ON THE PARTIAL SUMS OF HARMONIC DEVELOPMENTS AND OF POWER SERIES

BY

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1. **Introduction.** Consider the class E of power series $f(z)=\sum_0^\infty c_\nu z^\nu$, convergent for $|z|<1$ and such that $|f(z)|\leqq 1$. The following result is due to I. Schur and G. Szegö [5](1).

For any series of the class E,

$$|s_n(z)| \equiv \left|\sum_0^n c_\nu z^\nu\right| \leqq 1$$

in $|z|\leqq r_n$, but not always in $|z|<r_n+\epsilon$, $\epsilon>0$, where r_n is the largest r for which

$$T_n(r,\theta) = \frac{1}{2} + \sum_1^n r^\nu \cos \nu\theta \geqq 0 \qquad \text{for all } \theta.$$

The r_n are non-decreasing,

$$r_n > 1 - \frac{\log 2n}{n}, \qquad n = 1, 2, 3, \cdots,$$

$$r_n = 1 - \frac{\log 2n - \log\log 2n + \epsilon_n}{n}, \qquad \lim_{n\to\infty} \epsilon_n = 0.$$

We obtain the same constant r_n if we assume $Rf(z)\geqq 0$ and require $Rs_n(z)\geqq 0$. Here Ru means the real part of u; Iu will denote the imaginary part.

In what follows, we consider harmonic sine developments

$$H(r,\theta) = \sum_1^\infty b_\nu r^\nu \sin \nu\theta,$$

convergent for $0<r<1$, and non-negative for $0<\theta<\pi$. Evidently there exists an R_n with the following properties:

(a) Whenever

$$H(r,\theta) \geqq 0, \qquad 0 < r < 1; 0 < \theta < \pi, \tag{1.1}$$

then,

$$s_n(r,\theta) \equiv \sum_1^n b_\nu r^\nu \sin \nu\theta \geqq 0, \quad 0 < r \leqq R_n; 0 < \theta < \pi. \tag{1.2}$$

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(1) Numbers in brackets refer to the literature at the end of this paper.

(b) For any $\epsilon>0$ we can find an H satisfying (1.1) and such that $s_n(r, \theta)$ becomes negative for some θ and some $r<R_n+\epsilon$.

We denote the class of harmonic functions satisfying (1.1) by T. On writing $f(z)=\sum_1^\infty b_\nu z^\nu$, the power series $f(z)$ is regular in $|z|<1$, has all its coefficients real, and $If(z)\geqq 0$ in $|z|<1$, $Iz>0$. The class T has been discussed by Rogosinski [4]; the function $f(z)$ is called typically real. (Cf. also S. Mandelbrojt [2].)

One of the results of the present paper is

$$R_n = 1 - \frac{3\log n}{n} + \frac{\log\log n}{n} + O(1/n), \qquad \text{as } n \to \infty.$$

M. S. Robertson [3] gave the erroneous estimate

$$R_n \geqq 1 - 2\log n/n \qquad \text{for } n > 12.$$

His calculation yields however, as is seen easily, $R_n \geqq 1-4\log n/n$, for $n>n_0$(2). We then apply the properties of R_n to Fourier series of convex functions and to a certain class of power series.

Note that if $\phi(\theta)\sim\sum_1^\infty b_\nu \sin \nu\theta$, $\phi(\theta)\geqq 0$, $0<\theta<\pi$, then

$$\begin{aligned}
H(r, \theta) &= \frac{2}{\pi}\int_0^\pi \phi(x)\left(\sum_1^\infty r^\nu \sin \nu\theta \sin \nu x\right)dx \\
&= \frac{1}{\pi}\int_0^\pi \phi(x) \sum_1^\infty r^\nu[\cos \nu(\theta - x) - \cos \nu(\theta + x)]dx \\
&= \frac{1-r^2}{2\pi}\int_0^\pi \phi(x)\left(\frac{1}{1-2r\cos(\theta-x)+r^2} - \frac{1}{1-2r\cos(\theta+x)+r^2}\right)dx \\
&= \frac{2r(1-r^2)}{\pi}\sin\theta \int_0^\pi \frac{\phi(x)\sin x\, dx}{[1-2r\cos(\theta-x)+r^2][1-2r\cos(\theta+x)+r^2]}.
\end{aligned}$$

Hence $H(r, \theta)$ belongs to the class T. (Cf. Zygmund [8, p. 57].)

2. **Characterization of R_n.** We quote the following lemma, due to Fejér (Turán [7]).

LEMMA 1. *In order that*

$$\sum_{\nu=1}^n \lambda_\nu \sin \nu x \sin \nu y \geqq 0 \quad \textit{for } 0 < x < \pi,\ 0 < y < \pi,$$

it is necessary and sufficient that

$$\sum_1^n \nu\lambda_\nu \sin \nu\theta \geqq 0 \qquad \textit{for } 0 < \theta < \pi.$$

(2) Robertson, Annals of Mathematics, (2), vol. 42 (1941), pp. 829–838.

We now prove

THEOREM 1. *The quantity R_n as defined in* §1 *is the largest r for which*

$$S_n(r,\theta) \equiv \sum_1^n \nu r^\nu \sin \nu\theta \geqq 0 \qquad \textit{for} \quad 0 < \theta < \pi. \tag{2.1}$$

We have for $0<\rho<1$,

$$\rho^\nu b_\nu = \frac{2}{\pi}\int_0^\pi H(\rho, x) \sin \nu x\, dx, \qquad \nu = 1, 2, 3, \cdots;$$

hence

$$s_n(r,\theta) = \frac{2}{\pi}\int_0^\pi H(\rho, x)\left(\sum_1^n \left(\frac{r}{\rho}\right)^\nu \sin \nu\theta \sin \nu x\right) dx.$$

For any $r<R_n$ we can choose $\rho<1$ so that $r/\rho<R_n$; we then obtain by Lemma 1 (for $\lambda_\nu=r^\nu$) $s_n(r,\theta)\geqq 0$ for any $r<R_n$ and for $0<\theta<\pi$; hence (a) holds for $r\leqq R_n$. Conversely, for the function

$$H(r,\theta) = \sum_1^\infty \nu r^\nu \sin \nu\theta = r \sin\theta \frac{1-r^2}{(1-2r\cos\theta+r^2)^2} > 0$$

the function

$$s_n(r,\theta) = \sum_1^n \nu r^\nu \sin \nu\theta$$

becomes negative for any $r>R_n$ and for some θ in $(0, \pi)$. This proves Theorem 1. To estimate R_n we first give another characterization for it. An easy calculation yields

$$\begin{aligned}
\frac{(1-2r\cos\theta+r^2)^2}{r\sin\theta}&\sum_1^n \nu r^\nu \sin\nu\theta \\
&= 1 - r^2 - (n+1)r^{n+2}\cdot\frac{\sin(n-1)\theta}{\sin\theta} + r^{n+1}(2n+2+nr^2)\frac{\sin n\theta}{\sin\theta} \\
&\qquad - r^n(n+1+2nr^2)\frac{\sin(n+1)\theta}{\sin\theta} + nr^{n+1}\frac{\sin(n+2)\theta}{\sin\theta} \\
&\equiv C_n(r,\theta).
\end{aligned}$$

This furnishes

THEOREM 2. *R_n is the largest r for which*

$$C_n(r,\theta) \geqq 0 \qquad \textit{for all } \theta.$$

Evidently

$$\begin{aligned}C_n(r,\pi) &= 1 - r^2 + (n^2-1)r^{n+2}(-1)^{n-1} + nr^{n+1}(2n+2+nr^2)(-1)^{n-1}\\ &\quad + (n+1)r^n(n+1+2nr^2)(-1)^{n-1} + n(n+2)r^{n+1}(-1)^{n-1}\\ &= 1 - r^2 + (-1)^{n-1}\{(n^2-1)r^{n+2} + nr^{n+1}(2n+2+nr^2)\\ &\qquad + (n+1)r^n(n+1+2nr^2) + n(n+2)r^{n+1}\}.\end{aligned}$$

Thus

$$\begin{aligned}C_n(r,\theta) \geqq 1 - r^2 - \{&(n^2-1)r^{n+2} + nr^{n+1}(2n+2+nr^2)\\ &+ (n+1)r^n(n+1+2nr^2) + n(n+2)r^{n+1}\}, \quad n \geqq 1,\end{aligned}$$

and equality holds if $n=2k$, and $\theta=\pi$. This yields

THEOREM 3. *Denote the unique positive root of the equation*

$$p_n(r) \equiv 1 - r^2 - (n+1)^2r^n - n(3n+4)r^{n+1} - (3n^2+2n-1)r^{n+2} - n^2r^{n+3} = 0$$

by ρ_n. Then $R_n \geqq \rho_n$, and equality holds for $n=2k$, $k \geqq 1$.

Note that $p_n(0)=1$, $p_n(1)<0$, $p_n'(r)<0$. Hence ρ_n is unique and

$$0 < \rho_n < 1. \tag{2.2}$$

Evidently $p_n(-1)=0$, hence $1+r$ can be factored out, and we get

$$\frac{p_n(r)}{1+r} = 1 - r - (n+1)^2r^n - (2n^2+2n-1)r^{n+1} - n^2r^{n+2} \equiv q_n(r), \tag{2.3}$$

so that $q_n(\rho_n)=0$.

3. **Estimation of ρ_n and R_n.** Direct calculation gives

$$R_1 = 1; \rho_1 = 0.182\cdots.$$

Also $\rho_2=R_2$, and

$$S_2(r,\theta) = r\sin\theta + 2r^2\sin 2\theta = r\sin\theta(1+4r\cos\theta),$$

which yields by Theorem 1: $R_2=1/4=\rho_2$. A similar calculation yields $R_3=2^{1/2}/3$.

We shall prove

$$\rho_n = 1 - \frac{3\log n}{n} + \frac{\log\log n + \log 3/4 + \epsilon_n}{n}, \qquad \epsilon_n \to 0 \text{ as } n \to \infty. \tag{3.1}$$

Let c be a constant, and

$$r_n(c) = 1 - \frac{3\log n}{n} + \frac{\log\log n + c}{n}; \tag{3.2}$$

then from

$$\log(1-x) = -x + O(x^2) \qquad \text{as} \quad x \to 0,$$

we conclude

$$(3.3)\qquad \begin{aligned}\{r_n(c)\}^n &= \exp\{-3\log n + \log\log n + c + O(n^{-1}\log^2 n)\}\\ &= n^{-3}\log n\cdot e^c\{1 + O(n^{-1}\log^2 n)\}\quad \text{as}\quad n\to\infty.\end{aligned}$$

Furthermore, from (2.3), (3.2), and (3.3)

$$q_n\{r_n(c)\} = \frac{3\log n}{n} - \frac{\log\log n + c}{n} - \frac{4\log n}{n}\cdot e^c\{1 + o(1)\},$$

hence

$$\frac{nq_n\{r_n(c)\}}{\log n}\to 3 - 4e^c \qquad \text{as } n\to\infty.$$

Thus for

$$c = \log 3/4 + \epsilon,$$

ϵ a given small number, and for sufficiently large values of n

$$\operatorname{sgn} q_n\{r_n(c)\} = \operatorname{sgn}\epsilon,$$

from which follows (3.1).

We have thus proved

THEOREM 4. *If* $\rho_n > 0$ *and* $p_n(\rho_n) = 0$, *then*

$$\rho_n = 1 - \frac{3\log n}{n} + \frac{\log\log n + \log 3/4 + \epsilon_n}{n}, \qquad \textit{where } \epsilon_n\to 0 \textit{ as } n\to\infty.$$

4. **Derivation of an asymptotic estimate for R_n.** On writing

$$R_n = 1 - \frac{3\log n}{n} + \frac{\log\log n + \delta_n}{n}, \qquad n > 1,$$

it follows from Theorem 3 that

$$\delta_n \geqq \log 3/4 + \epsilon_n,$$

and equality holds for $n = 2k$, $k\geqq 1$; hence from Theorem 4

$$\liminf_{n\to\infty}\delta_n = \log 3/4, \qquad \lim_{k\to\infty}\delta_{2k} = \log 3/4.$$

It remains to give an estimate for R_{2k-1} from above.

If for a particular value of θ and r, $C_{2k-1}(r,\theta) < 0$, then by Theorem 2, evidently $R_{2k-1} < r$. We now choose $\theta = \pi - (3\pi/4k)$; then

$$
\begin{aligned}
C_{2k-1}(r, \theta) &= 1 - r^2 + \frac{1}{\sin(3\pi/4k)} \left\{ 2kr^{2k+1} \sin \frac{3(k-1)\pi}{2k} \right. \\
&\quad + r^{2k}[4k + (2k-1)r^2] \sin \frac{3(2k-1)\pi}{4k} \\
&\quad + r^{2k-1}[2k + 2(2k-1)r^2] \sin \frac{3\pi}{2} \\
&\quad \left. + (2k-1)r^{2k} \sin \frac{3(2k+1)\pi}{4k} \right\} \\
&= 1 - r^2 - \frac{1}{\sin(3\pi/4k)} \left\{ 2kr^{2k+1} \cos \frac{3\pi}{2k} \right. \\
&\quad + r^{2k}[4k + (2k-1)r^2] \left[\cos \frac{3\pi}{4k} \right] \\
&\quad \left. + 2kr^{2k-1} + (4k-2)r^{2k+1} + (2k-1)r^{2k} \cos \frac{3\pi}{4k} \right\} \\
&< 1 - r^2 - \frac{4k}{3\pi} \left\{ 2kr^{2k+1} \left(1 - \frac{9\pi^2}{8k^2} \right) \right. \\
&\quad + r^{2k}[4k + (2k-1)r^2] \left(1 - \frac{9\pi^2}{32k^2} \right) \\
&\quad \left. + 2kr^{2k-1} + (4k-2)r^{2k+1} + (2k-1)r^{2k} \left(1 - \frac{9\pi^2}{32k^2} \right) \right\}, \qquad k \geqq 3
\end{aligned}
$$

(since $\cos x > 1 - x^2/2$ for all x). Hence

$$
\begin{aligned}
C_{2k-1}(r, \theta) < 1 - r^2 - (2k/5)\{ & kr^{2k+1} + 2kr^{2k} + (k - 1/2)r^{2k+2} + 2kr^{2k-1} \\
& + (4k-2)r^{2k+1} + (k - 1/2)r^{2k} \}, \qquad k \geqq 5,
\end{aligned}
$$

thus

$$
C_{2k-1}(r, \theta) < 1 - r^2 - (2k/5)(11k - 3)r^{2k+2} < 1 - r^2 - 4k^2 r^{2k+2}.
$$

Choosing r so that

$$
1 - r^2 - 4k^2 r^{2k+2} \leqq 0, \tag{4.1}
$$

we get

$$
C_{2k-1}(r, \theta) < 0, \qquad R_{2k-1} < r.
$$

To find an upper bound for r, we put

$$
r = 1 - \frac{3 \log (2k-1)}{2k-1} + \frac{\log \log (2k-1) + c}{2k-1}; \tag{4.2}
$$

we obtain as in (3.3)

$$
\begin{aligned}
r^{2k-1} &= \exp\left\{-3\log(2k-1)+\log\log(2k-1)+c+O(k^{-1}\log^2 k)\right\} \\
&= (2k-1)^{-3}\log(2k-1)\cdot e^c\left\{1+O(k^{-1}\log^2 k)\right\}.
\end{aligned}
$$

Thus, using (4.2),

$$
\begin{aligned}
\frac{4k^2r^{2k+2}}{1-r^2} &= \left(\frac{2k}{2k-1}\right)^2\cdot\frac{r^3}{1+r}\cdot\frac{r^{2k-1}(2k-1)^2}{1-r} \\
&= \frac{1+o(1)}{2+o(1)}\cdot\frac{1}{3}e^c\left\{1+o(1)\right\}\to\frac{1}{6}e^c \qquad \text{as } k\to\infty.
\end{aligned}
$$

Hence (4.1) is satisfied for all sufficiently large k provided $e^c/6>1$, that is, $c>\log 6$. It now follows that $\limsup_{k\to\infty}\delta_{2k-1}\leqq 6$. Summarizing we have

THEOREM 5. *Let*

$$
R_n = 1-\frac{3\log n}{n}+\frac{\log\log n+\delta_n}{n}, \qquad n>1;
$$

then $\lim_{k\to\infty}\delta_{2k}=\log 3/4$, *and*

$$
\log 3/4 \leqq \liminf_{k\to\infty}\delta_{2k-1}\leqq\limsup_{k\to\infty}\delta_{2k-1}\leqq 6.
$$

5. **Application to Fourier series.** Consider the roof-function

$$
\frac{2b}{a(\pi-a)}\sum_1^\infty\frac{\sin\nu a\sin\nu\theta}{\nu^2}=\begin{cases}\dfrac{b\theta}{a} & \text{for } 0\leqq\theta\leqq a,\\[2ex] b\,\dfrac{\pi-\theta}{\pi-a} & \text{for } a\leqq\theta\leqq\pi,\end{cases}
$$

where $0<a<\pi$, $0<b$, and the corresponding harmonic function

$$
\frac{2b}{a(\pi-a)}\sum_1^\infty r^\nu\frac{\sin\nu a\sin\nu\theta}{\nu^2}=H(r;\,a,\,b).
$$

Denote its partial sums by

$$
H_n(r,\theta)=\frac{2b}{a(\pi-a)}\sum_1^n r^\nu\frac{\sin\nu a\sin\nu\theta}{\nu^2};
$$

then

$$
\frac{\partial^2 H_n(r,\theta)}{\partial\theta^2}=-\frac{2b}{a(\pi-a)}\sum_1^n r^\nu\sin\nu a\sin\nu\theta\leqq 0
$$

for $0<r\leqq R_n$, $0<\theta<\pi$, by Lemma 1 and Theorem 1. Hence $H_n(r,\theta)$ is convex upwards for $0<\theta<\pi$, $r\leqq R_n$; but not convex for $r>R_n$. The same is true for the limiting cases $a\to 0$ and $a\to\pi$. In which cases

$$H(r;\,0,\,b) = \frac{2b}{\pi}\sum_{1}^{\infty} r^{\nu}\,\frac{\sin\nu\theta}{\nu},$$

$$H(r;\,\pi,\,b) = \frac{2b}{\pi}\sum_{1}^{\infty} r^{\nu}\,\frac{\sin\nu(\pi-\theta)}{\nu}.$$

Moreover every polygon convex upwards and lying above the axis of abscissae is expressible as a finite sum with positive coefficients of roof-functions. Hence the partial sums of the corresponding harmonic development are convex upwards for $r\leqq R_n$. Finally any function positive in $0<\theta<\pi$, and convex upwards can be approximated uniformly by such polygons; hence we have

THEOREM 6. *If $f(\theta)>0$ in $0<\theta<\pi$, and is convex upwards, and if $f(\theta)\sim\sum_1^\infty b_\nu\sin\nu\theta$, then $\sum_1^n r^\nu b_\nu\sin\nu\theta$ is convex upwards in $0<\theta<\pi$, $r\leqq R_n$; but not always for $r<R_n+\epsilon$, $\epsilon>0$.*

6. **Cosine series.** We now consider the cosine series of the step function

$$\frac{2b}{\pi}\left\{\frac{\pi-a}{2}-\sum_{1}^{\infty}\frac{\sin\nu a\cos\nu\theta}{\nu}\right\} = \begin{cases} 0 & \text{for } 0\leqq\theta<a,\\ b & \text{for } a<\theta\leqq\pi,\end{cases}$$

where $0<a<\pi$, $b>0$; and the corresponding harmonic development

$$K(r,\,\theta) = \frac{b}{\pi}(\pi-a) - \frac{2b}{\pi}\sum_{1}^{\infty} r^{\nu}\,\frac{\sin\nu a\cos\nu\theta}{\nu}.$$

For the partial sums $K_n(r,\theta)$ of this series we have

$$\frac{\partial K_n(r,\,\theta)}{\partial\theta} = \frac{2b}{\pi}\sum_{1}^{n} r^{\nu}\sin\nu a\sin\nu\theta \geqq 0 \qquad \text{for } 0<r\leqq R_n,\ 0<\theta<\pi;$$

hence $K_n(r,\theta)$ is monotonic increasing in the same domain; R_n cannot be replaced by $R_n+\epsilon$, $\epsilon>0$. The same statement for any monotonic increasing function follows now in an obvious way. Hence we have

THEOREM 7. *If $f(\theta)$ is monotonic in $0<\theta<\pi$, and*

$$f(\theta)\sim a_0/2+\sum_{1}^{\infty} a_\nu\cos\nu\theta,$$

then the nth partial sum of $a_0/2+\sum_1^\infty a_\nu r^\nu\cos\nu\theta$ is monotonic in the same sense for $0<r\leqq R_n$, and here R_n cannot be replaced by $R_n+\epsilon$, $\epsilon>0$.

7. **Curves convex in direction of the v-axis.** We say that a curve in the (u,v)-plane is convex in the direction of the v-axis if any parallel to the v-axis

has at most two points in common with the curve. This class of mappings was considered by L. Fejér [1] and the author [6]. We now prove

THEOREM 8. *Suppose the power series* $\sum_0^\infty a_\nu z^\nu = f(z) = w = u + iv$ *is regular in* $|z| < 1$, *and all* a_ν *are real. Suppose further that the images* K_r *of the circles* $|z| = r$, $0 < r < 1$, *are convex in the direction of the v-axis (thus* $f(z)$ *is univalent). Then the partial sum* $\sum_0^n a_\nu z^\nu$ *has the same property in* $|z| \leqq R_n$, *but—in general—not in a larger circle.*

For the proof we may assume without loss of generality that the upper half of the circle $|z| < 1$ is mapped onto the upper half of the image in the w-plane. On writing $w(e^{i\theta}) = u(\theta) + iv(\theta) \sim \sum_0^\infty a_\nu \cos \nu\theta + i\sum_0^\infty a_\nu \sin \nu\theta$, we find that $v(\theta)$ is positive for $0 < \theta < \pi$, and (from the assumption) $u(\theta)$ is decreasing in the same interval. Our theorem follows now from Theorems 5 and 7.

8. **Conclusion.** Suppose $f(z) = \sum_1^\infty b_\nu z^\nu$ is a typically real function, that is,

$$\sum_1^\infty b_\nu r^\nu \sin \nu\theta \geqq 0 \qquad \text{for} \quad 0 < r < 1, \ 0 < \theta < \pi.$$

Then the Riesz means of second order

$$P_n(z) = (n+1)^{-2} \sum_{\nu=1}^n (n - \nu + 1)^2 b_\nu z^\nu, \qquad n \geqq 1,$$

are typically real in $|z| \leqq 1$ (Szász [6]; cf. Theorem 1). Evidently $\lim_{n\to\infty} P_n(z) = f(z)$ in $|z| < 1$, uniformly in $|z| \leqq r$, $r < 1$. Another such sequence of polynomials is

$$s_n(R_n z) = \sum_{\nu=1}^n b_\nu R_n^\nu z^\nu, \qquad n \geqq 1.$$

These polynomials are typical real in $|z| \leqq 1$ by property (a) of §1. Furthermore for $|z| \leqq r < 1$

$$\begin{aligned} |f(z) - s_n(R_n z)| &\leqq \sum_1^n |b_\nu| r^\nu (1 - R_n^\nu) + \sum_{n+1}^\infty |b_\nu| r^\nu \\ &< (1 - R_n) \sum_1^n \nu |b_\nu| r^\nu + \sum_{n+1}^\infty |b_\nu| r^\nu \to 0, \text{ as } n \to \infty. \end{aligned}$$

Hence

$$\lim_{n\to\infty} s_n(R_n z) = f(z)$$

uniformly in $|z| \leqq r < 1$.

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