THE TRANSFORMATION OF DOUBLE INTEGRALS

BY

R. G. HELSEL AND T. RADÓ

CHAPTER I. INTRODUCTION

1.1. Let there be given a continuous transformation T of the form

$$T: \quad x = x(u, v), \qquad y = y(u, v),$$

where $x(u, v)$, $y(u, v)$ are defined and continuous on the unit square

$$S_0: \quad 0 \leqq u \leqq 1, \qquad 0 \leqq v \leqq 1.$$

Designate the continuous image of the boundary of S_0 under T by C. We shall be concerned with the validity of the *topological transformation formula* (apparently first studied by Schauder [1](1))

$$(1) \qquad \iint_{S_0} F(x(u, v), y(u, v))J(u, v; T)dudv = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(x, y)\mu(x, y; S_0, T)dxdy,$$

where $F(x, y)$ is any measurable function in the xy-plane; $J(u, v; T)$ is the Jacobian, $x_u(u, v)y_v(u, v) - x_v(u, v)y_u(u, v)$; and $\mu(x, y; S_0, T)$ is equal to the topological index of the point (x, y) with respect to C if (x, y) is not on C; otherwise, $\mu(x, y; S_0, T) = 0$. The topological index of a point (x, y) with respect to the continuous curve C is defined as follows:

DEFINITION. *As (u, v) describes the boundary of S_0 once in the counterclockwise sense, $(x(u, v), y(u, v))$ describes in the xy-plane the directed closed continuous curve C, and, if (x, y) is a point not on C, the change of the continuously varying argument of the vector from (x, y) to $(x(u, v), y(u, v))$ is of the form $2k\pi$, where k is an integer (positive, negative, or zero). The integer k is called the topological index of the point (x, y) with respect to C.*

If $F(x, y) \equiv 1$, formula (1) becomes the *area formula*

$$(2) \qquad \iint_{S_0} J(u, v; T)dudv = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mu(x, y; S_0, T)dxdy.$$

If the transformation T is biunique, $|\mu(x, y; S_0, T)|$ assumes only two values, 0 or 1, and formula (1) reduces to the *ordinary transformation formula*

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(1) Numbers in brackets refer to bibliography at end of the paper.

$$\iint_{S_0} F(x(u, v), y(u, v)) \,|\, J(u, v; T) \,|\, du dv = \iint_{T(S_0^0)} F(x, y) dx dy,$$

where $T(S_0^0)$ is the image under T of S_0^0, the set of interior points of S_0.

1.2. Radó and Reichelderfer have established formula (1), under the assumption that the integral on the left exists and the locus of C is a set of planar measure zero, for a wide class of continuous transformations which they call K_3. (Many other results hold as well in class K_3. See Radó and Reichelderfer [1].) The purpose of the present paper is to show that this result implies all of the results on the transformation of double integrals in the literature of which we are aware (see McShane [1], Morrey [1], Rademacher [1], Radó [1, 2, 3], Schauder [1], Young [1, 2, 3]). It is not difficult to verify, by means of the two theorems of Radó and Reichelderfer quoted in 1.6 and 1.7, that the transformations considered by McShane [1], Morrey [1], Rademacher [1], Radó [1, 2, 3], and Schauder [1] belong to class K_3 and that the formulas established by these authors follow from formula (1). On the other hand, it is not apparent that Young's results are implied by those of Radó and Reichelderfer. We shall devote our attention, then, to the work of Young [1, 2, 3], ultimately showing that all of his results can be accounted for in terms of the results of Radó and Reichelderfer. (It is interesting to note that our methods will place many of the transformations considered by Young in Morrey's class L. Morrey [1] has established formula (2) for transformations in class L.)

1.3. The two transformation formulas which Young develops are the same on the left as formulas (1) and (2); however, the right sides of Young's formulas are not Lebesgue integrals, but the limits of Lebesgue integrals. We shall now give a brief description of these limits. Let σ be the generic notation for a subdivision of the boundary of S_0 by points $P_1, \cdots, P_m$, numbered consecutively in the positive sense around S_0. Let Π_σ be the directed closed polygon "inscribed" in C which is formed by straight segments connecting in order the images of $P_1, \cdots, P_m, P_1$ under T. Define $\mu_\sigma(x, y)$ to be equal to the topological index of the point (x, y) with respect to Π_σ if (x, y) is not on Π_σ; otherwise, set $\mu_\sigma(x, y)=0$. Then if $F(x, y)$ is any measurable function in the xy-plane, Young defines the *integral of $F(x, y)$ over the area of the polygon* Π_σ as follows:

$$(Y) \iint_{\Pi_\sigma} F(x, y) dx dy = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} F(x, y) \mu_\sigma(x, y) dx dy,$$

providing the Lebesgue integral on the right exists. Young then defines the *integral of $F(x, y)$ over the area of the curve C* in the following way:

$$(Y) \iint_C F(x, y) dx dy = \lim_{\|\sigma\| \to 0} (Y) \iint_{\Pi_\sigma} F(x, y) dx dy,$$

providing the limit on the right exists and is independent of the manner in which $\|\sigma\| \to 0$. ($\|\sigma\|$ is the norm of the subdivision σ.)

1.4. For certain classes of continuous transformations

$$T: \quad x = x(u, v), \qquad y = y(u, v), \qquad (u, v) \in S_0,$$

Young has established the *area formula*

$$\iint_{S_0} J(u, v; T)dudv = (Y)\iint_C dxdy, \tag{3}$$

where C is the image of the boundary of S_0 under T and $(Y)\iint_C dxdy$ is assumed to exist. In a few instances, he also asserts that the *general transformation formula*

$$\iint_{S_0} F(x(u, v), y(u, v))J(u, v; T)dudv = (Y)\iint_C F(x, y)dxdy \tag{4}$$

holds for any measurable function $F(x, y)$ in the xy-plane as soon as the integral on the right exists. Throughout his work, Young assumes that the coordinate functions $x(u, v)$ and $y(u, v)$ of the transformation T are absolutely continuous on every horizontal and on every vertical in S_0; hence, the continuous curve C is rectifiable and its locus is a set of planar measure zero.

1.5. A comparison of formulas (1) and (4) leads to this engaging question: When does

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(x, y)\mu_\sigma(x, y)dxdy \xrightarrow[\|\sigma\|\to 0]{} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(x, y)\mu(x, y; S_0, T)dxdy?$$

Young neither raises nor answers this question. Instead, he always assumes the existence of the limit on the left and defines it to be the new integral, $(Y)\iint_C F(x, y)dxdy$. It is clear that an improvement is achieved if in formula (4) we can replace $(Y)\iint_C F(x, y)dxdy$ by $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(x, y)\mu(x, y; S_0, T)dxdy$. (Schauder [1] has already enunciated the desideratum of replacing $(Y)\iint_C F(x, y)dxdy$ by something more tangible.) In fact, in every case considered by Young but one, we shall accomplish this replacement by showing that the transformation belongs to class K_3 and, hence, formula (1) holds as soon as the integral on the left exists (cf. 1.2, 1.4). In the one exceptional case just mentioned, it would seem that Young did not establish anything (cf. 3.10). (The preceding statements should not be construed to mean that we have succeeded in identifying $(Y)\iint_C F(x, y)dxdy$ and $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(x, y)\mu(x, y; S_0, T)dxdy$ in all the cases of Young. Actually, it appears to us that a general study of the relation of these integrals would lead to questions of great difficulty and interest.)

1.6. To place Young's transformations in class K_3, we shall utilize the two following theorems of Radó and Reichelderfer [1].

THEOREM. *If the transformation T is continuous on the domain $\mathfrak{D}$ of the uv-plane; the partial derivatives $x_u(u, v)$, $x_v(u, v)$, $y_u(u, v)$, $y_v(u, v)$ exist and are continuous on $\mathfrak{D}$; and $J(u, v; T)$ is summable on $\mathfrak{D}$, then $T \in K_3(\mathfrak{D})$.*

1.7. CLOSURE THEOREM. *Let there be given bounded domains $\mathfrak{D}$ and $\mathfrak{D}_n$ in the uv-plane and continuous transformations*

$$T: \quad x = x(u, v), \qquad y = y(u, v), \qquad (u, v) \in \mathfrak{D},$$
$$T_n: \quad x = x_n(u, v), \qquad y = y_n(u, v), \qquad (u, v) \in \mathfrak{D}_n,$$

with the following properties: (i) *$\mathfrak{D}_n \subset \mathfrak{D}$ and, for every closed set $F \subset \mathfrak{D}$, there exists an $n_0 = n_0(F)$ such that $F \subset \mathfrak{D}_n$ for all $n > n_0$*; (ii) *$J(u, v; T)$ exists a.e. (almost everywhere) and is summable on $\mathfrak{D}$*; (iii) *$T_n \in K_3(\mathfrak{D}_n)$ for $n = 1, 2, \cdots$* ; (iv) *for every closed set $F \subset \mathfrak{D}$, $x_n(u, v) \rightarrow_n$ uniformly to $x(u, v)$, $y_n(u, v) \rightarrow_n$ uniformly to $y(u, v)$, on F and*

$$\lim_n \iint_F | J(u, v; T_n) - J(u, v; T) | \, dudv = 0.$$

Then $T \in K_3(\mathfrak{D})$.

(The Closure Theorem is still valid if the condition

$$\lim_n \iint_F | J(u, v; T_n) - J(u, v; T) | \, dudv = 0$$

is replaced by the weaker condition

$$\lim_n \iint_F | J(u, v; T_n) | \, dudv = \iint_F | J(u, v; T) | \, dudv.)$$

1.8. The method employed in Chapter III to place each transformation considered by Young in class K_3 is this: We approximate to the given transformation by a sequence of transformations which belong to class K_3 (to begin, we use transformations which belong to class K_3 because of 1.6) and satisfy the conditions of the Closure Theorem, thus placing the given transformation in class K_3. The feature of this paper is the utilization of integral means to obtain the approximating transformations.

The second chapter of this paper is a systematic presentation of general theorems in analysis and properties of integral means which we apply in Chapter III to prove our results. Since integral means give rise to non-denumerable sequences of approximating functions, many of the theorems in Chapter II are concerned with such sequences; however, we work only with denumerable sequences in Chapter III. The intrinsic interest of our results on non-denumerable sequences of integral means justifies their inclusion.

1.9. The cases which we treat in Chapter III are numerous and contain many assumptions so we shall not list them here, but we shall summarize the essential differences between our results and those of Young.

1. By placing Young's transformations in class K_3, we do more than establish formula (1) of 1.1 for them, since many other results hold as well for transformations in class K_3 (cf. 1.2).

2. In most of his cases, Young did not establish formula (4) of 1.4, but only the area formula (3). On the other hand, we shall establish the general topological transformation formula (1) for all of Young's transformations. Young established formula (3) for the transformations considered in 3.6, 3.7, 3.9, 3.12, and 3.14; formula (4) for the transformations of 3.8 and 3.14.1. The transformation of 3.14.1 is the most general case for which Young claims to have established formula (4).

3. Throughout Chapter III we assume there is given a continuous transformation T: $x=x(u, v)$, $y=y(u, v)$, $(u, v)\in S_0$, which satisfies the standard hypothesis H_0: $x(u, v)$ and $y(u, v)$ are absolutely continuous in the Tonelli sense (cf. 2.17) on S_0 and the image of the boundary of S_0 under T is a set of planar measure zero. The additional restrictions on T which we use in Chapter III do not differ from restrictions used by Young; however, Young always makes the standard assumption that $x(u, v)$ and $y(u, v)$ are of bounded variation in the Tonelli sense (cf. 2.17) on S_0 and are absolutely continuous on every horizontal and on every vertical in S_0. Clearly our hypothesis H_0 is less restrictive than Young's standard assumption (cf. 1.4).

It must be remembered that in all cases our theorems are not identical with those of Young (cf. 1.3, 1.5). The references to Young's work which appear in 3.6, 3.7, 3.8, 3.9, 3.12, 3.14, and 3.14.1 direct the reader to the theorems of Young which correspond to our theorems of those sections.

1.10. Young's theorems on the transformation of double integrals are not final in character; hence, we have attempted to find a general theorem which would at once account for all of his results. The following proposition, which we have not been able to establish or deny, seems a likely generalization (cf. 3.15):

If T satisfies H_0 (cf. 1.9) and $J(u, v; T)$ is summable on S_0, then $T\in K_3(S_0^0)$ and formula (1) *holds if the integral on the left exists.*

Chapter II. Auxiliary theorems and integral means

2.1. Most of the results listed in this chapter are known; however, a few of them are new—for example, see 2.12, 2.21, 2.21.1. Proofs are included whenever convenient references are not available.

2.2. The following concept will be used extensively in the present paper:

Definition. *A family $\mathcal{J}$ of measurable functions defined on S_0 is said to have the Vitali property—briefly, property ($\mathcal{V}$)—on a measurable subset E of S_0^0 if the following condition is satisfied. To every $\epsilon>0$ there corresponds a $\delta=\delta(\epsilon)>0$ such that*

$$\left|\iint_e f(u, v)dudv\right| < \epsilon$$

for every function $f(u, v)$ of $\mathcal{J}$ and every measurable set $e \subset E$ with $|e| < \delta$. (If e is any set, $|e|$ denotes the exterior measure of e.)

2.2.1. COROLLARY. *If a family $\mathcal{J}$ of functions $f(u, v)$ has property $(\mathcal{V})$ on a measurable subset E of S_0, then the family $\mathcal{J}^*$ of functions $|f(u, v)|$ also has, and conversely.*

2.2.2. COROLLARY. *If $\mathcal{J}$ is a family of functions possessing property $(\mathcal{V})$ on a measurable subset E of S_0 and $\mathcal{G}$ is a family of measurable functions which are uniformly bounded on E, then the family of functions $f(u, v)\cdot g(u, v)$, $f(u, v) \in \mathcal{J}$ and $g(u, v) \in \mathcal{G}$, has property $(\mathcal{V})$ on E.*

2.2.3. COROLLARY. *If a family $\mathcal{J}$ of functions has property $(\mathcal{V})$ on a measurable subset E of S_0, then $\iint_E f(u, v)dudv$ exists for every function $f(u, v)$ of $\mathcal{J}$ and these integrals are uniformly bounded.*

2.3. THEOREM. *If on a measurable subset E of S_0, $f_n(u, v) \geqq 0$ and $f_n(u, v) \rightarrow_n f(u, v)$ a.e., a necessary and sufficient condition for $f(u, v)$ to be summable on E and*

$$\iint_E f_n(u, v)dudv \underset{n}{\rightarrow} \iint_E f(u, v)dudv$$

is that $\{f_n(u, v)\}$ have property $(\mathcal{V})$ on E.

Proof. See de la Vallée Poussin [1, p. 477]. This theorem may be extended to a non-denumerable sequence of continuous functions $f_\alpha(u, v)$, $0<\alpha<1$, such that $f_\alpha(u, v) \rightarrow_{\alpha\rightarrow 0} f(u, v)$ a.e. on E.

2.3.1. COROLLARY. *If on a measurable subset E of S_0, $f_n(u, v) \rightarrow_n f(u, v)$ a.e., a necessary and sufficient condition for $f(u, v)$ to be summable on E and*

$$\iint_E |f_n(u, v) - f(u, v)|\, dudv \underset{n}{\rightarrow} 0$$

is that $\{f_n(u, v)\}$ have property $(\mathcal{V})$ on E.

2.4. Many of the properties of integral means which we shall utilize depend upon the next two theorems concerning partial differentiation of an indefinite integral.

THEOREM. *If $f(u, v)$ is summable on S_0 and*

$$F(u, v) = \int_0^u \int_0^v f(\xi, \eta)d\xi d\eta, \qquad (u, v) \in S_0,$$

then, for every v, $F_u(u, v)=\int_0^v f(u, \eta)d\eta$, provided u does not belong to a set of measure zero which is independent of v.

Proof. See Fubini. [1]. (Similarly, for every u, $F_v(u, v)=\int_0^u f(\xi, v)d\xi$, provided v does not belong to a set of measure zero which is independent of u. A majority of the theorems of this chapter are stated in an unsymmetric form as regards u and v. In each such instance, a second theorem may be obtained by interchanging the rôles of u and v.)

2.5. THEOREM. *If $f(u, v)$ is summable on S_0 and*

$$g(u, v) = \int_0^u f(\xi, v)d\xi, \qquad (u, v) \in S_0,$$

(for a.e. v, $g(u, v)$ exists for every u) then $g_u(u, v)=f(u, v)$ a.e. on S_0.

Proof. See Helsel and Young [1].

2.6. We shall have occasion to employ the following types of integral means.

DEFINITION. *If $f(u, v)$ is summable on S_0 and $0<h<1/2$ is fixed, then*

$$\overset{h}{f_h}(u, v) = \frac{1}{4h^2}\int_{-h}^{h}\int_{-h}^{h} f(u+\alpha, v+\beta)d\alpha d\beta = \frac{1}{4h^2}\int_{u-h}^{u+h}\int_{v-h}^{v+h} f(\xi, \eta)d\xi d\eta,$$

defined on the square S_h: $h\leqq u\leqq 1-h$, $h\leqq v\leqq 1-h$, is called the h-h-integral mean of $f(u, v)$.

DEFINITION. *If $f(u, v)$ is summable on S_0 and $0<h<1/2$ is fixed, then*

$$f_h(u, v) = \frac{1}{2h}\int_{-h}^{h} f(u+\alpha, v)d\alpha = \frac{1}{2h}\int_{u-h}^{u+h} f(\xi, v)d\xi,$$

defined for a.e. v on the rectangle R_{h0}: $h\leqq u\leqq 1-h$, $0\leqq v\leqq 1$, is called the h-integral mean of $f(u, v)$.

2.7. THEOREM. *If $f(u, v)$ is continuous on S_0 and R (a closed oriented rectangle) comprised in S_0^0 is fixed, then, as $h\to 0$, $\overset{h}{f_h}(u, v)\to$ uniformly to $f(u, v)$ on R.*

Proof. Well known properties of integral means such as this are established in Bray [1], Morrey [1], and Radó [2]. When theorems on integral means are stated without proofs or references, the reader should consult the above papers.

2.8. THEOREM. *If $f(u, v)$ is continuous on S_0, then on S_h*

$$\frac{\partial \overset{h}{f_h}(u, v)}{\partial u} = \frac{1}{4h^2}\int_{v-h}^{v+h}\{f(u+h, \eta) - f(u-h, \eta)\}d\eta.$$

2.9. THEOREM. *If $f(u, v)$ is summable on S_0, then $f_h^h(u, v) \rightarrow_{h\to 0} f(u, v)$ a.e. on S_0.*

Proof. The result is immediate from Lebesgue's theorem on absolutely continuous set functions. See de la Vallée Poussin [1].

2.10. THEOREM. *If $f(u, v)$ is summable on S_0 and $R \subset S_0^0$ is fixed, then $\{f_h^h(u, v)\}$ has property $(\mathcal{V})$ on R.*

Proof. Let e be any measurable set comprised in R. Then for h small enough that $R \subset S_h$, we have

$$\left| \iint_e f_h^h(u, v)dudv \right| \leqq \frac{1}{4h^2}\int_{-h}^{h}\int_{-h}^{h}\left\{ \iint_e |f(u+\alpha, v+\beta)|\, dudv \right\} d\alpha d\beta$$

$$\leqq \frac{1}{4h^2}\int_{-h}^{h}\int_{-h}^{h}\left\{ \iint_{e_\alpha^\beta} |f(u, v)|\, dudv \right\} d\alpha d\beta$$

$$\leqq \frac{1}{4h^2}\int_{-h}^{h}\int_{-h}^{h} \Omega(|f|, S_0, |e|)d\alpha d\beta = \Omega(|f|, S_0, |e|),$$

where

$$e_\alpha^\beta = \underset{(u,v)}{E}\,[(u-\alpha, v-\beta) \in e],$$

$$\Omega(|f|, S_0, |e|) = \max_{E \text{ meas} \subset S_0, |E| \leqq |e|} \iint_E |f(u, v)|\, dudv.$$

Because of the absolute continuity of $\iint_E |f(u, v)|\,dudv$ as a set function on S_0, $\Omega(|f|, S_0, |e|) \rightarrow_{|e|\to 0} 0$; hence, $\{f_h^h(u, v)\}$ has property $(\mathcal{V})$ on R.

2.11. THEOREM. *If $f(u, v)$ and $g(u, v)$ belong to associated Lebesgue classes L^p and L^q respectively, where $p>0$, $q>0$, $1/p+1/q=1$, and if $R \subset S_0^0$ is fixed, then $\{f_h^h(u, v) \cdot g_h^h(u, v)\}$ has property $(\mathcal{V})$ on R.*

Proof. The result follows from Hölder's inequality and reasoning similar to that employed in 2.10.

2.12. THEOREM. *If $f(u, v)$ is summable on S_0 and the closed interval (a, b) is strictly interior to $(0, 1)$, then*

$$\int_a^b |f_h^h(u, v)|\, du \xrightarrow[h\to 0]{} \int_a^b |f(u, v)|\, du,$$

provided v does not belong to a set of measure zero which is independent of a and b.

Proof. By 2.9, $f_h^h(u, v) \rightarrow_{h\to 0} f(u, v)$ a.e. on S_0; hence from the Theorem of Fubini, $f_h^h(u, v) \rightarrow_{h\to 0} f(u, v)$ a.e. on a.e. line $v=$constant in S_0. Let E_v^1 be the exceptional set of v values. Also, by the Theorem of Fubini, $\int_0^1 |f(u, v)|\,du$

exists for v not belonging to an exceptional set E_v^2 of measure zero.

Choose two rational numbers α and β such that $0<\alpha<\beta<1$. Then define

$$g(v) = \int_{\alpha-\delta}^{\beta+\delta} |f(u, v)| \, du, \qquad v \text{ not} \in E_v^2,$$

where $\delta>0$ is rational and $0\leqq\alpha-\delta<\beta+\delta\leqq 1$. Since $g(v)$ is summable on $0\leqq v\leqq 1$, it follows from Lebesgue's theorem on absolutely continuous set functions that

$$\frac{1}{2h}\int_{-h}^{h} g(v+\eta)d\eta \underset{h\to 0}{\longrightarrow} g(v),$$

for v not belonging to an exceptional set $E_v^3(\alpha, \beta, \delta)$ of measure zero. Put

$$E_v = E_v^1 + E_v^2 + \sum_{\alpha,\beta,\delta \text{ rat.}} E_v^3(\alpha, \beta, \delta).$$

Then $|E_v| = 0$.

For v not$\in E_v$ and for a fixed rational $\delta>h$,

$$\begin{aligned}\int_\alpha^\beta |\overset{h}{f}_h(u, v)| \, du &\leqq \frac{1}{4h^2}\int_{-h}^{h}\int_{-h}^{h}\left\{\int_\alpha^\beta |f(u+\xi, v+\eta)| \, du\right\} d\xi d\eta \\ &\leqq \frac{1}{4h^2}\int_{-h}^{h}\int_{-h}^{h}\left\{\int_{\alpha-\delta}^{\beta+\delta} |f(u, v+\eta)| \, du\right\} d\xi d\eta \\ &\leqq \frac{1}{2h}\int_{-h}^{h} g(v+\eta)d\eta.\end{aligned}$$

Therefore

$$\limsup_{h\to 0}\int_\alpha^\beta |\overset{h}{f}_h(u, v)| \, du \leqq g(v) = \int_{\alpha-\delta}^{\beta+\delta} |f(u, v)| \, du, \quad v \text{ not} \in E_v.$$

As $\delta\to 0$ through positive rational values, we obtain

$$\limsup_{h\to 0}\int_\alpha^\beta |\overset{h}{f}_h(u, v)| \, du \leqq \int_\alpha^\beta |f(u, v)| \, du, \qquad v \text{ not} \in E_v.$$

This inequality holds for any two real numbers $0<a<b<1$, since for v not $\in E_v$

$$\begin{aligned}\limsup_{h\to 0}\int_a^b |\overset{h}{f}_h(u, v)| \, du &\leqq \limsup_{h\to 0}\int_{\alpha_n}^{\beta_n} |\overset{h}{f}_h(u, v)| \, du \\ &\leqq \int_{\alpha_n}^{\beta_n} |f(u, v)| \, du \underset{n}{\rightarrow} \int_a^b |f(u, v)| \, du,\end{aligned} \tag{1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of rational numbers converging to a and b respectively and satisfying $0<\alpha_n<a<b<\beta_n<1$.

By the Lemma of Fatou, for any two real numbers $0<a<b<1$

$$\begin{aligned}\int_a^b |f(u,v)|\,du &= \int_a^b \liminf_{h\to 0} |\overset{h}{f_h}(u,v)|\,du \\ &\leqq \liminf_{h\to 0}\int_a^b |\overset{h}{f_h}(u,v)|\,du, \qquad v \text{ not } \in \overset{1}{E_v}+\overset{2}{E_v}.\end{aligned} \tag{2}$$

The result follows from (1) and (2).

2.13. Lemma. *On $a\leqq u\leqq b$, let there be given a family of functions $f_\alpha(u)$, $0<\alpha<1$, which converge a.e. to $f(u)$ as $\alpha\to 0$. If, for $0<\alpha<1$ and $a\leqq u\leqq b$, $|f_\alpha(u)|\leqq g(u)$, where $g(u)$ is summable on $a\leqq u\leqq b$, then $\int_a^b f_\alpha(u)du\to_{\alpha\to 0}\int_a^b f(u)du$.*

Proof. Deny the truth of the assertion. Then there exists a denumerable subsequence of functions $\{f_{\alpha_n}(u)\}$ such that $f_{\alpha_n}(u)\to_n f(u)$ a.e. on $a\leqq u\leqq b$ and $\int_a^b f_{\alpha_n}(u)du$ does not converge to $\int_a^b f(u)du$. This contradicts Lebesgue's theorem on termwise integration.

2.14. Theorem. *If $f(u, v)\geqq 0$ is summable on S_0, $\lambda(u)\geqq 0$ is summable on $0\leqq u\leqq 1$, and $V(u)=\int_0^1 f(u, v)dv$ is a bounded function of u (on the set where it exists), then for a fixed rectangle R: $0<a\leqq u\leqq b<1$, $0<c\leqq v\leqq d<1$,*

$$\iint_R \overset{h}{f_h}(u,v)\lambda(u)\,du\,dv \xrightarrow[h\to 0]{} \iint_R f(u,v)\lambda(u)\,du\,dv.$$

Proof. Choose h small enough that $R\subset S_h$. By 2.12, for a.e. u, $a\leqq u\leqq b$,

$$\lambda(u)\int_c^d \overset{h}{f_h}(u,v)dv \xrightarrow[h\to 0]{} \lambda(u)\int_c^d f(u,v)dv.$$

Also

$$\begin{aligned}\lambda(u)\int_c^d \overset{h}{f_h}(u,v)dv &= \frac{\lambda(u)}{4h^2}\int_{-h}^h\int_{-h}^h\left\{\int_c^d f(u+\alpha, v+\beta)dv\right\}d\alpha d\beta \\ &\leqq \frac{\lambda(u)}{4h^2}\int_{-h}^h\int_{-h}^h V(u+\alpha)d\alpha d\beta \leqq M\cdot\lambda(u),\end{aligned}$$

where M is a bound for $V(u)$. By the lemma of 2.13

$$\int_a^b\left\{\lambda(u)\int_c^d \overset{h}{f_h}(u,v)dv\right\}du \xrightarrow[h\to 0]{} \int_a^b\left\{\lambda(u)\int_c^d f(u,v)dv\right\}du.$$

Hence, by the Theorem of Tonelli,

$$\iint_R \overset{h}{f_h}(u,v)\lambda(u)\,du\,dv \xrightarrow[h\to 0]{} \iint_R f(u,v)\lambda(u)\,du\,dv.$$

2.15. THEOREM. *If $f(u, v)$ is continuous on S_0, then on R_{h0} (cf. 26)*

$$\frac{\partial f_h(u, v)}{\partial u} = \frac{1}{2h}\left\{f(u + h, v) - f(u - h, v)\right\}.$$

2.16. THEOREM. *If $f(u, v)$ is summable on S_0, then $f_h(u, v) \to_{h\to 0} f(u, v)$ a.e. on S_0.*

Proof. The result is immediate from 2.5.

2.17. For the purpose of Chapter III, we shall need certain theorems pertaining to functions which are *absolutely continuous in the sense of Tonelli*—briefly, A.C.T.—and to their integral means.

DEFINITION. *The function $f(u, v)$ is* A.C.T. *on S_0 if* (i) *$f(u, v)$ is continuous*; (ii) *$f(u, v)$ is of bounded variation in the sense of Tonelli*; (iii) *for a.e. $v=\eta$, $0 \leqq \eta \leqq 1$, the function $f(u, \eta)$ is absolutely continuous in u, and, for a.e. $u=\xi$, $0 \leqq \xi \leqq 1$, the function $f(\xi, v)$ is absolutely continuous in v. Denote by $V(\eta; f)$ the total variation of $f(u, \eta)$ as a function of u on $0 \leqq u \leqq 1$; denote by $V(\xi; f)$ the total variation of $f(\xi, v)$ as a function of v on $0 \leqq v \leqq 1$. Then $f(u, v)$ is of bounded variation in the sense of Tonelli if $V(\eta; f)$ and $V(\xi; f)$ are summable functions of η and ξ respectively on the interval* (0, 1).

2.18. If $f(u, v)$ is A.C.T. on S_0, the partial derivatives $f_u(u, v)$ and $f_v(u, v)$ exist a.e. and are summable on S_0; moreover, for a.e. v, $V(v; f) = \int_0^1 |f_u(u, v)|\, \partial u$ and, for a.e. u, $V(u; f) = \int_0^1 |f_v(u, v)|\, dv$ (see Morrey [1]).

2.19. THEOREM. *If $f(u, v)$ is A.C.T. on S_0, then $\partial f_h^h(u, v)/\partial u = (\partial f(u, v)/\partial u)_h^h$ on S_h.*

2.19.1. COROLLARY. *If $f(u, v)$ is A.C.T. on S_0, then $\partial f_h^h(u, v)/\partial u \to_{h\to 0} \partial f(u, v)/\partial u$ a.e. on S_0.*

Proof. The result is immediate from 2.19 and 2.9.

2.20. THEOREM. *If $f(u, v)$ is A.C.T. on S_0, then $\partial f_h(u, v)/\partial u = (\partial f(u, v)/\partial u)_h$ on R_{h0} providing v does not belong to a set of measure zero which is independent of h.*

Proof. By 2.15, $\partial f_h(u, v)/\partial u = (1/2h)\{f(u+h, v) - f(u-h, v)\}$ on R_{h0} Since $f(u, v)$ is A.C.T., it is absolutely continuous in u for a.e. v, $0 \leqq v \leqq 1$. Fix a value of v for which $f(u, v)$ is absolutely continuous in u. Then

$$\begin{aligned}\left(\frac{\partial f(u, v)}{\partial u}\right)_h &= \frac{1}{2h}\int_{u-h}^{u+h} \frac{\partial f(\xi, v)}{\partial \xi}\, d\xi \\ &= \frac{1}{2h}\left\{f(u + h, v) - f(u - h, v)\right\} = \frac{\partial f_h(u, v)}{\partial u}.\end{aligned}$$

2.20.1. COROLLARY. *If $f(u,v)$ is A.C.T. on S_0, then $\partial f_h(u,v)/\partial u \to_{h\to 0} \partial f(u,v)/\partial u$ a.e. on S_0.*

Proof. The result is immediate from 2.20 and 2.16.

2.21. THEOREM. *If $f(u, v)$ is A.C.T. on S_0, then $\partial f_h(u, v)/\partial v = (\partial f(u, v)/\partial v)_h$ on R_{h0} providing v does not belong to a set of measure zero which is independent of h.*

Proof. For $0 \leqq v \leqq 1$,

$$\int_0^v \left(\frac{\partial f(u,\eta)}{\partial \eta}\right)_h d\eta = \frac{1}{2h}\int_{-h}^{h}\left\{\int_0^v \frac{\partial f(u+\alpha,\eta)}{\partial \eta}\, d\eta\right\} d\alpha$$
$$= \frac{1}{2h}\int_{-h}^{h}\{f(u+\alpha, v) - f(u+\alpha, 0)\}\, d\alpha$$
$$= f_h(u, v) - f_h(u, 0).$$

Therefore, on R_{h0},

$$f_h(u, v) = f_h(u, 0) + \frac{1}{2h}\int_0^v \int_{u-h}^{u+h} \frac{\partial f(\xi,\eta)}{\partial \eta}\, d\xi d\eta.$$

By 2.4,

$$\frac{\partial f_h(u, v)}{\partial v} = \frac{1}{2h}\int_{u-h}^{u+h}\frac{\partial f(\xi, v)}{\partial v}\, d\xi = \left(\frac{\partial f(u, v)}{\partial v}\right)_h$$

on R_{h0} providing v does not belong to a set of measure zero which is independent of h.

2.21.1. COROLLARY. *If $f(u,v)$ is A.C.T. on S_0, then $\partial f_h(u,v)/\partial v \to_{h\to 0} \partial f(u,v)/\partial v$ a.e. on S_0.*

2.22. THEOREM. *If $f(u, v)$ is A.C.T. on S_0, then $f_h(u, v)$ is A.C.T. on R_{h0}.*

Proof. Because of the absolute continuity of the indefinite Lebesgue integral, $f_h(u, v)$ is absolutely continuous in u for every v, $0 \leqq v \leqq 1$; hence,

$$V(v; f_h) = \int_h^{1-h}\left|\frac{\partial f_h(u, v)}{\partial u}\right| du$$
$$= \frac{1}{2h}\int_h^{1-h} |f(u+h, v) - f(u-h, v)|\, du \leqq \frac{M}{h},$$

where M is a bound for $|f(u, v)|$ on S_0. Thus $V(v; f_h)$ is summable on $0 \leqq v \leqq 1$.

From the proof of 2.21, on R_{h0},

$$f_h(u, v) = f_h(u, 0) + \frac{1}{2h}\int_0^v \left\{\int_{u-h}^{u+h}\frac{\partial f(\xi,\eta)}{\partial \eta}\, d\xi\right\} d\eta.$$

Therefore, $f_h(u, v)$ is absolutely continuous in v for every u, $h \leqq u \leqq 1-h$, and

$$V(u; f_h) = \int_0^1 \left| \frac{\partial f_h(u, v)}{\partial v} \right| dv = \int_0^1 \left| \left(\frac{\partial f(u, v)}{\partial v} \right)_h \right| dv$$
$$\leqq \frac{1}{2h} \int_0^1 \left\{ \int_{u-h}^{u+h} | f_v(\xi, v) | \, d\xi \right\} dv$$
$$\leqq \frac{1}{2h} \iint_{S_0} | f_v(u, v) | \, dudv.$$

Thus $V(u; f_h)$ is summable on $h \leqq u \leqq 1-h$ and the proof is complete.

CHAPTER III. TRANSFORMATION FORMULAS FOR DOUBLE INTEGRALS

3.1. In this final chapter we shall take each transformation for which W. H. Young established either formula (3) or (4) of 1.4 and, by placing it in class K_3, show that formula (1) of 1.1 holds if the integral on the left exists (cf. 1.2). Throughout the chapter we make the assumption that there is given a transformation

$$T: \quad x = x(u, v), \qquad y = y(u, v), \qquad (u, v) \in S_0,$$

which satisfies hypothesis H_0 of 1.9. Additional restrictions on the transformation T will be stated in the sequel. In the last three theorems of this chapter (cf. 3.9, 3.12, 3.14), the conditions imposed on the coordinate functions $x(u, v)$ and $y(u, v)$ are not symmetric. A second theorem may be obtained in each of these cases by interchanging the rôles of $x(u, v)$ and $y(u, v)$.

3.2. We shall now show that certain transformations obtained from T by approximating to the functions $x(u, v)$, $y(u, v)$ with h-h-integral means belong to class K_3. First we introduce the following notations:

$$\overset{n}{{}_nf}(u, v) = f_{1/n}^{1/n}(u, v), \qquad {}_nf(u, v) = f_{1/n}(u, v), \qquad {}_nS = S_{1/n}, \qquad n = 3, 4, \cdots,$$

where $f(u, v)$ is any summable function defined on S_0.

3.3. LEMMA. *If T satisfies H_0, the transformation*

$$\overset{n}{{}_n}T_m^m: \quad x = \overset{n}{{}_nx}(u, v), \qquad y = \overset{m}{{}_my}(u, v), \qquad n, m > 2, \ (u, v) \in {}_nS \cdot {}_mS,$$

belongs to class K_3 on ${}_nS^0 \cdot {}_mS^0$.

Proof. By 2.8, $\partial \overset{n}{{}_nx}(u, v)/\partial u$, $\partial \overset{n}{{}_nx}(u, v)/\partial v$, $\partial \overset{m}{{}_my}(u, v)/\partial u$, $\partial \overset{m}{{}_my}(u, v)/\partial v$ are continuous on ${}_nS \cdot {}_mS$; hence, ${}^nJ^m = J(u, v; \overset{n}{{}_n}T_m^m)$ is summable there. By 1.6, $\overset{n}{{}_n}T_m^m \in K_3({}_nS^0 \cdot {}_mS^0)$.

3.3.1. COROLLARY. *If T satisfies H_0, the transformation*

$$\overset{n}{{}_n}T_n^n: \quad x = \overset{n}{{}_nx}(u, v), \qquad y = \overset{n}{{}_ny}(u, v), \qquad n > 2, \ (u, v) \in {}_nS,$$

belongs to class K_3 *on* ${}_nS^0$.

3.3.2. COROLLARY. *If* T *satisfies* H_0, *the transformation* ${}^n_nT^m_m\,(1/n+1/m<1/2)$ *belongs to class* K_3 *on* $S^0_{1/n+1/m}$.

Proof. From the definition of class $K_3(\mathcal{D})$, if $T\in K_3(\mathcal{D})$ and $\mathcal{D}^*$ is any domain within $\mathcal{D}$, then $T\in K_3(\mathcal{D}^*)$. See Radó and Reichelderfer [1].

3.4. LEMMA. *If* T *satisfies* H_0, *the transformation*

$$ {}^n_nT:\quad x = {}^n_nx(u, v), \qquad y = y(u, v), \qquad\qquad n > 2,\ (u, v) \in {}_nS, $$

belongs to class K_3 *on* ${}_nS^0$.

Proof. Fix a value of $n>2$. For each positive integer m, satisfying $1/n+1/m<1/2$, define the transformation ${}^n_nT^m_m$ on $S_{1/n+1/m}$. We shall verify that the transformations ${}^n_nT$ and ${}^n_nT^m_m$ satisfy the conditions of the Closure Theorem (cf. 1.7).

(i) Clearly $S^0_{1/n+1/m}\subset {}_nS^0$ and, for every closed set $F\subset {}_nS^0$, there exists an $m_0=m_0(F)$ such that $F\subset S^0_{1/n+1/m}$ for all $m>m_0$.

(ii) By 2.8, $\partial^n_n x(u, v)/\partial u$ and $\partial^n_n x(u, v)/\partial v$ are continuous on ${}_nS$ and, therefore, bounded. The summability of ${}^nJ=J(u, v; {}^n_nT)$ on ${}_nS^0$ follows then from the summability of $y_u(u, v)$ and $y_v(u, v)$ on S_0 (cf. 2.18).

(iii) By 3.3.2, ${}^n_nT^m_m\in K_3(S^0_{1/n+1/m})$ for $1/n+1/m<1/2$.

(iv) For every closed set $F\subset {}_nS^0$ there exists a closed oriented rectangle R such that $F\subset R\subset {}_nS^0$. By 2.7, ${}^m_my(u, v)\rightarrow_m$uniformly to $y(u, v)$ on R. To show that ${}^nJ^m$ converges strongly to ${}^nJ$ on F, we shall verify that ${}^nJ^m\rightarrow_m {}^nJ$ a.e. on R and that $\{{}^nJ^m\}$ has property $(\mathcal{V})$ on R for fixed n; hence, by 2.3.1, $\iint_R|{}^nJ^m-{}^nJ|\,dudv\rightarrow_m 0$.

It follows directly from 2.19.1 that ${}^nJ^m\rightarrow_m {}^nJ$ a.e. on R. To show property $(\mathcal{V})$ for $\{{}^nJ^m\}$ on R, it suffices to consider the product

$$ \frac{\partial^n_n x(u, v)}{\partial u}\,\frac{\partial^m_m y(u, v)}{\partial v} = \frac{\partial^n_n x(u, v)}{\partial u}\;{}^m_m\!\left(\frac{\partial y(u, v)}{\partial v}\right) $$

since the other product of ${}^nJ^m$ may be handled in the same way. By 2.8, the first factor is bounded for fixed n and, by 2.10, $\{{}^m_m(\partial y(u, v)/\partial v)\}$ has property $(\mathcal{V})$ on R; hence, from 2.2.2, $\{\partial^n_n x(u, v)/\partial u\cdot\partial^m_m y(u, v)/\partial v\}$ has property $(\mathcal{V})$ on R for fixed n.

The conditions of the Closure Theorem being satisfied, we conclude that ${}^n_nT\in K_3({}_nS^0)$.

Exchanging the rôles of $x(u, v)$ and $y(u, v)$, we have the lemma:

3.4.1. LEMMA. *If* T *satisfies* H_0, *the transformation*

$$ T^n_n:\quad x = x(u, v), \qquad y = {}^n_ny(u, v), \qquad\qquad n > 2,\ (u, v) \in {}_nS, $$

belongs to class K_3 *on* ${}_nS^0$.

3.5. Our investigation of the transformations considered by Young will proceed according to the following scheme: First we approximate to the given transformation T by a suitable one of the transformations ${}^n_nT^n_n$, ${}^n_nT$, T^n_n. Then, as the reader will easily verify, all of the conditions of the Closure Theorem are satisfied except strong convergence of the Jacobians. Since the Jacobians of the approximating transformations converge a.e. to $J(u, v; T)$ (cf. 2.19.1), a necessary and sufficient condition for strong convergence is that the Jacobians of the approximating transformations have property $(\mathcal{V})$ (cf. 2.2, 2.3.1). We shall now investigate property $(\mathcal{V})$ for the Jacobians of the approximating transformations.

3.6. THEOREM 1 (see Young [2, pp. 80–85], cf. 1.9). *If T satisfies H_0 and one factor in each product $x_u(u, v)\cdot y_v(u, v)$, $x_v(u, v)\cdot y_u(u, v)$ is bounded on S_0, then $T\in K_3(S_0^0)$ and formula* (1) *holds whenever the integral on the left exists.*

Proof. Consider any fixed $R\subset S_0^0$. Choose n large enough that $R\subset {}_nS$. Approximate to T by ${}^n_nT^n_n$. To show property $(\mathcal{V})$ for $\{{}^nJ^n\}$ on R, it is sufficient to consider the product

$$\frac{\partial^n_n x(u, v)}{\partial u}\,\frac{\partial^n_n y(u, v)}{\partial v} = {}^n_n(x_u(u, v))\cdot {}^n_n(y_v(u, v))$$

since the other product in ${}^nJ^n$ may be handled in the same way. By assumption, one factor of $x_u(u, v)\cdot y_v(u, v)$, say $y_v(u, v)$, is bounded on S_0; hence, ${}^n_n(y_v(u, v))$ is uniformly bounded on R. By 2.10, $\{{}^n_n(x_u(u, v))\}$ has property $(\mathcal{V})$ on R. Therefore, from 2.2.2, $\{\partial^n_n x(u, v)/\partial u\cdot\partial^n_n y(u, v)/\partial v\}$ has property $(\mathcal{V})$ on R. By the Closure Theorem, $T\in K_3(S_0^0)$ and, by the result of Radó and Reichelderfer mentioned in 1.2, the theorem follows.

3.7. THEOREM 2 (see Young [2, pp. 80–85], cf. 1.9). *If T satisfies H_0 and in each product $x_u(u, v)\cdot y_v(u, v)$, $x_v(u, v)\cdot y_u(u, v)$, the factors belong to associated Lebesgue classes on S_0, then $T\in K_3(S_0^0)$ and formula* (1) *holds providing the integral on the left exists.*

Proof. Consider any fixed $R\subset S_0^0$. Choose n large enough that $R\subset {}_nS$. Approximate to T by ${}^n_nT^n_n$. On R

$$\left|{}^n_nJ^n_n\right| \leqq \left|{}^n_n(x_u(u, v))\cdot {}^n_n(y_v(u, v))\right| + \left|{}^n_n(x_v(u, v))\cdot {}^n_n(y_u(u, v))\right|.$$

It now follows from 2.11 and 2.2.1 that $\{{}^nJ^n\}$ has property $(\mathcal{V})$ on R. By the Closure Theorem, $T\in K_3(S_0^0)$; hence, the theorem follows.

3.8. THEOREM 3 (see Young [3, p. 163], cf. 1.9). *If T satisfies H_0; $|x_u(u, v)|$, $|y_u(u, v)| \leqq \lambda(u)$ and $|x_v(u, v)|$, $|y_v(u, v)| \leqq \mu(v)$, where $\lambda(u)$ and $\mu(v)$ are summable on* (0, 1), *then $T\in K_3(S_0^0)$ and formula* (1) *holds as soon as the integral on the left exists.*

Proof. Consider any fixed $R \subset S_0^0$. Choose n large enough that $R \subset {}_nS$. Approximate to T by ${}_n^nT$. To show property $(\mathcal{V})$ for $\{{}^nJ\}$ on R, it suffices to consider the product $\partial_n^n x(u, v)/\partial u \cdot \partial y(u, v)/\partial v$ since the other product of ${}^nJ$ may be handled in the same way. Let e be any measurable set comprised in R. Then

$$\iint_e \left|\frac{\partial_n^n x(u, v)}{\partial u}\right| \cdot \left|\frac{\partial y(u, v)}{\partial v}\right| dudv$$
$$\leqq \frac{n^2}{4}\int_{-1/n}^{1/n}\int_{-1/n}^{1/n}\left\{\iint_e |x_u(u+\alpha, v+\beta)| \cdot |y_v(u, v)|\, dudv\right\} d\alpha d\beta$$
$$\leqq \frac{n^2}{4}\int_{-1/n}^{1/n}\int_{-1/n}^{1/n}\left\{\iint_e \lambda(u+\alpha)\mu(v)dudv\right\} d\alpha d\beta$$
$$\leqq \frac{n^2}{4}\int_{-1/n}^{1/n}\int_{-1/n}^{1/n}\left\{\iint_{e_\alpha} \lambda(u)\mu(v)dudv\right\} d\alpha d\beta$$
$$\leqq \frac{n^2}{4}\int_{-1/n}^{1/n}\int_{-1/n}^{1/n}\Omega(\lambda(u)\mu(v), S_0, |e|)d\alpha d\beta$$
$$\leqq \Omega(\lambda(u)\mu(v), S_0, |e|).$$

Ω has been defined in 2.10 and

$$e_\alpha = \underset{(u,v)}{E}[(u-\alpha, v) \in e]$$

By the Theorem of Tonelli, $\lambda(u)\mu(v)$ is summable on S_0; hence, $\Omega(\lambda(u)\mu(v), S_0, |e|) \to_{|e| \to 0} 0$. Thus, by the last inequality and 2.2.1, $\{\partial_n^n x(u, v)/\partial u \cdot \partial y(u, v)/\partial v\}$ has property $(\mathcal{V})$ on R. Finally, by the Closure Theorem, $T \in K_3(S_0^0)$ and the theorem follows.

3.9. Theorem 4 (cf. 3.1). *If T satisfies H_0; $|x_u(u, v)| \leqq \lambda(u)$, $|x_v(u, v)| \leqq \mu(v)$, where $\lambda(u)$ and $\mu(v)$ are summable on* (0, 1); *and the total variations $V(v; y)$ and $V(u; y)$ are bounded on* (0, 1), *then $T \in K_3(S_0^0)$ and formula* (1) *holds whenever the integral on the left exists.*

Proof. Consider any fixed $R \subset S_0^0$. Choose n large enough that $R \subset {}_nS$. Approximate to T by T_n. On R, $|J^n| \leqq \lambda(u) \cdot {}_n^n|y_v(u, v)| + \mu(v) \cdot {}_n^n|y_u(u, v)|$. This inequality, along with 2.14, 2.3, and 2.2.1, implies property $(\mathcal{V})$ for $\{J^n\}$ on R. By the Closure Theorem, $T \in K_3(S_0^0)$, so the theorem follows.

3.10. Young really does not discuss a situation corresponding to Theorem 4. Instead, he claims (see Young [2, p. 88]) to have established formula (3) of 1.4 under the conditions: $x(u, v)$ and $y(u, v)$ are absolutely continuous on every horizontal and on every vertical in S_0; $y_u(u, v)$, $y_v(u, v)$ are summable on S_0; $|x_u(u, v)| \leqq \lambda(u)$ and $|x_v(u, v)| \leqq \mu(v)$, where $\lambda(u)$, $\mu(v)$ are summable on (0, 1) and $\lambda(u)y_v(u, v)$, $\mu(v)y_u(u, v)$ are summable on S_0. In attempting to prove this assertion, Young erroneously assumes that $y(u, v)$ can be written as follows: $y(u, v) = y_1(u, v) - y_2(u, v)$, where $y_1(u, v)$ and $y_2(u, v)$ possess non-negative first partial derivatives in addition to all of the properties of $y(u, v)$ (see Young [2, p. 91]). Even if $y(u, v)$ could be split in this manner, $V(v; y)$ and $V(u; y)$ would be bounded (as we assumed in Theorem 4) since

$$V(v; y) \leqq V(v; y_1) + V(v; y_2) = \int_0^1 \frac{\partial y_1(u, v)}{\partial u} du + \int_0^1 \frac{\partial y_2(u, v)}{\partial u} du$$
$$\leqq y_1(1, v) - y_1(0, v) + y_2(1, v) - y_2(0, v)$$

and, similarly, $V(u; y) \leqq y_1(u, 1) - y_1(u, 0) + y_2(u, 1) - y_2(u, 0)$.

3.11. For the purpose of the next theorem, we require the following lemma:

LEMMA. *If $f(u, v) \geqq 0$ and $g(u, v) \geqq 0$ are summable on S_0 and if $0 < \delta < 1/2$ is fixed, then*

$$\phi(\alpha, \beta) = \iint_{S_\delta} f(u + \alpha, v + \beta) g(u, v) du dv$$

exists a.e. and is summable in r: $-\delta \leqq \alpha \leqq \delta$, $-\delta \leqq \beta \leqq \delta$.

Proof. See Young [2, p. 91].

3.12. THEOREM 5 (see Young [2, pp. 92–93]; cf. 1.9, 3.1). *For T satisfying H_0 and $0 < \delta < 1/2$ fixed, define*

$$T(\alpha, \beta; 0, 0): \quad x = x(u + \alpha, v + \beta), \qquad y = y(u, v), \qquad (\alpha, \beta) \in r, \ (u, v) \in S_\delta.$$

Then for a.e. (α, β) in r, $T(\alpha, \beta; 0, 0) \in K_3(S_\delta^0)$ and the transformation formula

$$\iint_{S_\delta} F(x(u + \alpha, v + \beta), y(u, v)) J(u, v; T(\alpha, \beta; 0, 0)) du dv$$
$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} F(x, y) \cdot \mu(x, y; S_\delta, T(\alpha, \beta; 0, 0)) dx dy$$

holds if the integral on the left exists.

Proof. Consider any fixed $R \subset S_\delta^0$. Choose n_0 large enough that $R \subset S_{\delta+1/n}$ for $n > n_0$. Then for $(\alpha, \beta) \in r$ and $n > n_0$, define

$$ {}_n^n T(\alpha, \beta; 0, 0): \quad x = {}_n^n x(u + \alpha, v + \beta), \qquad y = y(u, v), \qquad (u, v) \in S_{\delta+1/n}.$$

We wish to show that $\{J(u, v; {}_n^n T(\alpha, \beta; 0, 0)\}$ has property $(\mathcal{V})$ on R for a.e. (α, β) in r. To this end, we need only consider the product $\partial_n^n x(u+\alpha, v+\beta)/\partial u \cdot \partial y(u, v)/\partial v$ since the other product of the Jacobian may be handled in the same way.

For $(\alpha, \beta) \in r$, define

$$G(u, v, \alpha, \beta) = | x_u(u+\alpha, v+\beta) | \cdot | y_v(u, v) |, \qquad (u, v) \in S_\delta,$$

and

$$G_n(u, v, \alpha, \beta) = {}_n^n| x_u(u+\alpha, v+\beta) | \cdot | y_v(u, v) |, \qquad n > n_0, (u, v) \in S_{\delta+1/n}.$$

By 3.11,

$$H(\alpha, \beta) = \iint_R G(u, v, \alpha, \beta) du dv$$

exists a.e. and is summable on r. Let E be the exceptional set of points (α, β) in r. From 2.9,

$$(1) \qquad {}_n^nH(\alpha, \beta) = \iint_R G_n(u, v, \alpha, \beta) du dv \xrightarrow[n]{} \iint_R G(u, v, \alpha, \beta) du dv = F(\alpha, \beta)$$

for a.e. $(\alpha, \beta) \in r$. Let E_2 be the exceptional set of points (α, β) in r. Consider a fixed point (α_0, β_0) in r and not belonging to E_1+E_2. By 2.9, $G_n(u, v, \alpha_0, \beta_0) \rightarrow_n G(u, v, \alpha_0, \beta_0)$ a.e. on R; hence, it follows from (1) and 2.3 that $\{G_n(u, v, \alpha_0, \beta_0)\}$ has property $(\mathcal{V})$ on R. For $(u, v) \in R$ and $n > n_0$,

$$G_n(u, v, \alpha_0, \beta_0) \geqq \left| \frac{\partial_n^n x(u+\alpha_0, v+\beta_0)}{\partial u} \right| \cdot \left| \frac{\partial y(u, v)}{\partial v} \right|;$$

therefore, $\{\partial_n^n x(u+\alpha_0, \ v+\beta_0)/\partial u \cdot \partial y(u, \ v)/\partial v\}$ has property $(\mathcal{V})$ on R. Finally, by the Closure Theorem, $T(\alpha, \beta; 0, 0)$ belongs to class K_3 on S_δ^0 for a.e. (α, β) in r.

To complete the theorem, we must show that for a.e. (α, β) in r the image of the boundary of S_δ under $T(\alpha, \beta; 0, 0)$ is a set of planar measure zero. Since $x(u, v)$ is A.C.T. on S_0, for a.e. (α, β) in r the function $x(u+\alpha, v+\beta)$ is absolutely continuous on the boundary of S_δ; hence, for such (α, β), the continuous image of the boundary of S_δ under $T(\alpha, \beta; 0, 0)$ is a semi-rectifiable curve and its locus is a set of planar measure zero.

(For the proof of Theorem 5 we need not assume that the image of the boundary of S_0 under T is a set of planar measure zero.)

3.13. To prove the final theorem, we approximate to the given transformation T by the transformation

$${}_nT_n\colon \quad x = {}_nx(u, v), \qquad y = {}_ny(u, v), \qquad n > 2, (u, v) \in R_{1/n\,0}.$$

Thus, to utilize the Closure Theorem, we must establish the lemma:

LEMMA. *If T satisfies H_0, the transformation ${}_nT_n$ belongs to class K_3 on $R^0_{1/n\,0}$.*

Proof. By 2.22, ${}_nx(u, v)$ and ${}_ny(u, v)$ are A.C.T. on $R_{1/n\,0}$. From 2.15, $\partial_n x(u, v)/\partial u$ and $\partial_n y(u, v)/\partial u$ are continuous and, therefore, bounded on $R_{1/n\,0}$. The result now follows from reasoning similar to that employed in 3.6.

3.14. THEOREM 6 (see Young [1, pp. 365–366; 2, p. 87]; cf. 1.9, 3.1). *If T satisfies H_0 and $|x_v(u, v)|$, $|y_v(y, v)| \leqq \mu(v)$, where $\mu(v)x_u(u, v)$ and $\mu(v)y_u(u, v)$ are summable on S_0, then $T \in K_3(S_0^0)$ and formula* (1) *holds providing the integral on the left exists.*

Proof. Consider any fixed $R \subset S_0^0$. Choose n large enough that $R \subset R_{1/n\ 0}$. Approximate to T by ${}_nT_n$. To show property $(\mathcal{U})$ for $\{J(u, v; {}_nT_n)\}$ on R, it suffices to consider the product $\partial_n x(u, v)/\partial u \cdot \partial_n y(u, v)/\partial v$ since the other product of the Jacobian may be handled in the same way. Let e be any measurable set comprised in R. Then

$$\begin{aligned}
\iint_e \left|\frac{\partial_n x(u, v)}{\partial u}\right| \cdot \left|\frac{\partial_n y(u, v)}{\partial v}\right| dudv \\
&\leqq \frac{n^2}{4}\int_{-1/n}^{1/n}\int_{-1/n}^{1/n}\left\{\iint_e |x_u(u+\alpha, v)| \cdot |y_v(u+\beta, v)|\, dudv\right\} d\alpha d\beta \\
&\leqq \frac{n^2}{4}\int_{-1/n}^{1/n}\int_{-1/n}^{1/n}\left\{\iint_e \mu(v)\, |x_u(u+\alpha, v)|\, dudv\right\} d\alpha d\beta \\
&\leqq \frac{n^2}{4}\int_{-1/n}^{1/n}\int_{-1/n}^{1/n}\left\{\iint_{e_\alpha} \mu(v)\, |x_u(u, v)|\, dudv\right\} d\alpha d\beta \\
&\leqq \frac{n^2}{4}\int_{-1/n}^{1/n}\int_{-1/n}^{1/n} \Omega(\mu(v)\, |x_u(u, v)|, S_0, |e|) d\alpha d\beta \\
&\leqq \Omega(\mu(v)\, |x_u(u, v)|, S_0, |e|),
\end{aligned}$$

where $\Omega(\mu(v)|x_u(u, v)|, S_0, |e|) \rightarrow_{|e| \to 0} 0$. Thus, by the last inequality and 2.2.1, $\{\partial_n x(u, v)/\partial u \cdot \partial_n y(u, v)/\partial v\}$ has property $(\mathcal{U})$ on R.

From 2.20.1 and 2.21.1, $J(u, v; {}_nT_n) \rightarrow_n J(u, v; T)$ a.e. on R; hence, as the reader can now easily verify, all of the conditions of the Closure Theorem are fulfilled. Therefore, $T \in K_3(S_0^0)$ and the theorem follows.

3.14.1. COROLLARY (see Young [3, pp. 189–190], cf. 1.9). *If T satisfies H_0; $|x_v(u, v)|$, $|y_v(u, v)| \leqq \mu(v)$, where $\mu(v)$ is summable on* (0, 1); *and the total variations $V(v; x)$ and $V(v; y)$ are bounded on* (0, 1), *then $T \in K_3(S_0^0)$ and formula* (1) *holds as soon as the integral on the left exists.*

Proof. Because of the Theorem of Tonelli, the relation

$$\int_0^1 \left\{\int_0^1 \mu(v)\, |x_u(u, v)|\, du\right\} dv = \int_0^1 \mu(v) V(v; x) dv$$

implies the summability of $\mu(v)x_u(u, v)$ on S_0. Likewise, $\mu(v)y_u(u, v)$ is summable on S_0.

3.15. The preceding six theorems account for all of Young's results on the transformation of double integrals. Theorem 5 seems to offer a method of approach to the conjectured proposition (cf. 1.10):

If T satisfies H_0 and $J(u, v; T)$ is summable on S_0, then $T \in K_3(S_0^0)$ and formula (1) *holds if the integral on the left exists.*

Along this line, we have considered the question of deducing Theorems 1, 2, 3, 4, and 6 from Theorem 5. The first four theorems follow readily from the fifth, but we have not been able to deduce Theorem 6 from Theorem 5.

Bibliography

H. E. Bray

1. *Proof of a formula for an area*, Bull. Amer. Math. Soc. vol. 29 (1923) pp. 264–270.

G. Fubini

1. *Sulla derivata seconda mista di un integrale doppio*, Rend. Cir. Mat. Palermo vol. 40 (1915) pp. 297– 98

R. G. Helsel and P. M. Young

1. *Characterization theorems for integral means*, to appear soon in Duke Math. J.

E. J. McShane

1. *Integrals over surfaces in parametric form*, Ann. of Math. (2) vol. 34 (1933) pp. 815–838.

C. B. Morrey

1. *A class of representations of manifolds*. I, Amer. J. Math. vol. 55 (1933) pp. 683–707.

H. Rademacher

1. *Über partielle und totale Differenzierbarkeit von Funktionen mehrerer Variabeln*. I, Math. Ann. vol. 79 (1919) pp. 340–359.

T. Radó

1. *On continuous transformations in the plane*, Fund. Math. vol. 27 (1936) pp. 201–211.
2. *A remark on the area of surfaces*, Amer. J. Math. vol. 58 (1936) pp. 598–606.
3. *On absolutely continuous transformations in the plane*, Duke Math. J. vol. 4 (1938) pp. 189–221.

T. Radó and P. Reichelderfer

1. *A theory of absolutely continuous transformations in the plane*, Trans. Amer. Math. Soc. vol. 49 (1941) pp. 258–307.

J. Schauder

1. *Über stetige Abbildungen*, Fund. Math. vol. 12 (1928) pp. 47–74.

C. de la Vallée Poussin

1. *Sur l'intégrale de Lebesgue*, Trans. Amer. Math. Soc. vol. 16 (1915) pp. 435–501.

W. H. Young

1. *On a formula for an area*, Proc. London Math. Soc. vol. 18 (1920) pp. 339–374.
2. *On a new set of conditions for a formula for an area*, Proc. London Math. Soc. vol. 21 (1923) pp. 75–94.
3. *Integration over the area of a curve and transformation of the variables in a multiple integral*, Proc. London Math. Soc. vol. 21 (1923) pp. 161–190.

The Ohio State University,
Columbus, Ohio.