BÉZOUT'S THEOREM AND ALGEBRAIC DIFFERENTIAL EQUATIONS[1]

BY

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The problem of determining by inspection the number of solutions of a system of algebraic equations finds its solution in Bézout's theorem and in important complements to that theorem obtained in recent years by van der Waerden[2]. The corresponding problem for a system of algebraic differential equations is that of determining bounds for the numbers of arbitrary constants which enter into the irreducible manifolds which the system yields. This problem has been considered by us in two previous papers[3].

In the present paper, we study the intersections of the *general solutions* of two algebraically irreducible forms A and B in the unknowns y and z. The statement of our results depends on some definitions which we proceed to give.

Let F be a form in several unknowns. F has an order in each of its unknowns. The maximum of these orders will be called the *order of F.*

Let Σ be a non-trivial prime ideal of forms in any unknowns. Σ has a certain number $q \geqq 0$ of arbitrary unknowns. We shall call q the *dimension* of the manifold of Σ.

By the order of an irreducible manifold $\mathfrak{M}$ of dimension zero, we mean the order of any resolvent for the prime ideal of which $\mathfrak{M}$ is the manifold.

An irreducible manifold $\mathfrak{N}$ which is part of a manifold $\mathfrak{M}$ will be called an *irreducible component* (often simply component) of $\mathfrak{M}$ if $\mathfrak{M}$ contains no irreducible manifold of which $\mathfrak{N}$ is a proper part[4].

Let us return now to A and B as above, which we suppose to have the respective orders m and n. Let the general solutions of A and B have a non-vacuous intersection $\mathfrak{M}$. It is a most natural conjecture that, if $\mathfrak{M}$ has one or more irreducible components of dimension zero, their orders do not exceed $m+n$. This conjecture is verified below for the cases in which neither of m and n exceeds unity. It was not without surprise that we found our conjecture to lapse into default for larger values of the orders. We shall show how to construct, for every $n \geqq 4$, a form of order n whose general solution intersects

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(1) For indications in regard to the general theory to which this paper attaches, one may consult the author's paper in the second volume of the Semicentennial Publications of the American Mathematical Society.

(2) van der Waerden, *Einführung in die algebraische Geometrie*, Berlin, 1939, chap. 6.

(3) *Systems of algebraic differential equations*, Ann. of Math. (2) vol. 36 (1935) p. 293; *Jacobi's problem on the order of a system of differential equations*, ibid. p. 303. The second of these papers will be denoted below by J.

(4) In other words, $\mathfrak{N}$ is essential in $\mathfrak{M}$.

the manifold of $y=0$ in the manifold of $y=0$, $z_{2n-3}=0$, a manifold of order $2n-3$.

Forms of orders not exceeding unity

1. We prove the statement made, for the cases with $m \leqq 1$, $n \leqq 1$, in the introduction.

When $m=n=0$, there is nothing to prove.

Let $m=0$, $n=1$. Let $\mathfrak{N}$ be a component of $\mathfrak{M}$ of dimension zero. We consider first the intersection $\mathfrak{M}'$ of the complete manifolds of A and B. Every component of $\mathfrak{M}'$ of dimension zero has an order not exceeding unity(5). Then, by Gourin's theorem(6), if $\mathfrak{N}$ is not contained in a component of $\mathfrak{M}'$ of dimension unity, $\mathfrak{N}$ is of order not greater than unity.

We have now to consider the case in which $\mathfrak{N}$ is contained in a component $\mathfrak{M}''$ of $\mathfrak{M}'$ of dimension unity. $\mathfrak{M}''$ is the general solution of a form C. Because A, which is of order zero, holds $\mathfrak{M}''$, C must be of order zero; this implies that $\mathfrak{M}''$ is the manifold of A. Then $\mathfrak{M}''$ must be a component of the manifold of B. Otherwise $\mathfrak{M}''$ would be contained in the general solution of B and $\mathfrak{N}$ would not be a component of $\mathfrak{M}$(7).

We suppose, as we may, that A involves z effectively. As $\mathfrak{M}''$ is a proper part of the manifold of B, B must be of order unity in z. Let $S=\partial A/\partial z$. Then some S^tB has a representation(8)

$$S^t B = C_0 A^p + C_1 A^{p_1} A_1^{q_1} + \cdots + C_r A^{p_r} A_1^{q_r}.$$

Here A_1 is the derivative of A and, for every i, $p_i+q_i>p$. The orders of the C_i in z and in y do not exceed 0 and 1, respectively, and no C_i is divisible by A. As $\mathfrak{N}$ is in the intersection of $\mathfrak{M}''$ and the general solution of B, C_0 must hold $\mathfrak{N}$(8). The manifold of the system C_0, A has components which are all of dimension zero and none of order greater than unity(8). This disposes of the case of $m=0$, $n=1$.

Now let $m=n=1$. We use $\mathfrak{N}$ and $\mathfrak{M}'$ as above. We take up immediately the case in which $\mathfrak{N}$ is contained in a component $\mathfrak{M}''$ of $\mathfrak{M}'$ of dimension unity; when $\mathfrak{N}$ is not so contained, its order cannot exceed 2(9). As $\mathfrak{N}$ is a component of $\mathfrak{M}$, $\mathfrak{M}''$ is not part of $\mathfrak{M}$. Let, then, $\mathfrak{M}''$ fail to be contained in the general solution of B. Then some other component of the manifold of B, indeed the manifold of a form of order zero, contains $\mathfrak{M}''$ and is thus identical with $\mathfrak{M}''$. By the case of $m=0$, $n=1$, the components of the inter-

(5) This is proved in J.

(6) Bull. Amer. Math. Soc. vol. 39 (1933) p. 593.

(7) The components of B other than its general solution are manifolds of forms of order zero. See *On certain points in the theory of algebraic differential equations*, Amer. J. Math. vol. 60 (1938) pp. 1–43, §30. This paper will be denoted by C. P.

(8) C. P. §31.

(9) By J.

section of $\mathfrak{M}''$ with the general solution of B are of dimension zero and of order at most unity. This completes the proof.

A FORM OF ORDER FOUR

2. In what follows, K_1 will represent, for any form K, the derivative of K. We let

$$A = y_1 - z_3 y^2, \tag{1}$$

$$B = A^4 - y_3^8, \tag{2}$$

$$C = y_3 A_1 - 2y_4 A, \tag{3}$$

$$F = B - y^6 C^2 = A^4 - y_3^8 - y^6 C^2. \tag{4}$$

We use the field of all constants. Let us see first that F is algebraically irreducible. If we consider the equation $F=0$ as an algebraic equation for y_4, we secure a function y_4 of two branches. Thus, if F were factorable, it would have a factor of positive degree free of y_4. Such a factor would have to be a factor of y^6A^2. As F is not divisible by y or by A, F is algebraically irreducible.

Let us determine now the components of the manifold of F other than the general solution.

Let $\mathfrak{N}$ be such a component. As $\partial F/\partial y_4 = 4y^6AC$, $\mathfrak{N}$ must be held by yC or by A. Suppose that A holds $\mathfrak{N}$. By (3) and (4), y_3 holds $\mathfrak{N}$. In every case then, B holds $\mathfrak{N}$.

Now B is the product of the four forms

$$E^{(j)} = y_1 - z_3 y^2 - j y_3^2, \qquad j = \pm 1, \pm (-1)^{1/2}, \tag{5}$$

each of which is algebraically irreducible. For what follows, it is important to know that the manifold of each $E^{(j)}$ is irreducible. From the manner in which z_3 figures in (5), one sees that a component of the manifold of an $E^{(j)}$ distinct from the general solution is held by y. Such a component, being of dimension unity(10), must be the manifold of y. But the *low power theorem*(11) shows that the manifold of y is not a component. This proves the irreducibility of the manifolds of the $E^{(j)}$.

We have, for every j,

$$C = y_3 E_1^{(j)} - 2y_4 E^{(j)}.$$

Referring to (4) and applying the low power theorem, we find that the manifold of each $E^{(j)}$ is a component of the manifold of F(12).

(10) C. P. §1.

(11) So we designate the theorem of C. P. §29.

(12) Technically, in applying the low power theorem, we have to multiply F by y_3^2 and to effect a reduction. Actually, on considering the proof of the low power theorem, one sees that one may dispense with this process of preparation. For instance, if one replaces y_4 in the coeffi-

Thus the manifold of F has five components, the general solution and the manifolds of the $E^{(j)}$.

3. In what follows it will be proved that the intersection of the general solution of F with the manifold of $y=0$, is the manifold of the system $y=0$, $z_5=0$. The latter manifold is of dimension zero and of order 5. The proof employs some general results, bearing on ideals of differential polynomials, which will now be set forth.

Deductions from Levi's theorem on power products

4. In what follows P is a power product in y and derivatives of y, d the degree of P, w the weight of P and p a positive integer.

Modifying a theorem due to Howard Levi[13], we derive the following result: *If*

$$d > \frac{p-1}{2} + \left((p-1)w + \frac{(p-1)^2}{4}\right)^{1/2} \tag{6}$$

then[14]

$$P \equiv 0, \qquad [y^p].$$

We suppose, as we may, that $p>1$. Let (6) be satisfied. Then

$$(p-1)w < d^2 - d(p-1). \tag{7}$$

Let

$$d = a(p-1) + b \tag{8}$$

where a and b are integers such that $a \geqq 0$, $0 < b \leqq p-1$. As $b(p-1-b) \geqq 0$, (7) gives

$$(p-1)w < d^2 - d(p-1) + (p-1-b)b. \tag{9}$$

We replace d in (9) by its expression in (8), finding that

$$w < a(a-1)(p-1) + 2ab. \tag{10}$$

By Levi's theorem, $P \equiv 0$, $[y^p]$.

We denote by $\delta(p, w)$ the second member of (6).

5. Representing y^p by u, we prove the following result, which holds for any power product P as in §4 and for any values of d, w, p.

P has a representation as a homogeneous polynomial in u and derivatives of u,

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cient of $E^{(j)}$ in C by a new unknown u, $E^{(j)}$ is seen immediately to furnish a component of the manifold of the form in u, y, z into which F is converted.

(13) Trans. Amer. Math. Soc. vol. 51 (1942) p. 545.

(14) The notation, as regards congruences, is due to E. R. Kolchin, Ann. of Math. (2) vol. 42 (1941) p. 740.

whose coefficients are homogeneous polynomials(15) *in* y *and derivatives of* y *of a common degree not greater than* $\delta(p, w)$.

If $d \leqq \delta(p, w)$, P itself is the representation sought. Otherwise, by §4, P is a linear combination of the u_i, with coefficients all of degree $d-p$ and none of weight greater than w. If $d-p \leqq \delta(p, w)$, we have the desired representation. Otherwise the coefficients of the u_i will be in $[u]$. Continuing in this manner, we find P expressed as in our statement.

Multipliers of a form

6. Let Σ be an ideal (differential) of forms in y and z; M a form in y and z; α a non-negative number. We shall say that *M admits α as a multiplier with respect to* Σ if for every $\epsilon > 0$ there exists an integer $n_0(\epsilon)$ such that, for every $n > n_0(\epsilon)$,

$$M^n \equiv P, \qquad [\Sigma]$$

where P is a form depending on n which, arranged as a polynomial in the y_i(16), contains no term of degree less than $n(\alpha - \epsilon)$. P may be zero. If α is a multiplier for M and if $0 \leqq \gamma < \alpha$, γ is also a multiplier.

We prove the following properties of multipliers:

(a) Let M and N admit α and β, respectively, as multipliers with respect to Σ. Let $\gamma = \min(\alpha, \beta)$. Then $M+N$ admits γ as a multiplier.

(b) For M and N as in (a), MN admits $\alpha+\beta$ as a multiplier.

(c) Let M^p, where p is a positive integer, admit α as a multiplier. Then M admits α/p.

(d) Let M admit α as a multiplier. Then M_1, the derivative of M, also admits α.

(e) If $M \equiv N$, $[\Sigma]$, M and N admit the same multipliers.

Proving (a), we take an $\epsilon > 0$. Let $n_0(\epsilon/2)$ serve as above for both M and N with respect to $\epsilon/2$. We consider $(M+N)^n$ for any $n \geqq 1$. Let $R = M^a N^b$ where $a+b=n$. If a and b both exceed $n_0(\epsilon/2)$, we have $R \equiv P$, $[\Sigma]$ where no term of P is of degree less than

$$a(\alpha - \epsilon/2) + b(\beta - \epsilon/2),$$

which quantity is not less than $n(\gamma - \epsilon/2)$. If $b \leqq n_0(\epsilon/2) < a$, we have $R \equiv P$, $[\Sigma]$ with no term of P of degree less than

$$[n - n_0(\epsilon/2)](\alpha - \epsilon/2).$$

This last quantity, if n is large in comparison with $n_0(\epsilon/2)$, exceeds $n(\alpha - \epsilon)$. The truth of (a) is now clear.

(15) The coefficients of the polynomials in the y_i are rational numbers.

(16) When P is thus arranged, its coefficients will be forms in z. The definition of multiplier gives a special role to y.

The proofs of (b), (c) and (e) are trivial.

Proving (d), we take an $\epsilon > 0$ and, relative to M, an $n_0(\epsilon/2)$. Let m be a fixed integer which exceeds $n_0(\epsilon/2)$. We consider an $n > 0$ and use $\delta(m, n)$ as in §4. Then M_1^n is a polynomial in M^m and its derivatives with coefficients which are forms in M of degree not greater than $\delta(m, n)$. In this expression for M_1^n, every power product in M^m and its derivatives is of degree not less than

$$q = [n - \delta(m, n)]/m. \tag{11}$$

Now, if n is large, $\delta(m, n)$ as one sees from (6), is small in comparison with n, so that q is only slightly less than n/m. Each power product in M^m and its derivatives is congruent to a form whose terms have degrees in the y_i not less than $qm(\alpha - \epsilon/2)$. If n is large, this last quantity exceeds $n(\alpha - \epsilon)$, q.e.d.

The form F. First operation

7. We return to F of §2, denoting the general solution of F by $\mathfrak{M}$. We show now that a solution in $\mathfrak{M}$ with $y = 0$ satisfies $z_5 = 0$. Later, we shall prove that every z with $z_5 = 0$ is admissible.

We determine first a form G which holds $\mathfrak{M}$ but none of the other four components.

We have by (2) and (3),

$$AB_1 - 4A_1B = 4y_3^7 C. \tag{12}$$

Thus by (4) (first representation of F), we have, when $F = 0$,

$$4y_3^7 B^{1/2} = y^3(AB_1 - 4A_1B). \tag{13}$$

Again, letting $K = y^3 C$, we have by (4), when $F = 0$, the relation $B^{1/2} = K$. Thus, for $F = 0$, $B \neq 0$,

$$B^{-1/2}B_1 = 2K_1. \tag{14}$$

Substituting into (13) the expression which (14) furnishes for B_1, and simplifying, we find for $F = 0$, $B \neq 0$,

$$4y_3^{14} + L = 0 \tag{15}$$

where

$$L = -4y^3 y_3^7 A K_1 + y^6 A^2 K_1^2 - 4y^6 A_1^2 B. \tag{16}$$

We designate the first member of (15) by G. Then G holds $\mathfrak{M}$.

8. In what follows, all multipliers will operate with respect to $[F, G]$, the differential ideal generated by F and G.

In (4), y_3^8 and y^6C^2 contain no terms of degree less than 8 in the y_i. Thus A^4 admits 8 as a multiplier so that, by (c) of §6, A admits 2. Now $z_3 y^2$ admits 2. By (a) of §6, y_1 admits 2. Then, by (d), every y_i with $i \geq 1$ admits 2. From

(3), using (a), (b), (d), we find that C admits 4. Referring to (4) and using (e), we see now that A^4 admits 14, so that A admits 3. By (3), now, C admits 5 and we find from (4) that A admits 4. We return to (3) and see that C admits 6. Also by (4), B admits 18. Finally K of §7 admits 9.

By (16), L admits 30. By (15), y_3 admits 15/7. Now $y_2-z_4y^2-2z_3yy_1$, which is A_1, admits 4. As y_1 admits 2, $y_2-z_4y^2$ admits 3. Then $y_3-z_5y^2-2z_4yy_1$ admits 3, so that $y_3-z_5y^2$ admits 3. As y_3 admits 15/7, z_5y^2 admits 15/7.

We infer that $[F, G]$ contains a form of the type $(z_5y^2)^m+M$ where every term of M is of degree greater than $2m$ in the y_i. It follows from the low power theorem that a solution in $\mathfrak{M}$ cannot have $y=0$ unless $z_5=0$.

Second operation

9. Let α be any polynomial of effective degree 4. We shall prove that $\mathfrak{M}$ contains $y=0$, $z=\alpha$. This will imply that every z for which $z_5=0$ appears in $\mathfrak{M}$ with $y=0$ and our investigation of F will be completed.

Representing by c an arbitrary constant and by v a new unknown, we put $z=\alpha$ in F and then make in F the substitution(17)

$$y = \sum_{j=1}^{6} c^{j} \alpha_2^{j-1} + c^{6} v. \tag{17}$$

We represent by A', A_1', B', C', F' the expressions into which A, A_1, B, C, F are, respectively, transformed when z is replaced by α and y by the second member of (17).

We find from (17)

$$A' = c^6v_1 + c^7P \tag{18}$$

with P a polynomial in x, c, v. Then we may write

$$A_1' = c^6v_2 + c^7Q. \tag{19}$$

In (17), the coefficient of c^2 is of the second degree in x; that of c^3 is of the fourth degree. We have thus

$$y_3 = c^3\beta + \cdots; \qquad y_4 = c^3\gamma + \cdots \tag{20}$$

with β of the first degree and γ constant. By (18), (19), (20),

$$C' = c^9(\beta v_2 - 2\gamma v_1) + c^{10}R$$

with R a polynomial in x, c and the v_j with $j \leqq 4$. We find thus

$$F' = c^{24}\left[v_1^4 - \beta^8 - (\beta v_2 - 2\gamma v_1)^2\right] + c^{25}T \tag{21}$$

with T of the type of R.

(17) Subscripts of α indicate differentiation.

10. Let V represent the coefficient of c^{24} in F'. As $\beta \neq 0$, the differential equation $V=0$ is effectively of the second order. Let then $v=\xi$ be a solution of $V=0$ with

$$\xi_1^4 - \beta^8 \neq 0. \tag{22}$$

We wish to show that F' is formally annulled by a series

$$v = \xi + \phi_2 c^{\rho_2} + \phi_3 c^{\rho_3} + \cdots \tag{23}$$

of the following description. The ρ_i are positive rational numbers, with a common denominator, which increase with their subscripts. The ϕ_i are analytic functions of x, all analytic at some point at which ξ is analytic(18).

It will suffice to show that $G=F'/c^{24}$ is annulled by a series (23). If G vanishes identically in x and c for $v=\xi$, then $v=\xi$ is an acceptable series (23). In what follows, we assume that such vanishing does not occur.

Introducing a new unknown u_1, we put, in G, $v=\xi+u_1$. Then G goes over into an expression H' in x, c and u_1,

$$H' = a'(c) + \sum b_i'(c) u_{10}^{\alpha_{0i}} \cdots u_{14}^{\alpha_{4i}}. \tag{24}$$

Here $\sum$ contains the terms of H' which are not free of the u_{1j} and, in $\sum$, i ranges from unity to some positive integer. As to $a'(c)$ and the $b_i'(c)$, they are polynomials in c with analytic functions of x for coefficients. Because ξ does not annul G identically, $a'(c)$ is not identically zero. On the other hand, because G vanishes for $v=\xi$, $c=0$, the lowest power of c in $a'(c)$ is positive. Because the bracketed terms in (21) contribute effectively to $\sum$ in (24), certain of the $b_i'(c)$ contain terms of power zero in c.

Let σ' be the least exponent of c in a' and σ_i' the least exponent of c in b_i'. Let

$$\rho_2 = \max \frac{\sigma' - \sigma_i'}{\alpha_{0i} + \cdots + \alpha_{4i}}$$

where i has the range which it has in $\sum$. As $\sigma' > 0$ and certain σ_i' equal 0, $\rho_2 > 0$.

We now take over §§12–16 of our paper *On the singular solutions of algebraic differential equations*(19), putting $m=4$ in that discussion. We are brought to the series (23) for v.

11. We have shown, all in all, that F, for $z=\alpha$, is annulled by a series

$$y = c + c^2\alpha_2 + \cdots + c^5\alpha_2^4 + c^6(\alpha_2^5 + \xi) + \cdots \tag{25}$$

where the unwritten terms have rational exponents greater than 6. The series (25) does not annul B for $z=\alpha$. Indeed,

(18) One may suppose that $\phi_1=\xi$, $\rho_1=0$.

(19) Ann. of Math. (2) vol. 37 (1936) p. 541.

$$B' = c^{24}(v_1^4 - \beta^8) + \cdots$$

and, because of (22), the coefficient of c^{24} in B' does not vanish for $v=\xi$.

It follows that every form which holds $\mathfrak{M}$ vanishes for $z=\alpha$ and for y as in (25). This means that $y=0$, $z=\alpha$ is in $\mathfrak{M}$.

Remarks

12. If in (1) to (4), we replace z_3, y_3, y_4 wherever they appear by z_{n-1}, y_{n-1}, y_n, respectively, where $n \geqq 4$, we obtain a form F with a general solution which intersects the manifold of $y=0$ in that of $y=0$, $z_{2n-3}=0$; the proofs require only the slightest changes.

In F of §2, if one replaces z_3 by z, one obtains a form which is of the first order in z and has a general solution which intersects the manifold of $y=0$ in that of $y=0$, $z_2=0$. This in itself is sufficiently anomalous. However, if it is desired to secure a form F whose order in z cannot be reduced, it suffices to replace y_3 and y_4 in (2), (3), (4) by zy_3 and its derivative, respectively.

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