

ON THE PARTIAL SUMS OF FOURIER SERIES AT POINTS OF DISCONTINUITY

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1. **Introduction.** Consider a Fourier sine series

$$(1.1) \quad f(\theta) \sim \sum_1^{\infty} b_\nu \sin \nu\theta, \quad b_\nu = (2/\pi) \int_0^\pi f(\theta) \sin \nu\theta d\theta, \quad 0 < \theta < \pi,$$

and write

$$(1.2) \quad s_n(\theta) = \sum_1^n b_\nu \sin \nu\theta, \quad n = 1, 2, 3, \dots$$

Fejér proved (cf. Zygmund [5, p. 181])⁽¹⁾ that if $f(\theta)$ is of bounded variation, and if $n\theta_n \rightarrow \alpha$ as $\theta_n \rightarrow 0$, then

$$(1.3) \quad s_n(\theta_n) \rightarrow (2/\pi)f(+0) \int_0^\alpha \frac{\sin t}{t} dt \equiv (2/\pi)f(+0)I(\alpha).$$

In particular, choosing α so that $I(\alpha) = \pi/2 = \int_0^\infty t^{-1} \sin t dt$ (thus $\int_\alpha^\infty t^{-1} \sin t dt = 0$), we get $s_n(\theta_n) \rightarrow f(+0)$, which is half of the jump of $f(\theta)$ at $\theta = 0$.

On the other hand for $\alpha = \pi$, which gives $I(\alpha)$ its maximal value

$$s_n(\theta_n) \rightarrow (2/\pi)f(+0) \int_0^\pi \frac{\sin t}{t} dt = f(+0) \times 1.08949 \dots$$

Thus the limit points of the partial sums as $\theta_n \rightarrow 0$ cover an interval which extends beyond $f(+0)$, if $f(+0) \neq 0$. This is called Gibbs' phenomenon.

It was also proved by Fejér and Csillag (for references and further results see Szász [4]) that for functions of bounded variation

$$(1.4) \quad n^{-1} \sum_1^n \nu b_\nu \rightarrow (2/\pi)f(+0), \quad \text{as } n \rightarrow \infty.$$

These facts suggest the consideration of

$$s_n(\theta_n) = \sum_1^n \nu b_\nu \frac{\sin \nu\theta_n}{\nu}, \quad \theta_n \rightarrow 0,$$

as a transform of the sequence $\{nb_n\}$, that is, as a special case of the triangular type transform

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⁽¹⁾ Numbers in brackets refer to the literature at the end of this paper.

$$(1.5) \quad T_n = \sum_1^n a_{nv} \tau_v,$$

where now $\tau_v = \nu b_v$, $a_{nv} = \nu^{-1} \sin \nu \theta_n$. We shall not restrict ourselves to regularity conditions, and we shall not assume convergence of the sequence $\{\tau_n\}$, but merely Cesàro summability of some order. We then seek simple necessary and sufficient conditions for the convergence of the transform T_n (in general to a different limit). The application to Fourier sine series yields a generalized Gibbs' phenomenon, and also a new device to determine the generalized jump of a function. Our results are in close relationship with some results of Rogosinski [1, 2].

We consider more generally the transform

$$(1.6) \quad T(\rho_n, \theta_n) = \sum_1^n \tau_v \rho_n \nu^{-1} \sin \nu \theta_n, \quad \rho_n \rightarrow 1, \theta_n \rightarrow 0,$$

which in the case $\tau_v = \nu b_v$, becomes $\sum_1^n \rho_n \nu b_v \sin \nu \theta_n = s_n(\rho_n, \theta_n)$, where $s_n(\rho, \theta)$ is the n th partial sum of the harmonic series $\sum_1^\infty \rho^v b_v \sin \nu \theta$.

2. Permanency with respect to convergent sequences. It is well known that the convergence of the sequence $\{\tau_n\}$ implies the convergence of the transform T_n , if and only if

$$\begin{aligned} \lim_{n \rightarrow \infty} a_{nv} &= 0, & \text{for } v = 1, 2, 3, \dots; \\ \sum_{v=1}^n |a_{nv}| &= O(1), & \text{as } n \rightarrow \infty; \\ \lim_{n \rightarrow \infty} \sum_1^n a_{nv} &= \sigma \text{ exists.} \end{aligned}$$

We then have $\lim T_n = \sigma \lim \tau_n$. If we restrict ourselves to sequences $\tau_n \rightarrow 0$, then the last condition can be omitted. Applied to (1.6) this yields the necessary and sufficient conditions:

$$(2.1) \quad \sum_1^n \rho_n \nu^{-1} |\sin \nu \theta_n| = O(1), \quad \text{as } n \rightarrow \infty;$$

$$(2.2) \quad \lim \sum_1^n \rho_n \nu^{-1} \sin \nu \theta_n = \sigma.$$

In particular the last condition is $s_n(\rho_n, \theta_n) \rightarrow \sigma$ for the harmonic series $\sum_1^\infty \rho^v \nu^{-1} \sin \nu \theta = \arctan \{(\rho \sin \theta)/(1 - \rho \cos \theta)\}$.

We first assume

$$(2.3) \quad 0 < \liminf \rho_n^n \leq \limsup \rho_n^n < \infty;$$

in this case for some $c_1 > 0$, $c_2 > 0$

$$c_1 \sum_1^n \nu^{-1} |\sin \nu \theta| < \sum_1^n \rho_n \nu^{-1} |\sin \nu \theta| < c_2 \sum_1^n \nu^{-1} |\sin \nu \theta|,$$

thus (2.1) reduces to

$$(2.4) \quad \sum_1^n \nu^{-1} |\sin \nu \theta_n| = O(1), \quad \text{as } n \rightarrow \infty.$$

Now for any $\theta > 0$

$$(2.5) \quad \sum_1^n \nu^{-1} |\sin \nu \theta| < n\theta,$$

hence $n\theta_n = O(1)$ implies (2.4). To prove the converse let $\theta_n < 1 < \theta_n(n-1)$, and put $[\theta_n^{-1}] = \kappa_n = \kappa$, so that $\kappa \leq \theta_n^{-1} < \kappa + 1 \leq n$. Now $\sum_1^n \nu^{-1} |\sin \nu \theta_n| > (1/2) \sum_1^n \nu^{-1} (1 - \cos 2\nu \theta_n)$, and

$$\begin{aligned} \left| \sum_1^n \nu^{-1} \cos 2\nu \theta_n \right| &< \sum_1^\kappa \nu^{-1} + \left| \sum_{\kappa+1}^n \nu^{-1} \cos 2\nu \theta_n \right| \\ &< 1 + \log \kappa + (1/(\kappa + 1)) \max_{\kappa < \lambda \leq n} \left| \sum_{\kappa+1}^\lambda \cos 2\nu \theta_n \right| \\ &< 1 + \log \theta_n^{-1} + \theta_n / \sin \theta_n < 3 - \log \theta_n. \end{aligned}$$

Thus

$$2 \sum_1^n \nu^{-1} |\sin \nu \theta_n| > \log n + \log \theta_n - 3 = -3 + \log(n\theta_n);$$

hence (2.4) implies $n\theta_n = O(1)$. For null sequences only this is required.

To satisfy (2.2) consider the case that 0 is a limit point of the sequence $\{n\theta_n\}$; for a subsequence of indices n : $n\theta_n \rightarrow 0$, and for that subsequence, using (2.5)

$$\sum_1^n \rho_n \nu^{-1} \sin \nu \theta_n = O\left(\sum_1^n \nu^{-1} |\sin \nu \theta_n|\right) = O(n\theta_n) = o(1).$$

Hence σ , if it exists, is 0 and then every convergent sequence is transformed into a null sequence. Next assume $\liminf n\theta_n > 0$. We choose a subsequence of integers $n = n'$ for which ρ_n^n and $n\theta_n$ have limits $n'\theta_{n'} \rightarrow \beta > 0$, $\rho_{n'}^{n'} \rightarrow e^\gamma$ say; by (2.3) γ is finite. Furthermore from $\log \rho/(\rho-1) \rightarrow 1$ as $\rho \rightarrow 1$, $n'(\rho_{n'} - 1) \rightarrow \gamma$.

Suppose first $\gamma = 0$, that is $\rho_{n'}^{n'} \rightarrow 1$, and $n'(\rho_{n'} - 1) \rightarrow 0$. Now, as n runs through the sequence $\{n'\}$

$$\left| \sum_1^n (\rho_n - 1) \nu^{-1} \sin \nu \theta_n \right| < |\rho_n - 1| O(n) \sum_1^n \nu^{-1} |\sin \nu \theta_n| = o(1) O(n\theta_n) = o(1),$$

hence

$$\lim \sum_1^n \rho_n^{\nu} \nu^{-1} \sin \nu \theta_n = \lim \sum_1^n \nu^{-1} \sin \nu \theta_n$$

for $n = n' \rightarrow \infty$, if either side exists. But

$$\begin{aligned} \sum_1^n \nu^{-1} \sin \nu \theta &= \int_0^\theta \left(\sum_1^n \cos \nu t \right) dt = - (1/2)\theta + \int_0^\theta \frac{\sin (n + 1/2)t}{2 \sin (t/2)} dt \\ &= - (1/2)\theta + \int_0^{(n+1/2)\theta} \frac{\sin u du}{(2n+1) \sin (u/(2n+1))}, \end{aligned}$$

hence

$$\begin{aligned} \sum_1^n \nu^{-1} \sin \nu \theta_n &= - (1/2)\theta_n + \int_0^\beta \frac{\sin u}{u} \cdot \frac{u du}{(2n+1) \sin (u/(2n+1))} + o(1) \\ &\rightarrow \int_0^\beta \frac{\sin u}{u} du \end{aligned}$$

as $n \rightarrow \infty$ through the sequence $\{n'\}$. The consideration of the case $\gamma \neq 0$ remains; we write

$$\begin{aligned} \sum_1^n \rho^{\nu} \nu^{-1} \sin \nu \theta &= \int_0^\theta \left(\sum_1^n \rho^{\nu} \cos \nu t \right) dt \\ &= \int_0^\theta \frac{\cos t - \rho^2 + \rho^{n+2} \cos nt - \rho^{n+1} \cos (n+1)t}{1 - 2\rho \cos t + \rho^2} dt \\ &= \int_0^\theta \frac{1 - \rho^2 - (1 - \cos t) + \rho^{n+1} [\cos nt - \cos (n+1)t] - (1 - \rho) \rho^{n+1} \cos nt}{(1 - \rho)^2 + 2\rho(1 - \cos t)} dt, \end{aligned}$$

thus

$$\begin{aligned} \sum_1^n \rho^{\nu} \nu^{-1} \sin \nu \theta &= (1 - \rho^2) \int_0^\theta \frac{dt}{(1 - \rho)^2 + 4\rho \sin^2 (t/2)} \\ &\quad - 2 \int_0^\theta \frac{\sin^2 (t/2) dt}{(1 - \rho)^2 + 4\rho \sin^2 (t/2)} \\ &\quad - (1 - \rho) \rho^{n+1} \int_0^\theta \frac{\cos nt dt}{(1 - \rho)^2 + 4\rho \sin^2 (t/2)} \\ &\quad + 2\rho^{n+1} \int_0^\theta \frac{\sin (t/2) \sin (n + 1/2)t dt}{(1 - \rho)^2 + 4\rho \sin^2 (t/2)}. \end{aligned} \tag{2.6}$$

Now

$$\int_0^{\theta_n} \frac{\sin^2(t/2) dt}{(1 - \rho_n)^2 + 4\rho_n \sin^2(t/2)} < \frac{\theta_n}{4\rho_n} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Next

$$\begin{aligned} (\rho_n^2 - 1) \int_0^{\theta_n} \frac{dt}{(1 - \rho_n)^2 + 4\rho_n \sin^2(t/2)} \\ = n(\rho_n - 1)(\rho_n + 1) \int_0^{n\theta_n} \frac{du}{n^2[(\rho_n - 1)^2 + 4\rho_n \sin^2(u/2n)]} \\ \rightarrow 2\gamma \int_0^\beta \frac{du}{\gamma^2 + u^2} = 2 \arctan(\beta/\gamma). \end{aligned}$$

Similarly

$$\begin{aligned} (\rho_n - 1)\rho_n^{n+1} \int_0^{\theta_n} \frac{\cos ntdt}{(1 - \rho_n)^2 + 4\rho_n \sin^2(t/2)} \\ = n(\rho_n - 1)\rho_n^{n+1} \int_0^{n\theta_n} \frac{\cos udu}{n^2[(\rho_n - 1)^2 + 4\rho_n \sin^2(u/2n)]} \rightarrow \gamma e^\gamma \int_0^\beta \frac{\cos udu}{\gamma^2 + u^2}, \end{aligned}$$

and

$$\begin{aligned} \rho_n^{n+1} \int_0^{\theta_n} \frac{\sin(t/2) \sin(n + 1/2)tdt}{(1 - \rho_n)^2 + 4\rho_n \sin^2(t/2)} \\ = \rho_n^{n+1} \int_0^{(n+1/2)\theta_n} \frac{(2n + 1) \sin\{u/(2n + 1)\} \cdot \sin udu}{n(2n + 1)[(\rho_n - 1)^2 + 4\rho_n \sin^2\{u/(2n + 1)\}]} \\ \rightarrow (1/2)e^\gamma \int_0^\beta \frac{u \sin udu}{\gamma^2 + u^2}. \end{aligned}$$

Summarizing

$$\begin{aligned} \sum_1^n \rho_n^{\nu-1} \sin \nu \theta_n \rightarrow \int_0^\beta \frac{\gamma e^\gamma \cos u + e^\gamma u \sin u - 2\gamma}{\gamma^2 + u^2} du \\ = \int_0^{\beta/\gamma} \frac{e^\gamma(t \sin \gamma t + \cos \gamma t) - 2}{1 + t^2} dt. \end{aligned}$$

The case (2.3) is now completely discussed. We next assume

$$\limsup_{n \rightarrow \infty} \rho_n^n = \infty,$$

so that for a subsequence $n = n': \rho_{n'}^{n'} \rightarrow \infty$. We first prove that (2.1) implies $n'\theta_{n'} \rightarrow 0$. Otherwise for a subsequence n'' of $n': n''\theta_{n''} \rightarrow \beta > 0$. For these indices

$$\sum_1^n \rho_n \nu^{-1} |\sin \nu \theta_n| > \sum_{\nu \leq \alpha/\theta_n} \rho_n \nu^{-1} |\sin \nu \theta_n|,$$

where α is so chosen that $0 < \alpha < \beta$ and $\alpha \leq \pi/2$. Now

$$\sum_{\nu \leq \alpha/\theta_n} \rho_n \nu^{-1} |\sin \nu \theta_n| > (2/\pi) \theta_n \sum \rho_n = \frac{2\rho_n \theta_n}{\pi} \frac{\rho_n^{[\alpha \theta_n^{-1}]} - 1}{\rho_n - 1},$$

hence (2.1) implies

$$\theta_n \rho_n^{\alpha n/2\beta} = O(\rho_n - 1),$$

which by virtue of $\log \rho_n/(\rho_n - 1) \rightarrow 1$ yields $n\theta_n = o(1)$. Furthermore

$$(2/\pi) \theta_n \sum \rho_n < \sum_1^n \rho_n \nu^{-1} |\sin \nu \theta_n| < \theta_n \sum_1^n \rho_n,$$

thus, if for a subsequence of indices $\rho_n^n \rightarrow \infty$, then for these indices (2.1) is equivalent to

$$\theta_n \rho_n^n = O(\rho_n - 1).$$

If this condition is satisfied, then in view of $n\theta_n \rightarrow 0$

$$\begin{aligned} 0 &< \theta_n \sum_1^n \rho_n - \sum_1^n \rho_n \nu^{-1} \sin \nu \theta_n = \theta_n \sum_1^n \rho_n \left(1 - \frac{\sin \nu \theta_n}{\nu \theta_n}\right) \\ &< \theta_n \left(1 - \frac{\sin n\theta_n}{n\theta_n}\right) \sum_1^n \rho_n \rightarrow 0, \end{aligned}$$

hence (2.2) holds if and only if $\lim \theta_n \rho_n^n/(\rho_n - 1)$ exists, which is then the value of σ .

Finally assume that $\liminf \rho_n^n = 0$; thus for a subsequence of indices $\rho_n^n \rightarrow 0$ ($\gamma = -\infty$). If $n\theta_n = O(1)$, then $\sum_1^n \rho_n \nu^{-1} |\sin \nu \theta_n| = O(\sum_1^n \nu^{-1} |\sin \nu \theta_n|) = O(1)$, which is (2.1). If on the other hand for a subsequence $n\theta_n \rightarrow \infty$, then

$$\sum_1^n \rho_n \nu^{-1} |\sin \nu \theta_n| > \sum_{\theta_n^{-1} < \nu}^{\leq n} \rho_n \nu^{-1} \frac{1 - \cos 2\nu \theta_n}{2};$$

but $\rho_n \nu^{-1} \downarrow$ as $\nu \uparrow$, hence for $\theta_n < 1$

$$\begin{aligned} \left| \sum_{\theta_n^{-1} < \nu}^{\leq n} \rho_n \nu^{-1} \cos 2\nu \theta_n \right| &< \rho_n^{1+[\theta_n^{-1}]} \frac{1}{1 + [\theta_n^{-1}]} \max_{\lambda \leq n} \left| \sum_{\theta_n^{-1} < \nu}^{\leq \lambda} \cos 2\nu \theta_n \right| \\ &< \frac{\theta_n}{\sin \theta_n} < \pi/2. \end{aligned}$$

Furthermore

$$\begin{aligned}\sum_{\theta_n^{-1} < \nu}^{\leq n} \rho_n \nu^{-1} &= \sum_{\theta_n^{-1} < \nu}^{\leq n} \nu^{-1} \exp(\nu \log \rho_n) > \sum \int_{\nu}^{\nu+1} u^{-1} \exp(-u \log \rho_n^{-1}) du \\ &= \int_{1+[\theta_n^{-1}]}^{\nu+1} u^{-1} \exp(-u \log \rho_n^{-1}) du = \int_{(1+[\theta_n^{-1}]) \log \rho_n^{-1}}^{(n+1) \log \rho_n^{-1}} t^{-1} e^{-t} dt;\end{aligned}$$

thus in this case (2.1) implies $\theta_n = O(\log 1/\rho_n)$, or $\theta_n = O(1 - \rho_n)$. If this condition is satisfied, then

$$\sum_1^n \rho_n \nu^{-1} |\sin \nu \theta_n| = O\left(\theta_n \sum_1^n \rho_n\right) = O(\theta_n/(1 - \rho_n)) = O(1),$$

hence (2.1) holds. To satisfy (2.2) now, we note that

$$\sum_{n+1}^{\infty} \rho_n \nu^{-1} \sin \nu \theta_n = O\left(\theta_n \frac{\rho_n^n}{1 - \rho_n}\right) = O(\rho_n^n) = o(1),$$

hence (2.2) holds if and only if

$$\sum_1^{\infty} \rho_n \nu^{-1} \sin \nu \theta_n = \arctan \frac{\rho_n \sin \theta_n}{1 - \rho_n \cos \theta_n}$$

has a limit, and σ is then this limit. But

$$\frac{\rho_n \sin \theta_n}{1 - \rho_n \cos \theta_n} = \frac{\rho_n \sin \theta_n}{1 - \rho_n + \rho_n(1 - \cos \theta_n)} \sim \frac{\theta_n}{1 - \rho_n} \cdot \frac{1}{1 + O(1 - \rho_n)} \sim \frac{\theta_n}{1 - \rho_n},$$

hence σ exists, if and only if $\lim \theta_n/(1 - \rho_n) = \delta < +\infty$. We then have $\sigma = \lim \arctan \{\theta_n/(1 - \rho_n)\} = \arctan \delta$. To summarize our results put

$$\begin{aligned}\sigma(\beta, 0) &= \int_0^{\beta} \frac{\sin u}{u} du, \\ (a) \quad \sigma(\beta, \gamma) &= \int_0^{\beta} \frac{\gamma e^{\gamma} \cos u + e^{\gamma} \sin u - 2\gamma}{\gamma^2 + u^2} du, \text{ for finite } \gamma \neq 0,\end{aligned}$$

$$(b) \quad \sigma(0, \infty) = \lim \frac{\theta_n \rho_n^n}{\rho_n - 1},$$

$$(c) \quad \sigma(\delta, -\infty) = \lim \arctan \frac{\theta_n}{1 - \rho_n} = \arctan \delta < \pi/2.$$

We then have

THEOREM 1. *Necessary and sufficient conditions that for every convergent sequence $n b_n \rightarrow \tau$ the transform $\sum_1^n \rho_n \nu b_\nu \sin \nu \theta_n$ has a limit, are that one of the following three cases holds:*

- (a') $n(\rho_n - 1) \rightarrow \gamma$ finite, $n\theta_n \rightarrow \beta < \infty$,
 (b') $n(\rho_n - 1) \rightarrow +\infty$, $\lim \theta_n \rho_n^n (\rho_n - 1)^{-1}$ exists,
 (c') $n(\rho_n - 1) \rightarrow -\infty$, $\lim \theta_n (1 - \rho_n)^{-1} = \delta$ exists, $0 \leq \delta < \infty$.

The limit of the transform is then $\tau\sigma$, where σ is defined above for the respective cases. Different subsequences may belong to different cases (β, γ) if only the corresponding σ attain the same value, and with the restriction $n\theta_n = O(1)$ in case (a').

3. **Permanency with respect to (C, κ) summability.** Given the sequence $\{\tau_n\}$, write

$$\tau_n^0 = \tau_n, \quad \tau_n^\kappa = \sum_{r=1}^n \tau_r^{\kappa-1}, \quad n, \kappa = 1, 2, 3, \dots;$$

also

$$(3.1) \quad A_n^\kappa = C_{n+\kappa, n} = \frac{(\kappa+1) \cdots (\kappa+n)}{n!} \sim \frac{n^\kappa}{\kappa!}, \quad \text{as } n \rightarrow \infty.$$

The sequence $\{\tau_n\}$ is summable (C, κ) to the value τ , if $\tau_n^\kappa / A_n^\kappa \rightarrow \tau$ as $n \rightarrow \infty$; $(C, 0)$ is evidently convergence.

We write

$$\Delta^0 \tau_n = \tau_n, \quad \Delta^1 \tau_n = \Delta \tau_n = \tau_n - \tau_{n+1}, \quad \Delta^\kappa \tau_n = \Delta(\Delta^{\kappa-1} \tau_n);$$

then by induction

$$(3.2) \quad \Delta^\kappa \tau_n = \sum_{r=0}^{\kappa} (-1)^r C_{\kappa, r, \tau_{n+r}}, \quad \kappa = 0, 1, 2, \dots$$

Abel's transformation yields for finite sums

$$\sum_1^n \alpha_r \tau_r = \sum_1^n \tau_r \Delta^1 \alpha_r = \sum_1^n \tau_r \Delta^2 \alpha_r = \dots,$$

where $\alpha_{n+1} = 0, \alpha_{n+2} = 0, \dots$. Applying this to (1.5) we get

$$T_n = \sum_{r=1}^n \tau_r \Delta^\kappa a_{nr},$$

where $a_{nr} = 0$ for $r > n$. Thus the transform converges for every (C, κ) summable sequence if in addition to the conditions of §2

$$\sum_{r=1}^n A_r^\kappa |\Delta^\kappa a_{nr}| = O(1) \quad \text{as } n \rightarrow \infty.$$

In particular for the transform (1.6) we have the conditions (2.1), (2.2) and

$$(3.3) \quad \sum_{r=1}^{n-\kappa} A_r^\kappa |\Delta^\kappa \rho_n^r|^{-1} \sin r\theta_n| + \sum_{r=n-\kappa+1}^n A_r^\kappa |\delta_r| = O(1) \quad \text{as } n \rightarrow \infty,$$

where, from (3.2)

$$\delta_\lambda = \sum_{\nu=0}^{n-\lambda} (-1)^\nu C_{\kappa, \nu} \frac{\lambda + \nu}{\lambda + \nu} \frac{\sin(\lambda + \nu)\theta_n}{\lambda + \nu}, \quad n - \kappa < \lambda \leq n.$$

We first consider $(C, 1)$ summability ($\kappa=1$). Now (3.3) becomes

$$\sum_1^{n-1} (\nu + 1) |\Delta \rho_n^\nu \nu^{-1} \sin \nu \theta_n| + (n + 1) \rho_n^n n^{-1} |\sin n \theta_n| = O(1),$$

or

$$(3.4) \quad \sum_1^{n-1} \nu |\rho_n^\nu \nu^{-1} \sin \nu \theta_n - \rho_n^{\nu+1} (\nu + 1)^{-1} \sin (\nu + 1) \theta_n| + \rho_n^n |\sin n \theta_n| = O(1).$$

We consider in succession the different cases of Theorem 1.

(a') For a sequence of indices $n\theta_n \rightarrow \beta < \infty$, $n(\rho_n - 1) \rightarrow \gamma$ finite, that is, $\rho_n^n \rightarrow e^\gamma > 0$. Thus $\rho_n^n \sin n\theta_n$ is $O(1)$, and

$$\begin{aligned} & \sum_1^{n-1} \nu \rho_n^\nu |\nu^{-1} \sin \nu \theta_n - \rho_n (\nu + 1)^{-1} \sin (\nu + 1) \theta_n| \\ & \leq \sum_1^{n-1} \nu \rho_n^\nu |\nu^{-1} \sin \nu \theta_n - (\nu + 1)^{-1} \sin (\nu + 1) \theta_n| \\ & \quad + |1 - \rho_n| \sum_1^{n-1} \rho_n^\nu |\sin (\nu + 1) \theta_n|. \end{aligned}$$

Now

$$|1 - \rho_n| \sum_1^{n-1} \rho_n^\nu |\sin (\nu + 1) \theta_n| < |1 - \rho_n| \sum_1^n \rho_n^\nu < \rho_n |1 - \rho_n| = O(1),$$

and

$$\begin{aligned} & \nu |\nu^{-1} \sin \nu \theta_n - (\nu + 1)^{-1} \sin (\nu + 1) \theta_n| \\ & = |(\nu + 1)^{-1} \sin (\nu + 1) \theta_n - 2 \sin (1/2) \theta_n \cos ((2\nu + 1)/2) \theta_n| < 2\theta_n; \end{aligned}$$

hence

$$\sum_1^{n-1} \nu \rho_n^\nu |\nu^{-1} \sin \nu \theta_n - (\nu + 1)^{-1} \sin (\nu + 1) \theta_n| < 2\theta_n \sum_1^n \rho_n^\nu = O(n\theta_n) = O(1).$$

Hence in this case no additional condition results.

(b') $n(\rho_n - 1) \rightarrow +\infty$, $\theta_n \rho_n^n (\rho_n - 1)^{-1} \rightarrow \sigma$. Hence $n\theta_n \rightarrow 0$, and now $\rho_n^n \sin n\theta_n = O(1)$ is equivalent to $n\theta_n \rho_n^n = O(1)$. Thus $\theta_n \rho_n^n (\rho_n - 1)^{-1} = n\theta_n \rho_n^n n^{-1} (\rho_n - 1)^{-1} \rightarrow 0$, that is, $\sigma = 0$. Now

$$\begin{aligned} |1 - \rho_n| \sum_1^{n-1} \rho_n^\nu |\sin (\nu + 1) \theta_n| &= O \left[(\rho_n - 1) \theta_n \sum_1^n \nu \rho_n^\nu \right] \\ &= O[\theta_n \rho_n^n (\rho_n - 1)^{-1}] = o(1); \end{aligned}$$

In case (a'') the first condition becomes $\sin n\theta_n = O(n^{1-\kappa})$, as $n \rightarrow \infty$; in particular $\sin n\theta_n \rightarrow 0$, thus in view of (a'') $n\theta_n \rightarrow \lambda\pi$, λ a positive integer or zero. On putting $n\theta_n = \lambda\pi + \epsilon_n$, we get $\sin \epsilon_n = O(n^{1-\kappa})$, or $n\theta_n - \lambda\pi = O(n^{1-\kappa})$. From the second condition now $\cos n\theta_n \sin \theta_n = O(n^{1-\kappa})$, as $n \rightarrow \infty$, or $\lambda\pi + \epsilon_n = O(n^{2-\kappa})$; hence for $\kappa=2$, (3.6) reduces to

$$(3.7) \quad n\theta_n = \lambda\pi + O(n^{-1}).$$

For $\kappa > 2$ we must have

$$n\theta_n - \lambda\pi = \epsilon_n = O(n^{1-\kappa}) \quad \text{and} \quad \lambda\pi + \epsilon_n = O(n^{2-\kappa}),$$

hence $\lambda=0$, and

$$(3.7') \quad n\theta_n = O(n^{1-\kappa}).$$

It then follows that

$$n^{\kappa-1} \sin(n - \nu)\theta_n = O(1) \quad \text{for } \nu = 0, 1, \dots, \kappa - 1.$$

Furthermore, for the rest of (3.3)

$$\sum_1^{n-\kappa} A_\nu^\kappa |\Delta^\kappa \rho_n^{\nu-1} \sin \nu\theta_n| = O\left(\sum_1^{n-\kappa} \nu^\kappa |\Delta^\kappa \rho_n^{\nu-1} \sin \nu\theta_n|\right).$$

Now

$$\Delta^\kappa \rho^{\nu-1} \sin \nu\theta = \Delta^\kappa \rho^\nu \int_0^\theta \cos \nu t dt = R \int_0^\theta \Delta^\kappa z^\nu dt, \quad z = \rho e^{it},$$

and, using (3.2)

$$\Delta^\kappa \rho^{\nu-1} \sin \nu\theta = R \int_0^\theta \sum_{\lambda=0}^{\kappa} (-1)^\lambda C_{\kappa, \lambda} z^{\nu+\lambda} dt = R \int_0^\theta z^\nu (1-z)^\kappa dt,$$

hence

$$\begin{aligned} |\Delta^\kappa \rho^{\nu-1} \sin \nu\theta| &< \rho^\nu \int_0^\theta |1 - \rho e^{it}|^\kappa dt < \rho^\nu \int_0^\theta \{(1-\rho)^2 + \rho t^2\}^{\kappa/2} dt \\ &< \theta \rho^\nu \{(1-\rho)^2 + \rho \theta^2\}^{\kappa/2}. \end{aligned}$$

Thus

$$\begin{aligned} \sum_1^{n-\kappa} \nu^\kappa |\Delta^\kappa \rho_n^{\nu-1} \sin \nu\theta_n| &< \left(\sum_1^n \nu^\kappa \rho_n^\nu\right) \{(1-\rho_n)^2 + \rho_n \theta_n^2\}^{\kappa/2} \theta_n \\ (3.8) \quad &< \{n^2(1-\rho_n)^2 + \rho_n n^2 \theta_n^2\}^{\kappa/2} \theta_n \sum_1^n \rho_n^\nu \\ &= O\left(\theta_n \sum_1^n \rho_n^\nu\right), \end{aligned}$$

and, from $\rho_n^\kappa = O(1)$,

$$\theta_n \sum_1^n \rho_n' = O(n\theta_n) = O(1).$$

Hence in case (a'') the additional condition is (3.7) for $\kappa=2$, and (3.7') for $\kappa>2$.

In case (b''): $n(\rho_n-1) \rightarrow +\infty$, $n\theta_n\rho_n^n = O(1)$, as $n \rightarrow \infty$; hence $\rho_n^n \rightarrow +\infty$, and $n\theta_n \rightarrow 0$. Now (3.6) becomes

$$(3.9) \quad n^\kappa \rho_n^n \theta_n = O(1).$$

For large n evidently $\rho_n > 1$, and

$$\theta_n \sum_1^n \rho_n' < n\theta_n \rho_n^n = O(1)$$

(from (3.9)). In view of (3.8) now (3.3) holds. Thus in this case the additional condition is (3.9) (for $\kappa \geq 2$).

Finally, in case (c''): $n(\rho_n-1) \rightarrow -\infty$ (that is $\rho_n^n \rightarrow 0$), and $\lim \theta_n/(1-\rho_n) = \delta < \infty$ exists. Now $\rho_n^n < 1/(n(1-\rho_n))$, hence $n\theta_n\rho_n^n = O(1)$; thus for $\kappa=2$ condition (3.6) reduces to $n\rho_n^n \sin n\theta_n = O(1)$. While for $\kappa>2$ (3.6) reduces to $n^{\kappa-1}\rho_n^n \sin n\theta_n = O(1)$ and $n^{\kappa-1}\theta_n\rho_n^n = O(1)$. Furthermore, as $\rho_n < 1$, $\theta_n \sum_0^n \rho_n' < \theta_n/(1-\rho_n) = O(1)$, hence, in view of (3.8) now (3.3) is satisfied.

We summarize our results in

THEOREM 3. *In order that $\lim \sum_1^n \rho_n' b_r \sin v\theta_n = \tau\sigma$ exists, whenever (C, κ) $\lim nb_n = \tau$ for some $\kappa \geq 2$, necessary and sufficient conditions are the alternatives:*

(a''') $n(\rho_n-1) \rightarrow \gamma$, finite, and for $\kappa=2$: $n\theta_n = \lambda\pi + O(n^{-1})$, λ an integer, for $\kappa>2$: $\theta_n = O(n^{-\kappa})$;

(b''') $n(\rho_n-1) \rightarrow +\infty$, $n^\kappa \rho_n^n \theta_n = O(1)$;

(c''') $n(\rho_n-1) \rightarrow -\infty$, $\lim \theta_n/(1-\rho_n) = \delta < \infty$ exists, and for $\kappa=2$: $n\rho_n^n \sin n\theta_n = O(1)$, for $\kappa>2$: $n^{\kappa-1}\rho_n^n(\theta_n + |\sin n\theta_n|) = O(1)$.

The value of σ is given in case (a''') by (a), where for $\kappa=2$: $\beta = \lambda\pi$, for $\kappa>2$: $\beta=0$, $\sigma=0$; in case (b'''): $\sigma=0$; in case (c'''): $\sigma = \arctan \delta$.

4. Application to Fourier series. First consider a function of bounded variation and its Fourier sine series (1.1). It follows from the introduction that $\lim nb_n$, if it exists, is $(2/\pi)f(+0)$. Under the assumptions of Theorem 1 on ρ_n and θ_n , $s_n(\rho_n, \theta_n) \rightarrow \tau\sigma = (2\sigma/\pi)f(+0)$. In particular whenever $\sigma > \pi/2$, then we have an analogue of Gibbs' phenomenon. It is known that for functions of bounded variation

$$(1/n) \sum_1^n vb_r \rightarrow (2/\pi)f(+0);$$

more generally if (cf. Szász [3, Lemma 6])

$$(4.1) \quad 2f_1(\theta) = (2/\theta) \int_0^\theta f(t) dt \rightarrow j, \quad \text{as } \theta \downarrow 0,$$

and

$$(4.2) \quad \lim_{\delta \downarrow 0} \liminf_{n \rightarrow \infty} \min_{0 < \kappa < \delta_n} \sum_n^{n+\kappa} b_\nu \geq 0,$$

then

$$(1/n) \sum_1^n \nu b_\nu \rightarrow j/\pi.$$

Hence, applying Theorem 2 we have

$$s_n(\rho_n, \theta_n) \rightarrow (j/\pi)\sigma(\beta, \gamma), \quad \text{as } n(\rho_n - 1) \rightarrow \gamma \text{ and } n\theta_n \rightarrow \beta;$$

j is the generalized jump of $f(\theta)$ at $\theta=0$. For $\gamma=0$ this yields a generalization of formula (1.4). Note that

$$f_1(\theta) = \theta^{-1} \int_0^\theta f(t) dt = \sum_1^\infty b_\nu \frac{1 - \cos \nu\theta}{\nu\theta} = (\theta/2) \sum_1^\infty \nu b_\nu \left(\frac{\sin(\nu\theta/2)}{\nu\theta/2} \right)^2;$$

$(2\theta/\pi) \{s_0/2 + \sum_1^\infty ((\sin \nu\theta)/\nu\theta)^2 s_\nu\}$ is called the Riemannian mean of the second kind corresponding to the sequence $\{s_n\}$. It is a regular transform, as is seen from the identity

$$\frac{2\theta}{\pi} \left\{ 1/2 + \sum_1^\infty \left(\frac{\sin \nu\theta}{\nu\theta} \right)^2 \right\} = 1.$$

If we assume only that $(C, 2) \lim nb_n = j/\pi$ exists, then Theorem 3 yields again a Gibbs' phenomenon in the case (a''') and $\lambda > 0$.

In this connection we introduce two lemmas.

LEMMA 1. *If*

$$(4.3) \quad (1-r) \sum_1^\infty \tau_n r^n \rightarrow \tau \quad \text{as } r \uparrow 1,$$

and

$$(4.4) \quad \tau'_n = \sum_1^n \tau_\nu > -pn,$$

for some $p > 0$, and all $n > 0$, then

$$(4.5) \quad (C, 2) \lim \tau_n = \tau.$$

We have from (4.3)

$$(1-r)^2 \sum_1^{\infty} \tau'_n r^n \rightarrow \tau \quad \text{as } r \uparrow 1,$$

hence

$$(1-r)^2 \sum_1^{\infty} (\tau'_n + pn) r^n \rightarrow \tau + p \quad \text{as } r \uparrow 1;$$

in view of (4.4) a theorem of Hardy and Littlewood yields

$$\sum_1^n (\tau'_v + pv) \sim (1/2)(\tau + p)n^2,$$

or

$$\sum_1^n \tau'_v \sim (1/2)\tau n^2, \quad \text{as } n \rightarrow \infty,$$

which is (4.5).

LEMMA 2. *If (4.1) holds, then $(1-r)\sum_1^{\infty} nb_n r^n \rightarrow j/\pi$. [3, Lemma 5].*

Combining these two lemmas it is seen that (4.1) and the assumption

$$(4.6) \quad \sum_1^n vb_v > -pn \quad \text{for some } p > 0 \text{ and all } n > 0,$$

imply $(C, 2) \lim nb_n = j/\pi$. With reference to Theorem 3 the assumptions (4.1) and (4.6) again yield a Gibbs' phenomenon.

In closing we remark that the existence of $f(+0)$ implies itself $(C, 2) \lim nb_n = (1/2)f(+0)$. A more general result will be given elsewhere.

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