a y_3 such that $y_3 > y_1$ and $y_3 > y_2$ (composition).

Probably the most used directed set is the set of integers ordered by magnitude. Other examples are (1) the neighborhoods of a point in a topological space ordered by inclusion, and (2) lattices.

A subset Y' is *cofinal* in Y if each y in Y has a successor y' in Y'. It can easily be shown that the cofinal subsets of Y satisfy the following conditions: (3) If y_0 is any element of Y, then the set(2) $\{y \mid y > y_0\}$ is cofinal in Y and its complement $\{y \mid y \not> y_0\}$ is not. (4) If Y_1 is not cofinal in Y and $Y_2 \subset Y_1$, then Y_2 is not cofinal in Y. (5) If Y_1 and Y_2 are not cofinal in Y, then $Y_1 + Y_2$ is not cofinal in Y. (6) If the order relation in a subset of Y is that imposed upon it by the order relation in Y, then every cofinal subset of Y is a directed set.

If f is a real-valued function defined on a directed set Y, let $\limsup_y f(y)$ be the least upper bound of those numbers K for which $\{y \mid f(y) > K\}$ is cofinal in Y; let $\liminf_y f(y) = -\limsup_y (-f(y))$; if $\limsup_y f(y) = \liminf_y f(y)$, then call this common value $\lim_y f(y)$. The Cauchy criterion is a necessary and sufficient condition for existence of $\lim_y f(y)$ and the limit defines an additive, homogeneous functional on those f's for which it is defined.

G. Birkhoff [5] first extended this definition of convergence to a topological space. Let S be a neighborhood space, satisfying, say, the axioms defining a Hausdorff space [2]; if f is a function defined on the directed set Y with

(2) $\in$, $\subset$, and so on will have the usual set-theoretical meanings; $\{y \mid \cdots\}$ will mean the class of those y satisfying the condition following the vertical bar.

values in the space S, $s=\lim_y f(y)$ if and only if for each neighborhood N of s there is a y_N in Y such that $f(y)\in N$ if $y>y_N$. If S is a complete metric space and if the neighborhoods of a point are the spheres about that point, then the Cauchy criterion is again necessary and sufficient for the existence of a limit; if S is a linear space with a uniform topology—that is, a topology in which addition of elements and multiplication of elements by real numbers are continuous operations—then this limit defines an additive, homogeneous function over the linear set of f's on Y to S for which it exists. In particular if S is a Banach space in any of the usual topologies, this is the case.

Tukey [27] has shown the importance of certain directed sets (first used by Moore) and has defined an order relation among directed sets which will be useful in later sections. For any ordinal number $\nu>0$, let D^ν be a set of power $\aleph_\nu$; the *stack* Δ^ν is the directed set whose elements are the finite subsets δ of D^ν, where $\delta>\delta'$ means $\delta\supset\delta'$; D^ν is the *base* of the stack Δ^ν. It is clear that if two stacks have bases of the same power, then there is an isomorphism—that is, a 1-1 order-preserving correspondence—between the two stacks.

A directed set X is a *cofinal part* of a directed set Y if there is an isomorphism between X and a cofinal subset Y' of Y. X and Y are *cofinally similar* (symbol: $X\sim Y$) if there is a directed set Z of which X and Y are both cofinal parts. X follows Y (symbol: $X>Y$) if and only if there exist two functions, h on X to Y and g on Y to X, such that if y is any point of Y and $x>g(y)$, then $h(x)>y$. Tukey showed that $X>Y$ and $Y>X$ if and only if $X\sim Y$, that $>$ is a reflexive and transitive ordering among directed sets, and that cofinal similarity is a reflexive, symmetrical and transitive relation, the equivalence relation associated with $>$.

The reader can easily see that ω, the class of integers ordered by magnitude, is a cofinal part of Δ^0, the stack on a countable base, and that the stack Δ^ν follows every directed set of power less than or equal to $\aleph_\nu$; also $\Delta^\nu>\Delta^\mu$ if and only if $\nu\geqq\mu$, so $\Delta^\nu\sim\Delta^\mu$ if and only if $\nu=\mu$. From this it follows that $\Delta^0>X$ if X is any countable directed set. If $\Delta^0>X$ either X has a last element—that is, an x_0 such that $x_0>x$ for each x in X—or $\Delta^0\sim X$. In all of what follows the trivial case will be explicitly rejected; that is, no directed sets mentioned hereafter will have a last element.

If X is a directed set, let $\lambda(X)$ be the smallest ordinal number μ such that a subset of X of power $\aleph_\mu$ has no upper bound in X; that is, $\lambda(X)$ is the smallest ordinal μ not satisfying the following condition: If $X'\subset X$ and the power of $X'\leqq\aleph_\mu$, there is an x_0 in X such that $x_0>x'$ for every x' in X'. For example, $\lambda(\Delta^\nu)=0$ for every ν; $\lambda(\omega)=0$; $\lambda(\Omega_n)=n$ for any integer n if Ω_n is the set of all ordinals of power less than $\aleph_n$ ordered by magnitude. From the definition it can readily be seen that if $X>Y$, then $\lambda(X)\leqq\lambda(Y)$, so $\lambda(X)$ is invariant under cofinal similarity. The next lemma is useful in §3.

LEMMA 1.1. *If X is a directed set, then $\lambda(X)=0$ if and only if $X>\Delta^0$.*

If $X>\Delta^0$, then $0\leqq\lambda(X)\leqq\lambda(\Delta^0)=0$. If $\lambda(X)=0$, a countable set $\{x_n'\}$ exists with no upper bound; by induction and the composition property a sequence $\{x_n\}$ can be defined so that $x_{n+1}>x_n$ and x_n'; then $\{x_n\}$ has no upper bound and is monotone. Define h on X to ω and g on ω to X by letting $g(n)=x_n$ for each n in ω; $h(x)=n+1$ if $x_n<x\not< x_{n+1}$.

The interested reader can also prove that no X can be chosen for which $\lambda(X)=\omega$; this fact clarifies some steps of the proof of Theorem 3.6.

2. **Neighborhoods and convergence in operator spaces.** This section considers relations among two Banach spaces(3) A and B and the space $\mathfrak{U}=A:B$ of all linear(4) operators defined over all of A with values in B; $\mathfrak{U}$ is also a Banach space if $\|U\|=\sup_{\|a\|\leqq 1}\|U(a)\|$ for each U in $\mathfrak{U}$. In the special case in which B is B_0, the set of real numbers, $A:B_0$ is the space A^* of all linear functionals on A. There are three natural ways in which a topology can be imposed on $\mathfrak{U}$; by analogy with the case in which $A=B=$ Hilbert space(5) these will be called norm, s^*, and w^* topologies in $\mathfrak{U}$. It is sufficient (see Wehausen, [29]) to define the neighborhoods of θ(6); the neighborhoods of the other points of $\mathfrak{U}$ are defined by translating the neighborhoods of θ.

NORM: *For any* $\epsilon>0$ *let* $N=N(\epsilon)=\{U\mid \|U\|<\epsilon\}$.

S^*: *For any integer* k, *any* $a_1,\cdots,a_k$ *in* A, *and* $\epsilon>0$ *let* $S=S(a_1,\cdots,a_k;\epsilon)=\{U\mid \|U(a_i)\|<\epsilon$ *for* $i=1,\cdots,k\}$.

W^*: *For any integer* k, *any* $a_1,\cdots,a_k$ *in* A, *and* $\beta_1,\cdots,\beta_k$ *in* B^* *and any* $\epsilon>0$ *let* $W=W(a_1,\cdots,a_k;1,\cdots,k;\epsilon)=\{U\mid |\beta_i(U(a_i))|<\epsilon$ *for* $i=1,\cdots,k\}$.

The families $\mathfrak{N}$, $\mathfrak{S}$, and $\mathfrak{W}$ of these sets N, S, and W are, respectively, the norm, s^*, and w^* neighborhoods of θ in $\mathfrak{U}$; in the special case $B=B_0$ both s^* and w^* topologies reduce to the ordinary weak* topology in A^*(7). If X is any directed set and if $U_x\in\mathfrak{U}$, the notations $U_0=n-\lim_x U_x$, $U_0=s^*-\lim_x U_x$ and $U_0=w^*-\lim_x U_x$ mean that U_x converges to U_0 in the corresponding topology.

The first half of the next theorem is used in §3. Two neighborhood systems $\mathfrak{N}'$ and $\mathfrak{N}''$ of θ in $A:B$ will be called equivalent (symbol: $\mathfrak{N}'\simeq\mathfrak{N}''$) if each N' contains an N'' and each N'' an N'. Clearly each W contains an S and each S an N. A Banach space B is called finite-dimensional (symbol: fd) if there exist a finite subset $b_1,\cdots,b_k$ in B such that every b in B is a linear combination of these b_i.

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(3) A Banach space [see 3] is a complete normed vector space. In all that follows the trivial space consisting of just one point will be ruled out and all spaces considered will be at least one-dimensional.

(4) Linear is used in Banach's sense, to mean additive and continuous.

(5) See J. von Neumann [20], for this case; others who have considered topologies in a Banach space are A. E. Taylor [25], and Alaoglu [1].

(6) θ will be used for the zero element of any linear space under discussion.

(7) See Taylor [25].

THEOREM 2.1. *$\mathfrak{N}\simeq\mathfrak{S}$ if and only if A is* fd; *$\mathfrak{S}\simeq\mathfrak{W}$ if and only if B is* fd; *hence $\mathfrak{N}\simeq\mathfrak{W}$ if and only if both A and B are* fd(8).

If A is fd, there is a basis $a_1, \cdots, a_k$ of linearly independent points of A with $a=\sum_{i\leqq k} t_{ai}a_i$, t_{ai} real, for each a in A. Since every two k-dimensional Banach spaces are isomorphic there is a $K>0$ such that $\sum_{i\leqq k}|t_{ai}|\leqq K$ if $\|a\|\leqq 1$. Recall that $\|U\|=\sup_{\|a\|\leqq 1}\|U(a)\|$; if $U\in S(a_1, \cdots, a_k; \epsilon/K)$, then

$$\|U(a)\| \leqq \sum_{i\leqq k} |t_{ai}| \, \|U(a_i)\| < \frac{K\epsilon}{K} = \epsilon$$

if $\|a\|\leqq 1$. Therefore $N(\epsilon)\supset S(a_1, \cdots, a_k; \epsilon/K)$ and $\mathfrak{N}\simeq\mathfrak{S}$ when A is fd. If A is not fd and $S=S(a_1, \cdots, a_k; \epsilon)$ is any s^* neighborhood of θ, there is an α in A^* such that $\alpha(a_i)=0$ for each $i\leqq k$, while $\|\alpha\|>0$. If $\|b\|\neq 0$, the element U_n of U defined by $U_n(a)=n\alpha(a)b$ is in S for every n while $\|U_n\|=n\|\alpha\|\,\|b\|\to\infty$ as n increases, so S is not contained in any sphere.

If B is fd, let $\beta_1, \cdots, \beta_q$ and $b_1, \cdots, b_q$ be conjugate bases in B^* and B, respectively; that is, choose them so that $\beta_i(b_i)=1$, $\beta_i(b_j)=0$ if $i\neq j$, and $\beta_1, \cdots, \beta_q$ and $b_1, \cdots, b_q$ are linearly independent and are bases in their respective spaces; then any b in B is of the form $\sum_{j\leqq q}\beta_j(b)b_j$. If $S=S(a_1, \cdots, a_k; \epsilon)$ is given, the w^* neighborhood for which $|\beta_j(U(a_i))|<\epsilon/(q\sup_j\|b_j\|)$ for every $i\leqq k, j\leqq q$ lies in S since

$$\|U(a_i)\| = \left\|\sum_{j\leqq q}\beta_j(U(a_i))b_j\right\| < \sum_{j\leqq q}|\beta_j(U(a_i))|\,\|b_j\| < \epsilon$$

if $|\beta_j(U(a_i))|<\epsilon/(q\sup_j\|b_j\|)$ for all i, j; hence $\mathfrak{S}\simeq\mathfrak{W}$. If B is not fd and if $W=W(a_1, \cdots, a_k; \beta_1, \cdots, \beta_k; \epsilon)$ is given, there is a point b in B with $\beta_i(b)=0$ for all $i\leqq k$ while $\|b\|>0$. If $a\in A$, if $\|a\|\neq 0$ and if α is any element of A^* for which $\alpha(a)\neq 0$, each U_n defined by letting $U_n(a')=n\alpha(a')b$ for each a' in A is in W since $\beta_i(U_n(a))=n\alpha(a)\beta_i(b)=0$; $\|U_n(a)\|=n|\alpha(a)|\,\|b\|\to\infty$ as n increases so W cannot lie in any $S(a, \epsilon)$ for which $\|a\|\neq 0$.

3. **The boundedness problem.** The theorem of Banach [3, p. 80, Theorem 5] already mentioned asserts that if $\{U_n\}$ is any sequence of elements of $A:B$ such that $\limsup_n\|U_n(a)\|<\infty$ for every a in A, then $\limsup_n\|U_n\|<\infty$. The boundedness problem is to characterize those triples A, B, X such that $\limsup_x\|U_x\|<\infty$ if $\limsup_x\|U_x(a)\|<\infty$ for each a. Some unsettled questions connected with this problem are collected at the end of this section.

Consider the following conditions:

(a) $\limsup_x\|U_x(a)\|<\infty$ for each a in A.

(b) $\lim_x\|U_x(a)\|=0$ for each a in A.

(8) Even in the unit sphere $\mathfrak{U}_1$ in $\mathfrak{U}$, the s^* and w^* topologies are generally different; this can be seen from the result, more general than one of Alaoglu [1], that $\mathfrak{U}_1$ is bicompact in the s^* topology if and only if B is fd, while $\mathfrak{U}_1$ is bicompact in the w^* topology if and only if B is reflexive.

(c) $\lim\sup_x \|U_x\| = \infty$.
(d) $\lim_x \|U_x\| = \infty$.

The first step in the solution of the boundedness problem is to show that the nature of B is unimportant.

THEOREM 3.1. *If A and X are given and if a B exists such that linear operators U_x can be defined on A to B so as to satisfy any combination of the conditions* (a)–(d), *then for any B', U_x' can be chosen in $A:B'$ to satisfy the same conditions.*

If the linear operators U_x on A to B are given, for each x let β_x be an element of B^* such that $\|\beta_x\| = 1$ and

$$\|U_x\| = \sup_{\|a\|\leqq 1} \|U_x(a)\| \leqq 2 \sup_{\|a\|\leqq 1} |\beta_x(U_x(a))|.$$

Let α_x in A^* be defined by $\alpha_x(a) = \beta_x(U_x(a))$; then $|\alpha_x(a)| \leqq \|U_x(a)\|$ while $\|U_x\| < 2\|\alpha_x\|$ so the α_x satisfy those conditions which are satisfied by the U_x. If B' is any other space, let b' be any point in B' for which $\|b'\| = 1$ and define U_x' by the equation $U_x'(a) = \alpha_x(a)b'$; then $\|U_x'\| = \|\alpha_x\|$ and $\|U_x'(a)\| = |\alpha_x(a)|$ so the U_x' have the same properties.

For any combination of the conditions (a)–(d) let $\mathfrak{P}$ with those subscripts be the class of all pairs $[A, X]$, where A is a Banach space and X a directed set, such that α_x in A^* exist satisfying that set of conditions; for example, $\mathfrak{P}_{ac}$ is the set of all $[A, X]$ such that α_x in A^* exist with $\lim\sup_x |\alpha_x(a)| < \infty$ for each a in A and $\lim\sup_x \|\alpha_x\| = \infty$. The problem of boundedness is to characterize $\mathfrak{P}_{ac}$; related to this are the problems of characterizing the sets $\mathfrak{P}_{ad}$, $\mathfrak{P}_{bc}$, and $\mathfrak{P}_{bd}$. There are several obvious relations among these classes; $\mathfrak{P}_{bd} \subset \mathfrak{P}_{ad} \subset \mathfrak{P}_{ac}$; $\mathfrak{P}_{bd} \subset \mathfrak{P}_{bc} \subset \mathfrak{P}_{ac}$, and $\mathfrak{P}_d \supset \mathfrak{P}_{ad}$; to be proved later (Theorem 3.7) is the fact that $\mathfrak{P}_{ad} = \mathfrak{P}_{bd}$.

Consider first some "monotony" properties of these sets. A Banach space A' will be called a *linear image* of a Banach space A if there is a linear operator U on A to B whose values fill up B. If U is such an operator and if $A_0 = \{a \mid U(a) = \theta\}$, then A' is isomorphic to the Banach space A/A_0(9).

THEOREM 3.2. *If $[A, Y] \in \mathfrak{P}_{bd}$, $\mathfrak{P}_{bc}$, or $\mathfrak{P}_{ac}$ and $X > Y$, then $[A, X]$ is in the same class. If A' is a linear image of A and if $[A', X] \in \mathfrak{P}_{bd}$, $\mathfrak{P}_{bc}$, or $\mathfrak{P}_{ac}$, then $[A, X]$ is in the same class.*

If $X > Y$, there are functions g on Y to X and h on X to Y such that $h(x) > y$ if $x > g(y)$. If $[A, Y] \in \mathfrak{P}_{bd}$, there exist α_y in A^* such that $\lim_y |\alpha_y(a)| = 0$ for each a in A and $\lim_y \|\alpha_y\| = \infty$. Let $\alpha_x = \alpha_{h(x)}$; then for each a in A and

(9) If A_0 is a closed linear subset of A, the elements of A/A_0 are the cosets $E_a = \{a_1 \mid a - a_1 \in A\}$ where $\|E_a\| = \inf_{a_1 \in E_a} \|a_1\|$; with the usual definitions of the vector operations A/A_0 is a Banach space. If $A_0 = U^{-1}(\theta)$, the elements of A/A_0 are the sets $E = U^{-1}(a')$, a' in A'; the transformation T on A/A_0 to A' defined by $T(U^{-1}(a')) = a'$ is linear and 1-1 so [3, p. 41, Theorem 5] it is an isomorphism.

$\epsilon>0$ there is a y_ϵ such that $|\alpha_y(a)|<\epsilon$ if $y>y_\epsilon$. Let $x_\epsilon=g(y_\epsilon)$; if $x>x_\epsilon$, then $|\alpha_x(a)|=|\alpha_{h(x)}(a)|<\epsilon$ since $h(x)>y_\epsilon$, so $\lim_x|\alpha_x(a)|=0$; similarly $\lim_x\|\alpha_x\|=\infty$.

If $[A, Y]\in\mathfrak{P}_{ac}$ or $\mathfrak{P}_{bc}$, let α_y in A^* have the corresponding properties. Define h_1 on X to Y as follows: Suppose that h_1 is already defined on a subset X' of X so that (1) X' contains every predecessor of each of its elements, (2) for each x' in X' there is a sequence $\{x_n'\}\subset X'$ such that $x_n'>x'$ and $\lim_n\|\alpha_{h_1(x_n')}\|=\infty$, and (3) $h_1(x')>h(x')$ for every x' in X'. If $X'\neq X$, let x be any element of $X-X'$ and let $\{x_n\}$ be any sequence of points of $X-X'$ such that $x_{n+1}>x_n$ for each n while $x_1>x$. Since $\lim\sup_y\|\alpha_y\|=\infty$, for each n there exists a point $h_1(x_n)$ in Y such that $h_1(x_n)>h(x_n)$ and $\|\alpha_{h_1(x_n)}\|>n$. Let $X''=X'+\{x''|$ an n exists for which $x_n>x''\}$; for each x'' in X'' for which $h_1(x'')$ is not already defined let $h_1(x'')=h(x'')$; then h_1 is defined over X'' with the properties (1)–(3). Starting with X' equal to the empty set and applying transfinite induction defines h_1 over all X with the properties (2) and (3). From (3) and a repetition of the argument in the preceding paragraph it follows that the α_x defined by $\alpha_x=\alpha_{h_1(x)}$ satisfy (a) or (b) if the α_y do; (2) implies that α_x satisfy (c).

Suppose that the linear operator U maps A onto all of B, let $A_0=U^{-1}(\theta)$, and construct A/A_0. If $\alpha_x'\in A'^*$, define α_x in A^* by $\alpha_x(a)=\alpha_x'(U(a))$; clearly the α_x satisfy (a) or (b) if the α_x' do. For each x.

$$\|\alpha_x\| = \sup_a(|\alpha_x(a)|/\|a\|) = \sup_a(|\alpha_x'(U(a))|/\|a\|).$$

For each $\epsilon>0$ there is an a' of norm one such that $\alpha_x'(a')>\|\alpha_x'\|-\epsilon$; if[9] $E=T^{-1}(a')$, $\|E\|\leqq\|T^{-1}\|$ so there is an a in E of norm less than $\|T^{-1}\|+\epsilon$; hence $\|\alpha_x\|>(\|\alpha_x'\|-\epsilon)/(\|T^{-1}\|+\epsilon)$ for every $\epsilon>0$ or $\|\alpha_x\|\geqq\|\alpha_x'\|/\|T^{-1}\|$. Therefore the α_x satisfy (c) or (d) if the α_x' do.

We now consider a case in which $[A, X]$ can be shown to be in the smallest of these classes.

THEOREM 3.3. *If A is not* fd, *if B is any Banach space, and if $\mathfrak{S}$, the s^* neighborhood system of θ in $A:B$, is directed by the relation $S>S'$ if $S\subset S'$, then* $[A, \mathfrak{S}]\in\mathfrak{P}_{bd}$.

If $S\in\mathfrak{S}$, there is a least integer k such that $S=S(a_1, \cdots, a_k; \epsilon)$. If A is not fd, by Theorem 2.1, S contains a point U_S for which $\|U_S\|>k$; then U_S defined in this way have the properties (b) and (d). If $a\in A$ and $\epsilon>0$ is given, $U_S\in S\subset S(a;\epsilon)$ if $S>S(a;\epsilon)$, so $\|U_S(a)\|<\epsilon$ if $S>S(a;\epsilon)$ or $\lim_S\|U_S(a)\|=0$. If $S=S(a_1, \cdots, a_k;\epsilon)\subset S'=S(a_1', \cdots, a_q';\epsilon')$, then each a_j' is linearly dependent on the a_i. For suppose that some a_j' does not depend on the a_i; then [3, p. 57, lemma] there is an α in A^* such that $\alpha(a_i)=0$ for all i while $\alpha(a_j')=1$. Take $b\neq\theta$ in B and let $U(a)=\alpha(a)b$; then $kU\in S$ for every k, but if $k\|b\|>\epsilon'$, $kU\notin S'$; this contradicts the assumption that $S\subset S'$. If $a_1', \cdots, a_q'$ are chosen linearly independent, then $S(a_1, \cdots, a_k; \epsilon) \subset S(a_1', \cdots, a_q'; 1/q)$ implies that $k \geqq q$, so $\|U_S\| > q$ if $S > S(a_1', \cdots, a_q'; 1/q)$, and $\lim_S \|U_S\| = \infty$.

Banach's theorem asserts that $[A, \omega] \notin \mathfrak{P}_{ac}$ for any A; a converse of this is contained in the first half of the next theorem.

THEOREM 3.4. *$\Delta^0 > X$ if and only if there is no A for which $[A, X]$ is in $\mathfrak{P}_{ac}$ (or $\mathfrak{P}_{bc}$). A is* fd *if and only if there is no X for which $[A, X]$ is in $\mathfrak{P}_{ac}$ (or $\mathfrak{P}_{bd}$).*

If $[A, X] \in \mathfrak{P}_{ac}$ and $\Delta^0 > X$, then $[A, \omega] \in \mathfrak{P}_{ac}$, by Theorem 3.2; this contradicts Banach's theorem. If $\Delta^0 \not> X$, no countable subset of X is cofinal in X. Let $A = c_0(X, B_0)$(10), where B_0 is the space of real numbers, and define α_x in A^* in a way similar to that used in the proof of Theorem 3.3. Suppose that the α_x have been defined on a subset X' of X to satisfy the conditions (1) X' contains every predecessor of each element of X', (2) for each x' in X', there is a monotone sequence $\{x_n'\} \subset X'$ such that $\lim_n \|\alpha_{x'_n}\| = \infty$, and (3) for each x' in X', $\alpha_{x'}$ is defined by $\alpha_{x'}(a) = k_{x'} a(x')$ for each a in A, and some constant $k_{x'}$. Then take any x not in X' and any monotone sequence $\{x_n\} \subset X - X'$ such that $x_1 > x$ and for each n define α_{x_n} in A^* by $\alpha_{x_n}(a) = na(x_n)$ for each a in A; define $\alpha_{x'}$ in A^* for those predecessors x' of any x_n for which it is not already defined by setting $\alpha_{x'} = \theta$. As before, this and transfinite induction serve to define α_x for every x in X so that $\limsup_x \|\alpha_x\| = \infty$. For each a in A the set $E_a = \{x \mid |a(x)| > 0\}$ is countable, hence not cofinal in X; therefore there is an x_a in X such that no x in E_a follows x_a, so $\lim_x \alpha_x(a) = 0$ for each a since $\alpha_x(a) = 0$ if $x > x_a$. This shows that $[c_0(X, B_0), X] \in \mathfrak{P}_{bc}$ unless $\Delta^0 > X$.

If A is fd, by Theorem 2.1, norm and strong neighborhoods systems are equivalent; using this fact it is clear that no $[A, X] \in \mathfrak{P}_{bc}$. To show that no $[A, X]$ can be in $\mathfrak{P}_{ac}$ requires only an application of the method of proof used in that theorem. If A is not fd, Theorem 3.3 asserts that an X exists with $[A, X]$ in $\mathfrak{P}_{bd}$.

We turn now to a characterization of the nature of $\mathfrak{S}$ considered as a directed set rather than as a neighborhood system.

THEOREM 3.5. *A is not* fd *if and only if for every B there is an ordinal number $\nu > 0$ such that Δ^ν is a cofinal part of the directed set $\mathfrak{S}$ of strong neighborhoods of θ in $A:B$; ν is unique and depends cnly on A.*

Proof. If A is not fd, let A' be a vector basis in A; that is, a set of points a' of norm one, such that no a' in A' is linearly dependent on any of the others, while every element of A is a linear combination of elements of A'. Let ν be the ordinal for which the power of A' is $\aleph_\nu$. If B is any Banach space, we shall show that Δ^ν with this choice of ν is a cofinal part of $\mathfrak{S}$, the strong neighborhood system in $A:B$; clearly ν does not depend on B but only on A.

Let f be any 1-1 correspondence between the class of neighborhoods $S(a; 1)$, a in A', and D^ν, the base of the stack Δ^ν; extend f to all of Δ^ν by letting

(10)This is defined before Corollary 3.3.

$f(\delta)=S(f(d_1), \cdots, f(d_k); 1/k)$ if $d_1, \cdots, d_k$ are the elements of δ. Then f defines a 1-1 correspondence between Δ^ν and a certain subset $\mathfrak{S}'$ of $\mathfrak{S}$.

$\mathfrak{S}'$ is cofinal in $\mathfrak{S}$, for if $S(a_1, \cdots, a_k; \epsilon)\in\mathfrak{S}$, there exist a_{ij} in A' and real numbers t_{ij} such that $a_i=\sum_{j\leq k_i} t_{ij}a_{ij}$. Then S_1, the neighborhood such that $|U(a_{ij})| < \epsilon/(\sum_{ij}|t_{ij}|)$, is contained in S. If enough additional elements of A' are used to make $1/k$ smaller than $\epsilon/(\sum_{ij}|t_{ij}|)$, S_1 contains some $S'\in\mathfrak{S}'$, so $\mathfrak{S}'$ is cofinal in $\mathfrak{S}$.

f preserves order between $\mathfrak{S}'$ and Δ^ν. Obviously $f(\delta)>f(\delta')$ if $\delta>\delta'$. Suppose that $S=S(a_1, \cdots, a_k; 1/k)$ and $S'=S(a_1', \cdots, a_q'; 1/q)$ are in $\mathfrak{S}'$ and that $S\subset S'$. By the argument used in Theorem 3.3, each a_j' is a linear combination of the $a_i, i\leq k$; hence each a_j' must be an a_i, so $q\leq k$ and $f^{-1}(S)>f^{-1}(S')$.

This shows that Δ^ν is a cofinal part of $\mathfrak{S}$ if A is not fd; $\nu\neq 0$ since $\Delta^0>S$ if $\nu=0$ and no function on $\mathfrak{S}$ to U can exist satisfying Theorem 3.3. ν is unique since $\mathfrak{S}$ and Δ^ν are cofinally similar and (see §1) no set can be cofinally similar to two different stacks. If A is fd, $\mathfrak{S}\sim\mathfrak{N}\sim\Delta^0$, so no Δ^ν with $\nu>0$ can be a cofinal part of $\mathfrak{S}$ in this case.

If A is fd, let $\nu(A)=0$; if A is not fd, let $\nu(A)$ be the ordinal greater than 0 whose existence is asserted by this theorem.

COROLLARY 3.1. *If $\eta(A)=\min \nu(A')$ where the minimum is taken over all non-*fd *linear images A' of A, and if $X>\Delta^{\eta(A)}$, then $[A, X]\in\mathfrak{P}_{bd}$.*

Since the set of ordinals $\nu(A')$ is well-ordered by magnitude, there is a smallest one, so $\eta(A)$ is defined; let A' be an image of A for which $\nu(A')=\eta(A)$. By Theorems 3.3, 3.5, and 3.2, $[A', \Delta^{\nu(A')}]\in\mathfrak{P}_{bd}$, so, by 3.2, $[A, X]\in\mathfrak{P}_{bd}$ if $X>\Delta^{\eta(A)}$.

COROLLARY 3.2. *If $A=B^{(2n+1)}$, $[A, X]\in\mathfrak{P}_{bd}$ if $X>\Delta^{\nu(B^*)}$; if $A=B^{(2n+2)}$, $[A, X]\in\mathfrak{P}_{bd}$ if $X>\Delta^{\nu(B^{**})}$*(11).

This follows from Corollary 3.1 and this theorem: *Let A be isomorphic to a conjugate space and let A_1 be the subset of A^{**} consisting of all those points $\mathfrak{a}_a$ defined for each a in A by $\mathfrak{a}_a(\alpha)=\alpha(a)$ for every α in A^*; then there is a projection of A^{**} into A_1*(12).

If Y is any set of points y and B is any Banach space, there are certain easily defined Banach spaces of functions f on Y to B(13). Let $m(Y, B)$ be the

(11) $B^{(n)}$ is defined by induction from $B^{(0)}=B$, $B^{(n+1)}=(B^{(n)})^*$.

(12) This need only be proved if $A=B^*$ for some B; in this case reducing each $\mathfrak{a}$ defined over B^{**} to a function defined only over B_1 defines a transformation of A^{**} into A: mapping back to A_1 by the usual method gives the desired projection. Phillips [23] has shown that c_0 is not the range of a projection of $m=c_0^{**}$; so some restriction on A is needed; it is not known whether A is isomorphic to a conjugate space if A_1 is the range of a projection in A^{**}.

(13) Most of the results given in this paragraph for these spaces of functions on Y to B can be adapted to the more general spaces of functions f on Y for which the value $f(y)$ always lies in some fixed space B_y; if all $B_y=B$, this reduces to the case discussed in the text. For example, see [9] for one case where Y is countable.

space of those f's such that $\|f\| = \sup_y \|f(y)\| < \infty$; for any p with $1 \leq p < \infty$ let $l_p(Y, B)$ be the space of those f's for which $\|f\| = (\sum_y \|f(y)\|^p)^{1/p} < \infty$ (14); let $c_0(Y, B)$ be the set of f's for which $\{y \mid |f(y)| > \epsilon\}$ is a finite set for every $\epsilon > 0$. In the special case $Y = \omega$, $B = B_0$, these spaces reduce to the well known spaces m, l_p, and c_0. It may be noted that the conjugate spaces of $c_0(Y, B)$, $l_1(Y, B)$, and $l_p(Y, B)$ with $1 < p < \infty$ are, respectively, equivalent to $l_1(Y, B^*)$, $m(Y, B^*)$ and $l_{p'}(Y, B^*)$ where $1/p + 1/p' = 1$.

If Y' is any subset of Y, let T be the operation which takes a function f on Y to B into the function Tf defined by $Tf(y) = f(y)$ if $y \in Y'$, $Tf(y) = \theta$ if $y \notin Y'$. Then it is clear that T defines a projection of norm 1 in each of the spaces $m(Y, B)$, $l_p(Y, B)$ and $c_0(Y, B)$ and that the range of T in these cases is equivalent to $m(Y', B)$, $l_p(Y', B)$ and $c_0(Y', B)$. Also, if Y is an infinite set, the spaces m, l_p, and c_0 are, respectively, linear images of $m[Y, B]$, $l_p[Y, B]$ and $c_0[Y, B]$.

COROLLARY 3.3 *If Y is any infinite set and B is any Banach space, $[m(Y, B), X]$, $[l_p(Y, B), X]$ and $[c_0(Y, B), X] \in \mathfrak{P}_{bd}$ if $X > \Delta^\gamma$ where $\aleph_\gamma$ is the power of the continuum.*

This follows from 3.2 and 3.5 since the power of a vector basis in m, l_p, or c_0, is that of the continuum.

The next result gives some conditions involving $\lambda(X)$; if A is not fd, let $\mu(A)$ be the smallest ordinal such that a fundamental set(15) of power $\aleph_\mu$ exists in A.

THEOREM 3.6. *If $\lambda(X) > \mu(A) > 0$ and if $\lim_x U_x(a)$ exists for every a in A, then there is an x_0 such that $U_x = U_{x_0}$ if $x > x_0$, so $[A, X] \notin \mathfrak{P}_{bc}$. If $\lambda(X) > \nu(A)$, $[A, X] \notin \mathfrak{P}_{ac}$. If $\lambda(X) = \mu(A) > 0$, then $[A, X] \in \mathfrak{P}_{bc}$.*

Let A' be a fundamental set in A of power $\aleph_{\mu(A)}$; if $\lim_x U_x(a)$ exists for each a, then for each a in A' and integer k there is an x_{ak} in X such that $\|U_x(a) - U_{x'}(a)\| < 1/k$ if $x, x' > x_{ak}$. If $\lambda(X) > \mu(A)$, for each k there is an x_k which follows all x_{ak} so $\|U_x(a) - U_{x'}(a)\| < 1/k$ for all a in A' if $x, x' > x_k$. Since $\lambda(X) > 0$ there exists an x_0' following all x_k; so, if $x_0 > x_0'$, $U_x(a) = U_{x_0}(a)$ for all a in A' if $x > x_0$; hence $U_x(a) = U_{x_0}(a)$ for all a in A.

A similar argument if $\lambda(X) > \nu(A)$ and if $\lim\sup_x |\alpha_x(a)| < \infty$ for every a shows that there is an x_0 in X such that, for each a, a $k_a > 0$ exists with $|\alpha_x(a)| < k_a$ if $x > x_0$. That $\lim\sup_x \|\alpha_x\| < \infty$ follows from a theorem of Hildebrandt [15] which has as a special case this theorem(16): *If A is a Banach*

(14) If ϕ is a real-valued, non-negative function defined over Y, $\sum_y \phi(y)$ is defined to be $\sup_\eta \sum_{y \in \eta} \phi(y)$, where the supremum is taken over all finite subsets η of Y. Hence the assumption that $\sum_y \phi(y) < \infty$ implies that $\{y \mid \phi(y) > 0\}$ is at most countable.

(15) A set A' is fundamental in A if the set of linear combinations of elements of A' is dense in A. It is known [18] that $\mu(A) = \nu(A)$ if and only if $\aleph_\mu$ is at least as great as the power of the continuum.

(16) Banach's theorem is a corollary of Hildebrandt's in its more general form; the reader

space, if X is a directed set and if $\alpha_x \in A^*$, then $\lim\sup_x \|\alpha_x\| < \infty$ *if a sequence $\{x_n\} \subset X$ exists with the following property: For each a in A there exist integers k_a and m_a such that $|\alpha_x(a)| < k_a$ if $x > x_{m_a}$.*

If $\lambda(X) = \mu(A) > 0$, transfinite sequences $\{x_\nu\} \subset X$ and $\{a_\nu\} \subset A$ can be defined as follows: (1) ν ranges over all ordinals $< \omega_{\mu(A)}$, the first ordinal of power $\aleph_{\mu(A)}$; (2) $x_\nu > x_\rho$ if $\nu > \rho$ and $\{x_\nu\}$ has no upper bound; (3) the set $\{a_\nu \mid \nu < \omega_{\mu(A)}\}$ is fundamental in A. Let $n(\nu)$ be the largest integer such that $\nu - n$ is defined and define α_ν in A^* so that $\alpha_\nu(a_\rho) = 0$ if $\rho < \nu$ while $\|\alpha_\nu\| = n(\nu)$. If $\Omega = \{\nu \mid \nu < \omega_{\mu(A)}\}$, then $[A, \Omega] \in \mathfrak{P}_{bc}$, for if $a \in A$, there is a sequence $\{a_n'\}$ of linear combinations of the a_ν such that $\|a_n' - a\| \to 0$; hence there is a $\nu_a < \omega_{\mu(A)}$ such that a is in the smallest closed linear manifold containing $\{a_\nu \mid \nu < \nu_a\}$; hence $\alpha_\nu(a) = 0$ if $\nu > \nu_a$ so $\lim_\nu \alpha_\nu(a) = 0$ for every a in A; clearly $\lim\sup_\nu \|\alpha_\nu\| = \infty$. Also $X > \Omega$, for defining $g(\nu) = x_\nu$ and $h(x) = \inf \nu$ such that $x \not> x_\nu$ gives two functions with the desired properties. Therefore $[A, X] \in \mathfrak{P}_{bc}$ when $\lambda(X) = \mu(A) > 0$.

COROLLARY 3.4. *If $A = l_p(Y, B)$ or $\epsilon_0(Y, B)$, if Y is uncountable, and if $X > \Omega_1$, the set of denumerable ordinals, $[A, X] \in \mathfrak{P}_{bc}$. Under the hypothesis of the continuum $[m(Y, B), X] \in \mathfrak{P}_{bc}$ if $X > \Omega_1$ and Y is any infinite set.*

We conclude this section with a theorem giving some relationships among the sets $\mathfrak{P}$.

THEOREM 3.7. *If A exists for which $[A, X] \in \mathfrak{P}_d$, then $\lambda(X) = 0$. If $\lambda(X) = 0$, $[A, X] \in \mathfrak{P}_{ac}$ if and only if $[A, X] \in \mathfrak{P}_{bd}$, so $\mathfrak{P}_{ad} = \mathfrak{P}_{bd}$. $\mathfrak{P}_{bc} \neq \mathfrak{P}_{bd}$.*

If $[A, X] \in \mathfrak{P}_d$, there exist α_x in A^* and a sequence $\{x_n\} \subset X$ such that $\|\alpha_x\| > n$ if $x > x_n$; if $\lambda(X) > 0$, then an x_0 must exist following all x_n so that $\|\alpha_{x_0}\| = \infty$, which is impossible. If $\lambda(X) = 0$ and $[A, X] \in \mathfrak{P}_{ac}$, let $\{x_n\}$ be a monotone sequence with no upper bound in X and let α_x in A^* satisfy (a) and (c). Define h on X to X so that $h(x) > x$ for every x while $\|\alpha_{h(x)}\| > n^2$ if $x_n < x \not> x_{n+1}$; let $h(x) = x$ if $x \not> x_1$. Let $\alpha_x' = (1/n)\alpha_{h(x)}$ if $x_n < x \not> x_{n+1}$, $\alpha_x' = \alpha_x$ if $x \not> x_1$; then $\lim_x \alpha_x'(a) = 0$ for every a while $\lim_x \|\alpha_x\| = \infty$. Since $\mathfrak{P}_{bd} \subset \mathfrak{P}_{ad} \subset \mathfrak{P}_{ac}\mathfrak{P}_d$, $\mathfrak{P}_{bd} = \mathfrak{P}_{ad}$. If X is the set of denumerable ordinals, or any other set such that $\lambda(X) > 0$, $[c_0(X, B_0), X] \in \mathfrak{P}_{bc}$ by the construction in Theorem 3.4; by the first statement of this theorem $[c_0(X, B_0), X] \notin \mathfrak{P}_{bd}$.

COROLLARY 3.5. *If $[A, Y] \in \mathfrak{P}_{ac}$, if $X > Y$, and if $X > \Delta^0$, then $[A, X] \in \mathfrak{P}_{bd}$.*

The major problem remaining here is to reduce the number of pairs whose class is unknown. Other problems are these: (1) Is $\mathfrak{P}_{bc} = \mathfrak{P}_{ac}$? (2) A corollary of Theorem 4.2 is that if A' is the range of a projection in A and if $[A', X]$ is in some class, then $[A, X]$ is in the same class; is this true if A' is any closed linear subset of A? (3) A special case of (2) is to decide whether or not

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will note that the characterization sought in this section is not to involve the special choice of α_x; in that sense Hildebrandt's theorem is a useful tool but not a result of the desired type.

there is an X such that $[c_0, X]$ is in some class while $[m, X]$ is not. This might also be settled by the answer to (4). Is $[A^*, X]$ in one of these classes if $[A, X]$ is?

4. **Totally measurable functions and real operators.** In this section let Y be an abstract set, let Υ be a field(17) of subsets of Y, and let B be a Banach space. If E is any set in Υ, let ϕ_E, the characteristic function of E, be 1 on E and 0 on its complement. A function f on Y with values in B is called *simple* if there exist a finite number of sets E_i in Υ and points b_i in B such that $f=\sum_{i\leq k}\phi_{E_i}b_i$. Let V be the space of all functions f on Y to B for which there exists a sequence $\{f_n\}$ of simple functions which converges uniformly to f; if $\|f\|=\sup_{y\subset Y}\|f(y)\|$, V is a Banach space. In the special case in which $B=B_0$, the space of real numbers, call the space V_r. If β is any element of B^* and $f\in V$, the function βf defined by $\beta f(y)=\beta(f(y))$ is in V_r and $\|\beta f\|\leq\|\beta\|\,\|f\|$; if $\phi\in V_r$ and $b\in B$, the function ϕb defined by $\phi b(y)=\phi(y)b$ is in V and $\|\phi b\|=\|\phi\|\,\|b\|$; moreover these functions of the form ϕb, with ϕ in V_r and b in B, form a fundamental set in V, since every $\phi_E b$ is of this form. Also if $\phi\in V_r$ there exist β in B^* and b in B such that $\beta\phi b=\phi$; in fact any choice such that $\beta(b)=1$ will do.

Let $\mathfrak{U}$ be $V:B$, the space of linear operators on V to B. Gowurin [13] has shown that each U in $\mathfrak{U}$ can be defined by means of a certain integral: For each $E\in\Upsilon$ define $\Phi(E)$ in $B:B$ by the relation $\Phi(E)b=U(\phi_E b)$ for every b in B; then (1) each $\Phi(E)$ is a linear operator on B to B, (2) Φ is *additive*; that is, $\Phi(E_1)+\Phi(E_2)=\Phi(E_1+E_2)$ if E_1 and E_2 are disjoint sets in Υ and (3) Φ is *limited*; that is,

$$W\Phi(Y) = \sup\left\|\sum_{i\leq k}\Phi(E_i)b_i\right\| < \infty$$

where the supremum is taken over all choices of b_i with $\|b_i\|\leq 1$ and all partitions of Y into a finite number of disjoint sets $E_1, \cdots, E_k$ in Υ(18). On the other hand each Φ satisfying these three conditions defines a U in $\mathfrak{U}$ by means of the Gowurin integral: If $f=\sum_{i\leq k}\phi_{E_i}b_i$, let $\int fd\Phi=\sum_{i\leq k}\Phi(E_i)b_i$; then $\|\int fd\Phi\|\leq W\Phi(Y)\|f\|$ so $\|\int(f_n-f_m)d\Phi\|\to 0$ if $\|f_n-f_m\|\to 0$. If $f\in V$, let $\{f_n\}$ be a sequence of simple functions converging to f and let $\int fd\Phi=\lim_n\int f_n d\Phi$. If $U(f)=\int fd\Phi$, then $U\in\mathfrak{U}$, $\|U\|=W\Phi(Y)$, and Φ is derived from U by the relation $U(\phi_E b)=\Phi(E)b$ for every b in B.

Some of these set functions are of the form $\Phi(E)=\Psi(E)I$ where I is the

(17) Υ is a field if finite sums of sets in Υ are in Υ, if $Y\in\Upsilon$ and if complements of sets in Υ are in Υ. The reader is asked to distinguish between Υ, used here, and Υ used later in this section.

(18) It is easily verified that if Φ is limited and additive and if $W\Phi(E)=\sup\|\sum_{i\leq k}\Phi(E_i)b_i\|$, where the supremum is taken over all finite partitions of E into disjoint sets $E_1, \cdots, E_k$ in Υ, and sequences of points $b_1, \cdots, b_k$ of norm less than or equal to 1, then $W\Phi$ is a bounded, real-valued non-negative function on Υ such that $W\Phi(E_1)\leq W\Phi(E_1+E_2)\leq W\Phi(E_1)+W\Phi(E_2)$ for every pair of disjoint sets in Υ.

identity transformation in B and Ψ is a bounded, additive, real-valued function on Υ. For such a Φ write $U(f)=\int fd\Psi$ instead of $\int fd\Phi$; then $\|U\|=V\Psi(Y)=\sup\sum_{i\leq k}|\Psi(E_i)|$ where the supremum is taken over all partitions of Y into a finite number of disjoint sets of Υ. All the elements of V_r^* are of the form $\Upsilon(\phi)=\int\phi d\Psi$ for some bounded, additive real-valued Ψ defined on Υ([19]). The correspondence τ associating U in $V:B$ and $\tau U=\Upsilon$ in V_r^* if $\Upsilon(\phi)=\int\phi d\Psi$ for all ϕ in V_r and $U(f)=\int fd\Psi$ for all f in V is one-to-one and norm preserving between V_r^* and a subset $\mathfrak{U}_r$ of $V:B$. The reader can easily prove the following results:

(1) *If* $\beta\in B^*$, $U\in\mathfrak{U}_r$ *and* $f\in V$, *then* $\beta(U(f))=\tau U(\beta f)$.

(2) $U\in\mathfrak{U}_r$ *if and only if* $U(f)$ *is a linear combination of the values of* f *whenever* f *is a simple function in* V.

Since $\mathfrak{U}_r$ is a subset of $\mathfrak{U}$, the topologies defined in §2 impose three topologies in $\mathfrak{U}_r$. If $\mathfrak{N}$, $\mathfrak{S}$ and $\mathfrak{W}$ are the norm, s^* and w^* neighborhood systems of θ in $\mathfrak{U}$, let $\mathfrak{N}_r$, $\mathfrak{S}_r$ and $\mathfrak{W}_r$ be the intersection of these with $\mathfrak{U}_r$; that is, $N_r\in\mathfrak{N}_r$ if there is an N in $\mathfrak{N}$ such that $N_r=N\mathfrak{U}_r$, and similarly for $\mathfrak{S}_r$ and $\mathfrak{W}_r$.

THEOREM 4.1. $\mathfrak{N}_r\simeq\mathfrak{S}_r$ *if and only if* V_r *is* fd; $\mathfrak{S}_r\simeq\mathfrak{W}_r$ *if and only if one of the spaces* B *or* V_r *is* fd. *Hence* $\mathfrak{N}_r\simeq\mathfrak{W}_r$ *if and only if* V_r *is* fd.

If V_r is fd, there exist $\phi_1, \cdots, \phi_k$ in V_r which form a basis in V_r, so that for each $\epsilon>0$ there is a $\delta>0$ such that $\|\Upsilon\|<\epsilon$ if $|\Upsilon(\phi_i)|<\delta$ for $i=1, \cdots, k$. Take β_i in B^* and f_i in V, say $f_i=\phi_ib_i$, such that $\beta_if_i=\phi_i$ for $i=1, \cdots, k$. If U is in $W_r=W_r(f_1, \cdots, f_k, \beta_1, \cdots, \beta_k; \delta)$, then $|\beta_i(U(f_i))|<\delta$ for all i, but

$$|\beta_i(U(f_i))|=|\Upsilon(\beta_if_i)|=|\Upsilon(\phi_i)|$$

so $\|U\|=\|\Upsilon\|<\epsilon$ and $U\in N_r(\epsilon)$; that is, if V_r is fd, there is a W_r contained in each N_r, so $\mathfrak{N}_r\simeq\mathfrak{S}_r\simeq\mathfrak{W}_r$.

If B is fd, $\mathfrak{S}\simeq\mathfrak{W}$; so $\mathfrak{S}_r\simeq\mathfrak{W}_r$.

If V_r is not fd, consider two classes of neighborhoods $S_r(f_1, \cdots, f_k; \epsilon)$, first taking all f_i to be simple functions. Let $f_i=\sum_{j\leq k_i}\phi_{E_{ij}}b_{ij}$; then, for every β in B^*, $\beta f_i=\sum_{j\leq k_i}\phi_{E_{ij}}\beta(b_{ij})$, so the set of functions $\{\beta f_i \mid \beta \text{ in } B^*, i=1, \cdots, k\}$ can all be expressed as linear combinations of the characteristic functions of the finite number of sets E_{ij}. Hence the smallest linear manifold in V_r containing all the βf_i is fd and therefore does not fill up V_r; by a lemma of Banach [3, p. 57] there exists an Υ in V_r^* such that $\Upsilon(\beta f_i)=0$ for all f_i and β, while $\|\Upsilon\|>0$. Let $U=\tau^{-1}\Upsilon$, then $\beta(U(f_i))=\Upsilon(\beta f_i)=0$ so $U(f_i)=\theta$ for each i while $\|U\|=\|\Upsilon\|>0$. For each $K>0$ there is a point $U'=KU/\|U\|$ such that $\|U'\|=K$ and $U'\in S_r(f_1, \cdots, f_k; \epsilon)$.

This holds if the f_i are simple functions; if $f_1, \cdots, f_k$ are any functions in V, there exist k sequences of simple functions $\{f_{in}\}$ such that $\|f_{in}-f_i\|<2^{-n}$

([19]) See Kantorovich and Fichtenholz [11], or specialize Gowurin's integral.

for each $i \leqq k$. Hence, for any $U \in \mathfrak{U}_r$, $\|U(f_{in}-f_i)\| < 2^{-n}\|U\|$. For any $K>0$ and $\epsilon>0$ take n_0 so that $2^{-n_0} < \epsilon/4K$; then in $S(f_{in_0}, \cdots, f_{kn_0}; \epsilon/4)$ there is a point U with $\|U\| = K$. $\|U(f_{in_0}-f_i)\| < K2^{-n_0} < \epsilon/4$ so $\|U(f_i)\| < \epsilon/2$ and $U \in S_r(f_1, \cdots, f_k; \epsilon)$ while $\|U\| = K$. This construction can be carried through for any S_r in $\mathfrak{S}_r$; so no S_r lies in an N_r if V_r is not fd.

If neither V_r nor B is fd, there is an f_0 in V such that no fd subspace of B contains all the values of f_0; consider $S_r(f_0; \epsilon)$ and any $W_r = W_r(f_1, \cdots, f_k; \beta_1, \cdots, \beta_k; \delta)$. Let $y_1, \cdots, y_{k+1}$ be points of Y for which the $k+1$ points $f_0(y_i)$ in B are linearly independent. Then $k+1$ numbers t_j exist which are not all zero and which satisfy the k equations $\sum_{j \leqq k+1} t_j \beta_i(f_i(y_j)) = 0, i = 1, \cdots, k$. Let U in $\mathfrak{U}_r$ be defined by $U(f) = \sum_{j \leqq k+1} t_j f(y_j)$; then $U \in W_r$ since $\beta_i(U(f_i)) = \beta_i(\sum_{j \leqq k+1} t_j f_i(y_j)) = \sum_{j \leqq k+1} t_j \beta_i(f_i(y_j)) = 0$, while $\|U(f_0)\| > 0$ since the points $f_0(y_1), \cdots, f_0(y_{k+1})$ are linearly independent. For every K, $KU \in W_r$ but for K large enough $KU \notin S_r$; hence $\mathfrak{S}_r$ is not equivalent to $\mathfrak{W}_r$ in this case.

One remark on the problem of boundedness may be made for such spaces and operations.

COROLLARY 4.1. *For any B, V_r is a linear image of V; hence $\eta(V) \leqq \nu(V_r)$. If $X > \mathfrak{S}_r$ and V_r is not* fd, *U_x in $\mathfrak{U}_r$ can be chosen so that* $\lim_x \|U_x(f)\| = 0$ *for every f while* $\lim_x \|U_x\| = \infty$. *If V_r is* fd, *then no X and U_x in $\mathfrak{U}_r$ exist which satisfy these conditions.*

Note that V_r is fd if and only if Y has only a finite number of distinct elements; V is fd if and only if both B and V_r are fd.

The next lemma is used in §6.

LEMMA 4.1. *If U_x and $U_0 \in \mathfrak{U}_r$ and if $\Upsilon_x = \tau U_x$, then $U_0 = w^* - \lim_x U_x$ if and only if $\Upsilon_0 = \tau U_0 = w^* - \lim_x \Upsilon_x$.*

$U_0 = w^* - \lim_x U_x$ if and only if $\lim_x \beta(U_x(f)) = \beta(U_0(f))$ for every β in B^* and f in V; that is, if and only if $\Upsilon_0(\beta f) = \lim_x \Upsilon_x(\beta f)$ for every β in B^* and f in V. But the set of such βf fills up V_r; so this is true if and only if $\Upsilon_0(\phi) = \lim_x \Upsilon_x(\phi)$ for every ϕ in V_r; that is, if and only if $\Upsilon_0 = w^* - \lim_x \Upsilon_x$[20].

5. **A completely additive integral.** That the class of totally measurable functions is rather limited is clear from the fact that the set of values of such a function f is a totally bounded subset of B; that is, for each $\epsilon > 0$ there is a finite set of spheres of radius ϵ which together cover the set of values of f. If a measurable function is defined to be a function which is the limit of a pointwise convergent sequence of simple functions, then the class of bounded, measurable functions includes the class of totally measurable functions; the two classes are the same only if B is fd or if Y has only a finite number of

[20] This lemma is a restatement of the fact that the equivalence τ of V_r^* and $\mathfrak{U}_r$ carries $\mathfrak{W}_r$ into the set of weak* neighborhoods of θ in V_r^*. Note that the proof of (2) of Theorem 6.2 shows that the w^* and s^* topologies are the same in the unit sphere of $\mathfrak{U}_r$ although they are different in the unit sphere of $\mathfrak{U}$ itself.

distinct elements. Birkhoff [4], Bochner [6], Dunford [10], Gelfand [12], Pettis [21], Phillips [22], and Price [24], among others, have defined and studied integrals of a Lebesgue-like nature for functions with values in a Banach space; Gowurin [13] and Bochner and Taylor [7] have considered a "Riemann-Stieltjes" integral; as far as I know, no attempt except in [24] has been made to define a completely additive integral similar to Gowurin's.

In this section Y is any set and B is a Banach space(21); Υ is further restricted to be a σ-field(22). A function f on Y to B will be called *half-simple* if it is bounded; that is, if there is a $K>0$ such that $\|f(y)\| \leqq K$ for all y, and if there exist a countable number of disjoint sets $\{E_i\}$ in Υ such that f has the constant value b_i on E_i.

LEMMA 5.1. *$\mathfrak{B}$, the class of bounded measurable functions on Y to B, is the class of all functions on Y to B which can be uniformly approximated by half-simple functions; hence $\mathfrak{B}$ is a Banach space if $\|f\| = \sup_y \|f(y)\|$.*

Every half-simple function is in $\mathfrak{B}$; so every f which can be approximated uniformly by half-simple functions $\{f_n\}$ is in $\mathfrak{B}$; the construction of a sequence of simple functions converging to f is as follows: Suppose that $f_n(y) = b_{in}$ if $y \in E_{in}$ and enumerate $\{E_{in}\}$ and $\{b_{in}\}$ as single sequences $\{E_j'\}$ and $\{b_j'\}$. For each j let E_{kj}'' be any enumeration of the disjoint sets obtained by intersecting all possible combinations of the E_i', $i \leqq j$, and their complements. Define f_j' by $f_j'(y) = \theta$ on $Y - \sum_{i \leqq j} E_i'$, $f_j'(y) = b_{kj}''$ on E_{kj}'' where b_{kj}'' is a b_i', $i \leqq j$, for which $\sup_{y \in E_{kj}''} \|b_i' - f(y)\|$ is a minimum; then the f_j' are simple functions and converge pointwise to f. If $f \in \mathfrak{B}$, there is a sequence $\{f_n\}$ of simple functions such that $\|f_n(y) - f(y)\| \to 0$ for each y in Y and $\|f_n\| \leqq \|f\|$; enumerate the values of these f_n in a sequence $\{b_i\}$. For any $\epsilon > 0$ let $E_i' = \{y \mid \|f(y) - b_i\| < \epsilon\}$; then $\sum_i E_i' = Y$ and each $E_i' \in \Upsilon$, because $y \in E_i'$ if and only if there is an n such that, for every $k > n$, $\|f_k(y) - b_i\| < \epsilon$; that is, if and only if $y \in \sum_n \prod_{k>n} E_{ki}$ where $E_{ki} = \{y \mid \|f_k(y) - b_i\| < \epsilon\}$. Each E_{ki} is in Υ, so each E_i' is also in Υ; let $E_1 = E_1'$, $E_{i+1} = E_{i+1}' - \sum_{k \leqq i} E_k'$, and define f_ϵ by $f_\epsilon(y) = b_i$ if $y \in E_i$. Since the E_i are disjoint, are in Υ, and cover Y, f_ϵ is half-simple; clearly $\|f - f_\epsilon\| < \epsilon$ so every element of $\mathfrak{B}$ can be approximated uniformly by half-simple functions.

From this lemma it is clear that the difficulties of defining an integral for functions in $\mathfrak{B}$ are mostly concentrated in defining the integral of every half-simple function. Gelfand [12] calls a series $\sum_i b_i$ of points of a Banach space B *unconditionally convergent* if $\sum_i |\beta(b_i)| < \infty$ for every β in B^*. If Δ^0 is the stack whose elements are the finite sets, δ, of positive integers, Alaoglu has shown

(21) Throughout this section we shall consider that B is imbedded in its second conjugate space B^{**} by the usual transformation associating b in B with $\mathfrak{b}_b$ in B^{**} if $\mathfrak{b}_b(\beta) = \beta(b)$ for every β in B^*.

(22) Υ is a σ-field (sometimes called a Borel field) if Υ is a field and if the sum of any countable collection of sets of Υ is a set of Υ.

that $\sum_i b_i$ converges unconditionally if and only if $\lim_\delta \sum_{i\in\delta} \beta(b_i)$ converges for every β in B^* and if and only if $\sup_\delta \|\sum_{i\in\delta} b_i\| < \infty$. Let the *sum* of the unconditionally convergent series $\sum_i b_i$ be that point $\mathfrak{b}$ of B^{**} for which $\mathfrak{b}(\beta) = \lim_\delta \beta(b_\delta)$ for every β in B^*, where $b_\delta = \sum_{i\in\delta} b_i$; then $\|\mathfrak{b}\| \leqq \lim\sup_\delta \|b_\delta\|$.

If Φ is a limited, additive function defined over the σ-field Υ with values in $B:B$, if $\{E_i\}$ is any sequence of disjoint sets in Υ, and if $\{b_i\}$ is any sequence of points of norm not exceeding 1, then $\sum_i \Phi(E_i)b_i$ is unconditionally convergent because $\|\sum_{i\in\delta} \Phi(E_i)b_i\| \leqq W\Phi(Y)$ for every δ. Φ is called completely additive (symbol: ca) if for each b in B and sequence $\{E_i\}$ of disjoint sets of Υ, $\sum_i \Phi(E_i)b = \Phi(\sum_i E_i)b$; complete additivity clearly implies finite additivity. A theorem of Orlicz [3, p. 240, (3)] asserts that if every subseries of a given series converges in Gelfand's sense to a point of B, then the series converges in Orlicz' sense; that is, $\lim_\delta \|\sum_{i\in\delta} b_i - \sum_i b_i\| = 0$. Hence, if Φ is ca,

$$\left\| \sum_{i\in\delta} \Phi(E_i)b - \Phi\left(\sum_i E_i\right)b \right\| \to 0$$

for every b in B and sequence $\{E_i\}$ of disjoint sets of Υ.

If Φ is ca and limited and if f is a half-simple function with the values b_i on the disjoint sets E_i of Υ, let $\int f d\Phi$ be the sum of the series $\sum_i \Phi(E_i)b_i$; from the complete additivity of Φ it is easily shown that this sum is independent of the decomposition $\{E_i\}$ as long as f is constant over each set E_i. The argument used to show that $\sum_i \Phi(E_i)b_i$ is unconditionally convergent also shows that $\|\int f d\Phi\| \leqq \|f\| W\Phi(Y)$ if f is half-simple. If f is any element of $\mathfrak{V}$, let $\{f_n\}$ be a sequence of half-simple functions converging uniformly to f; then the points $\mathfrak{b}_n = \int f_n d\Phi$ form a Cauchy sequence in B^{**} and must converge to some point of B^{**}; let $\int f d\Phi = \lim_n \int f_n d\Phi$. This value is easily shown to be independent of the choice of the sequence $\{f_n\}$ converging uniformly to f. If $U(f) = \int f d\Phi$, then U is a linear operator on V with values in B^{**} and $\|U\| = W\Phi(Y)$.

In many cases it is desirable to have $\int f d\Phi$ in B for every f in $\mathfrak{V}$. This is equivalent to requiring that $\int f d\Phi$ be in B for every half-simple function f; that is, to requiring that $\sum_i \Phi(E_i)b_i$ be in B for every sequence $\{E_i\}$ of disjoint sets of Υ and every bounded sequence $\{b_i\}$ of points of B. By the theorem of Orlicz mentioned before, this is equivalent to requiring that $\sum_i \Phi(E_i)b_i$ converge in Orlicz' sense for every such choice of $\{E_i\}$ and $\{b_i\}$; Φ will be called *convergent* if this last condition holds. From this we have the following result.

THEOREM 5.1. *If Φ is limited and* ca, *$\int f d\Phi \in B$ for each f in $\mathfrak{V}$ if and only if Φ is convergent; hence $\int f d\Phi \in B$ if B is weakly sequentially complete*(23) *or is*

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(23) B is weakly sequentially complete if the existence of $\lim_n \beta(b_n)$ for every β in B^* implies that a b_0 in B exists such that $\lim_n \beta(b_n) = \beta(b_0)$ for every β in B^*.

reflexive or if Φ *is of bounded variation*(24) *or if* $\Phi = \Psi T$ *where* Ψ *is a real-valued,* ca *set-function and* T *is any element of* $B:B$.

THEOREM 5.2. *If* Φ *is additive and convergent,* $W\Phi(Y) < \infty$; *if* Φ *is* ca, *then* $W\Phi(\sum_i E_i) = \lim_n W\Phi(\sum_{i<n} E_i)$; *if* Φ *is* ca *and convergent, then* $W\Phi(E_k)$ *decreases to zero for every decreasing sequence* $\{E_k\}$ *of sets of* Υ *with empty intersection.*

Suppose that Φ is additive and convergent and that $W\Phi(Y) = \infty$; say that a set Y_0 of Υ has property (A) if Y_0 has two disjoint subsets Y_0' and Y_0'' such that $W\Phi(Y_0') = \infty$ and $W\Phi(Y_0'') = \infty$. Then only a finite number of disjoint sets of Υ can have property (A) for if an infinite number have this property, then there would exist a sequence $\{Y_i\}$ of disjoint sets of Υ such that $W\Phi(Y_i) = \infty$ for each i. Choice of E_{ij} in Υ and b_{ij} of norm not greater than one could then be made so that the series $\sum_{ij} \Phi(E_{ij})b_{ij}$ reordered in any way as a simple series would have unbounded partial sums, thus contradicting convergence of Φ.

Therefore there exist disjoint sets $Y_1, \cdots, Y_k$, $k \geqq 1$, whose sum is Y, such that $E \subset Y$, and $W\Phi(Y_i - E) = \infty$ imply that $W\Phi(E) < \infty$; let Y_0 represent any one of these Y_i. Define the sequence $\{E_i\} \subset \Upsilon$ of subsets of Y_0 as follows: If among the sets $E \subset Y_0$ such that $W\Phi(Y_0 - E) = \infty$ there are any such that $W\Phi(E) \neq 0$, let n be the smallest integer such that such an E_1 exists with $W\Phi(E_1) > 1/n$; if E_i, $i<k$, are defined and disjoint, let n_k be the smallest integer such that a subset E_k of $Y_0 - \sum_{i<k} E_i$ has $W\Phi(E_k) > 1/n_k$ and $W\Phi(Y_0 - E_k) = \infty$.

If an n_0 exists such that $W\Phi(E_i) > 1/n_0$ for an infinite sequence of these E_i, then a series $\sum_{ij} \Phi(E_{ij}')b_{ij}'$ could be found with partial sums not tending to zero which again contradicts convergence, so $W\Phi(E_i) \to 0$ as $i \to \infty$. Consider $E_0 = Y - \sum_i E_i$, and $\sum_i E_i$; by the definition of Y_0 either $W\Phi(E_0)$ or $W\Phi(\sum_i E_i) < \infty$. If $W\Phi(\sum_i E_i) < \infty$, then $W\Phi(E_0) = \infty$ since $W\Phi(Y_0) = \infty$; hence there exist a sequence $\{E_{kj}\}$ of partitions of E_0 and a sequence $\{b_{kj}\}$ of points of norm not exceeding 1 such that

$$\left\| \sum_{j \leqq n} \Phi(E_{kj})b_{kj} \right\| > k;$$

the partition $\{E_{k+1j}\}$ may be made a refinement of the partition $\{E_{kj}\}$. By definition of E_0, if $E \subset E_0$ and $W\Phi(E_0 - E) = \infty$, then $W\Phi(E) = 0$, since otherwise E would have been chosen among the E_i at some stage. Hence for each k there is a j_k such that $\|\Phi(E_{kj})\| = 0$ if $j \neq j_k$, so, letting $E_k' = E_{kj_k}$ and $b_k' = b_{kj_k}$, $\|\Phi(E_k')b_k'\| > k$ and $W\Phi(E_k') = \infty$. Therefore $E_{k+1}' \subset E_k'$ and $W\Phi(E_k' - E_{k+1}') = 0$, so $\Phi(E_{k+1}') = \Phi(E_k')$ for every k. It follows that $\|\Phi(E_1')b_k'\| \to \infty$ while

(24) Φ is of bounded variation if $\sup \sum_{i \leqq k} \|\Phi(E_i)\| < \infty$ where the supremum is taken over all partitions of Y into a finite number of disjoint sets of Υ.

$\|b_k'\| \leqq 1$, but this is impossible since $\Phi(E_1)$ is a linear operator on B, so $W\Phi(E_0) < \infty$.

This leaves the alternative hypothesis that $W\Phi(\sum_i E_i) = \infty$. A more careful repetition of the preceding argument, letting $\{E_{kj}\}$ be a partition of $\sum_{i>k} E_i$, leads to a contradiction here. This shows that $W\Phi(Y_0)$ must be finite, but Y_0 was any one set in a finite partition of Y so $W\Phi(Y) < \infty$ also.

Next assume that Φ is ca. Clearly the sets E_i mentioned can be taken to be disjoint; suppose then that

$$W\Phi\left(\sum_i E_i\right) > K = \lim_n W\Phi\left(\sum_{i\leqq n} E_i\right);$$

then there exists a partition $E^1, \cdots, E^k$ of $\sum_i E_i$ into disjoint sets of Υ and a set of points $b_1, \cdots, b_k$ of norm not greater than 1 such that

$$\left\| \sum_{j\leqq k} \Phi(E^j)b_j \right\| > K + 2\epsilon$$

for some $\epsilon > 0$. By complete additivity of Φ, n can be chosen so large that

$$\left\| \Phi(E^j)b_j - \sum_{i\leqq n} \Phi(E_iE^j)b_j \right\| < \frac{\epsilon}{2k}$$

for each $j \leqq k$, so $\|\sum_{j\leqq k} \sum_{i\leqq n} \Phi(E_iE^j)b_j\| > K + \epsilon$. The sets E_iE^j, $i \leqq n$, $j \leqq k$, form a partitition of $\sum_{i\leqq n} E_i$ so $K + \epsilon < W\Phi(\sum_{i\leqq n} E_i) \leqq K$; this contradiction shows that $W\Phi(\sum_i E_i) \leqq \lim_n W\Phi(\sum_{i\leqq n} E_i)$. Since $W\Phi(E)$ increases with E, the conclusion holds.

The other conclusion is a simple consequence of these two; the assumptions that $W\Phi(E_k) \downarrow 2\epsilon > 0$ and that Φ is ca show that there exists a sequence $\{k_i\}$ such that $W\Phi(E_{k_i} - E_{k_{i+1}}) > \epsilon$ for every i; this contradicts convergence.

If $B = B_0$, the space of real numbers, each $\Phi(E)$ is a real number and Φ is convergent if and only if $\sum_i \Phi(E_i)t_i$ converges for every bounded sequence $\{t_i\}$ of real numbers and every sequence $\{E_i\}$ of disjoint sets of Υ; that is, if and only if $\sum_i |\Phi(E_i)| < \infty$ for each such sequence $\{E_i\}$.

COROLLARY 5.1. *If and only if the real-valued, additive set-function Ψ on Υ has the property that $\sum_i \Psi(E_i)$ converges for every sequence $\{E_i\}$ of disjoint sets of Υ, $V\Psi(Y) < \infty$.*

This is true since $V\Psi(Y) = W\Psi(Y)$ in this case. Since a ca real-valued set-function has the property that

$$\sum_i |\Psi(E_i)| = \Psi(E') + |\Psi(E'')|$$

where E' is the sum of those E_i such that $\Psi(E_i) \geqq 0$, and E'' is the sum of those E_i such that $\Psi(E_i) < 0$; from this follows a well known theorem.

COROLLARY 5.2. *A* ca *real-valued set-function is of bounded variation.*

If B is any fd space and the values of Φ are in $B:B$, both these corollaries hold for such a Φ, although $V\Phi(Y)$ and $W\Psi(Y)$ are no longer so simply related.

THEOREM 5.3. *A linear operator U on V to B can be expressed in the form $U(f)=\int fd\Phi$ where Φ is* ca *and convergent if and only if $U(f)=\lim_n U(f_n)$ whenever $\|f_n\|$ is uniformly bounded and f_n converges pointwise to f.*

If f_n converges pointwise to f, if $E_{kn}=\{y \mid \|f(y)-f_m(y)\|>1/k \text{ if } m>n\}$, then for each k, $E_{kn}\downarrow 0$, so by the preceding lemma $\lim_n W\Phi(E_{kn})=0$ for each k. Then

$$\|U(f)-U(f_n)\| \leqq \left\|\int_{E_{kn}}(f-f_n)d\Phi\right\| + \left\|\int_{Y-E_{kn}}(f-f_n)d\Phi\right\|.$$

For given $\epsilon>0$ take $k>1/\epsilon$ and take n so large that $W\Phi(E_{kn})<\epsilon$; then

$$\|U(f)-U(f_n)\| < \epsilon\|f-f_n\| + \epsilon W\Phi(Y).$$

If U satisfies the last condition of the theorem, let $\Phi(E)$ be the operator on B to B defined by $\Phi(E)b=U(\phi_E b)$ for each b in B. Then Φ is additive and limited since $\|U\|=W\Phi(Y)$. If f is a simple function, $U(f)=\int fd\Phi$ is an element of B. If f is half-simple with values b_i on sets E_i,

$$\left\|\sum_{i\in\delta}\Phi(E_i)b_i\right\| = \left\|U\left(\sum_{i\in\delta}\phi_{E_i}b_i\right)\right\| \leqq \|f\|\,\|U\|,$$

but no matter in what order the sets E_i are arranged $\lim_n\sum_{i\leqq n}U(\phi_{E_i}b_i)=U(f)$ so $\sum_i \Phi(E_i)b_i$ is convergent in Orlicz' sense and Φ is convergent.

$$\sum_i \Phi(E_i)b = \lim_n \sum_{i\leqq n} U(\phi_E b_i) = \lim_n U\left(\sum_{i\leqq n}\phi_{E_i}b\right) = \lim_n U\left(\phi_{\sum_{i\leqq n} E_i}b\right)$$
$$= \Phi\left(\sum_i E_i\right)b$$

so Φ is ca.

In case Φ is equal to ΨI, where Ψ is real-valued, this integral is consistent with, say, Dunford's integral for bounded, measurable functions.

A desirable property for an integral is this: If f is in $\mathfrak{B}$ and T is in $B:B$, then $T\int fd\Phi=\int Tfd\Phi$, where Tf is the function in $\mathfrak{B}$ such that $Tf(y)=T(f(y))$. With this integral this does not hold for all T and f for two reasons, $\int fd\Phi$ may not lie in B, and T may not commute with all $\Phi(E)$. The first difficulty can be avoided; if T is an operator on B to B, define T^*, the adjoint[25] of T, to

[25] Banach [3, p. 99] calls T^* the conjugate of T and represents it by $\overline{T}$.

be that operator on B^* to B^* such that $T^*\beta(b)=\beta(Tb)$ for every b in B and β in B^*. Then (1) $\|T^*\|=\|T\|$, (2) $(T_1T_2)^*=T_2^*T_1^*$ and (3) if $T^{**}=(T^*)^*$, the operator T^{**} agrees with T over B.

THEOREM 5.4. *If Φ is limited and* ca, *T commutes with all $\Phi(E)$ if and only if $\int Tfd\Phi=T^{**}\int fd\Phi$ for every f in $\mathfrak{V}$.*

If there is an E in Υ such that $\Phi(E)$ does not commute with T, let b be a point such that $T\Phi(E)b\neq\Phi(E)Tb$ and let $f=\phi_Eb$; then $T^{**}\int fd\Phi=T\Phi(E)b \neq \Phi(E)Tb=\int Tfd\Phi$.

If f is half-simple with values b_i on disjoint sets E_i, Tf has values Tb_i on the same sets E_i. $\int fd\Phi$ is that point $\mathfrak{b}$ of B^{**} for which

$$\mathfrak{b}(\beta) = \lim_\delta \sum_{i\in\delta}\beta(\Phi(E_i)b_i) = \lim_\delta \beta\left(\sum_{i\in\delta}\Phi(E_i)b_i\right).$$

$$\begin{aligned} T^{**}\mathfrak{b}(\beta) &= \mathfrak{b}(T^*\beta) = \lim_\delta T^*\beta\left(\sum_{i\in\delta}\Phi(E_i)b_i\right) = \lim_\delta \beta\left[T\left(\sum_{i\in\delta}\Phi(E_i)b_i\right)\right] \\ &= \lim_\delta \beta\left(\sum_{i\in\delta}T\Phi(E_i)b_i\right) = \lim_\delta \beta\left(\sum_{i\in\delta}\Phi(E_i)Tb_i\right) \\ &= \lim_\delta \sum_{i\in\delta}\beta(\Phi(E_i)Tb_i). \end{aligned}$$

But $\int Tfd\Phi$ is the point $\mathfrak{b}_1$ of B^{**} for which

$$\mathfrak{b}_1(\beta) = \lim_\delta \sum_{i\in\delta}\beta(\Phi(E_i)Tb_i) = T^{**}\mathfrak{b}(\beta)$$

for every β in B^*.

Since all the operators involved are continuous and since the half-simple functions are dense in $\mathfrak{V}$, the conclusion follows.

COROLLARY 5.3. *If Ψ is* ca *and real-valued, then $T\int fd\Psi=\int Tfd\Psi$ for every f in $\mathfrak{V}$ and every linear operator T on B to B.*

In this case $V\Psi(Y)<\infty$ so $\int fd\Psi\in B$ for each f in $\mathfrak{V}$. Every T in $B:B$ commutes with multiplication by real numbers.

This section closes with some examples. Let Y be the class of integers and let Υ be the class of all subsets of Y; for every real-valued function f on Y and every set E in Υ let $\Phi(E)f=\phi_Ef$; that is, $\Phi(E)f(n)=f(n)$ if $n\in E$, $\Phi(E)f(n)=0$ if $n\notin E$. If B is $l_p=l_p(Y, B_0)$, $1\leqq p<\infty$, then $\Phi(E)f\in B$ if f does, so each $\Phi(E)\in B:B$ as $\|\Phi(E)\|\leqq 1$ for every E. This function is ca since $\sum_i \Phi(E_i)b=\Phi(\sum_i E_i)b$ for every b and sequence $\{E_i\}$ of disjoint sets in Y. However, $W\Phi(Y)=\infty$ so the theorem that a real-valued ca set-function is of bounded variation is not true if the words "real-valued" are deleted, even if "limited" replaces "of bounded variation."

If $B=c_0$ instead of l_p, each $\Phi(E)$ again defines a linear operator of norm less than or equal to 1 on B to B. Since $W\Phi(E)=1$ on every non-empty set E in Y, this gives an example of a function such that $W\Phi(Y)<\infty$ while Φ is not convergent. For example, defining b_i in B by $b_n(n)=1$, $b_i(n)=0$ if $i\neq n$, gives a sequence of points such that[26] $\sum_n \Phi((n))b_n$ is in m instead of in c_0 since the sum is that f for which $f(n)=1$ for every n.

For one more example, take $B=m$ and let Φ be defined by $\Phi(E)b=\phi_E f_0 b$, where f_0 is the function for which $f_0(n)=1/n$. Then $\|\Phi(E)\|=W\Phi(E)$ $=\sup_{n\in E} 1/n$, so Φ is convergent, but Φ is not of bounded variation since $\sum_n \|\Phi((n))\| =\sum_n 1/n=\infty$.

A more general integral is easily defined with properties almost precisely the same as those discussed here. If B and B' are two Banach spaces and if the set-function Φ has values in $B:B'$, then limited, ca, and convergent set-functions Φ can be defined almost as before; in this case $\int f d\Phi$ is a point in B'^{**} or, if Φ is convergent, in B'. An illustration is furnished by the last example above if the values of Φ are interpreted as transformations of m into c_0. Theorem 5.4 does not carry over to this case.

6. **General summability theorems.** Silverman and Toeplitz and others have given conditions on a matrix $\{a_{mn}\}$ of real numbers which are necessary and sufficient that it transform every convergent sequence $\{t_m\}$ into another convergent sequence $\{s_m\}$, where $s_m=\sum_n a_{mn}t_n$, which converges to the same limit. The theorem has a great many generalizations; one of these arises naturally from using functions on a directed set instead of sequences, another from letting the values of these functions be points of a Banach space instead of real numbers. The form of the theorem to be stated is suggested by the fact that c, the space of convergent sequences of real numbers, is a Banach space if $\|\{t_n\}\|=\sup_n |t_n|$; in fact it is a space of the form considered in §4 if the field is the smallest field containing all the finite sets of integers.

Let Y be any directed set and B any Banach space; let A be a Banach space whose elements are functions f on Y to B with the property that $\lim_y f(y)$ exists (in the norm topology) for each A. Define the operator L on A to B by setting $L(f)=\lim_y f(y)$ for each f in A; then L is additive and homogeneous but need not be continuous[27]. A set $A'\subset A$ is *dense in limit* in A if for each f in A and $\epsilon>0$ there is an f' in A' such that $\|f-f'\|<\epsilon$ and $\|L(f)-L(f')\|<\epsilon$.

Note that in the simple case $A=c$, above, the set A' of sequences which are ultimately constant is dense in c, and hence dense in limit in c because L is continuous in this case. The conditions (a) and (b) of Silverman-Toeplitz

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(26) (n) is the set whose only element is n.

(27) The referee quite justly remarks that any additive homogeneous function L' on any Banach space A to B could be considered with similar results; for example, letting $L'(f)$ be the weak limit instead of the norm limit of f would give analogous results.

assure that every ultimately constant sequence will be taken into a sequence with the same limit.

If X is a directed set, for each x in X let U_x be a linear operator on A to B; the transformation $\{U_x\}$ thus defined on A to a class of functions on X to B is called *regular* on a subset A' of A if $\lim_x U_x(f) = L(f)$ for every f in A'. Clearly if $\{U_x\}$ is regular on A' and $A' \subset A''$, $\{U_x\}$ is regular on A''.

THEOREM 6.1. (1) *$\{U_x\}$ is regular on A if it satisfies the conditions* (a′) *there exists a set A' dense in limit in A such that $\{U_x\}$ is regular on A', and* (b′) $\lim\sup_x \|U_x\| < \infty$. (2) *If L is discontinuous on A and $\{U_x\}$ is regular on A,* $\lim\sup_x \|U_x\| = \infty$. (3) *If A is* fd, *if $\Delta^0 > X$, if $\lambda(X) > \mu(A)$, or if for any other reason $[A, X]$ is not in $\mathfrak{P}_{bc}$, and if $\{U_x\}$ is regular on A, then* $\lim\sup \|U_x\| < \infty$ *and L is continuous.* (4) *If L is continuous on A, $\{U_x\}$ is regular on A if and only if $L = s^* - \lim_x U_x$.* (5) *If L is continuous and $X > \Delta^{\eta(A)}$ or if for any other reason $[A, X] \in \mathfrak{P}_{bc}$, there exists a $\{U_x\}$ regular on A such that* $\lim\sup_x \|U_x\| = \infty$ (*in the first case $\{U_x\}$ exists such that* $\lim_x \|U_x\| = \infty$).

(1) is a minor adaptation of a standard theorem on convergence of linear operators [3, p. 79, Theorem 3]. (2) is obvious since $\|L\| \leqq \lim\sup_x \|U_x\|$. (3) follows from various results of §3. (4) is a restatement of regularity on A. (5) follows from the definition of $\mathfrak{P}_{bc}$ and, for the last part, from Corollary 3.1 and Theorem 3.2.

In the special case in which A is a space V, as considered in §4, $\lim_y f(y)$ can exist for a simple function if and only if f is ultimately constant; in particular $\lim_y \phi_E b(y)$ exists if and only if either E or $Y - E$ is not cofinal in Y. The properties of cofinality mentioned after the definition show that if Υ_0 is a field of subsets of Y, and if Υ is the subclass of those sets E of Υ_0 for which either E or $Y - E$ is not cofinal in Y, then Υ is also a field. Use subscripts to indicate the field involved.

LEMMA 6.1. *If f can be uniformly approximated by functions simple Υ_0, and if* $\lim_y f(y)$ *exists, then f can be uniformly approximated by functions simple Υ.*

If $\epsilon > 0$ is given, there exists a function $f_\epsilon = \sum_{i \leqq k} \phi_{E_i} b_i$ where the $E_i \in \Upsilon_0$ such that $\|f_\epsilon(y) - f(y)\| < \epsilon/3$ for all y in Y. Also there is a y_ϵ in Y such that $\|f(y) - b_0\| < \epsilon/3$ if $y > y_\epsilon$, where $b_0 = \lim_y f(y)$. Let $E' = \sum E_i$ where the sum is taken over those E_i which contain a successor of y_ϵ. Define f'_ϵ on Y to B by $f'_\epsilon(y) = b_0$ if $y \in E'$, $f'_\epsilon(y) = f_\epsilon(y)$ if $y \notin E'$. Then $Y - E'$ is not cofinal in Y so $E' \in \Upsilon$; no E_i disjoint from E' can be cofinal in Y so the other E are also in Υ. Hence f'_ϵ is simple Υ, but $\|f'_\epsilon - f\| < \epsilon$.

From this lemma it follows that for this section it suffices to assume that Υ is a field of this special sort; that is, Υ satisfies (C): for each E in Υ either E or its complement is not cofinal in Y. In this case L is continuous on V, in fact $\|L\| \leqq 1$. It is clear from the criterion (2) of §4 that $L \in \mathfrak{U}_r$. Since the simple functions are dense in V, the condition (a′) of (1), Theorem 6.1, for this special space can be replaced by (a'') $\lim_x U_x(\phi_Y b)=b$ for each b in B; $\lim_x U_x(\phi_E b)=\theta$ for each b in B and each E in Υ such that E is not cofinal in Y. (2) of that theorem can not occur in this case; it is known that $\eta(V)\leqq\nu(V_r)$.

The special case in which A is a space V of this type while each U_x is in $\mathfrak{U}_r$ presents a situation more general than one studied by Vulich [28]. He considers convergent sequences $\{b_n\}$ of points of a Banach space and transformations U_m defined by means of a matrix of real numbers $\{a_{mn}\}$ so that $U_m(\{b_n\})=\sum_n a_{mn}b_n$ and this series converges absolutely for each $\{b_n\}$ so that $\sum_n |a_{mn}|<\infty$ for each m. Vulich proves that $\lim_m U_m(\{b_n\})=\lim_n b_n$ for every convergent sequence $\{b_n\}$ of points of B, if and only if the matrix satisfies the Toeplitz conditions; that is, if and only if the matrix defines a transformation regular on real sequences.

From Lemma 4.1 we have, letting $\Upsilon=\tau U$, as in §4.

LEMMA 6.2. *If each $U_x\in\mathfrak{U}_r$, $\{U_x\}$ is regular on V if and only if $L=s^*-\lim_x U_x$; $\{\Upsilon_x\}$ is regular on V_r if and only if $L=w^*-\lim_x U_x$.*

We use this to derive the following extension of Vulich's theorem.

THEOREM 6.2. *Let Y be a directed set, Υ a field of subsets of Y satisfying* (C), *and V the Banach space of functions totally measurable with respect to this field; Let X be a directed set, for each x in X let U_x be in $\mathfrak{U}_r$, and let $\Upsilon_x=\tau U_x$. (1) If $\{U_x\}$ is regular on V, $\{\Upsilon_x\}$ is regular on V_r. (2) If $\{\Upsilon_x\}$ is regular on V_r and* $\limsup_x \|U_x\|=\limsup_x\|\Upsilon_x\|<\infty$, *then $\{U_x\}$ is regular on V. (3) If B is* fd *and $\{\Upsilon_x\}$ is regular on V_r, then $\{U_x\}$ is regular on V. (4) If V_r is* fd, *or if $\Delta^0>X$, or if $\lambda(X)>\mu(V_r)$, or if for any other reason $[V_r, X]\notin\mathfrak{P}_{bc}$, then* $\limsup_x\|U_x\|<\infty$ *if $\{\Upsilon_x\}$ is regular on V_r, so, by (2), $\{U_x\}$ is regular on V. (5) If neither V_r nor B is* fd *and if $X>\Delta^{\nu(V_r)}$, then U_x can be chosen from $\mathfrak{U}_r$ so that $\{\Upsilon_x\}$ is regular on V_r while $\{U_x\}$ is not regular on V.*

(1) By (2) of §4, $L\in\mathfrak{U}_r$; if $\Lambda=\tau L$ and $L=s^*-\lim_x U_x=x^*-\lim_x U_x$, then $\Lambda=w^*-\lim_x\Upsilon_x$; clearly $\Lambda(\phi)=\lim_y\phi(y)$ for each ϕ in V_r.

(2) If $\phi\in V_r$ and $b\in B$, then $\phi b\in V$ and

$$\begin{aligned}\|U_x(\phi b)-L(\phi b)\| &= \sup_{\|\beta\|\leqq 1}|\beta(U_x(\phi b))-\beta(L(\phi b))| \\ &= \sup_{\|\beta\|\leqq 1}|\Upsilon_x(\beta(b)\phi)-\Lambda(\beta(b)\phi)| \\ &= |\Upsilon_x(\phi)-\Lambda(\phi)|\sup_{\|\beta\|\leqq 1}|\beta(b)| = \|b\|\,|\Upsilon_x(\phi)-\Lambda(\phi)|.\end{aligned}$$

Hence $\|U_x(\phi b)-L(\phi b)\|\to 0$ for every ϕ in V_r and b in B if $\{\Upsilon_x\}$ is regular on V_r. But the set of all ϕb is fundamental in V, so, by (1) of Theorem 6.1, $\{U_x\}$ is regular on V.

(3) is true by Lemma 6.2 and Theorem 4.1. (4) follows from Theorems 4.1, 3.4 (the known half) and 3.6.

(5) If neither V_r nor B is fd, s^* and w^* topologies in $\mathfrak{U}_r$ are different; hence there is a neighborhood S_r of L which contains no w^* neighborhood of L. For each w^* neighborhood W of L let U_W in $\mathfrak{U}_r$ be in $W_r - S_r$; directing $\mathfrak{W}_r$ by inclusion gives $L = w^* - \lim_W U_W$ while L can not be $s^* - \lim_W U_W$. Since the weak* neighborhood system in V_r^* is isomorphic to $\mathfrak{W}_r$ as a directed set, $\Delta^{\nu(V_r)} \sim \mathfrak{W}_r$; in the usual manner if $X > \Delta^{\nu(V_r)}$, $\{U_x\}$ can be defined in terms of the $\{U_W\}$ to have the same properties.

As an example let us consider multiple sequences. Let Y be the set of n-tuples $y = y_1, \cdots, y_n$ of positive integers, directed by $y > y'$ if $y_i \geqq y_i'$, $i = 1, \cdots, n$, $n > 1$. Let B be any Banach space and let $bc(Y, B)$ be the set of those bounded functions f on Y to B for which $\lim_y f(y)$ exists. Then $bc(Y, B)$ is a subclass of $m(Y, B)$, which, since Y is countable, is a space $\mathfrak{B}$ of the sort considered in §5. Let X be the set of m-tuples $x = x_1, \cdots, x_n$ of positive integers, directed as Y is, and for each x in X let Φ_x be a convergent, ca set-function defined over all subsets of Y. Then $\Delta^0 > X$ and each Φ_x defines a linear operator U_x on $bc(Y, B)$ to B by the relation $U_x(f) = \int f d\Phi_x$ using the integral of §5.

THEOREM 6.3. *Under these conditions* $\{U_x\}$ *is regular on* $bc(Y, B)$ *if and only if* (1) $s^* - \lim_x \Phi_x(Y) = I$, (2) $\| U_x(f\phi_E)\| \to 0$ *if* $f \in bc(Y, B)$ *and if* E *is any set not cofinal in* Y, *and* (3) $\limsup_x W\Phi_x(Y) < \infty$.

If B is fd, (2) can be replaced by (2′) $s^* - \lim_x \Phi_x(E) = \theta$ (or $\lim_x \|\Phi_x(E)\| = 0$) if E is not cofinal in Y.

A smaller class of functions on this Y is the class $rc(Y, B)$; let Y' be a subset of Y which is directed by the same order relation holding in Y itself; then $f \in rc(Y, B)$ if and only if $\lim_{y'} f(y')$ exists for every such directed subset Y' of Y. A simple investigation shows that any such Y' has the following characteristics:

(a) There exists a set of integers $i_1, \cdots, i_j, \cdots, i_p$, $0 \leqq p \leqq n$, $i_j \leqq n$ for all $j = 1, \cdots, p$ and a set of integers $n_1, \cdots, n_p$, such that $y_{i_j} \leqq n_j$ for every $j = 1, \cdots, p$.

(b) The set Y'' of those y in Y' such that $y_{i_j} = n_j$ for $j = 1, \cdots, p$ is cofinal in Y' while $Y' - Y''$ is not.

Define a *slice* Y' of Y to be any set of the form $\{y \mid y_{i_j} = n_j \text{ for } j = 1, \cdots, p\}$ for any choice of $0 \leqq p \leqq n$, and $i_j \leqq n$: for example, if $n = 3$, the slices obtained by fixing 3, 2, 1 and 0 elements of each y are, respectively, single elements, columns, layers, and all of Y. Then each slice is a directed subset of Y (in case n elements of y are fixed we have the trivial directed set with one element) and the characteristics (a) and (b) show that Y' is a directed subset of Y if and only if there is a slice Y'' such that the intersection $Y''Y'$ is cofinal in both Y'' and Y' while $Y' - Y''$ is not cofinal in Y'. Letting Υ be the smallest field containing all the slices in Y, it is easily seen that $rc(Y, B)$ is the space V associated with this Υ.

Taking X as before to be the set of m-tuples $x_1, \cdots, x_m$, let Φ_x be any limited additive set-function on Υ to $B:B$ and define $U_x(f) = \int f d\Phi_x$ for each f in $rc(Y, B)$, using the Gowurin integral.

THEOREM 6.4. *Under these last conditions* $\{U_x\}$ *is regular on* $rc(Y, B)$ *if and only if* (1) $s^* - \lim_x \Phi_x(Y) = I$, (2) $s^* - \lim_x \Phi_x(E) = \theta$ *for every slice E not cofinal in Y, and* (3) $\lim \sup_x W\Phi_x(Y) < \infty$.

The usual modification if Φ is real-valued can be made.

These examples suffice to show something of the generality of the theorems of this section; Theorem 6.1 contains as special cases a number of theorems due to Toeplitz, Hamilton [14], Hill [16], the writer [8] and others. Its use is restricted by the requirement that the class of functions under discussion is a Banach space under some norm adapted to the problem; this is not the case, for example, of the class of all convergent double sequences(28). Further information about the problem of boundedness would also improve the results here.

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