QUASI-ANALYTICITY AND ANALYTIC CONTINUATION—A GENERAL PRINCIPLE

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1. All the results of this paper are based on a general principle (Theorem I) of an asymptotic character. We show that this theorem leads to results in branches of the theory of functions generally regarded as quite distinct from each other. All the results on Watson's problem, on the theory of quasi-analyticity, on one hand, and many known results on the distribution of the singularities of a Dirichlet series and the distribution of the values taken by its analytic continuation, on the other hand, may be considered as particular cases of this theorem. As a matter of fact many new results far more general in both branches are obtained. Having established the main theorem we show, by a certain number of lemmas, cases in which the hypotheses of this theorem are satisfied. We then establish theorems with the same conclusions, their hypotheses being however more specific and easier to handle than those of the main theorem.

Theorem I can be considered as a generalization of the principle of Cauchy's theorem which gives estimates of the coefficients of a Taylor series, or more generally of a Dirichlet series, by the maximum of the absolute value of the corresponding function on a circle around the origin (or on a vertical line for a Dirichlet series). The idea is to estimate the coefficients of expressions $\sum_{m=1}^{n} d_m e^{-\lambda_m s}$ $(n \geqq 1)$ which represent a function only "asymptotically," this estimate being made, on the other hand, only by means of the maximum of the modulus of the function on circles situated in any part of the plane, the radii of these circles diminishing with the increase of the upper density of the sequence $\{\lambda_n\}$ (lim sup n/λ_n). The closeness of approach is given by the evaluation of certain integrals.

2. Throughout this paper $\{\lambda_n\}$ $(1 \leqq n < N \leqq \infty)$ will be a finite (if $N < \infty$) or infinite (if $N = \infty$) sequence of positive increasing numbers. The number $N(x)$ $(x > 0)$ of quantities λ_n smaller than x will be called the *distribution function*, or shorter the *distribution* of the sequence $\{\lambda_n\}$; clearly $N(x) = 0$ for $x \leqq \lambda_1$. The quantities $d = \liminf N(x)/x$ $(x \to \infty)$, $D = \limsup N(x)/x$ $(x \to \infty)$ will be called respectively *lower* and *upper densities* of $\{\lambda_n\}$. If $N(x) = Dx + n(x)$, the function $n(x)$ will be called the *excess distribution function* or briefly the *excess distribution* of the sequence $\{\lambda_n\}$. Clearly $0 \leqq d \leqq D$. We shall moreover suppose $D < \infty$. A sequence $\{\lambda_n\}$ will be characterized by its upper density and its excess distribution. As a matter of fact $n(x)$ characterizes completely $\{\lambda_n\}$ by itself, since in every interval $(\lambda_k, \lambda_{k+1})$ $(1 \leqq k < N - 1$

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if $N>2$) (and in (λ_1, ∞) if $N=2$) we have $N'(x)=D+n'(x)=0$, that is to say $n'(x)=-D$, which equality defines D when $n(x)$ is given.

It is obvious that $\limsup n(x)/x=0$ $(x\to\infty)$. But the behavior of the function $n(x)/x$ is irregular; it takes both positive and negative values, with the only limitation on the magnitude of the numerical value of this quantity when it is negative furnished by $\liminf_{x=\infty} n(x)/x \geqq -D$. It is necessary therefore for the considerations to be developed later to introduce a function of a regular character which characterizes the excess distribution of a sequence $\{\lambda_n\}$. A sequence $\{\lambda_n\}$ of positive increasing numbers being given, and $n(x)$ being its excess distribution, we shall call the function

$$\Lambda^*(u) = \int_0^\infty \exp\left(-\pi ur + 2r^2\int_0^\infty (n(x)/x(x^2+r^2))dx\right)dr \tag{1}$$

the *growth function of the excess distribution* $n(x)$ (of the sequence $\{\lambda_n\}$). The introduction of this function $\Lambda^*(u)$ is justified by the following considerations. Let us first note that, since for $0<x\leqq\lambda_1$, $n(x)=-Dx$, the integral $\int_0^\infty(n(x)/x(x^2+r^2))dx$ $(r>0)$ exists. It is moreover obvious that for any $\epsilon>0$, $2r^2\int_0^\infty(n(x)/x(x^2+r^2))dx<\epsilon r$ if $r>r(\epsilon)$, and therefore $\Lambda^*(u)$ is defined, continuous and analytic for $u>0$. $\Lambda^*(u)$ increases as $u\to 0+$. On the other hand $\Lambda^*(u)$ can also be expressed in another way which will furnish interesting properties of this function and will show that it determines in a unique manner the function $n(x)$, at least when the upper density of $\{\lambda_n\}$ is given. Let

$$\Lambda(z) = \prod (1 - z^2/\lambda_n^2) \qquad (1 \leqq n < N). \tag{2}$$

Since $D<\infty$, $\Lambda(z)$ represents an entire function of $z=x+yi=re^{i\phi}$ with

$$\max_{|z|=r} |\Lambda(z)| = \Lambda(ir) = \prod (1 + r^2/\lambda_n^2).$$

It is also well known [1](1) that

$$\limsup_{r=\infty} (\log \Lambda(ir)/r) \leqq \pi D.$$

The Laplace transform of $\Lambda(ir)$, $L(\zeta)=\int_0^\infty e^{-\pi r\zeta}\Lambda(ir)dr$, represents a function holomorphic for $\Re(\zeta)>D$, and bounded for $\Re(\zeta)\geqq D+\epsilon$, where $\epsilon>0$ is arbitrary.

We have by integration by parts the well known equality [11]:

$$\log \Lambda(ir) = \sum_{n=1}^\infty \log(1+r^2/\lambda_n^2) = \int_0^\infty \log(1+r^2/x^2)dN(x)$$
$$= 2r^2\int_0^\infty (N(x)/x(x^2+r^2))dx.$$

(1) Numbers in brackets refer to the bibliography at the end of the paper.

It is then obvious that

$$\log \Lambda(ir) - \pi D r = 2r^2 \int_0^\infty ((N(x)/x - D)/(x^2 + r^2))dx$$
$$= 2r^2 \int_0^\infty (n(x)/x(x^2 + r^2))dx.$$

In other words, if $u > 0$,

$$(3) \quad \begin{aligned} L(D+u) &= \int_0^\infty \exp\,(-\pi(D+u)r)\Lambda(ir)dr \\ &= \int_0^\infty \exp\left(-\pi u r + 2r^2 \int_0^\infty (n(x)/x(x^2+r^2)dx\right)dr = \Lambda^*(u). \end{aligned}$$

The integral $\int_0^\infty \exp\,(-\pi(D+u)r)\Lambda(ir)dr$ can therefore also serve for the definition of $\Lambda^*(u)$. Let

$$(4) \quad \Lambda(z) = \sum_{k=0}^{\infty} (-1)^k c_k z^{2k}.$$

Obviously $c_k \geqq 0$ $(k \geqq 0)$, $c_0 = 1$, and

$$(5) \quad \Lambda(ir) = \sum_{k=0}^{\infty} c_k r^{2k}.$$

It is readily seen (it follows also from a well known theorem [12]) that, for $u > 0$,

$$(6) \quad \begin{aligned} \Lambda^*(u) &= \int_0^\infty \exp\,(-\pi(D+u)r)\Lambda(ir)dr \\ &= \int_0^\infty \exp\,(-\pi(D+u)r)(\sum c_k r^{2k})dr \\ &= \sum c_k (2k)!/(\pi(D+u))^{2k+1}. \end{aligned}$$

This formula shows, among other things, that if $\Lambda^*(u)$ is the growth function of the excess distribution $n(x)$ of a sequence $\{\lambda_n\}$, with upper density D, then there exists no other sequence with upper density D of which the excess distribution $n(x)$ admits $\Lambda^*(u)$ as its growth function. Indeed, by (6), if $\Lambda^*(u)$ and D are given, the Taylor coefficients $(-1)^k c_k$ of the canonical product $\Lambda(z)$ are defined, and so are the zeros $\{\lambda_n\}$ of this product.

Let $A(u)$ be a positive integrable function for $u > 0$(²). If the growth function $\Lambda^*(u)$ of the excess distribution $n(x)$ of $\{\lambda_n\}$ is such that there exists a

(²) That is to say integrable in every interval (a, b) with $0 < a < b < \infty$, with the possible value of the integral $+\infty$. $A(u)$ may take the value $+\infty$, or even be identically equal to $+\infty$.

positive constant p such that for u sufficiently small $\Lambda^*(pu) \leqq A(u)$, we shall say that $n(x)$ *belongs to* $A(u)$. In particular $n(x)$ belongs to $\Lambda^*(u)$.

We shall show later on (Lemma III) that, under simple conditions on $A(u)$, there exists a sequence $\{\lambda_n\}$ with a given upper density of which the excess distribution $n(x)$ is a function tending to infinity with x and belonging to $A(u)$.

To every sequence $\{\lambda_n\}$ of positive increasing numbers of finite upper density, we shall associate the sequence $\{\Lambda_n\}$ defined by the equality

$$\Lambda_n = (\lambda_n | \Lambda'(\lambda_n) |)^{-1} \qquad (1 \leqq n < N) \tag{7}$$

where $\Lambda'(\lambda_n)$ is the derivative of $\Lambda(z)$, given by (2), at $z=\lambda_n$. Let us put, for $1 \leqq j < N$,

$$\Lambda_j(z) = \Lambda(z)(1 - z^2/\lambda_j^2)^{-1}. \tag{8}$$

Since $\Lambda(\lambda_j)=0$, we see that

$$\Lambda_j(\lambda_j) = \lim_{z=\lambda_j} \Lambda(z)(1 - z^2/\lambda_j^2)^{-1} = - 2^{-1}\Lambda'(\lambda_j)\lambda_j,$$

that is to say,

$$\Lambda_n = 2^{-1} | \Lambda_j(\lambda_j) |^{-1}. \tag{9}$$

The sequence $\{\Lambda_n\}$ shall be called the *sequence associated with the sequence* $\{\lambda_n\}$.

Let now $\{\phi_n(x)\}$ $(1 \leqq n < N \leqq \infty)$ be a sequence of non-negative continuous functions defined for $x \geqq 0$, each of these functions being non-decreasing, with $\phi_1(0)=0$, and satisfying for every n, $1 \leqq n < N-1$, the equality $\phi_{n+1}(x) = o(\phi_n(x))$ $(x \to 0)$. The function $\phi(x) = \text{g.l.b.}_{1 \leqq n < N}\, \phi_n(x)$, $x \geqq 0$, will be called the lower envelope of the sequence $\{\phi_n(x)\}$. On writing $\phi_n^*(x) = \min \phi_k(x)$ $(1 \leqq k \leqq n)$, we see that $\phi_n^*(x)$ is a non-negative, nonincreasing function and that these functions tend monotonically to $\phi(x)$ as n increases. It is therefore clear that for every a, b, $0 < a < b < \infty$, the integral $\int_a^b \log \phi(x)dx$ has either a finite value or the value $-\infty$. If its value is $-\infty$ the same will hold if a is replaced by a', $0 < a' < a$; the integral can have the value $-\infty$ only if for a certain c with $a \leqq c$, $\phi(x)=0$ for $0 < x < c$. It is obvious that $\phi_n(0)=0$ for $n \geqq 1$, and that $\phi(0)=0$. Since the sequences $\{\phi_n(x)\}$ described above play a fundamental role in defining and in studying asymptotic series, we shall call such a sequence $\{\phi_n(x)\}$ *an asymptotic sequence.*

In the complex plane $s=\sigma+it$ we shall denote by $C(W, R)$ $(0 < R < \infty)$ the open circle $|s-W| < R$. If $V=\sigma'+it'$ we shall denote by $S(V, R')$ $(0 < R' < \infty)$ the region $|t-t'| < R'$, $\sigma > \sigma'$, and we shall call it a *horizontal half-strip of width* $2R'$. We shall call the region $|t-t'| < R'$, $\sigma > -\infty$, a *horizontal strip of width* $2R'$.

If L is a Jordan arc, we shall call the union of circles $\cup C(s', R_0)$

($0<R_0<\infty$, $s'\in L$), where R_0 is fixed, a *channel of width* $2R_0$. The curve L will be called the *central line of the channel*. If V and W are the extremities of L *and if* $R_0<R$, $R_0<R'$, the corresponding channel will be said to *connect the circle* $C(W, R)$ *to the horizontal half-strip* $S(V, R')$.

All the sequences which are mentioned in the paper are either finite or infinite. At any rate the index in the sequences $\{\lambda_n\}$, $\{d_n\}$, $\{\phi_n(x)\}$, $\{\Lambda_n\}$ (denoted by this same index) will vary, in each statement, between the same limits: $1<N\leqq\infty$, with $N<\infty$ if the sequences are finite, $N=\infty$ if the sequences are infinite.

3. In all the theorems established in this paper there appears a sequence $\{\lambda_n\}$ of positive increasing numbers. In some of these theorems such a sequence is characterized only by its upper (or lower) density, in others an estimation of the excess distribution is involved. In the main theorem itself two kinds of hypotheses are made: in one part of it only the upper density is involved, in the second one the excess distribution plays a role. The theorems, or parts of theorems, in which the excess distribution is involved are of much more delicate nature than the others.

We are now in a position to state our main theorem:

THEOREM I. *Let* $\{\lambda_n\}$ ($1\leqq n<N\leqq\infty$) *be a sequence of positive increasing numbers of upper density* D, *let* $\{d_n\}$ *be a sequence of complex numbers, and let* $\{\phi_n(x)\}$ *be an aymptotic sequence with lower envelope* $\phi(x)$. *Let* $F(s)$ *be a holomorphic and bounded function in a region* Δ, *composed of a circle* $C(s_1, \pi a)$, *of a horizontal half-strip of width* $2\pi a_2$, *and of a channel connecting them of width* $2\pi a_1$(3) *and let in* Δ, *for* $\sigma>\sigma_0$,

$$\left|F(s)-\sum_{m=1}^{n}d_me^{-\lambda_m s}\right|\leqq\phi_n(e^{-\sigma})\qquad(1\leqq n<N). \tag{10}$$

If $D<a_1$ *and if, on setting* $\omega=(2(a_2-D))^{-1}$, *one of the following two conditions*, (I) *or* (II), *is satisfied*:

(I) *The excess distribution of* $\{\lambda_n\}$ *belongs to a function* $A(u)$ *such that*

$$A(u^\omega)\phi(u)=O(1)\qquad(u\to 0), \tag{11}$$

$$\liminf_{t=0+}\int_t^1\log\left(A(u^\omega)\phi(u)\right)u^{\omega-1}du=-\infty\,(4). \tag{12}$$

(II) *There exists a constant* $\omega'>\omega$ *such that*

$$\int_0^1\log\phi(u)u^{\omega'-1}du=-\infty, \tag{13}$$

then

(3) We recall that by the definition of a connecting channel $a_1\leqq a$, $a_1\leqq a_2$.

(4) If $A(u_1^\omega)=\infty$, $\phi(u_1)=0$, we shall set $A(u_1^\omega)\phi(u_1)=0$.

(14) $$|d_n| \leqq KM(s_1)\Lambda_n \exp(\lambda_n \mathcal{R}(s_1)) \qquad (1 \leqq n < N)\text{[5]}$$

where $K<\infty$ depends only on a and D, where $\{\Lambda_n\}$ is the sequence associated with $\{\lambda_n\}$, and where $M(s_1)$ is such that in $C(s_1, \pi a)$, $|F(s)| \leqq M(s_1)$.

And every function $F_0(s)$ holomorphic and bounded in Δ and satisfying there, for $\sigma > \sigma'$, the inequalities

(15) $$\left| F_0(s) - \sum_{m=1}^{n} d_m e^{-\lambda_m s} \right| \leqq \phi_n(e^{-\sigma}) \qquad (1 \leqq n < N),$$

is identically equal to $F(s)$.

REMARK. *The value of the constant K is given by $K = 2\pi a\Lambda^*(a-D)$ where $\Lambda^*(u)$ is the growth function of the excess distribution $n(x)$ of $\{\lambda_n\}$.*

Let us note that the condition (I) can hold (since $A(u)$ is such that an excess distribution belongs to it, that is to say $A(u) \geqq \Lambda^*(pu)$ $(0<u<u_0)$, the last function being positive and increasing as $u \to 0$) only if $\int_0^1 \log \phi(u) u^{\omega-1} du = -\infty$. But if this condition is satisfied it is usually possible to choose a function $A(u)$ increasing to ∞ as $u \to 0$ slowly enough, in order to satisfy (11) and (12), and rapidly enough, in order to assure the existence of a sequence $\{\lambda_n\}$ with a given upper density D, the excess distribution $n(x)$ of $\{\lambda_n\}$ tending to infinity with x and belonging to $A(u)$. For instance we may choose $A(u) = (\phi(u^{1/\omega}))^{-\alpha}(1/2<\alpha<1)$, if this function $A(u)$ satisfies the conditions of Lemma III. (This follows immediately from Lemma III.) If in the preceding integral ω is replaced by $\omega' > \omega$, as in (13), the condition on $\phi(x)$ becomes much more restrictive, and this is the reason why no condition on the excess distribution is then any longer necessary.

4. For the proof of Theorem I it is necessary to prove some preliminary lemmas.

LEMMA I. *Let $0 \leqq \pi\alpha < 2e^{\beta}$, and let $\Phi(s)$ be a function not identically zero, holomorphic and bounded in the region $R(\alpha, \beta)$ defined by*

(16) $$|t| < (\pi/2)(1 - \alpha e^{-\sigma}), \qquad \sigma \geqq \beta.$$

The curve C given by the equation $e^{\sigma} \cos t = e^{\beta}$ belongs to $R(\alpha, \beta)$, and l being any segment of C on which $t \geqq 0$, we have

(17) $$\underset{l \subset C}{\text{g.l.b.}} \int_l \log |\Phi(s)| e^{-\sigma} d\sigma > -\infty,$$

the integral being taken in the sense of increasing σ.

For the points of the boundary of $R(\alpha, \beta)$ on which $\sigma > \beta$ we have

$$\cos t = \sin(\pi/2 - |t|) \leqq \pi/2 - |t| = \pi\alpha/2e^{\sigma} < e^{\beta-\sigma}.$$

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[5] $\mathcal{R}(z)$ is the real part of z.

This shows that $C\subset R(\alpha, \beta)$. Let D be the region, bounded by C, in which $\cos t > e^{\beta-\sigma}(\sigma>\beta, |t| < \pi/2)$. Let us put $z=x+iy=e^s$. From $x=e^\sigma \cos t$ it follows that D is mapped, by this transformation, on the half-plane $x>e^\beta=\gamma$. Let $f(z)=\Phi(s)$; $f(z)$ is not identically zero, holomorphic and bounded in $x\geqq\gamma$. If $s\in C$, $t\geqq 0$, then $x=\gamma$, $y\geqq 0$, and $|z|^2e^{-\sigma}d\sigma/dy\to 1$ when $s\in C$, $\sigma\to\infty$, $t>0$ or, what amounts to the same thing, when $x=\gamma$, $y\to\infty$. By a well known theorem [7, 10] it follows from the conditions on $f(z)$ that

$$\int_{-\infty}^{\infty}(\log |f(\gamma+iy)|/(\gamma^2+y^2))dy = B > -\infty.$$

If $|f(\gamma+iy)| < M$ $(1<M<\infty, -\infty<y<\infty)$ we have for every segment m on the line $x=\gamma$, on integrating in the sense of increasing y,

$$\int_m (\log |f(z)|/|z^2|)dy \geqq B - \log M\int_{-\infty}^{\infty} dy/(y^2+\gamma^2) = D_1 > -\infty.$$

By this inequality and the relation above involving $d\sigma/dy$ for $s\in C$ $(t>0$, $\sigma\to\infty)$ our lemma is established.

LEMMA II. *Let $\Phi(s)$ be holomorphic in a circle $C(s', \pi R)$ and in this circle let $|\Phi(s)|\leqq M$. Let $\{\lambda_n\}$ be of upper density $D<R$, let $\Lambda(z)$ be defined by* (2) *and $\Lambda_j(z)$ by* (8), *let*

$$\Lambda_j(z) = \sum_{k=0}^{\infty}(-1)c_k^{(j)}z^{2k}, \tag{18}$$

and let $0<u<R-D$. The series $\sum(-1)c_k^{(j)}\Phi^{(2k)}(s)$ converges uniformly in $C(s', \pi(R-D-u))$ and represents there a holomorphic function $\Phi_j(s)$ satisfying the inequality

$$|\Phi_j(s)| \leqq \pi RM\Lambda^*(u), \tag{19}$$

where $\Lambda^*(u)$ is the growth function of the excess distribution $n(x)$ of $\{\lambda_n\}$.

By Cauchy's theorem we have for $s\in C(s', \pi(R-D-u))$

$$\begin{aligned}|\Phi^{(2k)}(s)| &\leqq ((2k)!/2\pi)\left|\oint_{|W-s'|=R'}(\Phi(w)/(w-s)^{2k+1})dw\right| \\ &\leqq (2k)!R'M/(R'-\pi(R-D-u))^{2k+1},\end{aligned}$$

where R' is any quantity such that $\pi(R-D-u)<R'<\pi R$. Thus

$$|\Phi^{(2k)}(s)| \leqq \pi(2k)!RM/(\pi(D+u))^{2k+1}. \tag{20}$$

Obviously, if the c_k are given by (4), $0\leqq c_k^{(j)}\leqq c_k (k\geqq 0)$ and we have, by (20) and (6), $|\Phi_j(s)| \leqq \sum_{k=0}^{\infty}c_k|\Phi^{(2k)}(s)| \leqq \pi RM\sum\epsilon_k(2k)!/(\pi(D+u))^{2k+1}$ $=\pi RM\Lambda^*(u)$, and the lemma is proved.

5. We now proceed to the proof of the main theorem. Let $S(s_2, \pi a_2)$, $s_2=\sigma_2+it_2$, be the horizontal half-strip, part of Δ, and let L be the central line of the channel connecting $C(s_1, \pi a)$ to $S(s_2, \pi a_2)$ and belonging to Δ. Let $F_j(s)=\sum_{k=0}^{\infty}(-1)^k c_k^{(j)} F^{(2k)}(s)$ where the $c_k^{(j)}$ are given by (18). By the hypotheses of the theorem and by Lemma II, $F_j(s)$ is holomorphic in each of the circles $C(s_1, \pi(a-D-u))$, $C(s', \pi(a_1-D-u))(s'\in L)$, $C(\sigma+it_2, \pi(a_2-D-u))(\sigma_2<\sigma\leqq\sigma_2+\pi a_2)$, $C(\sigma+it_2, \pi(a_2-D-u))(\sigma>\sigma_2+\pi a_2)$, where u is such that $0<u<a_1-D$. We have moreover, by the same Lemma II,

$$|F_j(s_1)| \leqq \pi a M(s_1)\Lambda^*(u),$$

where u is such that $0<u<a-D$. Therefore, since $\Lambda^*(u)$ is continuous for $u>0$,

$$|F_j(s_1)| \leqq \pi a M(s_1)\Lambda^*(a-D). \tag{21}$$

Let us now write

$$\Phi_n(s) = F(s) - \sum_{m=1}^{n} d_m e^{-\lambda_m s}. \tag{22}$$

By hypothesis $\Phi_n(s)$ is holomorphic in every circle $C(\bar\sigma+it, \pi a_2)$ with $\bar\sigma\geqq\sigma'=\max(\sigma_0+\pi a_2, \sigma_2)$, and in every such circle

$$|\Phi_n(s)| \leqq \phi_n(e^{-\bar\sigma+\pi a_2}). \tag{23}$$

It follows then from Lemma II that the function

$$\begin{aligned}
\Phi_{n,j}(s) &= \sum_{k=0}^{\infty}(-1)^k c_k^{(j)}\Phi_n^{(2k)}(s)\\
&= \sum_{k=0}^{\infty}(-1)^k c_k^{(j)}F^{(2k)}(s) - \sum_{k=0}^{\infty}(-1)^k c_k^{(j)}\sum_{m=1}^{n}\lambda_m^{2k} d_m e^{-\lambda_m s}\\
&= F_j(s) - \sum_{m=1}^{n} d_m e^{-\lambda_m s}\sum_{k=0}^{\infty}(-1)^k c_k^{(j)}\lambda_m^{2k} = F_j(s) - \sum_{m=1}^{n} d_m\Lambda_j(\lambda_m)e^{-\lambda_m s}
\end{aligned}$$

is holomorphic in every circle $C(\bar\sigma+it_2, \pi(a_2-D-u))$, $\bar\sigma\geqq\sigma'$ with $0<u<a_2-D$, and in this circle

$$|\Phi_{n,j}(s)| \leqq \pi a_2\phi_n(e^{-\bar\sigma+\pi a_2})\Lambda^*(u). \tag{24}$$

Since $\Lambda_j(\lambda_m)=0$ for $m\neq j$, we see that for $n\geqq j$

$$\sum_{m=1}^{n} d_m\Lambda_j(\lambda_m)e^{-\lambda_m s} = d_j\Lambda_j(\lambda_j)e^{-\lambda_j s}.$$

We may therefore write, for $n\geqq j(n<N\leqq\infty)$,

$$\Phi_{n,j}(s) = \Psi_j(s) = F_j(s) - d_j\Lambda_j(\lambda_j)e^{-\lambda_j s}, \tag{25}$$

and by (24) we have

$$(26)\qquad |\Psi_j(s)| \leqq \pi a_2 \phi_n(e^{-\sigma+\pi a_2})\Lambda^*(u),$$

$(s = \sigma + it,\ |t - t_2| \leqq \pi(a_2 - D - u),\ 0 < u < a_2 - D,\ \sigma > \sigma',\ j \leqq n < N).$

From $\phi_{n+1}(x) = o(\phi_n(x))\,(x\to 0)$ it follows that for σ sufficiently large $\phi_k(e^{-\sigma+\pi a_2}) \geqq \phi_j(e^{-\sigma+\pi a_2})$, $1 \leqq k \leqq j$, j being fixed. Therefore there exists a quantity σ^* such that (26) holds for $|t-t_2| \leqq \pi(a_2-D-u)$, $\sigma \geqq \sigma^*$, $0<u<a_2-D$, $1 \leqq n < N$. We have for the same values of s

$$(27)\qquad |\Psi_j(s)| \leqq \pi a_2 \Lambda^*(u) \operatorname*{g\,l.b.}_{1\leqq n<N} \phi_n(e^{-\sigma+\pi a_2}) = \pi a_2 \Lambda^*(u)\phi(e^{-\sigma+\pi a_2}).$$

We shall now prove that $\Psi_j(s)$ is identically zero. For that purpose we shall consider the functions $\theta_j(\nu; s) = \Psi_j(s/\nu + it_2)$ for every $\nu \geqq \omega$. It follows from (27) that in the region $|t| \leqq \pi\nu(a_2-D-u) = (\pi/2)(\nu/\omega - 2\nu u)$, $\sigma \geqq \nu\sigma^*$, with $0<u<a_2-D$, the following inequality holds:

$$(28)\qquad |\theta_j(\nu; s)| \leqq \pi a_2 \Lambda^*(u)\phi(e^{-\sigma/\nu+\pi a_2}).$$

Let q be an arbitrary positive quantity and let us choose $\alpha = \alpha(q)$, $\beta = \beta(\nu, q)$ in the following manner: $\alpha = 2\omega q e^{\pi a_2 \omega}$, $\beta > \max\,(\nu\sigma^*, \log\,(\pi\omega q) + \pi a_2\omega)$. We have then $\beta > \nu\sigma^*$, $2e^\beta > \pi\alpha > 0$. Since in (28) u has only to satisfy the inequality $0<u<a_2-D$, but otherwise may vary with s, we choose, for $\sigma \geqq \beta$, $u = 1/2\omega - 1/2\nu + (\alpha/2\nu)e^{-\sigma}$; the condition $0<u<a_2-D = 1/2\omega$ is then satisfied. It follows from (28) that $\theta_j(\nu; s)$ is holomorphic in the region $R(\alpha, \beta)$, given by $|t| < (\pi/2)(1-\alpha e^{-\sigma})$, $\sigma \geqq \beta$, with $\alpha = \alpha(q)$, $\beta = \beta(\nu, q)$ chosen as above, and satisfies there the inequality

$$|\theta_j(\nu; s)| \leqq \pi a_2 \Lambda^*(1/2\omega - 1/2\nu + (\alpha/2\nu)e^{-\sigma})\phi(e^{-\sigma/\nu+\pi a_2}),$$

which, on putting $v = e^{-\sigma/\nu+\pi a_2}$, may also be written

$$(29)\qquad |\theta_j(\nu; s)| \leqq \pi a_2 \Lambda^*(1/2\omega - 1/2\nu + (\omega/\nu) q e^{\pi a_2(\omega-\nu)} v^\nu)\phi(v).$$

This inequality shows first that if $\nu > \omega$, $\theta_j(\nu; s)$ is bounded in $R(\alpha, \beta)$. Indeed as $\sigma \to \infty$, that is to say as $v \to 0$, the right member of (29) tends to $\pi a_2 \Lambda^*(1/2\omega - 1/2\nu)\phi(0) = 0$, since from $\nu > \omega$ it follows that $0 < \Lambda^*(1/2\omega - 1/2\nu) < \infty$, and we have $\phi(0) = 0$. If the condition (I) of our theorem is satisfied, then $\theta_j(\omega; s)$ is also bounded in the corresponding region $R(\alpha, \beta)$, $\alpha = \alpha(q)$, $\beta = (\omega, q)$, if q is chosen however in a special manner. Indeed if (I) is satisfied there exists, by definition of the function $A(u)$, a positive constant p such that $\Lambda^*(pu) \leqq A(u)$ for u sufficiently small. If we choose then $q = p$ we see from (29) that in $R(\alpha, \beta)$ with $\alpha = \alpha(p)$, $\beta = \beta(\omega, q)$, we have for v sufficiently small (σ large)

$$(30)\qquad |\theta_j(\omega; s)| \leqq \pi a_2 \Lambda^*(p v^\omega)\phi(v) \leqq \pi a_2 A(v^\omega)\phi(v) \qquad (v = e^{-\sigma/\omega+\pi a_2}).$$

But since from (11) of the condition (I) it follows that for v sufficiently small, that is to say for σ sufficiently large, the right-hand term of this last inequality is bounded, we see that, in $R(\alpha, \beta)$, $(\alpha=\alpha(p), \beta=\beta(\omega, p))$, $\theta_j(\omega; s)$ is bounded, if the condition (I) of the theorem is satisfied. On the other hand we see that if (12) is satisfied then (30) gives, since $v^{\omega-1}dv=-(e^{\omega\pi a_2-\sigma})d\sigma/\omega$,

$$\text{(31)} \qquad \underset{\beta\leqq c<d<\infty}{\text{g.l.b.}} \int_c^d \log |\theta_j(\omega; s)| e^{-\sigma}d\sigma = -\infty,$$

the integral being taken on the line $e^\sigma \cos t=e^\beta$, $t\geqq 0 (s=\sigma+it)$. Thus, if (I) is satisfied, $\theta_j(\omega; s)$ is bounded in $R(\alpha, \beta)$ and (31) holds on the specified line, and we see by Lemma I that $\theta_j(\omega; s)\equiv 0$, that is to say $\Psi_j(s)\equiv 0$.

If now (II) is satisfied, we see from (13) and (29), since $\lim \Lambda^*(1/2\omega-1/2\omega' +(\omega/\omega')e^{\pi a_2(\omega-\omega')}v^{\omega'})=\Lambda^*(1/2\omega-1/2\omega')(v\to 0)$, that on the line $e^\sigma \cos t=e^\beta$, $t\geqq 0$ lying in $R(\alpha, \beta)$ with $\alpha=\alpha(1)$, $\beta=\beta(\omega', 1)$, the relationship

$$\text{(32)} \qquad \underset{\beta\leqq c<d<\infty}{\text{g.l.b.}} \int_c^d \log |\theta_j(\omega'; s)| e^{-\sigma}d\sigma = -\infty$$

holds, and this together with the fact proved above that $\theta_j(\omega'; s)$ is bounded in $R(\alpha, \beta)$ (since $\omega'>\omega$) gives, by Lemma I, $\theta_j(\omega'; s)\equiv 0$, that is to say $\Psi_j(s)\equiv 0$. Thus each of the two conditions (I) or (II) of the theorem gives $\Psi_j(s)\equiv 0$. In other words if the conditions of our theorem are satisfied we have

$$\text{(33)} \qquad F_j(s) = d_j\Lambda_j(\lambda_j)e^{-\lambda_j s}.$$

It follows then from (21) that

$$\text{(34)} \qquad |d_j\Lambda_j(\lambda_j)| \exp(-\lambda_j\mathfrak{R}(s_1)) \leqq \pi a M(s_1)\Lambda^*(a-D) \qquad (1\leqq j<N).$$

By (9) we see then that (14) is satisfied with $K=2\pi a M(s_1)\Lambda^*(a-D)$.

Let us now prove that $F(s)=F_0(s)$ identically. From (10) and (15) it follows that there exists a quantity σ^* such that in the region $|t-t_2|<\pi a_2$, $\sigma>\sigma^*$,

$$|F(s)-F_0(s)| \leqq 2\phi_n(e^{-\sigma}) \qquad (1\leqq n<N).$$

On writing

$$\Omega(s) = F(2a_2 s+it_2) - F_0(2a_2 s+it_2),$$

we see that, for $|t|<\pi/2$, $\sigma\geqq\sigma^*/2a_2$, which region with the notation of the statement of Lemma I may also be written as $R(0, \sigma^*/2a_2)$, we have

$$\text{(35)} \qquad |\Omega(s)| \leqq 2\phi_n(e^{-2a_2\sigma}) \qquad (1\leqq n<N).$$

We have therefore also

$$\text{(36)} \qquad |\Omega(s)| \leqq 2\phi(e^{-2a_2\sigma}).$$

If $\Omega(s)$ were not identically zero, we would have, by Lemma I, on integrating along the line $e^\sigma \cos t=e^{\sigma^*/2a_2}=e^\beta$, $t\geqq 0$,

(37) $$\underset{\beta \leqq c<d<\infty}{\text{g.l.b.}} \int_c^d \log |\Omega(s)| e^{-\sigma} d\sigma > -\infty \quad (\beta = \sigma^*/2a_2),$$

and it would then follow from (36) that

$$\underset{\beta \leqq c<d<\infty}{\text{g.l.b.}} \int_c^d \log \phi(e^{-2a_2\sigma}) e^{-\sigma} d\sigma > -\infty,$$

that is to say

(38) $$\underset{0<\gamma<\delta\leqq\beta_1}{\text{g.l.b.}} \int_\gamma^\delta \log \phi(u) u^{1/2a_2-1} du > -\infty,$$

where β_1 is a constant.

Since $\omega' > \omega = (2(a_2-D))^{-1} \geqq 1/2a_2$, it would follow from (38) that

$$\underset{0<\gamma<\delta<\beta_1}{\text{g.l.b.}} \int_\gamma^\delta \log \phi(u) u^{\omega'-1} du > -\infty,$$

$$\underset{0<\gamma<\delta<\beta_1}{\text{g.l.b.}} \int_\gamma^\delta \log \phi(u) u^{\omega-1} du > -\infty.$$

The first of these inequalities is in contradiction with (13), the second is in contradiction with (12), because for u sufficiently small $A(u^\omega) \geqq \Lambda^*(pu^\omega)$, the last function being positive and increasing as $u \to 0$. Therefore if (I) or (II) is satisfied, $\Omega(s) \equiv 0$, that is to say $F(s) = F_0(s)$ identically. Our theorem is thus completely proved.

6. We shall now prove some lemmas which will either determine circumstances in which the hypotheses of the main theorem are satisfied, or will permit us to evaluate elements involved in the statement of the theorem if special conditions are satisfied.

For instance some of the lemmas will give conditions on a function $A(u)$ in order that interesting excess distributions $n(x)$ belong to it; other lemmas will enumerate hypotheses in order that conditions of the form (10) may be satisfied; still other lemmas will allow us to evaluate the terms of $\{\Lambda_n\}$ when the sequence $\{\lambda_n\}$ satisfies certain specific important conditions.

LEMMA III. *Let $A(u)$ be a positive continuous function for $u>0$, such that* $\log A(u)(\log u)^{-1} \to -\infty$ *as* $u \to 0$, $\log A(u)$ *being a convex function of* $\log u$. *Let* $0 \leqq D < \infty$. *There exists a sequence* $\{\lambda_n\}$ *with upper density D of which the excess distribution $n(x)$ tends to infinity with x and belongs to $A(u)$. If $D<1$, a sequence with the specified properties exists of which all the elements are integers.*

Let us write $\log A(1/r) = C(\log r)$ $(r>0)$. The function $C(t)$ is a convex function of t $(-\infty < t < \infty)$, and $C(t)/t \to \infty$ as $t \to \infty$. Let us set for $x \geqq 0$

$$\Pi(x) = \underset{-\infty<t<\infty}{\text{l.u.b.}} (xt - C(t)). \tag{39}$$

It is then known [7] that

$$C(t) = \underset{0\leqq x}{\text{l.u.b.}} (xt - \Pi(x)). \tag{40}$$

It follows from the definition of $\Pi(x)$ that $\Pi(x) < \infty$ and that $\Pi(x)/x \to \infty$. If we set $b_n = e^{-\Pi(2n)}$, it follows from (40) that $b_n r^{2n} \leqq e^{C(\log r)}$ $(n \geqq 1,\ r > 1)$. The series $\psi(z) = \sum_{n=1}^{\infty}(b_n/2^n)z^{2n}$ represents therefore an even entire function, different from a polynomial, satisfying for $r > 1$ the inequality

$$\psi(r) = \max_{|z|=r} |\psi(z)| \leqq e^{C(\log r)}.$$

It is obvious that for r sufficiently large

$$L(r) = r + \sum_{n=2}^{\infty} (b_n/2^n) r^{2n-1} < e^{C(\log r)} = A(1/r). \tag{41}$$

Let us write

$$S(r) = 1 + \sum_{n=2}^{\infty} (b_n/(2(n-1))!2^n) r^{2(n-1)} = \sum_{n=0}^{\infty} l_n r^{2n} \qquad (l_0 = 1).$$

Since $\Pi(x) < \infty$, $b_n > 0 (n \geqq 2)$ and $l_n > 0 (n \geqq 0)$. It is then possible to make correspond to every integer $m > 0$ a quantity μ_m such that $(1 + r^2/\mu_m^2)^{2^m} \leqq S(r)$ $(r \geqq 0)$. We shall then have

$$E(r) = \prod_{m=1}^{\infty} (1 + r^2/\mu_m^2) \leqq (S(r))^{\sum_1^{\infty} 2^{-m}} = S(r). \tag{42}$$

It follows from this inequality and from (41) that for u sufficiently small

$$\begin{aligned} \int_0^\infty e^{-\pi ur}E(r)dr &\leqq \int_0^\infty e^{-\pi ur}S(r)dr = \int_0^\infty e^{-\pi ur}\left(\sum_{n=0}^{\infty} l_n r^{2n}\right)dr \\ &= \sum_{n=0}^{\infty} l_n (2n)!/(\pi u)^{2n+1} = L(1/\pi u) \leqq A(\pi u). \end{aligned} \tag{43}$$

If $D > 0$, let us put $\nu_n = n/D (n \geqq 1)$. Let us denote by $\{m_n\}$ a sequence of increasing quantities such that $m_n \geqq \mu_n (n \geqq 1)$, $n/m_n \to 0$ as $n \to \infty$, and such that no m_n is equal to any ν_n. Let us denote by $\{\lambda_n\}$ the sequence composed of the quantities m_n and ν_n and written in increasing order. Let $N(x)$ be the distribution of $\{\lambda_n\}$, $n_1(x)$ the distribution of $\{m_n\}$, $n_2(x)$ the distribution of $\{\mu_n\}$, and $N_1(x)$ the distribution of $\{\nu_n\}$. We have obviously $N(x) = N_1(x) + n_1(x) = [Dx] + n_1(x)$ with $n_1(x) \leqq n_2(x)$, $[Dx]$ denoting the greatest integer smaller than Dx. If $D = 0$, we shall put $\lambda_n = m_n (n \geqq 1)$; we have in this case

$N(x)=n_1(x)\leqq n_2(x)$. It is seen that the excess distribution $n(x)$ of $\{\lambda_n\}$ is in each case not larger than $n_1(x)$. We see also, since $\lim n_1(x)/x=\lim n/m_n=0$, that the upper density of $\{\lambda_n\}$ is D. On the other hand, since $0\leqq n_1(x)-n(x)\leqq 1$, and since $n_1(x)\to\infty$ as $x\to\infty$, we have $n(x)\to\infty$. Since the upper density of $\{m_n\}$ is zero, its excess distribution, $n_1(x)$, is equal to its distribution. If we denote by $\Lambda_1^*(u)$ the growth function of this distribution, and by $\Lambda^*(u)$ the growth function of the excess distribution of $\{\lambda_n\}$, we see that $\Lambda^*(u)\leqq\Lambda_1^*(u)$, and on putting $E_1(r)=\prod_{n=1}^{\infty}(1+r^2/m_n^2)$ we have by what we have seen in §2, and by (43) for u sufficiently small,

$$\Lambda^*(u) \leqq \Lambda_1^*(u) = \int_0^\infty e^{-\pi ur}E_1(r)dr \leqq \int_0^\infty e^{-\pi ur}E(r)dr \leqq A(\pi u).$$

In other words, for u sufficiently small $\Lambda^*(u/\pi)\leqq A(u)$, that is to say $n(x)$ belongs to $A(u)$.

If we take for $\nu_n(n\geqq 1)$ the integral part of $n/D+1$, if $0<D<1$, and for $\{m_n\}$ a sequence of increasing integers with $m_n\geqq\mu_n(n\geqq 1)$, $n/m_n\to 0$ and moreover, if $0<D$, with the condition that no m_n is equal to any ν_n, then the sequence $\{\lambda_n\}$ of all the m_n and all the ν_n, if $0<D$, and composed only of all the m_n, if $D=0$, still satisfies the desired conditions. But now the λ_n are integers. Our lemma is thus proved.

Of special interest for us will be the structure of a sequence $\{\lambda_n\}$ of which the excess distribution belongs to a function $A(u)$ of the form $A(u)=(\text{g.l.b.}_{n\geqq 1}\, N_n u^n)^{-\alpha}$ where $\{N_n\}$ is a sequence of positive numbers with $N_n^{1/n}\to\infty$ as $n\to\infty$, and where $\alpha>0$. We have

$$\log A(1/r) = \alpha \operatorname*{l.u.b.}_{n\geqq 1} (n\log r - \log N_n). \tag{44}$$

The function

$$C(t) = \operatorname*{l.u.b.}_{n\geqq 1} (nt - \log N_n) = \log A(e^{-t})/\alpha \tag{45}$$

is a convex function of t[7], therefore $\log A(u)$ is a convex function of $\log u$. It is also obvious that $C(t)/t\to\infty$. Therefore $\log A(u)(\log u)^{-1}\to-\infty$ as $u\to 0$. Lemma III can therefore be applied to $A(u)$. But our next purpose is to study the character of a sequence $\{\lambda_n\}$ of which the excess distribution belongs to such a function $A(u)$. It is known [7] that the function

$$\Pi(x) = \operatorname*{l.u.b.}_{-\infty<t<\infty} (xt - C(t)) \qquad (x\geqq 0) \tag{46}$$

is convex, and, as we have already seen on page 107 (see also [7]) that

$$C(t) = \operatorname*{l.u.b.}_{x\geqq 0} (xt - \Pi(x)). \tag{47}$$

The curve given by $y=\Pi(x)$ is a convex curve on which lie points P_{n_i}

with coordinates $(n_i, \log N_{n_i})$, no point P_n with coordinates $(n, \log N_n)$ being below this curve. Every other convex curve given by $y=B(x)$ with the property that no point P_n is below it is such that $B(x) \leqq \Pi(x)$. This proves that $y=\Pi(x)$ represents a polygonal convex curve with vertices at points of the sequence $\{P_n\}$; the points of $\{P_n\}$ which are not vertices are above this curve. We shall call this polygonal convex curve the *base of the sequence* $\{n, \log N_n\}$ [7]. We shall write $N^c(x)=e^{\Pi(x)}$. We have $N^c(n)=N_n^c$, where $\{N_n^c\}$ is the sequence called the convex regularized sequence of $\{N_n\}$, regularized by means of logarithms [5]. The function $N^c(x)$ will be called the *convex base, by means of logarithms of the sequence* $\{N_n\}$. Let us note that from the definition of the functions $N^c(x)$, $C(t)$, and from (46) it follows that

$$
\begin{aligned}
\alpha \log N^c(n/\alpha) &= \alpha \operatorname*{l.u.b.}_{-\infty<t<\infty} (nt/\alpha - C(t)) = \operatorname*{l.u.b.}_{-\infty<t<\infty} (nt - \log A(e^{-t})) \\
&= \operatorname*{l.u.b.}_{r>0} (n \log r - \log A(1/r)) = \alpha\Pi(n/\alpha).
\end{aligned} \tag{48}
$$

We are now in a position to prove the following lemmas:

LEMMA IV. *Let $\{N_n\}$ be a sequence of positive quantities with $N_n^{1/n}\to\infty$, let $\alpha>0$, let*

$$A(u) = (\operatorname*{g.l.b.}_{n\geqq 1} N_n u^n)^{-\alpha} \qquad (u > 0), \tag{49}$$

and let $N^c(x)$ be the convex base, by means of logarithms of the sequence $\{N_n\}$. Let $\{m_n\}$ be a sequence of positive increasing integers such that $\sum m_n^{-1}\leqq 1$, and such that $m_n\geqq n$. The excess distribution of every sequence $\{\lambda_n\}$ of finite upper density, formed by the union of two sequences of positive increasing quantities without common term, $\{\lambda_n\}=\{\nu_n\}\cup\{\mu_n\}$; the excess distribution of $\{\nu_n\}$ being nonpositive, the μ_n satisfying the inequalities

$$\mu_n \geqq 2^{1/2} \max_{1\leqq k\leqq m_n} (C_{m_n}^k (2k)!(N^c(4k/\alpha))^{\alpha/2})^{1/2k}, \tag{50}$$

belongs to $A(u)$.

LEMMA V. *If $\{N_n\}$, α, $A(u)$, $N^c(x)$ and $\{m_n\}$ are defined as in Lemma* IV, *if* $\log N^c(x)/x\to\infty$ *monotonically as $x\to\infty$, then the excess distribution of every sequence $\{\lambda_n\}$ of finite upper density, formed by the union of two sequences of positive increasing quantities, $\{\lambda_n\}=\{\nu_n\}\cup\{\mu_n\}$, the excess distribution of $\{\nu_n\}$ being nonpositive, the μ_n satisfying the inequalities*

$$\mu_n \geqq 2m_n(N^c(4n/\alpha))^{\alpha/4n}, \tag{51}$$

belongs to $A(u)$.

REMARK. From $\log N^c(x)/x\to\infty$, and from the fact that $\log N^c(x)$ is convex in x, follows already that $\log N^c(x)/x\to\infty$ monotonically *for x sufficiently large.*

For the proof of our lemmas consider $C(t)$ defined by (45) and $\Pi(x)$ defined by (46), and set $\log b_n = -(\alpha/2)\Pi(4n/\alpha) = \log (N^c(4n/\alpha))^{-\alpha/2}$. It follows from (47) that

$$b_n r^{2n} \leqq e^{(\alpha/2)C(\log r)} = (A(1/r))^{1/2} \qquad (r > 0,\ n \geqq 1).$$

We have for r sufficiently large

$$(52) \qquad L_1(r) = r + \sum_1^\infty (b_n/2^n) r^{2n+1} \leqq A(1/r).$$

Let us write

$$S_1(r) = 1 + \sum_1^\infty (b_n/(2n)!2^n) r^{2n} = \sum_0^\infty L_n r^{2n} \qquad (L_0 = 1).$$

Here, as in the proof of Lemma IV, $\Pi(x) < \infty$, therefore $b_n > 0 (n \geqq 1)$. We have for $n \geqq 1$

$$(1 + r^2/\mu_n^2)^{m_n} = 1 + \sum_{k=1}^{m_n} C_{m_n}^k r^{2k}/\mu_n^{2k}.$$

But if (50) is satisfied, we have for $1 \leqq k \leqq m_n$

$$\mu_n^{2k} \geqq C_{m_n}^k (N^c(4k/\alpha))^{\alpha/2}(2k)!2^k = (C_{m_n}^k/b_k)(2k)!2^k = C_{m_n}^k/L_k \quad (n \geqq 1),$$

that is to say $C_{m_n}^k/\mu_n^{2k} \leqq L_k$, which proves that for $n \geqq 1$

$$1 + r^2/\mu_n^2 \leqq (S_1(r))^{1/m_n}.$$

We have thus

$$E(r) = \prod_{n=1}^\infty (1 + r^2/\mu_n^2) \leqq (S_1(r))^{\sum_1^\infty m_n^{-1}} \leqq S_1(r).$$

It follows from this inequality and from (52) that for u sufficiently small

$$(53) \qquad \int_0^\infty e^{-\pi u r} E(r) dr \leqq \int_0^\infty e^{-\pi u r} S_1(r) dr = \int_0^\infty e^{-\pi u r}\left(\sum L_n r^{2n}\right) dr$$
$$= \sum L_n (2n)!/(\pi u)^{2n+1} = L_1(1/\pi u) \leqq A(\pi u).$$

It follows from the fact that $C(t)/t \to \infty$, and from (46), that $\Pi(x)/x \to \infty$, thus $(N^c(4k/\alpha))^{\alpha/4k} \to \infty$ as $k \to \infty$. Therefore, since $m_n \geqq n$, and since, by (50), $\mu_n \geqq e^{-1} n C_{m_n}^n (N^c(4n/\alpha))^{\alpha/4n}$, we have $\mu_n/n \to \infty$. The excess distribution $n_1(x)$ of $\{\mu_n\}$ is its distribution itself, since its upper density is zero. The growth function of this distribution is therefore equal to the left-hand member of (53). The distribution $n_1(x)$ belongs thus to $A(u)$. On the other hand it is clear, by the definition of $\{\nu_n\}$, $\{\mu_n\}$ and $\{\lambda_n\}$, that the excess distribution of $\{\lambda_n\}$ is not larger than $n_1(x)$. Lemma IV is thus proved.

Lemma V is an immediate consequence of Lemma IV. Indeed

$$\max_{1\leq k\leq m_n} (C_{m_n}^k(2k)!)^{1/2k} \leqq 2^{1/2} m_n$$

and, therefore, from $\log N^c(x)/x\to\infty$ monotonically it follows that the right-hand expression in (50) is not larger than the right-hand expression in (51).

Let $\{N_n\}$ be any sequence of positive quantities. If $\liminf N_n^{1/n}=M<\infty$, then $\phi(u)=\text{g.l.b.}_{n\geqq 1} N_n u^n=0$ for $0<u<M^{-1}$. Indeed, if $N_{n_i}^{1/n_i}\to M(i\to\infty)$, we have $0\leqq\phi(u)\leqq\text{g.l.b.}_{i\geqq 1} N_{n_i}u^{n_i}\leqq\lim (M+\epsilon)^{n_i}u^{n_i}=0$, if $u<(M+\epsilon)^{-1}(0<\epsilon)$. We have then for $\alpha>0$, $0<u<M^{-1}$, $A(u)=(\text{g.l.b.}_{n\geqq 1} N_n u^n)^{-\alpha}=(\phi(u))^{-\alpha}=\infty$, and the excess distribution of every $\{\lambda_n\}$ belongs to $A(u)$.

Without supposing the restrictive condition $M<\infty$, we see that for $0<u<\min((N_1/N_q)^{1/(q-1)}, (N_2/N_q)^{1/(q-2)}, \cdots, (N_{q-1}/N_q))(q>1)$ $N_q u^q$ is smaller than each of the quantities N_1u, $N_2u^2, \cdots, N_{q-1}u^{q-1}$, and therefore $\phi(u)=\text{g.l.b.}_{n\geqq q} N_n u^n$. In other words the value of $\phi(u)$, for u sufficiently small, depends only on the N_n with n sufficiently large. This is therefore also true for $A(u)=(\phi(u))^{-\alpha}$. This allows us to change a finite number of N_n without altering $A(u)$ for u sufficiently small. Let now $N_n^{1/n}\to\infty$ and let n_0 be such that $\log N_{n_0}/n_0=\min_{n\geqq 1}\log N_n/n=a$. We shall replace the quantities $N_1, N_2, \cdots, N_{n_0-1}$ respectively by $e^a, e^{2a}, \cdots, e^{(n_0-1)a}$, conserving the other values of $N_n(n\geqq n_0)$. This new sequence will be denoted by $\{\overline{N}_n\}$. The function $\overline{N}^c(x)$ is formed by means of the $\overline{N}_n$ as $N^c(x)$ is formed by means of the N_n.

Let $\{N_n\}$ and $\{m_n\}$ be two sequences of positive quantities with $m_n\geqq 1(n\geqq 1)$ and let $\alpha>0$. Let, on the other hand, $\{\mu_n\}$ be a sequence of positive increasing quantities. If this sequence is finite, or if

$$\mu_n \geqq 2^{1/2} \max_{1\leq k\leq m_n} (C_{m_n}^k(2k)!(N^c(4k/\alpha))^{\alpha/2})^{1/2k} \qquad (n\geqq 1),$$

we shall say that $\{\mu_n\}$ is *an* $S(\{N_n\}, \{m_n\}, \alpha)$ *sequence*. If the sequence $\{\mu_n\}$ is finite, or if

$$\mu_n \geqq 2m_n(\overline{N}^c(4n/\alpha))^{\alpha/4n} \qquad (n\geqq 1),$$

we shall say that $\{\mu_n\}$ is *an* $\overline{S}(\{N_n\}, \{m_n\}, \alpha)$ *sequence*. If $\{\lambda_n\}$ is a sequence of positive increasing quantities of upper density $D(0\leqq D<\infty)$, formed by the union of two sequences of positive increasing quantities $\{\mu_n\}$, $\{\nu_n\}$, without common term, the excess distribution of $\{\nu_n\}$ being nonpositive, the sequence $\{\mu_n\}$ being an $S(\{N_n\}, \{m_n\}, \alpha)$ sequence, we shall say that $\{\lambda_n\}$ is *a* $\Sigma(D, \{N_n\}, \{m_n\}, \alpha)$ *sequence*. If the sequence $\{\mu_n\}$ is an $\overline{S}(\{N_n\}, \{m_n\}, \alpha)$ sequence, the sequence $\{\nu_n\}$ being defined as above, the upper density of $\{\lambda_n\}$ being D, we shall say that $\{\lambda_n\}$ is *a* $\overline{\Sigma}(D, \{N_n\}, \{m_n\}, \alpha)$ *sequence*.

The remarks made above together with Lemmas IV and V enable us to state the following lemmas, of which IV_1 and V_1 are other forms of IV and V respectively and are, thus, obvious.

LEMMA IV_1. *Let* $\{N_n\}$ *be a sequence of positive quantities, let* $\{m_n\}$ *be a sequence of positive increasing integers with* $m_n \geqq n, \sum m_n^{-1} \leqq 1$. *The excess distribution of every* $\Sigma(D, \{N_n\}, \{m_n\}, \alpha)$ *sequence* $(\alpha > 0,\ 0 \leqq D < \infty)$ *belongs to* $A(u) = (\text{g.l.b.}_{n \geqq 1} N_n u^n)^{-\alpha}$.

LEMMA V_1. *If* $\{N_n\}, \{m_n\}, \alpha, D$ *are defined as in Lemma* IV_1, *the excess distribution of every* $\overline{\Sigma}(D, \{N_n\}, \{m_n\}, \alpha)$ *sequence belongs to* $A(u) = (\text{g.l.b.}_{n \geqq 1} N_n u^n)^{-\alpha}$.

LEMMA VI. *Let* $n(x)$ *be the excess distribution of* $\{\lambda_n\}$. *If* $n(x) < A < \infty$ $(A > 0)$, $n(x)$ *belongs to* $A(u) = u^{-2A-1}$. *If* $n(x) = O(x^\alpha)$ *with* $0 < \alpha < 1$, $n(x)$ *belongs to* $A(u) = e^{u^{\alpha/(\alpha-1)}}$.

If the first hypothesis of Lemma VI is satisfied, we have, since $n(x) \leqq 0$, for $x \leqq \lambda_1$,

$$2r^2 \int_0^\infty (n(x)/x(x^2 + r^2))dx \leqq 2Ar^2 \int_{\lambda_1}^\infty dx/x(x^2 + r^2) < N_0 + 2A \log r.$$

It follows then from (1) that

$$\Lambda^*(u) \leqq \int_0^\infty e^{-\pi ur + N_0 + 2A \log r}\, dr = Mu^{-2A-1} \qquad (u > 0).$$

That is to say $\Lambda^*(uM^{1/(1+2A)}) \leqq u^{-2A-1}$.

Let us now suppose that $n(x) < Bx^\alpha (0 < \alpha < 1)(B > 0)$. We have then for $r > 0$

$$\begin{aligned} 2r^2 \int_0^\infty (n(x)/x(x^2 + r^2))dx &\leqq 2Br^2 \int_{\lambda_1}^\infty dx/x^{1-\alpha}(x^2 + r^2) \\ &< 2Br^2 \int_0^\infty dx/x^{1-\alpha}(x^2 + r^2) \\ &= (B\pi/\cos((1 - \alpha)\pi/2))r^\alpha = Cr^\alpha. \end{aligned}$$

It follows then from (1) that for u sufficiently small

$$\Lambda^*(u) \leqq \int_0^\infty e^{-\pi ur + Cr^\alpha} dr = (1/\pi u) \int_0^\infty e^{-t + C(t/\pi u)^\alpha} dt. \tag{54}$$

There exists a constant M such that for u sufficiently small $(u > 0)$, $t \geqq 1$,

$$e^{C(t/\pi u)^\alpha} < M \max_{1 \leqq n} P^n t^n n^{-n/\alpha},$$

where $P = (C\alpha e)^{1/\alpha}/\pi u$, and therefore

$$e^{C(t/\pi u)^\alpha} < M \sum_{n=1}^\infty P^n t^n n^{-n/\alpha}.$$

It follows from this inequality and from (54) that

$$\Lambda^*(u) \leqq (M/\pi u)\int_1^\infty e^{-t}\left(\sum_1^\infty P^n t^n n^{-n/\alpha}\right)dt + (1/\pi u)\int_0^1 e^{-t+C(t/\pi u)^\alpha}dt$$

$$\leqq (M/\pi u)\sum (P^n n^{-n/\alpha})\int_0^\infty e^{-t}t^n dt + (1/\pi u)e^{C/(\pi u)^\alpha}$$

$$\leqq (M/\pi u)\sum n^{n(\alpha-1)/\alpha}S^n(\pi u)^{-n} + (1/\pi u)e^{C/(\pi u)^\alpha},$$

where S is a positive constant. But since

$$R(z) = \sum n^{n(\alpha-1)/\alpha}z^n$$

is an entire function of finite order, we have by a well known theorem [11], as $r\to\infty$,

$$\log \max_{|z|=r} |R(z)| = \log R(r) \backsim \log \max_{n\leqq 1} (r^n n^{n(\alpha-1)/\alpha}) \backsim Kr^{\alpha/(1-\alpha)},$$

where K is a constant. And this shows that for u sufficiently small $\log \Lambda^*(u) < K_1 u^{\alpha/(\alpha-1)}$ when K_1 is a constant, that is to say $\log \Lambda^*(uK_1^{(1-\alpha)/\alpha}) < u^{\alpha/(\alpha-1)}$. Our lemma is thus proved.

7. The next few lemmas will specify conditions that the inequalities of the type (10) be fulfilled. The first of these lemmas (Lemma VII) will enable us, by passing through Theorem I, to prove important theorems on Dirichlet series.

LEMMA VII. *If the Dirichlet series $\sum_{m=1}^\infty d_m e^{-\lambda_m s}$ has an abscissa of absolute convergence, σ_A, then for every $\sigma \geqq \sigma' > \sigma_A$ the function $F(s)$ represented by this series satisfies the inequalities*

$$\left|F(s) - \sum_{m=1}^n d_m e^{-\lambda_m s}\right| \leqq A(\sigma')e^{\lambda_n(\sigma'-\sigma)} \qquad (n \geqq 1), \tag{55}$$

where

$$A(\sigma') = \sum_{m=1}^\infty |d_m| e^{-\lambda_m \sigma'}.$$

We have indeed for $\sigma \geqq \sigma'$

$$\left|F(s) - \sum_{m=1}^n d_m e^{-\lambda_m s}\right| \leqq \sum_{m=n+1}^\infty |d_m| e^{-\lambda_m s} = e^{-\lambda_n \sigma}\sum_{n+1}^\infty |d_m| e^{-(\lambda_m-\lambda_n)\sigma}$$

$$\leqq e^{-\lambda_n\sigma}\sum_{n+1}^\infty |d_m| e^{-(\lambda_m-\lambda_n)\sigma'}$$

$$\leqq \left(\sum_{m=1}^\infty |d_m| e^{-(\lambda_m-\lambda_n)\sigma'}\right)e^{-\lambda_n\sigma} = A(\sigma')e^{\lambda_n(\sigma'-\sigma)}.$$

REMARK. *The relationship* (55) *is of type* (10) *where* $\phi_n(x)=A(\sigma')e^{\lambda_n\sigma'}x^{-\lambda_n}$.

Other cases of relationships of form (10) are furnished by the analysis of classes of infinitely differentiable functions. The following lemmas, which concern such classes of functions, will be useful for establishing theorems on quasi-analyticity or even on far more general properties of classes of functions. As usual we shall denote by $C\{M_n\}$ a class of infinitely differentiable functions defined either on a finite closed interval $I\equiv[a, b]$, or on an interval $I\equiv[a, \infty)$, $I\equiv(-\infty, b]$, $I\equiv(-\infty, \infty)$, and such that on I $|f^{(n)}(x)|<K^nM_n$ $(n\geqq 1)$, where K is a finite constant depending on $f(x)$. We shall also denote by $\overline{C}\{M_n\}$ the class of infinitely differentiable functions on $[a, \infty)$ such that to every $f(x)$ belonging to $\overline{C}\{M_n\}$ there corresponds a finite constant g such that $\int_0^\infty|f^{(n)}(x)|dx<g^nM_n$ $(n\geqq 1)$ and a finite constant A such that $\int_0^\infty|f(x)|dx<A$. If $f\in C\{M_n\}$ the constant K defined above will be denoted by $K(f)$. If $f\in\overline{C}\{M_n\}$ the constants g, A defined above will be denoted respectively by $g(f)$, $A(f)$.

Let $\{p_k\}$ be a sequence of positive increasing integers. We shall denote by $\{q_n\}=\text{com }\{p_k\}$ the sequence of increasing non-negative integers distinct from the integers $\{p_k\}$. Each of the sequences $\{p_k\}$, $\{q_n\}$ may be finite or infinite, but obviously $\{q_n\}$ contains at least one term. As in §2 we shall write $\{\lambda_n\}$ $(1\leqq n<N\leqq\infty)$ which means that $\{\lambda_n\}$ is finite if $N<\infty$, and is infinite if $N=\infty$. In any case, $N\leqq\infty$, we shall understand by λ_N the value ∞. This convention is made for the following reason: we shall have to write expressions of the form $\text{g.l.b.}_{\lambda_n\leqq q<\lambda_{n+1}}A_q(r)$ $(1\leqq n<N)$; if $N=\infty$ the meaning of this expression is clear, but if $N<\infty$, that is to say if $\{\lambda_n\}$ is finite, the preceding expression for the last value of n, $n=N-1$, is equivalent, with our convention, to the expression $\text{g.l.b.}_{\lambda_{N-1}\leqq q<\infty}A_q(r)$.

LEMMA VIII. *Let* $f(x)\in\overline{C}\{M_n\}$ *in* $[0, \infty)$ *and let, for a sequence* $\{p_k\}$ *of positive integers,* $f^{(p_k)}(0)=0$. *Let* $\{q_n\}=\text{com }\{p_k\}$ $(1\leqq n<N\leqq\infty)$, *and* $\lambda_n=q_n+1$ $(1\leqq n<N\leqq\infty)$. *Let us set*

$$F(s)=\int_0^\infty e^{-xe^s}f(x)dx \qquad (s=\sigma+it). \tag{56}$$

The function $F(s)$ *is defined, holomorphic and bounded in the horizontal strip* $(-\infty<\sigma<\infty)$, $|t|<\pi/2$, *and satisfies there the inequalities*

$$|F(s)|<A \qquad (A=A(f)), \tag{57}$$

$$\left|F(s)-\sum_{m=1}^{n}d_me^{-\lambda_ms}\right|\leqq\phi_n(e^{-\sigma}) \qquad (1\leqq n<N), \tag{58}$$

where $d_m=f^{(q_m)}(0)$, *and where*

$$\phi_n(x)=\underset{\lambda_n\leqq q<\lambda_{n+1}}{\text{g.l.b.}}\ g^qM_qx^q \qquad (g=g(f);\ 1\leqq n<N). \tag{59}$$

LEMMA IX. *Let* $f(x)\in C\{M_n\}$ *in* $[0, \infty)$. *Let* $f^{(p_k)}(0)=0$, $\{q_n\}=\text{com}\{p_k\}$ $(1\leqq n<N\leqq\infty)$, $\lambda_n=q_n+1$ $(1\leqq n<N)$. *Let* $F(s)$ *be defined by* (56). $F(s)$ *is defined and holomorphic in the horizontal strip* $(-\infty<\sigma<\infty)$, $|t|<\pi/2$, *and for every* δ $(0<\delta<1)$ *the inequalities* (58) *are satisfied in the region* $\Delta(\delta)$ *given by* $\cos t>\delta e^{-\sigma}$, $\sigma>\log\delta$ $(|t|<\pi/2)$ *with* $d_m=f^{(q_m)}(0)$ *and with*

$$\phi_n(x) = \delta^{-1} \underset{\lambda_n\leqq q<\lambda_{n+1}}{\text{g.l.b.}} K^qM_qx^q \quad (K = K(f))(1 \leqq n <N). \tag{60}$$

In $\Delta(\delta)$, *moreover, the following inequality is satisfied*:

$$|F(s)| \leqq \delta^{-1}|f(0)| + \delta^{-2}KM_1. \tag{61}$$

If to the conditions on $f(x)$ *we add the inequality* $|f(x)|\leqq M$ $(0\leqq x<\infty)$ *then* (61) *can be replaced by*

$$|F(s)| < \delta^{-1}M. \tag{62}$$

We give a proof common to both lemmas. If $\Re(\zeta)>0$, $n\geqq 0$, we have in either case (Lemmas VIII, IX) $f^{(n)}(x)e^{-x\zeta}\to 0$ as $x\to\infty$. This is obvious, if $f(x)\in C\{M_n\}$, for $n\geqq 1$; and since

$$|f(x)| \leqq |f(0)| + \int_0^x |f'(t)|\,dt \leqq |f(0)| + KM_1x \qquad (K = K(f)), \tag{63}$$

we have also $f(x)e^{-x\zeta}\to 0$ as $x\to\infty$, $\Re(\zeta)>0$. If $f(x)\in\overline{C}\{M_n\}$, we have, for $n\geqq 0$, $|f^{(n)}(x)|\leqq|f^{(n)}(0)| + \int_0^x|f^{(n+1)}(t)|\,dt\leqq|f^{(n)}(0)| + \int_0^\infty|f^{(n+1)}(x)|\,dx\leqq|f^{(n)}(0)| + g^{n+1}M_{n+1}$ $(g=g(f))$, therefore, here too, we have, for every $n\geqq 0$, $f^{(n)}(x)e^{-x\zeta}\to 0$, as $x\to\infty$, $\Re(\zeta)>0$. Let us put $\zeta=e^s$, $F(s)=\Phi(\zeta)$.

From

$$\Phi(\zeta) = \int_0^\infty e^{-x\zeta}f(x)\,dx \qquad (\Re(\zeta) > 0) \tag{64}$$

it follows, by integration by parts q times, since $f^{(n)}(x)e^{-x\zeta}\to 0$ as $x\to\infty$, $\Re(\zeta)>0$, that

$$\begin{aligned}\Phi(\zeta) &= f(0)/\zeta + f'(0)/\zeta^2 + \cdots + f^{(q-1)}(0)/\zeta^q \\ &\quad + (1/\zeta)^q\int_0^\infty f^{(q)}(x)e^{-x\zeta}\,dx \qquad (\Re(\zeta) > 0).\end{aligned} \tag{65}$$

If $q_n+1=\lambda_n\leqq q<\lambda_{n+1}=q_{n+1}+1$, (65) gives, since $f^{(p_k)}(0)=0$ $(k\geqq 1)$,

$$\Phi(\zeta) = \sum_{m=1}^{n} f^{(q_m)}(0)/\zeta^{\lambda_m} + (1/\zeta)^q\int_0^\infty f^{(q)}(x)e^{-x\zeta}\,dx \qquad (\Re(\zeta) > 0).$$

That is to say for $\Re(\zeta)>0$,

$$(66)\qquad \left|\Phi(\zeta)-\sum_{m=1}^{n} f^{(q_m)}(0)/\zeta^{\lambda_m}\right| = (1/|\zeta|)^q \left|\int_0^\infty f^{(q)}(x)e^{-x\zeta}dx\right|.$$

The function $\Phi(\zeta)$ is obviously holomorphic for $\mathcal{R}(\zeta)>0$. If $f(x)\in\overline{C}\{M_n\}$ it follows immediately from (64) and (66) that for $\mathcal{R}(\zeta)>0$,

$$(67)\qquad |\Phi(\zeta)| \leqq \int_0^\infty |f(x)|\,dx = A(f),$$

$$(68)\qquad \left|\Phi(\zeta)-\sum_{m=1}^{n} f^{(q_m)}(0)/\zeta^{\lambda_m}\right| \leqq g^q M_q/|\zeta|^q \qquad (\lambda_n \leqq q < \lambda_{n+1}).$$

If $f(x)\in C\{M_n\}$ it follows from (64) and (63) that for $\mathcal{R}(\zeta)\geqq\delta>0$,

$$(69)\qquad \begin{aligned} |\Phi(\zeta)| &\leqq \int_0^\infty |f(x)|\,e^{-\delta x}dx = \int_0^\infty (|f(0)| + KM_1x)e^{-\delta x}dx \\ &= \delta^{-1}|f(0)| + \delta^{-2}KM_1 \qquad (K = K(f)), \end{aligned}$$

and if we suppose, moreover, that $|f(x)|\leqq M(x\geqq 0)$, then

$$|\Phi(\zeta)| \leqq \int_0^\infty |f(x)|\,e^{-\delta x}dx \leqq \delta^{-1}M \qquad (\mathcal{R}(\zeta)\geqq\delta>0).$$

It follows from (66) that if $f(x)\in C\{M_n\}$ then, for $\mathcal{R}(\zeta)\geqq\delta>0$,

$$(70)\qquad \left|\Phi(\zeta)-\sum_{m=1}^{n} f^{(q_m)}(0)/\zeta^{\lambda_m}\right| \leqq K^q M_q/\delta|\zeta|^q \qquad (\lambda_n \leqq q < \lambda_{n+1}).$$

It follows from (68), if $f(x)\in\overline{C}\{M_n\}$, and from (70), if $f(x)\in C\{M_n\}$, that, in both cases,

$$\left|\Phi(\zeta)-\sum_{m=1}^{n} f^{(q_m)}(0)/\zeta^{\lambda_m}\right| \leqq \phi_n(1/|\zeta|),$$

with $\phi_n(x)$ given by (59) and for $\mathcal{R}(\zeta)>0$, if $f(x)\in\overline{C}\{M_n\}$; with $\phi_n(x)$ given by (60) and for $\mathcal{R}(\zeta)\geqq\delta>0$, if $f(x)\in C\{M_n\}$. Since by the transformation $\zeta=e^s$ the half-plane $\mathcal{R}(\zeta)>0$ is mapped on the horizontal strip $(-\infty<\sigma<\infty)$, $|t|<\pi/2$, and the half-plane $\mathcal{R}(\zeta)\geqq\delta>0$ on $\Delta(\delta)$, Lemmas VIII and IX are proved.

8. The following lemma will serve to estimate the terms of the sequence $\{\Lambda_n\}$ associated with a given sequence $\{\lambda_n\}$. This estimate will be made when complementary conditions on the sequence $\{\lambda_n\}$ are imposed. The lemma which follows will allow us to establish, by means of the main theorem, precise theorems on singularities of functions defined by a Dirichlet series.

Lemma X. *If the upper density of the sequence $\{\lambda_n\}$ is $D<\infty$, and if*

$$(71)\qquad \liminf_{n=\infty}(\lambda_{n+1}-\lambda_n) = h > 0,$$

then to every $\epsilon>0$ *there corresponds an* $A(\epsilon)$ *such that, in denoting by* $\{\Lambda_n\}$ *the sequence associated with* $\{\lambda_n\}$, *we have*

$$\Lambda_n \leqq A(\epsilon)e^{(B(D,h)+\epsilon)\lambda_n} \qquad (n \geqq 1),$$

where

$$B(D, h) = 3D(6 - \log hD), \quad \text{if} \quad D > 0, \qquad B(0, h) = 0. \tag{72}$$

The function $\Lambda(z)$ being given by (2) it follows from a theorem of Ostrowski, of which a proof was also given by Vladimir Bernstein [1, p. 267], that in the region which is outside the circles $|z\pm\lambda_n| = q_1$, where q_1 is a constant, and outside the circle $|z| = R(\epsilon')$, the following inequality is satisfied:

$$|\Lambda(z)| > e^{-(B(D,h)+\epsilon')|z|} \qquad (\epsilon' > 0),$$

with $B(D, h)$ defined by (72).

Let us note that from the definition of D and h it follows that $Dh\leqq 1$, thus $B(D, h)>0$, if $D>0$. On chosing $q_1<h$, we see that for n sufficiently large the closed circle $\overline{C}(\lambda_n, q_1)$ given by $|z-\lambda_n|\leqq q_1$ contains no λ_k with $k\neq n$, and therefore, in this circle

$$|(z - \lambda_n)/\Lambda(z)| \leqq q_1 e^{(B(D,h)+\epsilon')(\lambda_n+q_1)}$$

and

$$\lambda_n\Lambda_n = 1/\Lambda'(\lambda_n) = \lim_{z=\lambda_n} |(z - \lambda_n)/\Lambda(z)| < q_1 e^{(B(D,h)+\epsilon')(\lambda_n+q_1)}.$$

Our lemma follows immediately from this inequality.

The following lemma will not be used in this paper, but it seems interesting to state it since it furnishes an estimate of the Λ_n with conditions on the λ_n much more general than those of the preceding lemma. Of course the estimate of Λ_n is then much less precise.

LEMMA XI. *If the upper density of the sequence* $\{\lambda_n\}$ *is finite, and if there exists a finite positive quantity* μ, *such that for n sufficiently large*

$$\lambda_{n+1} - \lambda_n > \lambda_n^{-\mu}, \tag{73}$$

then to every $\epsilon>0$ *there corresponds a quantity* $A(\epsilon)$ *such that*

$$\Lambda_n \leqq A(\epsilon)e^{\lambda_n^{1+\epsilon}} \qquad (1 \leqq n).$$

Let $p_n=\lambda_n^2$. Since the upper density of $\{\lambda_n\}$ is finite, there exists a constant $\lambda>0$ such that $p_n>\lambda n^2$, and the exponent of convergence of $\{p_n\}$ is not larger than 1/2(6). By a classical theorem of Borel on canonical products [11, p. 57], in the region which is outside of the circles $|z-p_n| = p_n^{-h}$ with $h>1/2$, $p_n>1$, the following inequality is satisfied:

(6) That is to say $\sum p_n^{-\alpha}$ converges with arbitrary $\alpha>1/2$.

$$\log \left| \prod_{n=1}^{\infty} (1 - \zeta/p_n) \right| > - |\zeta|^{1/2+\epsilon'}$$

with any $\epsilon' > 0$, if $|\zeta| > r(\epsilon')$.

Let $\nu = \max(2, \mu)$. Since for $n > n_0$, $\lambda_{n+1} - \lambda_n > \lambda_n^{-\nu}$, we have for $n > n_0 + 1$, $p_{n+1} - p_n > (\lambda_{n+1} + \lambda_n)\lambda_n^{-\nu} > 2p_n^{(1-\nu)/2}$ and $p_n - p_{n-1} > 2p_{n-1}^{(1-\nu)/2} > 2p_n^{(1-\nu)/2}$ $(p_n > 1)$. In other words the closed circle $\overline{C}(p_n, p_n^{(1-\nu)/2})$, given by $|\zeta - p_n| \leqq p_n^{(1-\nu)/2}$, contains no p_k with $k \neq n$. To every $\epsilon' > 0$ there corresponds therefore, by Borel's theorem, a quantity $n(\epsilon')$ such that for $n > n(\epsilon')$ we have on the circumference of $C(p_n, p_n^{(1-\nu)/2})$

$$\log \left| \prod_{k=1}^{\infty}{}' (1 - \zeta/p_k) \right| > - (p_n + p_n^{(1-\nu)/2})^{1/2+\epsilon'} > - (2p_n)^{1/2+\epsilon'} > - \lambda_n^{1+3\epsilon'}.$$

On the circumference of this circle we have then

$$\left| (\zeta - p_n) / \prod_{k=1}^{\infty}{}' (1 - \zeta/p_k) \right| \leqq p_n^{(1-\nu)/2} e^{\lambda_n^{1+3\epsilon'}}.$$

And if $\Lambda(z)$ is given by (2), we have for $n > n(\epsilon')$

$$\lim_{\zeta = p_n} |(\zeta - p_n)| / \prod_{k=1}^{\infty}{}' (1 - \zeta/p_k) = \lim_{z = \lambda_n} |(z^2 - \lambda_n^2)/\Lambda(z)| = 2\lambda_n^2 \Lambda_n \leqq \lambda_n^{1-\nu} e^{\lambda_n^{1+3\epsilon'}}$$

which proves our lemma.

9. In this section we shall use some of our lemmas in order to obtain results following from Theorem I but with hypotheses more specific than those of that theorem. However Theorem II is an immediate corollary of Theorem I and no use of the proved lemmas is required for its proof. The theorems of this section are all useful for the applications to come in the next sections.

Let $s(u) = \sigma(u) + it(u)$ be a continuous complex function of the real variable $u(-\infty < u < \infty)$. Let us suppose that for $u \geqq u_0$, $t(u) = t_0$ (constant), and that $\sigma(u) \to -\infty$ as $u \to -\infty$, and let $R > 0$ (constant). The region

$$\sum (s(u), R) = \bigcup_{-\infty < u < \infty} C(s(u), R),$$

formed by the union of the open circles with centers at $s(u)$ and radius R will be called a *curvilinear strip of width $2R$, horizontal at the right and extending to $-\infty$ at the left.* The curve composed of the points of which the affixes are given by $s(u)$ will be called the *central line of the strip.*

THEOREM II. *Let $\{\lambda_n\}$, $\{\phi_n(x)\}$, $\{d_n\}$, D, $\phi(x)$ have the same meaning as in Theorem* I. *Let $F(s)$ be a holomorphic and bounded function in a curvilinear strip of width $2\pi a$, horizontal at the right and extending to $-\infty$ at the left, and let in this region, for $\sigma > \sigma_0$,*

$$(74)\qquad \left|F(s) - \sum_{m=1}^{n} d_m e^{-\lambda_m s}\right| \leqq \phi_n(e^{-\sigma}) \qquad (1 \leqq n < N \leqq \infty).$$

If $D<a$, and if, in setting $\omega=(2(a-D))^{-1}$ one of the following two conditions is satisfied:

(I) *The excess distribution of $\{\lambda_n\}$ belongs to a function $A(u)$ such that*

$$(75)\qquad A(u^\omega)\phi(u) = O(1) \qquad (u \to 0),$$

$$(76)\qquad \liminf_{t=0+} \int_t^1 \log (A(u^\omega)\phi(u))u^{\omega-1}du = -\infty.$$

(II) *There exists a constant $\omega'>\omega$ such that*

$$(77)\qquad \int_0^1 \log \phi(u)u^{\omega'-1}du = -\infty;$$

then $d_n=0$ $(n\geqq 1)$, $F(s)=0$ identically.

It follows indeed from Theorem I that (14) is satisfied for every $n(1\leqq n<N)$ with $\Re(s_1)$ negative and numerically arbitrarily large. Thus $d_n=0(1\leqq n<N)$. On the other hand the functon $F_0(s)\equiv 0$ satisfies (74) with $d_m=0$; thus by Theorem I, $F(s)\equiv F_0(s)\equiv 0$.

Remark. The significance of Theorem II becomes clearer if we recall Lemma III, which gives conditions on $A(u)$ under which a sequence $\{\lambda_n\}$ exists with given upper density D and of which the excess distribution, $n(x)$, belongs to $A(u)$, $n(x)$ tending to infinity with x. This is particularly interesting if $D=0$, since then it shows that there exists a sequence $\{\lambda_n\}$ with upper density zero (density zero), which has nevertheless an infinity of elements, and which belongs to $A(u)$.

In all the theorems to come D is the upper density of $\{\lambda_n\}$, $0\leqq D<\infty$, $n(x)$ is the excess distribution of $\{\lambda_n\}$, $\{M_q\}$ is an infinite sequence of positive quantities, $T(r)=\text{l.u.b.}_{q\geqq 1}\, r^q/M_q$, $r>0$, $\{m_q\}$ an infinite sequence of positive increasing integers.

Theorem III. *Let $F(s)$ be holomorphic and bounded in the horizontal strip $|t|<\pi a$ $(-\infty<\sigma<\infty)$, let in this strip, for $\sigma>\sigma_0$,*

$$(78)\qquad \left|F(s) - \sum_{m=1}^{n} d_m e^{-\lambda_m s}\right| \leqq M_q e^{-q\sigma} \qquad (1 \leqq n < N,\ \lambda_n \leqq q < \lambda_{n+1}),$$

and let $D<a$[7]. The function $F(s)$ is identically zero and $d_n=0$ $(1\leqq n<N)$ if one of the following conditions is satisfied with $\omega=(2(a-D))^{-1}$:

(I)

$$(79)\qquad n(x) = O(1),$$

[7] We recall that if the sequence $\{\lambda_n\}$ is finite, that is to say if $N<\infty$, we put $\lambda_N=\infty$.

$$\int_1^\infty \log T(r) r^{-(1+\omega)} dr = \infty. \tag{80}$$

(II) *The condition* (80) *is satisfied and*

$$n(x) = O(x^\alpha) \qquad (0 < \alpha < 1/2), \tag{81}$$

$$e^{r^{\omega\alpha/(1-\alpha)}} = O(T(r)) \qquad (r \to \infty). \tag{82}$$

(III) $\{\lambda_n\}$ *is a* $\Sigma(D, \{M_q^\omega\}, \{m_q\}, \alpha)$ *sequence with* $m_q \geqq q$, $\sum m_q^{-1} \leqq 1$, $0 < \alpha < \omega^{-1}$, *and* (80) *is satisfied.*

(IV) $\{\lambda_n\}$ *is a* $\overline{\Sigma}(D, \{M_q^\omega\}, \{m_q\}, \alpha)$ *sequence with* $m_q \geqq q$, $\sum m_q^{-1} \leqq 1$, $0 < \alpha < \omega^{-1}$, *and* (80) *is satisfied.*

(V) *There exists a constant* $\omega' > \omega$ *such that*

$$\int_1^\infty \log T(r) r^{-(1+\omega')} dr = \infty. \tag{83}$$

REMARK. The fundamental condition on $\{M_q\}$ in each of the conditions (I), (II), (III), (IV) is (80). Since (83) is much more restrictive than (80), one sees how much more delicate the results are when conditions on the excess distribution are involved, as in the case (I), (II), (III), (IV). Among the conditions (I), (II), (III), (IV) the condition (I) is the most elementary. The most significant of all the conditions are (III) and (IV). They show that to every $\{M_q\}$ there corresponds (in many ways, depending on the choice of α and $\{m_q\}$) a sequence $\{\lambda_n\}$ of given upper density $D < \infty$, with $n(x)$ tending to infinity with x, and such that the conditions (78) and (80) are sufficient from which to draw the conclusions of the theorem.

Before we pass to the proof of the theorem let us also make the following remarks. Theorem III stated with the condition (I) contains Carleman's theorem on Watson's problem [2, 5]. Indeed if we put $a = 1/2$, $D = 0$, $N = 2$ (thus $n(x) \leqq 1$), $d_1 = 0$, $\lambda_1 = 1$, $z = e^s$, $F(s) = \Phi(z)$, we see that, if for $\mathcal{R}(z) > 0$

$$|\Phi(z)| < M_q / |z|^q \qquad (q \geqq 1), \tag{84}$$

with

$$\int_1^\infty \log T(r) r^{-2} dr = \infty, \tag{85}$$

$\Phi(z) \equiv 0$, which constitutes Carleman's theorem in a form given by Ostrowski [10], on Watson's problem.

But let us also note that the theorem stated with each of the conditions (III) or (IV) constitutes a fundamentally much more general theorem than Carleman's. Indeed, once more putting $a = 1/2$, $D = 0$, we see that to every sequence $\{M_q\}$ there correspond *infinite* sequences $\{\lambda_n\}$ which are $\Sigma(0, \{M_q\},$

$\{m_q\}, \alpha)$ sequences (or, what amounts to the same thing, $S(\{M_q\}, \{m_q\}, \alpha)$ sequences) and $\overline{\Sigma}(0, \{M_q\}, \{m_q\}, \alpha)$ sequences (or, what amounts to the same thing, $\overline{S}(\{M_q\}, \{m_q\}, \alpha)$ sequences) such that if for $\Re(z)>0$ ($|\arg z| <\pi/2$)

$$\left| \Phi(z) - \sum_{m=1}^{n} d_m z^{-\lambda_m} \right| \leqq M_q/|z|^q \qquad (\lambda_n \leqq q < \lambda_{n+1}), \tag{86}$$

and if (85) holds then $\Phi(z)\equiv 0$.

We note further that Theorem III stated with condition (V) contains, as a particular case, a result established in collaboration with F. E. Ulrich [9, Lemma VII], and which is by itself a generalization of a result of the author [6]. The result in [9] corresponds to Theorem III (with condition (V)) with $a=1/2$ but where (83) is replaced by a more restrictive condition which may be stated in the following manner: there exists an $\omega'>\omega$ such that

$$\int_1^\infty \log T_1(r) r^{-(1+\omega')} dr = \infty, \tag{87}$$

where $T_1(r) = \text{l.u.b.}_{n\geqq 1}\, r^{\lambda_n}/M_{\lambda_n}$ (the λ_n being integers). Since obviously $T_1(r) \leqq T(r)$, (87) is more restrictive than (83).

Let us now pass to the proof of Theorem III. We shall set $\phi_n(x) = \text{g.l.b. } M_q x^q$ ($\lambda_n \leqq q < \lambda_{n+1}$). It is then obvious that $\phi(x) = \text{g.l.b.}_{1\leqq n<N}\, \phi_n(x) = (T(1/x))^{-1}$. It is also obvious that $\{\phi_n(x)\}$ is an asymptotic sequence. Let us note that (80) and (83) are equivalent respectively to

$$\int_0^1 \log \phi(u) u^{\omega-1}\, du = -\infty, \tag{88}$$

$$\int_0^1 \log \phi(u) u^{\omega'-1} du = -\infty. \tag{89}$$

From Lemma VI it follows that if (79) is satisfied then $n(x)$ belongs to u^{-B} where $0<B<\infty$. By the definition of $\phi_n(x)$ given above it is obvious that, if we put $A(u)=u^{-B}$, it follows from (88) that the conditions (75) and (76) of Theorem II are satisfied. Thus $F(s)\equiv 0$, $d_m=0$ if (I) of Theorem III is satisfied, since the other conditions of Theorem II are clearly satisfied.

It follows also from Lemma VI that if (81) is satisfied then $n(x)$ belongs to $A(u)=e^{u^{\alpha/(\alpha-1)}}$. Thus, by (82), we have $A(u^\omega)\phi(u) = e^{u^{\omega\alpha/(\alpha-1)}}\phi(u) = O(1)$ ($u\to 0$), that is to say (75) is valid; since, if $0<\alpha<1/2$,

$$\int_0^1 u^{\omega\alpha/(\alpha-1)} u^{\omega-1} du < \infty,$$

we see that if (80) holds (that is if (88) holds) (76) holds also. Once more the conditions of Theorem II are satisfied, and $F(s)\equiv 0$, $d_n=0$.

Let us now suppose the condition (III) satisfied. The upper density of $\{\lambda_n\}$ is then D and, by Lemma IV, its excess distribution belongs to $A(u) = (\text{g.l.b.}_{q\geq 1} M_q^\omega u^q)^{-\alpha}$ $(0<\alpha<\omega^{-1})$. Since

$$A(u^\omega) = (\underset{q\geqq 1}{\text{g.l.b.}}\ M_q^\omega u^{\omega q})^{-\alpha} = (\phi(u))^{-\beta} \qquad (\beta = \alpha\omega < 1)$$

we see that from (80), that is from (88), it follows that

$$\int_0^1 \log (A(u^\omega)\phi(u))u^{\omega-1}du = (1-\beta)\int_0^1 \log \phi(u)u^{\omega-1}du = -\infty,$$

and (76) is satisfied. It is also obvious that $\lim A(u^\omega)\phi(u) = (\phi(u))^{1-\beta}\to 0$ as $u\to 0$, thus (75) is satisfied. The conditions of Theorem II are still satisfied and $F(s)\equiv 0$, $d_n = 0$.

Next let us suppose condition (IV) satisfied. If $\liminf M_n^{1/n} < \infty$, then, as we have already seen on page 111, $\phi(u) = 0$ for u sufficiently small, therefore (75) and (76) are satisfied, and once more, by Theorem II, $F(s)\equiv 0$, $d_n = 0$. Let us then suppose $M_q^{1/q}\to\infty$. We write $\min_{q\geq 1} (\log M_q)/q = (\log M_{q_0})/q_0 = b$ and set $\overline{M}_q = e^{bq}$ if $1\leqq q\leqq q_0$, $\overline{M}_q = M_q$ if $q > q_0$. We also write $\max M_q^{1/q}e^{-b} = e^\gamma$ $(q\leqq q_0)$; then $e^{-q\gamma}M_q\leqq e^{bq}$ $(q\leqq q_0)$ and, since obviously $\gamma\geqq 0$, we have generally $e^{-q\gamma}M_q\leqq \overline{M}_q$ $(q\geqq 1)$. Let us then set $F^*(s) = F(s+\gamma)$, $d_m e^{-\lambda_m\gamma} = d_m^*$. The conditions of Theorem III with (IV) are then satisfied if $F(s)$ is replaced by $F^*(s)$, d_m by d_m^*, M_q by $\overline{M}_q$, σ_0 by $\sigma_0+\gamma$. This is obvious for all the conditions except perhaps for (80), but since (80) is equivalent to (88) and since $\phi(u)$, for u sufficiently small $(u<\epsilon)$ depends only on the $\{M_q\}$ with $q>q(\epsilon)$, as we have shown on page 111, we see that (80) is still satisfied if $\{M_q\}$ is replaced by $\{\overline{M}_q\}$. If we set $\overline{N}_q = \overline{M}_q^\omega$ we see that the corresponding function $\overline{N}^c(x)$ (for its definition see page 111) is such that $\log \overline{N}^c(x)/x$ tends to infinity monotonically and

$$2m_n(\overline{N}^c(4n/\alpha))^{\alpha/4n} \geqq 2^{1/2} \max_{1\leqq k\leqq m_n} (C_{m_n}^k(2k)!(\overline{N}^c(4k/\alpha)^{\alpha/2})^{1/2k}.$$

In other words from the fact that $\{\lambda_n\}$ is a $\overline{\Sigma}(D, \{M_q^\omega\}, \{m_q\}, \alpha)$ sequence it follows that it is also a $\Sigma(D, \{\overline{M}_q^\omega\}, \{m_q\}, \alpha)$ sequence. Therefore, taking into account what we have just said, we see that the conditions of Theorem III with (III) are satisfied if $F(s)$, $\{d_n\}$, $\{M_q\}$, σ_0 are respectively replaced by $F^*(s)$, $\{d_n^*\}$, $\{\overline{M}_q\}$, $\sigma_0+\gamma$. Since Theorem III with condition (III) has already been proved we have $F^*(s)\equiv 0$, $d_n^* = 0$, that is to say $F(s)\equiv 0$, $d_n = 0$.

Let us now suppose that the condition (V) is satisfied. Then the condition (89), which is the same as (77), is satisfied, and obviously the condition (II) of Theorem II is satisfied, and once more, by Theorem II, $F(s)\equiv 0$, $d_n = 0$. Theorem III is therefore completely proved.

10. We shall now apply our results to the theory of quasi-analyticity.

Let $\{p_k\}$ be a sequence of positive increasing integers. Let $\{q_n\}$ be the sequence of all non-negative integers which are not contained in $\{p_k\}$. $\{q_n\}$ contains at least one term. As in §7 we shall write $\{q_n\} = \text{com}\ \{p_k\}$.

We shall now prove the following theorem.

THEOREM IV. *Let $f(x)$ belong to $\overline{C}\{M_n\}$ in $[0, \infty)$. Let $f^{(p_k)}(0)=0$ $(k \geqq 1)$ for a sequence $\{p_k\}$ of positive increasing integers with lower density $d>1/2$. Let $\{q_n\} = \text{com}\ \{p_k\}$, and let $\lambda_n = q_n+1$. Let $n(x)$ be the excess distribution of $\{\lambda_n\}$.*

The function $f(x)$ is identically zero if one of the following conditions is satisfied with $\omega=(2d-1)^{-1}$:

(I)

$$n(x) = O(1), \tag{90}$$

$$\int_1^\infty \log T(r) r^{-(1+\omega)} dr = \infty. \tag{91}$$

(II) *The condition* (91) *is satisfied and*

$$n(x) = O(x^\alpha) \qquad (0 < \alpha < 1/2), \tag{92}$$

$$e^{r^{\omega\alpha/(1-\alpha)}} = O(T(r)). \tag{93}$$

(III) *$\{\lambda_n\}$ is a $\Sigma(1-d, \{M_q^\omega\}, \{m_q\}, \alpha)$ sequence with $m_q \geqq q$, $\sum m_q^{-1} \leqq 1$, $0<\alpha<\omega^{-1}$, and* (91) *is satisfied.*

(IV) *$\{\lambda_n\}$ is a $\overline{\Sigma}(1-d, \{M_q^\omega\}, \{m_q\}, \alpha)$ sequence with $m_q \geqq q$, $\sum m_q^{-1} \leqq 1$, $0<\alpha<\omega^{-1}$, and* (91) *is satisfied.*

(V) *There exists a constant $\omega'>\omega$ such that*

$$\int_1^\infty \log T(r) r^{-(1+\omega')} dr = \infty. \tag{94}$$

The remark made after the statement of Theorem III should be repeated, mutatis mutandis, here. Let us note that if $d=1$, that is to say if $k/p_k \to 1$ as $k\to\infty$, then $\omega=1$, (91) becomes (85) ($\int_1^\infty \log T(r) r^{-2} dr = \infty$), and $\Sigma(1-d, \{M_q^\omega\}, \{m_q\}, \alpha)$ $(0<\alpha<\omega^{-1})$, $\overline{\Sigma}(1-d, \{M_q^\omega\}, \{m_n\}, \alpha)$ $(0<\alpha<\omega^{-1})$ become respectively $S(\{M_q\}, \{m_q\}, \alpha)$, $\overline{S}(\{M_q\}, \{m_q\}, \alpha)$ $(0<\alpha<1)$. This shows that the particular case of Theorem IV with $d=1$, stated with the most elementary of the four conditions (I), (II), (III), (IV), contains as particular case the theorem of Denjoy-Carleman on quasi-analytic classes of functions. Indeed let $\psi(x)$ belong to $C\{M_q\}$ in $[0, 1]$, and let $\psi^{(n)}(0) = \psi^{(n)}(1) = 0$ $(n \geqq 0)$. The function $f(x)$, defined as follows: $f(x) = \psi(x)$ in $[0, 1]$, and $f(x)=0$ in $(1, \infty)$, belongs obviously to $\overline{C}\{M_q\}$ in $[0, \infty)$, and the conditions (I) of Theorem IV are satisfied for $f(x)$ with $p_k = k$ $(k \geqq 1)$; $\{q_n\}$ is reduced to the single term 0, and $\{\lambda_n\}$ to the single term 1 (thus $n(x) \leqq 1$), $d=1$. Therefore $\psi(x)$ is zero identically if (85) is satisfied. But it is well known from elementary considerations [5] that if every function $\psi(x)$ of $C\{M_q\}$ with $\psi^{(n)}(0)=\psi^{(n)}(1)=0$ $(n\geqq 0)$ is identically zero, then every function $\psi(x)$ of $C\{M_q\}$ with $\psi^{(n)}(0)=0$ $(n\geqq 0)$ is identically zero. Thus the class $C\{M_q\}$ is quasi-analytic if (85) is satisfied. This is the theorem of Denjoy-Carleman [5]. But Theorem IV with the conditions (II), (IV), in which we suppose $d=1$, contains a fundamentally more general theorem than that of Denjoy-Carleman, since to every sequence $\{M_q\}$ there correspond, in an infinity of ways (depending on the choice of the sequence $\{m_q\}$ and the constant $0<\alpha<1$), infinite sequences $\{\lambda_n\}$ with $D=0$ which are $S(\{M_q\}, \{m_q\}, \alpha)$ sequences, as well as sequences $\{\lambda_n\}$ which are $\overline{S}(\{M_q\}, \{m_q\}, \alpha)$ sequences.

We now pass to the proof of Theorem IV. If $\int_0^\infty |f^{(n)}(x)|\,dx\leqq g^n M_n$ $(n\geqq 1)$, let us set $f_1(x)=g^{-1}f(x/g)$. We have then $\int_0^\infty |f_1^{(n)}(x)|\,dx\leqq M_n$ $(n\geqq 1)$. Let us consider the function

$$F(s) = \int_0^\infty e^{-xe^s} f_1(x)dx \qquad (s = \sigma + it). \tag{95}$$

By Lemma VIII, $F(s)$ is holomorphic and bounded in the horizontal strip $(-\infty<\sigma<\infty)$, $|t|<\pi/2$, and (58) is satisfied with $\lambda_n=q_n+1$, $d_m=f^{(q_m)}(0)$ and with $\phi_n(x)=\text{g.l.b.}_{\lambda_n\leqq q<\lambda_{n+1}} M_q x^q$. In other words if the conditions of Theorem IV are satisfied, $F(s)$ satisfies the conditions of Theorem III with $a=1/2$, $D=1-d$, and in such a way that to each of the five conditions of Theorem IV there corresponds the condition of the same number in Theorem III. Thus by Theorem III, $F(s)\equiv 0$ if one of the five conditions of Theorem IV is satisfied. In other words the function

$$\Phi(\zeta) = \int_0^\infty e^{-x\zeta} f_1(x)dx$$

is then identically zero. But it is well known from the theory of Laplace transforms [12] that from $\Phi(\zeta)\equiv 0$ follows $f_1(x)\equiv 0$, that is to say $f(x)\equiv 0$, and Theorem IV is proved.

Theorem III was inferred from Theorem II by using a certain number of lemmas which permitted us to specify conditions under which the hypotheses of Theorem II were satisfied, and Theorem IV follows from Theorem III by means of Lemma VIII. It is obviously possible to infer a theorem concerning quasi-analyticity directly from Theorem II, on using Lemma VIII. This theorem will then have a more general character than Theorem IV, but, on the other hand, it will be much less specific in its hypotheses. The theorem of which the proof is obvious if we combine Theorem II and Lemma VIII is the following:

THEOREM V. *Let $f(x)$ belong to $\overline{C}\{M_q\}$ in $[0, \infty)$ and let $f^{(p_k)}(0)=0$ $(k\geqq 1)$, the lower density d of $\{p_k\}$ being such that $d>1/2$. Let $\{q_n\}$ and $\{\lambda_n\}$ have the same meaning as in Theorem* IV. *Let $\phi(u)=(T(1/u))^{-1}$. The function $f(x)$ is* *identically zero if one of the following conditions is satisfied with* $\omega=(2d-1)^{-1}$:

(I) *The excess distribution of* $\{\lambda_n\}$ *belongs to a function* $A(u)$ *such that* (75) *and* (76) *hold.*

(II) *There exists a constant* $\omega'>\omega$ *such that* (77) *holds*(8).

It is useful here, in order to understand the meaning of this theorem, to recall Lemma III, especially the last part of it, which asserts in effect that for functions $A(u)$ with certain general properties there exists a sequence $\{\lambda_n\}$, with *integers* as elements, of which the excess distribution belongs to $A(u)$. Even if $d=1$, that is to say $D=0$, the sequence $\{\lambda_n\}$ still contains an infinity of elements.

Let us now prove the following theorem.

THEOREM VI. *Let* $f(x)$ *belong to* $C\{M_n\}$ *on* $[0, \infty)$, *and let* $f^{(p_k)}(0)=0$ $(k\geqq 1)$ *for a sequence* $\{p_k\}$ *with lower density* $d>1/2$. *If there exists a constant* $\omega'>(2d-1)^{-1}$ *such that*

$$\int_1^\infty \log T(r) r^{-(1+\omega')}dr = \infty \tag{96}$$

then $f(x)$ *is a linear function of* x.

Obviously, if we suppose moreover that $\int_0^\infty |f(x)|dx<\infty$, then by this theorem $f(x)\equiv 0$. This theorem is of the same nature as Theorem IV with the condition (V) which is in effect the most restrictive of all. As a matter of fact Theorem VI does not contain the Denjoy-Carleman theorem, since if $d=1$, the quantity $1+\omega'$ in (96) is greater than 2. The interest of this theorem consists however in the fact that d may take any value greater than 1/2, $(1/2<d\leqq 1)$. It contains as a particular case Theorem III of [9], in which (96) is replaced by the more restrictive condition (87) with $T_1(r)=\text{l.u.b.}_{n\geqq 1} r^{\lambda_n}/M_{\lambda_n}$.

If $|f^{(n)}(x)|<K^n M_n$ $(n\geqq 1)$, let us set $f_1(x)=f(x/2K)-f'(0)x/2K-f(0)$. Then $f_1(0)=f_1'(0)=f_1^{(p_k)}(0)=0$ $(k\geqq 1)$ and $|f_1^{(n)}(x)|\leqq M_n$ $(n\geqq 1)$. Let us then define $F(s)$ by (95). By Lemma IX, $F(s)$ is holomorphic in the horizontal strip $|t|<\pi/2$ $(-\infty<\sigma<\infty)$, and if $\Delta(\delta)$ has, for $0<\delta<1$, the same meaning as in Lemma IX, we have in this region, by Lemma IX,

$$|F(s)| \leqq \delta^{-1}|f_1(0)| + \delta^{-2}M_1 = \delta^{-2}M_1. \tag{97}$$

$$\left|F(s) - \sum_{m=1}^{n} d_m e^{-\lambda_m s}\right| \leqq \phi_n(e^\sigma) \tag{98}$$

where $\{\lambda_n\}$ and $\{d_n\}$ have the same meaning as in the statement of Lemma IX (f being replaced by f_1), and where $\phi_n(x)$ is given by (60) with $K=K(f_1)=1$.

(8) Condition (I) of this Theorem is more general than each of the conditions (I), (II), (III), (IV) of Theorem IV. Condition (II) of Theorem V is the same as condition (V) of Theorem IV.

There exists a positive quantity ξ such that

$$\pi/2 > \pi c = \cos^{-1}(e^{-\xi}) > \pi(1/2\omega' + D),$$

since the upper density D of $\{\lambda_n\}$ satisfies the relation $0 \leqq D = 1 - d < 1/2$, and $1/2\omega' + D < d + D - 1/2 = 1/2$. The region $D(\delta, \xi)$, given by $\sigma \geqq \log \delta + \xi$, $|t| \leqq \cos^{-1}(e^{-\xi})$, is contained in $\Delta(\delta)$. Since (96) is equivalent to (89) in which $\phi(u) = \text{g.l.b.}_{n \geqq 1} \phi_n(u)$, we see that (97) and (98) are satisfied in $D(\delta, \xi)$, (89) being satisfied.

Thus the conditions of Theorem I, with the condition (II), are satisfied with $a = a_1 = a_2 = c$, $s_1 = \log \delta + \xi + \pi c$, $M(s_1) = \delta^{-2} M_1$.

Thus, by Theorem I, it follows that

$$(99) \qquad |d_n| < L(\xi)\delta^{-2} M_1 \Lambda_n e^{\lambda_n(\log \delta + \xi + \pi c)} = L(\xi)\delta^{\lambda_n - 2} M_1 \Lambda_n e^{-\lambda_n(\xi + \pi c)}.$$

Therefore, in fixing ξ, and fixing n so as to have $\lambda_n > 2$, we see in making $\delta \to 0$ that $d_n = 0$ if $\lambda_n > 2$. If there exists a $\lambda_n \leqq 2$ the corresponding d_n is either $f_1(0)$ or $f_1'(0)$ (since the λ_n are positive integers and $\lambda_n = q_n + 1$), both of these quantities being zero, and we have $d_n = 0$. But since the function $F_0(s) \equiv 0$ also satisfies, in $D(\delta, \xi)$, the inequalities (98), with $d_m = 0$, we see, by Theorem I, that $F(s) \equiv 0$, and therefore by the theory of Laplace transforms $f_1(x) \equiv 0$, that is to say $f(x) = f'(0)x + f(0)$, which proves our theorem. The simple example $f(x) = x$ shows that from the hypotheses of the theorem it does not follow that $f(x) \equiv 0$.

Before closing this section we should note that theorems of the kind just proved, and particularly Theorem IV with condition (II), suggest a study of interesting classes of entire functions of finite order.

11. We shall study in this section the analytic continuation of a function holomorphic in a horizontal half-strip and satisfying, therein, inequalities of form (10).

THEOREM VII. *Let $F(s)$ be not identically zero, holomorphic in a horizontal half-strip $S(V, \pi a)$ of width $2\pi a$ in which* (10) *is satisfied, where $\phi_n(x)$ is an asymptotic sequence. If $D < a$, and if, on setting $\omega = (2(a - D))^{-1}$, one of the conditions* (I) *or* (II) *of Theorem* I *is satisfied, then in every curvilinear strip $\Sigma = \Sigma(s(u), 2\pi a)$ of width $2\pi a$, horizontal at the right and extending to $-\infty$ at the left, the horizontal part coinciding with $S(V, \pi a)$ for $\sigma > \sigma_0$, the analytic continuation of $F(s)$ satisfies one of the three conditions:*

(a) *$F(s)$ admits there at least one singularity.*

(b) *$F(s)$ tends to ∞ as $\sigma \to -\infty$, uniformly with respect to t, $s = \sigma + it$ belonging to the strip $\Sigma(s(u), 2\pi a - \epsilon)$ with ϵ arbitrary such that $0 < \epsilon < 2\pi a$.*

(c) *$F(s)$ takes in Σ each value, except at most two, infinitely many times.*

Suppose that (a) is not satisfied. Let then L be the central line of the strip Σ, let $\{s_i\}$ be a sequence of points on L with $\sigma_i = \Re(s_i) \to -\infty$, let $a > a_1 > D$, and let $|F(s)| \leqq M(s_i)$ in $C(S_i, \pi a_1)$. There exists an $n = n_0$ such that $d_{n_0} \neq 0$, since if this were not true the function $F_0(s)\equiv 0$ would satisfy, in $S(V, \pi a)$, the same inequalities (10) as $F(s)$, and it would follow by Theorem I that $F(s)\equiv 0$, contrary to the hypotheses. By Theorem I, where s_1 is replaced by any s_i, we have

$$M(s_i) \geqq |d_{n_0}| K^{-1}\Lambda_{n_0}^{-1}e^{-\lambda_{n_0}\sigma_i}$$

and since the right-hand member tends to ∞ as $i\to\infty$, we have $M(s_i)\to\infty$. If in $C(s_i, \pi a_1)$, $F(s)$ did not tend uniformly to infinity with i, it would follow, from the theory of normal families, that the family $F_i(s)=F(s+s_i)$ is not normal in $C(0, \pi a)$, and $F_i(s)$ $(i\geqq 1)$ would take in this circle every value, except possibly two, infinitely many times; $F(s)$ would then take these values in $C(s_i, \pi a)$ $(i\geqq 1)$. From this the theorem becomes immediately evident.

From Theorem VII follows immediately the less general but more specific theorem:

THEOREM VIII. *If $F(s)$ is not identically zero, holomorphic in $S(V, \pi a)$, and satisfies there* (78), *if $D<a$, and if one of the conditions* (I), (II), (III), (IV), (V) *of Theorem* III *is satisfied, then in every curvilinear strip Σ described in Theorem* VII, *$F(s)$ satisfies one of the conditions* (a), (b), (c) *of Theorem* VII.

Indeed, as we have seen in the proof of Theorem III, the condition (78) together with one of the five conditions of Theorem III constitutes, by our lemmas, a particular case of the hypotheses of Theorem I.

If $\{d_n\}$ is a sequence of complex numbers, and $\{\lambda_n\}$, as usual, a sequence of positive increasing quantities with $0\leqq D<\infty$, we shall set, if $N=\infty$,

$$\sigma_{d,\lambda} = \limsup_{n=\infty} (\log |d_n| - \log \Lambda_n)/\lambda_n,$$

where $\{\Lambda_n\}$ is the sequence associated with $\{\lambda_n\}$.

The following theorem can now be proved immediately.

THEOREM IX. *Let $\sigma_{d,\lambda}>-\infty$. If $F(s)$ is holomorphic in $S(V, \pi a)$ with $D<a$, and if in this region $F(s)$ satisfies either* (10), *one of the conditions* (I), (II) *of Theorem* I *being satisfied, or* (78), *one of the conditions* (I), (II), (III), (IV), (V) *of Theorem* III *being satisfied, then there exists a singularity of $F(s)$ in every channel of width $2\pi a$, connecting a circle $C(s_1, \pi a)$, with $\Re(s_1)<\sigma_{d,\lambda}$, to $S(V, \pi a)$.*

Let $a>a_1>D$. If the theorem were not true, $F(s)$ would be holomorphic and bounded in the region composed of the circle $C(s_1, \pi a_1)$, of the horizontal half-strip $S(V, \pi a_1)$, and a channel of width $2\pi a_1$ connecting them. It would then follow from (14) of Theorem I that

$$\Re(s_1) \geqq \limsup_{n=\infty} (\log |d_n| - \log \Lambda_n)/\lambda_n = \sigma_{d,\lambda},$$

contrary to the hypotheses of the theorem.

12. We shall next apply the results of §11 to the study of the analytic continuation of a Dirichlet series. Without loss of generality we may suppose that the series $\sum d_n e^{-\lambda_n s}$ does not contain a constant term, that is the λ_n are all positive, and that $N=\infty$. We shall suppose that the upper density of $\{\lambda_n\}$ is finite. It is then well known that the abscissa σ_C of ordinary convergence and the abscissa σ_A of absolute convergence coincide, and that this quantity is given by the expression

$$\sigma_C = \sigma_A = \limsup_{n=\infty} (\log |d_n| / \lambda_n).$$

We shall suppose that the series has a region of convergence, that is to say $\sigma_C < \infty$.

We shall write

$$F(s) = \sum_{1}^{\infty} d_n e^{-\lambda_n s}, \tag{100}$$

even if $F(s)$ is given only by the analytic continuation of the series, but we shall give to this continuation the following meaning: to say that $F(s)$, given by (100), is holomorphic in a region Δ containing points $s=\sigma+it$ with σ positive and arbitrarily large means that $F(s)$ is holomorphic in Δ and is given, *in this region*, for σ sufficiently large, by the sum of the Dirichlet series. Such a region Δ may be a horizontal strip ($|t-t_0|<R$, $-\infty<\sigma<\infty$), a horizontal half-strip ($|t-t_0|<R$, $\sigma>\sigma_0$), a curvilinear strip horizontal at the right, $\cdots$. A *channel connecting a circle* $C(s_1, R)$ *to the half-plane* $\sigma>\sigma_0$ is the union of circles with a given radius $R_0<R$ of which the centers are on a Jordan arc of which one extremity is s_1, the other extremity being a point of the line $\sigma=\sigma_0$. The quantity $2R_0$ is *the width of this channel.* By what we have said above, to say that $F(s)$, given by (100), is holomorphic in a circle $C(s_1, R)$, in the half-plane $\sigma>\sigma_C$, and in a channel of width $2R_0$ connecting them means that in the region $\Delta=\Delta(s_1, R_1, R_0)$ composed of these three regions[9] $F(s)$ is holomorphic and is given, for σ sufficiently large, by the sum of the series in (100).

As we have seen in Lemma VII, (55) holds for $\sigma\geqq\sigma'>\sigma_A$. We may therefor say that (10) is satisfied in a horizontal half-strip $S(V, R')$ ($V=\sigma'+it'$) with arbitrary width $2R'$ and with $\sigma'>\sigma_C=\sigma_A$, the asymptotic sequence $\{\phi_n(x)\}$ being given by $\phi_n(x)=A(\sigma')e^{\lambda_n\sigma'}x^{\lambda_n}$ where $A(\sigma')$ is constant with respect to n. The lower envelope, $\phi(x)$, is obviously zero for $0\leqq x<e^{-\sigma'}$. The equality (13) is therefore satisfied with an arbitrary value for ω'. It follows then immediately from Theorem I (with condition (II)) that the theorem stated below is true.

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(9) In future we shall omit, in the definition of $\Delta(s_1, R_1, R_0)$, the part "in the half-plane $\sigma>\sigma_C$."

THEOREM X. *Let the upper density of* $\{\lambda_n\}$ *be* $D<\infty$. *If*

$$F(s) = \sum_{1}^{\infty} d_n e^{-\lambda_n s} \tag{100}$$

is holomorphic and bounded in a circle $C(s_1, \pi a)$ *and in a curvilinear channel of width* $2\pi a_1$ *connecting this circle to the half-plane* $\sigma>\sigma_C$, *with* $D<a_1$, *then*

$$|d_n| \leqq KM(s_1)\Lambda_n \exp \lambda_n \Re(s_1), \tag{101}$$

where $M(s_1)$, $\{\Lambda_n\}$ *have the same meaning as in Theorem* I. K *is a constant depending only on* a *and* D.

REMARK. *The value of* K *is the same as in the remark which follows the statement of Theorem* I.

This theorem contains a theorem of the author [6] in which the curvilinear channel is assumed to be horizontal, and in which K is not as precise as in the remark which follows the statement of Theorem X.

There follows immediately from Theorem II (recalling that $\phi(u)=0$ for u sufficiently small, in case of Dirichlet series) the following theorem.

THEOREM XI. *If* $F(s)$, *given by* (100), *is holomorphic and bounded in a curvilinear strip of width* $2\pi a$, *horizontal at the right and extending to* $-\infty$ *at the left, and if* $D<a$, *then* $F(s)$ *is identically zero.*

The following theorem is a corollary of Theorem VII.

THEOREM XII. *If* $F(s)$, *given by* (100), *is not identically zero, then in the strip described in Theorem* XI *the only eventualities which are possible are* (a), (b), (c) *of Theorem* VII.

Theorem XII was first established for the case $\sigma_C=-\infty$ (the eventuality (a) is then automatically removed, since $F(s)$ is then an entire function) in a joint paper by J. J. Gergen and the author [8].

Theorem IX furnishes immediately the following result.

THEOREM XIII. *Let* $\sigma_{d,\lambda}>-\infty$. *The function* $F(s)$ *given by* (100) *admits a singular point in every curvilinear channel of width* $2\pi a$, $a>D$, *connected with the half-plane* $\sigma>\sigma_C$, *of which the central line contains at least one point* s_1 *such that* $\Re(s_1)<\sigma_{d,\lambda}$.

Let us set

$$\sigma_\lambda = \limsup_{n=\infty} \log \Lambda_n/\lambda_n,$$

where $\{\Lambda_n\}$ is the sequence associated with $\{\lambda_n\}$.

It is obvious that

$$\sigma_{d,\lambda} \geqq \limsup_{n=\infty} \log |d_n|/\lambda_n - \limsup_{n=\infty} \log \Lambda_n/\lambda_n = \sigma_C - \sigma_\lambda.$$

It follows, on the other hand, from Lemma X that if (71) is satisfied then $\sigma_\lambda \leqq B(D, h)$ $(=3D(6-\log hD)$, if $D>0$, $B(0, h)=0)$. Theorem XIII gives therefore the following result:

THEOREM XIV. *If $F(s)$ is given by* (100), *if $\sigma_C > -\infty$ and if*

$$\liminf (\lambda_{n+1} - \lambda_n) = h > 0, \tag{102}$$

then there exists a singularity of $F(s)$ in every curvilinear channel of width $2\pi a$, $a>D$, connected with the half-plane $\sigma>\sigma_C$, the central line containing at least one point s_1 such that $\mathcal{R}(s_1)<\sigma_C-B(D, h)$.

Theorem XIV contains a theorem of Ostrowski (see [6]) by which, if (102) is satisfied $(D<\infty)$, the series given by (100) admits a singularity in every circle $C(V, G(D, h))$ where V is any point on the axis of convergence, and where $G(D, h)$ tends to zero as $D\rightarrow 0$ (h being fixed). It contains therefore, as a particular case, Fabry's theorem on Taylor series (λ_n integers) by which $\sum_1^\infty d_n e^{-\lambda_n s}$ with $\lambda_n/n\rightarrow\infty$ admits the axis of convergence $(\sigma_C > -\infty)$ as a cut.

Let us set $\sigma'_{d,\lambda} = \limsup (\log |d_n| + \log \Lambda_n)/\lambda_n$.

From Theorem XIII follows the theorem:

THEOREM XV. *If $\sigma_C > -\infty$ the function $\Phi(s)$ given by the series*

$$\Phi(s) = \sum_1^\infty d_n \Lambda_n e^{-\lambda_n s}$$

admits a singularity in every curvilinear channel of width $2\pi a$, $a>D$, connected with the half-plane $\sigma>\sigma'_{d,\lambda}$, the central line containing at least one point s_1 such that $\mathcal{R}(s_1) < \limsup_{n\geq 1} \log |d_n|/\lambda_n = \sigma_C$.

For the proof it is sufficient to note that in this theorem $d_n\Lambda_n$ plays the same role as d_n in Theorem XIII, $\sigma'_{d,\lambda}$ the role of σ_C, and $\sigma_C = \limsup \log |d_n|/\lambda_n$ plays the role of $\sigma_{d,\lambda}$ in Theorem XIII.

This theorem is closely connected to a classical theorem of Cramér [1].

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