

# ON THE EXTENSION OF LINEAR TRANSFORMATIONS

BY

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**1. Introduction.** Suppose we are given a linear subspace  $X$  of a normed linear space  $Z$  (n.l.  $Z$ ), and a linear transformation (l.t.)  $T$  on  $X$  to a n.l.  $Y^{(1)}$ . A l.t.  $T_1$  defined on  $Z$ , such that  $T_1x = Tx$  for all  $x \in X$ , is called an *extension* of  $T$ . The well known Hahn-Banach theorem states that every linear functional (l.f.) on a linear subspace of a n.l. space may be extended to be a l.f. on the whole space, with preservation of the norm [1, p. 55]<sup>(2)</sup>. Thus in case  $Y$  is one-dimensional, an extension  $T_1$  always exists, with  $|T_1| = |T|$ , and without enlargement of the range  $Y$ . In his book on linear operations, Banach leaves open the question of under what conditions an extension  $T_1$  will exist, when the range is a general n.l.  $Y$  [1, p. 234]. It is this question which is studied in the present paper.

There is always an extension  $T_1$  with range  $Y$ , if there is a bounded<sup>(3)</sup> projection  $P$  on  $Z$  into  $X$ , namely  $T_1 = TP$ <sup>(4)</sup>. In §2 it is shown conversely, in case  $Z$  and  $X$  are Banach spaces and  $T$  is one-to-one, that if  $Y$  is strictly the n.l. range of  $T$ , the existence of a  $T_1$  necessarily requires the existence of a  $P$ . An example is given, on the other hand, where  $Y$  is the smallest Banach space containing the range of  $T$ , and  $T$  is one-to-one, in which a  $T_1$  with range  $Y$  exists, although there is no bounded projection on  $Z$  into  $X$ .

§3 is a discussion of completion of n.l. spaces, and of extensions which are uniquely determined by the closures of subspaces. It contains an example which shows that in general when  $Z$  is a not-closed linear subspace of Hilbert space, if  $X \subset Z$ , then for the existence of a bounded projection on  $Z$  into  $X$ , it is not sufficient merely that  $X$  be closed in  $Z$ .

In §4, it is shown that an extension  $T_1$  always exists, with  $|T_1| = |T|$ , provided that we are allowed to enlarge the range  $Y$  to a n.l. space  $W \supset Y$ . The proof of this theorem is based on the notions of convex functional or pseudo-norm, and of quotient space (difference group). In §5 a related theorem of R. S. Phillips is quoted and compared with our theorem.

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(<sup>1</sup>) Our terminology is for the most part standard, as given in Banach [1]. Thus "linear" transformation means a bounded and distributive transformation; a "Banach" space means a complete n.l. space. Throughout, "transformation  $T$  on  $X$  to  $Y$ " means that the domain of  $T$  is all of  $X$ , and that the proper range of  $T$  is included in  $Y$  but is not necessarily all of  $Y$ ; " $T$  on  $X$  into  $Y$ ," that the proper range is all of the space  $Y$ .

(<sup>2</sup>) Numbers in brackets refer to the list of References at the end of the paper.

(<sup>3</sup>) We specify *bounded* projection, that is, a linear transformation  $P$  such that  $P^2 = P$ , since later we consider also unbounded distributive projections.

(<sup>4</sup>) Bohnenblust [3, p. 301]; Phillips [8, p. 540].

Some algebraic properties of distributive transformations and extensions in linear spaces are collected in §6, in part preliminary to a classification of linear transformations into four fundamental types, which is given in §7. The classification is with regard to the character of the range (complete or incomplete), and to the existence or non-existence of a bounded projection into the closed subspace<sup>(6)</sup> or zeros of the linear transformation. The existence of all four types is shown by examples.

§8 contains theorems on one-to-one extension of one-to-one linear transformations, with applications to show a certain relationship between the four types of §7, and to show that there always exists a "maximal" extension with range the quotient space  $Z/X'$ , where  $X'$  is the zero subspace of  $T$ , and the norm in  $Z/X'$  is an extension of the norm on the original range space  $L \cong X/X'$ .

In a final section (and elsewhere through the paper), unsolved problems are indicated which I hope to deal with successfully in another paper after the war.

**2. Projections and extensions.** Let  $U$  be a l.t. on  $X \subset Z$  into a n.l.  $L$ . As already remarked in the introduction, if there is a bounded projection  $P$  on  $Z$  into  $X$ , then  $U_1 = UP$  is an extension on  $Z$  with range  $L$ . A modification of this is the following

*Remark 1.* If there is a bounded projection  $P$  on  $Z$  into  $X_1 \supset X$ , and if  $U_1$  is an extension on  $Z$  (or on  $X_1$ ) to  $W \supset L$ , then  $U' = U_1P$  is an extension on  $Z$  into  $Y = U_1X_1$ ,  $W \supset Y \supset L$ .

A parallel process of collapsing the range of an extension is indicated in

*Remark 2.* If there is an extension  $U_1$  with range  $W \supset L$ , and if there is a bounded projection  $Q$  on  $W$  into  $Y$ ,  $W \supset Y \supset L$ , then  $U' = QU_1$  is an extension with range  $Y$ .

If  $U$  is one-to-one and linear on a Banach space  $X$  into a Banach space  $Y$ , then  $U$  is an isomorphism of  $X$  with  $Y$  [1, pp. 41 and 180]. A one-to-one linear transformation  $U$  on a Banach space  $X$  is not necessarily an isomorphism, but if it is not, its range n.l. space  $L$  is not complete (and is a set of the first category in the completion<sup>(6)</sup>  $L^c$  of  $L$  [1, p. 35]).

If  $U$  is an isomorphism of any n.l.  $X$  with a n.l.  $Y$ , then the existence of an extension  $U_1$  on a n.l.  $Z \supset X$  into  $Y$  requires the existence of a bounded projection  $P$  on  $Z$  into  $X$  ( $P = U^{-1}U_1$ ). We may make a stronger statement:

**2.1. THEOREM.** *Suppose  $U$  is one-to-one and linear on a closed linear subspace  $X$  of a Banach space  $Z$ , with range  $L$  (not necessarily complete). Then the existence of an extension  $U_1$  on  $Z$  with range confined to  $L$  requires the existence of a bounded projection on  $Z$  into  $X$ .*

For the proof, we use a lemma of Murray, that for Banach spaces the

<sup>(6)</sup> Throughout this paper, "subspace" is an abbreviation for "linear subspace."

<sup>(6)</sup> For this notion, see §3.

existence of projections and of complementary manifolds<sup>(7)</sup> are equivalent [5, p. 140]. Let  $Z'$  be the closed linear subspace consisting of all elements  $z'$  of  $Z$  for which  $U_1 z' = 0$ , and divide  $Z$  into hyperplanes or cosets  $x + Z'$ . By hypothesis, each hyperplane contains one and only one  $x \in X$ , and every element of  $Z$  is included in some hyperplane since  $U_1$  is defined on all of  $Z$ . Thus each  $z \in Z$  has a unique decomposition  $z = x + (z - x)$ , where  $x \in X$  is the representative of the hyperplane to which  $z$  belongs, so that  $(z - x) \in Z'$ . Therefore  $X$  and  $Z'$  are complementary manifolds in  $Z$ , and by Murray's lemma the operation  $P$  defined by  $Pz = x$  is a projection on  $Z$  to  $X$ . The projection  $P$  is given by  $P = U^{-1}U_1$  (even though  $U^{-1}$  is unbounded in case  $L$  is not complete).

*Remark 3.* The hypothesis in the above theorem that  $Z$  is a Banach space is essential. This is shown by the following.

*Example.* Consider a space  $l_p$ ,  $p < 2$ , and a closed linear subspace  $l \subset l_p$  for which there is no projection, as constructed in [9, p. 83]. The space  $l$  is a sum of subspaces  $l_v$  of finite dimensional  $p$ -spaces  $l_{p,2^v}$ . Besides the  $p$ -norm, define an  $l_2$  norm in  $l_p$  by  $\|x\|_2 = (\sum |x_i|^2)^{1/2}$ , where  $x = \{x_i\} \in l_p$ . In each space  $l_{p,2^v}$ , let  $l'_v$  be the orthogonal complement of the space  $l_v$  entering into the definition of  $l$ . In a similar way to the definition of  $l$ , define a closed linear subspace  $l' \subset l_p$  as the sum of the subspaces  $l'_v$ . Then by Murray's lemma, the direct sum  $l \oplus l'$  is neither the whole of  $l_p$ , nor a closed subspace in  $l_p$ , since, by the construction of  $l$ , there is no bounded projection on  $l_p$  or even on  $l \oplus l'$  to  $l$ . The linear subspace  $l \oplus l'$  contains the linear subspace of all finite sequences<sup>(8)</sup> of  $l_p$ , since  $l_v$  and  $l'_v$  span each  $l_{p,2^v}$ . Hence  $l \oplus l'$  is dense in  $l_p$ .

Let  $X$  be the subspace  $l \subset l_p$  as above; let  $L$  be a copy of  $l$  with the norm  $\|x\|_2$ , and let  $U$  on  $X$  into  $L$  be the identity on  $l$  to  $l$ . Let  $Z$  be  $l \oplus l'$ , and let  $P$  be the (unbounded) projection on  $Z$  into  $l = X$  determined by the complementary subspaces  $l$  and  $l'$  in  $Z$ ;  $P$  is then the section of an orthogonal projection on  $l_2 \supset l_p$ . Then  $U_1 = UP$  is a norm-preserving extension on  $Z$  with range  $L$ , since  $\|U_1 x\| = \|Px\|_2 \leq \|x\|_2 \leq \|x\|_p$ . The extension is thus possible with range restricted to  $L$ , in spite of the fact that there is no bounded projection on  $Z$  into  $X$ .

*Remark 4.* If  $Z$  and  $X$  are Banach spaces,  $Y$  the smallest Banach space containing  $L$ , it is *not* true that the existence of an extension on  $Z$  with range  $Y$  always requires the existence of a bounded projection on  $Z$  into  $X$ . This is shown by essentially the same example as in Remark 3: Let  $Z$  be  $l_p$ . Then  $U_1$  as above on  $l \oplus l'$  has a unique continuous extension<sup>(9)</sup>  $U'$  on  $l_p$  with range included in the completion  $Y = L^c$  of  $L$ , although there is no bounded projec-

(7) Two linear subspaces  $X$  and  $X'$  in a n.l. space  $Z$  are *complementary manifolds* or *complementary closed linear subspaces* if: (a) they are closed; and (b) each  $z \in Z$  may be uniquely expressed in the form  $z = x + x'$ ,  $x \in X$ ,  $x' \in X'$ , that is,  $Z = X \oplus X'$ . Two linear subspaces satisfying condition (b) will be called *complementary subspaces*.

(8) A finite sequence is a sequence  $x = \{x_i\}$  in which all but a finite number of the coordinates  $x_i$  are zero. The linear subspace of all finite sequences is obviously dense in  $l_p$ .

(9) See §3, Theorem 3.1.

tion on  $Z=l_p$  into  $X=l$ . ( $Y$ =a closed linear subspace of  $l_2$ =a Hilbert space.)

**3. Completion of n.l. spaces; unique extensions.** Any n.l. space  $X$  can be completed in one and in only one way to be a Banach space  $X^c$  in which it is a dense linear subspace. We sketch the proof, since although the existence of a completion is well known, it is perhaps not so well known that there is a smallest linear completion which is unique to within equivalence.

Define a space  $X^c$  as follows. The elements of the space are the equivalence classes  $\mathcal{X} = [\{x_n\}]$  of Cauchy sequences  $\{x_n\}$  of elements of the original space  $X$ , two sequences  $\{x_n'\}$  and  $\{x_n''\}$  being equivalent if  $\lim_{n \rightarrow \infty} \|x_n' - x_n''\| = 0$ . With the zero-element,  $c\mathcal{X}$ ,  $\mathcal{X} + \mathcal{Y}$  defined in obvious fashion, it is evident that this space is a linear space. We define the norm  $\|\mathcal{X}\|$  for each class  $\mathcal{X}$  as the common value of  $\lim_{n \rightarrow \infty} \|x_n\|$  for the Cauchy sequences  $\{x_n\}$  of the class. It may then be easily verified that the space is a Banach space. If we let each  $x \in X$  correspond to the class  $\mathcal{X}$  of which a representative is the Cauchy sequence  $\{x_n\}$ , where  $x_n = x$  for all  $n$ , this correspondence defines an equivalence of the original space  $X$  with a dense subspace of the Banach space  $X^c$ .

The completion  $X^c$ , such that  $X$  is a dense subspace of  $X^c$ , is unique to within equivalence. For if  $X_1$  is any completion of  $X$ , if  $\rightarrow$  denotes convergence in the norm, then obviously  $\{x_n'\} \rightarrow x_0$ ,  $\{x_n''\} \rightarrow x_0$  imply  $\{x_n' - x_n''\} \rightarrow 0$ , and  $\{x_n'\} \rightarrow x_0$ ,  $\{x_n' - x_n''\} \rightarrow 0$  imply  $\{x_n''\} \rightarrow x_0$  (and  $\|x_0\| = \lim_n \|x_n'\| = \lim_n \|x_n''\|$ ). Thus if  $X$  is dense in  $X_1$ , then  $X_1$  is equivalent to  $X^c$  defined as above.

The completion  $X^c$  of  $X$  in various cases may be of finitely or infinitely greater dimension than  $X$ . This is illustrated in the following examples. (1) Consider  $l_2$ , the Hilbert space of sequences, and the dense linear subspace  $S_f$  of all finite sequences<sup>(10)</sup>. Then  $S_f^c = l_2$ , and obviously  $S_f$  is infinitely less dimensional than  $l_2$ <sup>(11)</sup>. (2) In (1), let  $S'$  be a complementary subspace<sup>(12)</sup> to  $S_f$  in  $l_2$ , and let  $l_n'$  be any finite dimensional subspace of  $S'$ ,  $l''$  a complementary subspace to  $l_n'$  in  $S'$ . Then  $S_f \oplus l''$  is finitely less dimensional than  $l_2$ , and  $(S_f \oplus l'')^c = l_2$ . (3) If  $l_n$  is any finite dimensional subspace of an infinite dimensional Banach space  $Z$ , there are not-closed complementary subspaces  $l'$  to  $l_n$  in  $Z$ , such that  $(l')^c$  may be either  $Z$  or a closed complementary subspace to a (proper) subspace of  $l_n$ .

The following theorem is well known, and we omit its proof<sup>(13)</sup>.

**3.1. THEOREM.** *Given any l.t.  $T$  on a n.l.  $X$  into a n.l.  $Y$ , there always exists*

<sup>(10)</sup> See footnote 8. The space  $S_f$  is algebraically the same as the linear space of all polynomials.

<sup>(11)</sup> In the sense that  $l_2$  cannot be obtained from  $S_f$  by the adjunction to  $S_f$  of a finite number of elements of  $l_2 - S_f$ ; that is, there is no finite dimensional complementary subspace to  $S_f$  in  $l_2$ .

<sup>(12)</sup> See footnote 7, and §6, Theorem 6.2.

<sup>(13)</sup> See, for example, [6, p. 10].

an extension  $T_1$  on  $X^c$  to  $Y^c$ , and  $|T_1| = |T|$ . The extension  $T_1$  is unique: if  $U_1$  is any (continuous) extension of  $T$  on  $X^c$  into  $W \supset Y$ , then  $W$  is equivalent to  $Y^c$  or to a linear subspace in  $Y^c$ , and  $T_1 = VU_1$ , where  $V$  is the equivalence on  $W$  to  $Y^c$  ( $V$  is an extension of the identity on  $Y$  to  $Y$ ).

If  $X$  is a n.l. subspace of a n.l.  $Z$ , the closure of  $X$  in  $Z$  will be denoted by  $X^c$ .

3.2. COROLLARY. Given a l.t.  $T$  on a n.l.  $Z$  to a n.l.  $W$ ,  $T^c$  on  $Z^c$  to  $W^c$  is unique. If  $U$  is  $T$  considered only on  $X \subset Z$  into  $Y \subset W$ , then  $U^c$  is unique and agrees with  $T^c$  on  $X^c$ . In particular,  $U^c$  on  $X^c = Z \cdot X^c$  agrees with  $T$  and is the unique (continuous) extension of  $U$  on  $X^c$ .

3.3. THEOREM. Let  $X$  be a Banach subspace of a n.l. space  $Z$ , and let  $Y$  be a closed linear subspace of  $Z$  which intersects  $X$  only in the origin. Suppose that the projection  $P$  on  $X \oplus Y$  to  $X$  through  $Y$  is bounded. Then  $X \oplus Y$  is closed in  $Z$ .

**Proof.** Since  $X$  is a Banach space,  $P$  may be extended by Theorem 3.1 to be defined on  $(X \oplus Y)^c$ . Then  $P$  as extended is a bounded projection on  $(X \oplus Y)^c$  into  $X$ . Let  $Y' \supset Y$  be the corresponding complementary subspace. Choose any  $y' \in Y'$ . Then  $Py' = 0$ , and there exists a sequence  $\{x_n + y_n\}$ ,  $x_n \in X$ ,  $y_n \in Y$ , such that  $x_n + y_n \rightarrow y'$ ; hence  $P(x_n + y_n) = x_n \rightarrow Py' = 0$ . Therefore  $y_n \rightarrow y'$ , and  $y' \in Y$ . Thus  $Y' = Y$ , and  $(X \oplus Y)^c = X \oplus Y$ .

The following example shows the necessity, in the above theorem, of the assumption that  $X$  is a Banach subspace, not merely a closed subspace of  $Z$ .

*Example.* Let  $Z$  be  $l_p$  with the  $l_2$  norm ( $p < 2$ ), and let  $X$  and  $Y$  be respectively the subspaces  $l$  and  $l'$  of  $l_p$  (see Remarks 3 and 4 of §2). Then by the construction of  $l$  and  $l'$  (see [9]),  $X$  and  $Y$  are closed in  $Z$ . For consider  $Z^c = l_2$ , and let  $l_{(2)} (= X^c)$  be the subspace of  $l_2$  of all sequences such that their coordinates are included in the characteristic subspaces of the spaces  $l_{p,2^v}$ . Then any sequences in the  $l_2$  closure of  $X = l$  in  $Z$  must be contained in  $l_{(2)} \cdot Z$ —but  $l$  by definition contains all sequences of  $l_p$  whose coordinates are included in the characteristic subspaces. Therefore  $X = l_{(2)} \cdot Z$ , and is closed. Similarly  $Y$  is closed. The (orthogonal) projection  $P$  on  $X \oplus Y$  into  $X$  through  $Y$  is of norm 1, but  $X \oplus Y$  is not closed. Moreover there is no bounded projection on  $(X \oplus Y)^c = Z$  into  $X$ . For if  $P$  were such a projection, the corresponding complementary subspace necessarily would be closed in  $Z$ , and hence a complementary manifold to  $l$  in  $l_p$  (since any subspace closed in the  $l_2$  norm is also closed in the  $l_p$  norm, because of the inequality  $\|x\|_p \geq \|x\|_2$ ). Thus we have also the

*Remark.* For a not-closed linear subspace  $Z$  of Hilbert space  $l_2$ , the closure in  $Z$  of a subspace  $X \subset Z$  is a necessary<sup>(14)</sup> but not in itself a sufficient condi-

<sup>(14)</sup> The condition is necessary, for if  $X$  is the subspace of  $P$ ,  $x_n \rightarrow x$ ,  $x_n \in X$ , implies  $Px_n = x_n \rightarrow Px = x$ , and  $x \in X$ .

tion for the existence of a bounded projection on  $Z$  into  $X$ .

**4. Extension by enlargement of the range-space.** If  $X$  is either a not-closed dense linear subspace of a n.l. space  $Z$ , or a closed linear subspace of  $Z$  for which there is no projection, if  $Y$  is a copy of  $X$ , and if  $U$  is the identity on  $X$  to  $Y$ , then clearly there is no extension  $U_1$  on  $Z$  with range restricted to  $Y$ . This fact suggested the following theorem.

**4.1. THEOREM.** *Given any linear transformation  $U$  on a n.l.  $X$  to a n.l.  $Y$ , and any n.l. space  $Z \supset X$ , there is always an extension  $U_1$  on  $Z$ , provided we are allowed to increase the range from  $Y$  to a n.l. space  $W \supset Y$ . Moreover  $W$  and  $U_1$  may be chosen so that  $|U_1| = |U|$ .*

Our proof will be clear when Theorem 4.3, below, is reached.

A *convex functional* on a n.l. space  $X$  is any (nonlinear) functional defined on  $X$  which has the pseudo-norm properties

- (1)  $f(x) \geq 0$ ,
- (2)  $f(tx) = |t| \cdot f(x)$ ,
- (3)  $f(x+x') \leq f(x) + f(x')$ .

These are the usual norm properties, with the omission of the property that  $\|x\| = 0$  necessarily implies  $x = 0$ . If for a convex functional  $f(x)$  there exists a constant  $K$  such that  $f(x) \leq K \cdot \|x\|$  for all  $x$ ,  $f(x)$  will be said to be a *bounded convex functional*. (For example, if  $\bar{x}(x)$  is a linear functional on  $X$ ,  $f(x) = |\bar{x}(x)|$  is a convex functional; also  $f(x) = \|x\|$  is a convex functional; these are extreme cases.) The smallest possible  $K$  in the inequality will be called the *norm* of  $f(x)$ , and will be denoted by  $|f| = K$ .

By (2) and (3),  $|f(x) - f(x')| \leq f(x - x')$ ; hence if  $f(x)$  is bounded,  $f(x)$  is continuous. If  $f(x)$  is continuous, it may easily be shown also that there is a finite  $K$  such that  $f(x) \leq K \cdot \|x\|$  for all  $x$ . Thus just as for distributive operations, for convex functionals boundedness is equivalent to continuity.

**4.2. THEOREM.** *If  $U(x)$  is any linear transformation on a n.l. space  $X$  into a n.l. space  $Y$ , then  $f(x) = \|Ux\|$  is a bounded convex functional on  $X$ , and  $|f| = |U|$ . Conversely, any bounded convex functional  $f(x)$  defined on  $X$  gives rise to a n.l. space  $Y$  and to a linear transformation  $y = U(x)$  on  $X$  into  $Y$ , of norm  $|U| = |f|$ .*

**Proof.** The first statement is immediately obvious. To prove the converse, let  $X'$  be the closed linear subspace consisting of those elements  $x' \in X$  such that  $f(x') = 0$ . (If  $U$  is one-to-one,  $X'$  consists of the single element 0.) Let  $X/X'$  be the quotient space (difference group) of which the elements are the hyperplanes (cosets)  $x + X'$  of  $X$ . As above  $|f(x) - f(x_1)| \leq f(x - x_1)$ . Hence if  $x$  and  $x_1$  are in the same hyperplane,  $x_1 = x + x'$ , then  $f(x - x_1) = f(x') = 0$  and  $f(x) = f(x_1)$ . We define a norm for each hyperplane-element of  $X/X'$  to be the common value of  $f(x)$  for the elements  $x$  of the hyperplane:  $\|x + X'\| = f(x)$ . Let  $Y$  be the normed linear space  $X/X'$ ; and let  $U$  be the linear transforma-

tion on  $X$  to  $Y$  which takes each  $x \in X$  into the hyperplane of  $X/X'$  of which it is a representative point. Then  $U$  and  $Y$  have the required properties.

If  $U$  and  $Y$  are given, and if  $f(x) = \|Ux\|$ , obviously  $X/X'$  with norm as above is equivalent to  $Y$ , the hyperplanes of  $X/X'$  being the inverse images of the elements of  $Y$ .

By Theorem 4.2, we see that if the range of  $U$  is all of  $Y$ , then Theorem 4.1 is equivalent to the following theorem.

4.3. THEOREM. *Let  $X$  be any linear subspace of a n.l. space  $Z$ . Let  $f(x)$  be any convex functional defined on  $X$ . Then there always exists a convex functional  $F(z)$  defined on  $Z$ , such that  $|F| = |f|$ , which coincides with  $f(x)$  on  $X$ .*

**Proof.** Let  $\{\bar{x}\}$  be any set of linear functionals on  $X$  such that  $\sup_{\bar{x} \in \{\bar{x}\}} |\bar{x}(x)| = f(x)$  for all  $x \in X$ <sup>(15)</sup>. (The existence of such a set will be proved presently.) For each  $\bar{x}$ , let  $\bar{z}$  be a norm-preserving extension to  $Z$ , as given by the Hahn-Banach theorem. Then evidently  $F(z) = \sup_{\bar{x} \in \{\bar{x}\}} |\bar{z}(z)|$  is a convex functional on  $Z$  which satisfies the requirements of the theorem.

We complete the proof by showing that there is a set of linear functionals  $\{\bar{x}\}$  as above. Without loss of generality we may suppose that  $|f| = 1$ .  $f(x)$  is a  $p$ -functional [1, p. 28]. For each  $x'_1 \in X$  for which  $f(x'_1) \neq 0$ , choose  $x_1 = tx'_1$  such that  $f(x_1) = 1$ . On the subspace  $\{tx_1\}$  of  $X$ , define a functional  $g(tx_1) = t$ . Then on this subspace  $|g(tx_1)| = |t| \cdot f(x_1) = f(tx_1)$ . Extend  $g(x)$  to the whole space  $X$  by Banach's  $p$ -functional extension theorem; then  $|g(x)| \leq f(x) \leq \|x\|$  for all  $x \in X$ . The set  $\{\bar{x}\}$  of all functionals  $\bar{x}(x) = g(x)$  so obtained, which includes one functional  $\bar{x}$  for each  $x_1 \in X$  for which  $f(x_1) = 1$ , clearly has the desired property in relation to  $f(x)$ .

For the case of complex n.l. spaces, the proof of the complex Hahn-Banach theorem [4] may be slightly modified to apply for a pseudo-normed complex linear space, and the resulting theorem used above in the place of the  $p$ -functional extension theorem.

In Theorem 4.1, if the range of  $U$  is not the whole of  $Y$ , let it be  $Y' \subset Y$ . Then by Theorems 4.2 and 4.3,  $U$  may be extended to be  $U_1$  on  $Z$  with range a n.l. space  $W' \supset Y'$ . In  $W' \supset Y'$  and  $Y \supset Y'$  there exist complementary subspaces<sup>(16)</sup>  $W''$  and  $Y''$  to  $Y'$ , so that each  $w' \in W'$ , and  $y \in Y$ , may be uniquely expressed respectively in the forms  $w' = y' + w''$ ,  $y' \in Y'$ ,  $w'' \in W''$ , and  $y = y' + y''$ ,  $y' \in Y'$ ,  $y'' \in Y''$ . Let  $W = Y \oplus W''$ , and identify  $(y, 0)$  with  $y$  and  $(0, w'')$  with  $w''$ . Let  $\bar{y}$  and  $\bar{w}$  be any pair of functionals on  $Y$  and  $W'$ , respectively, which coincide on  $Y'$  (that is,  $\bar{y}(y') = \bar{w}(y')$  for all  $y' \in Y'$ ). Let  $h(y, w'') = \bar{y}(y) + \bar{w}(w'') = \bar{y}(y'') + \bar{w}(w')$ . Define a norm in  $W$  by  $\|(y, w'')\| = \sup \|h(y, w'')\|$ , the sup being taken over all functionals  $\bar{y}, \bar{w}$  on  $Y, W'$ , such that  $|\bar{y}| \leq 1, |\bar{w}| \leq 1$ , which coincide on  $Y'$ . Then the n.l. space  $W$  con-

<sup>(15)</sup> In the special case that  $f(x) = \|x\|$ , a set  $\{\bar{x}\}$  is a "determining manifold" in the conjugate space  $\bar{X}$ .

<sup>(16)</sup> See Theorem 6.2, §6.

tains both  $W'$  and  $Y$  as subspaces. Thus we see that also in the case when the range of  $U$  is not the whole of  $Y$ , Theorem 4.1 is true as stated, though perhaps not with maximum economy in the dimension of  $W$ .

**5. Phillips' extension theorem.** Following is a theorem of R. S. Phillips [8, p. 538], restated in a different form for the purpose of comparison:

**5.1. THEOREM.** *Let  $U$  be any linear transformation on a n.l.  $X$  to a n.l.  $Y$ . There always exists a certain Banach space  $M_T$  containing  $Y$  as a subspace, with the following property: For every n.l.  $Z$  containing  $X$ , there is an extension  $U_Z$  on  $Z$  to  $M_T$ , with  $\|U_Z\| = \|U\|$ . (Furthermore  $M_T$  is independent of  $X$  and  $U$ .)*

In this theorem the space  $M_T$  is the set of all bounded functions  $a(t)$  on a class  $T$  of elements  $t$ , to the real numbers, with norm  $\|a\| = \sup_{t \in T} |a(t)|$ . For each  $t \in T$ , the functional  $\bar{x}_t$  defined by  $\bar{x}_t(x) = a(t)$ , where  $Ux = a$ , is linear, and  $\sup_{t \in T} |\bar{x}_t| = \|U\|$ . The extension  $U_Z$  is defined by  $U_Z z = \{\bar{z}_t(z)\}$ , where for each  $t$ ,  $\bar{z}_t$  is any extension of  $\bar{x}_t$  given by the Hahn-Banach theorem.

The set of functionals  $\{\bar{x}_t\}$  evidently directly corresponds with the set of functionals  $\{\bar{x}\}$  in the proof of Theorem 4.3, and just as  $\sup_{x \in \{x\}} |\bar{x}(x)| = f(x) = \|Ux\|$  for all  $x \in X$ , so

$$|\bar{x}_t(x)| = |a(t)| \leq \sup_{t \in T} |a(t)| = \|a\| = \|Ux\| = f(x) \leq \|x\|$$

and  $\sup_{t \in T} |\bar{x}_t(x)| = \|Ux\|$ . The zero subspace of the extension in Theorem 4.3 is the set of  $z$  for which  $\bar{z}(z) = 0$  for every  $\bar{z} \in \{\bar{z}\}$ , likewise the zero subspace of  $U_Z$  is the set of  $z$  for which  $\bar{z}_t(z) = 0$  for all  $t \in T$ . Also  $\|U_Z(z)\| = \sup_{t \in T} |\bar{z}_t(z)|$ , and  $\|U_1 z\| = \sup_{t \in \{t\}} |\bar{z}(z)| = F(z)$ . Therefore there exists an equivalence between the space  $W$  in Theorem 4.1 and the range of  $U_Z$  in  $M_T$ , and  $U_Z$  and the extension  $U_1$  are essentially the same.

The class  $T$  may be any class of sufficiently large cardinal number so that  $M_T$  will contain  $Y$ . In case  $Y$  is any separable n.l. space,  $T$  may be countable ( $M_T$  is then the space  $(m)$  of bounded sequences) [8, p. 524]. Thus any  $U$  with separable range  $Y$  can be extended to be  $U_1$  on any  $Z \supset X$ , with range  $(m)$ .

**5.2. THEOREM.** *Given any Banach space  $X$  which is isomorphic to a subspace  $Y \subset M_T$ , if there is a bounded projection on  $M_T$  into  $Y$ , then there is a bounded projection on  $Z$  into  $X$ , for every n.l.  $Z \supset X$ . Moreover, there is a constant  $K$  independent of  $Z$ , such that for every  $Z$  there is a projection  $P_Z$  of norm not greater than  $K$ .*

Let  $U$  be the isomorphism of  $X$  into  $Y$ , and let  $U_Z$  be its extension on  $Z$  to  $M_T$ , as given by Phillips' theorem. Let  $Q$  be the projection on  $M_T$  to  $Y$ . Then it may be easily verified that  $U^{-1}QU_Z$  is a projection on  $Z$  to  $X$ , of norm  $\leq |U^{-1}| \cdot |Q| \cdot |U_Z| = |U^{-1}| \cdot |Q| \cdot |U| = K$ .

If we are given any extension on  $W \supset Y$  as in §4, if  $T$  is a set in one-to-one correspondence with the set of functionals  $\{\bar{x}\}$  which determines  $U_1$ , then as has already been seen,  $M_T \supset W \supset Y$  ( $W$  is written for the equivalent subspace



in  $M_T$ ), and  $U_1$  could have been obtained by Theorem 5.1. Conversely any extension by Theorem 5.1 is realizable by Theorem 4.1—it determines a convex functional  $F(z)$  which might have been obtained in Theorem 4.3.

But for a fixed imbedding of  $Y$  in an  $M_T$  given in advance, some extension by Theorem 4.1 might not be possible with range an equivalent subspace in  $M_T$  by Theorem 5.1. This is certainly true for example if:  $X$  is a subspace of  $M_T$ ,  $Y \subset M_T$  is a copy of  $X$ ,  $U$  the identity on  $X$  to  $Y$ ,  $Z$  is  $M_{T'}$  where  $T$  is a proper subset of  $T'$ ,  $M_T \supset X$  being the subspace of bounded functions on  $T'$  which are 0 on  $T' - T$ , and the extension by §4 is the identity on  $M_T$  into  $M_{T'}$ .

Even if, for an extension by §4,  $W$  is imbeddable in  $M_T$ , it seems probable that  $W$  is not always necessarily imbeddable in  $M_T$  in such a way as to preserve the orientation or even the inclusion of  $Y$  in  $W$ , so that when  $Y$  is fixed in  $M_T$ , a particular extension by §4 may not be realizable in  $M_T$  by Theorem 5.1.

**6. Distributive transformations in linear spaces.** A linear space (without a topology) is of course a purely algebraic system. In this section we collect some purely algebraic properties of distributive transformations and extensions in linear spaces.

“Isomorphism” as used in this section will mean merely an algebraic isomorphism, that is, a one-to-one distributive homomorphism. As in footnote 7, two subspaces<sup>(17)</sup>  $X'$  and  $X''$  of a linear space  $X$ , intersecting only in the origin, are called *complementary subspaces* if each  $x \in X$  may be represented in the form  $x = x' + x''$ ,  $x' \in X'$ ,  $x'' \in X''$ .

**6.1. THEOREM.** *A projection  $P$  determines a decomposition of  $X$  into complementary subspaces  $X'$ ,  $X''$ ; any decomposition of  $X$  into complementary subspaces  $X'$ ,  $X''$  determines projections  $P$  through  $X''$  into  $X'$  and  $1 - P$  through  $X'$  into  $X''$ .*

**6.2. THEOREM.** *For every subspace  $X' \subset X$ , there exists a projection  $P$  having  $X'$  as its subspace, that is, there exists a complementary subspace  $X''$  to  $X'$ . The subspace  $X''$  is isomorphic with the quotient group  $X/X'$ , the one-to-one correspondence for the isomorphism being the correspondence of each coset or “hyperplane” with its representative element in  $X''$ . Similarly  $X/X''$  is isomorphic with  $X'$ .*

The proof of the first statement is by an argument similar to the algebraic part of the proof of the Hahn-Banach theorem<sup>(18)</sup>; the remainder of the theorem is clear from §4.

**6.3. THEOREM.** *Any distributive transformation  $T$  on  $X$  into  $L$  is of the form of a projection followed by an isomorphism on the subspace of the projection*

<sup>(17)</sup> As before, *subspace* means always a linear subspace.

<sup>(18)</sup> [1, pp. 28–29].

into  $L$ . ( $T = VP$ , where  $V$  is the isomorphism on  $X''$  into  $L$ .)

Let  $X'$  be the zero-subspace of  $T$ . Then as in §4,  $X/X'$  is isomorphic with  $L$ ; by Theorem 6.2 there is a complementary subspace  $X''$  to  $X'$  which is isomorphic with  $X/X'$  and hence with  $L$ .

6.4. THEOREM. *Let  $T$  be a distributive transformation on  $X \subset Z$  into  $L$ . Then there exists a "maximal" space  $\mathcal{W} \supset L$  and a unique "maximal" distributive extension  $T_1$  on  $Z$  into  $\mathcal{W}$ . Any possible distributive extension  $T_2$  of  $T$  on  $Z$  may be considered to have as range a subspace of  $\mathcal{W}$ .*

Let  $X'$  be the zero-subspace of  $T$ . The space  $\mathcal{W}$  is  $Z/X'$ ; as before  $L$  is isomorphic with  $X/X'$ , and through the isomorphism  $L$  is identified with the subspace  $X/X'$  of  $\mathcal{W}$ . The "maximal" extension  $T_1$  is defined as follows: for each  $z \in Z$ ,  $T_1 z$  is the element of  $\mathcal{W}$  corresponding to the hyperplane  $H'$  (parallel to  $X'$ ) of  $Z/X'$  such that  $z \in H'$ . The range of  $T_1$  is all of  $\mathcal{W}$ , and  $T_1$  clearly is unique to within isomorphism.

If  $T_2$  is any extension of  $T$  on  $Z$ , let  $Z' \subset Z$  be the zero-subspace of  $T_2$ . Then  $Z' \supset X'$ , and there is a complementary subspace  $Z''$  in  $Z$  to  $Z'$ , with  $Z'' \supset X''$ . Let  $\bar{Z}'$  be a complementary subspace to  $X'$  in  $Z'$ . Then  $\mathcal{W}$  is isomorphic with  $Z'' \oplus \bar{Z}'$ , and the range of  $T_2$ ,  $Z/Z'$ , with  $Z''$ . If  $\mathcal{U}$  is the subspace  $Z'/X'$  of  $\mathcal{W}$ , the range of  $T_2$  may be regarded as  $\mathcal{W}/\mathcal{U} = (Z/X')/(Z'/X')$ . (If  $H, H'$  are hyperplanes of  $\mathcal{W}$ ,  $H \sim H'$  if  $(H - H') \subset Z'$ , that is,  $(H - H') \in \mathcal{U}$ .)

6.5. THEOREM. *If  $T_2$  is any distributive extension,  $T_2$  may be regarded as a collapse of the maximal extension  $T_1$ , by either a projection in the maximal space  $\mathcal{W}$ , or a projection in  $Z$  (as in Remarks 1 and 2, §2).*

Consider the isomorphism of  $Z'' \oplus \bar{Z}'$  with  $\mathcal{W}$ , as in the preceding theorem. Through this isomorphism the projection  $Q$  in  $\mathcal{W}$  is determined by the projection on  $Z'' \oplus \bar{Z}'$  through  $\bar{Z}'$  into  $Z''$ ,  $T_2 = QT_1$ .

The projection  $P$ , such that  $T_2 = T_1 P$ , is the projection on  $Z = Z'' \oplus \bar{Z}' \oplus X'$  through  $\bar{Z}'$  into  $Z'' \oplus X'$ .

6.6. THEOREM. *Let  $T_2$  be a distributive extension with range  $W_2$ , and  $Q$  a projection in  $W_2$ . Let  $T_3$  be the collapsed extension  $QT_2$ . Then there always exist projections  $P$  in  $Z$  such that  $T_3$  may be regarded as a collapse of  $T_2$  by  $P$ ,  $T_3 = QT_2 = T_2 P$ .*

Let  $Z'_2$  be the zero-subspace of  $T_2$ ,  $Z''_2$  a complementary subspace in  $Z$  to  $Z'_2$ . Let  $Z'_0$  be the zero-subspace of  $QT_2$  in  $Z''_2$ ; let  $\bar{Z}''$  be a complementary subspace in  $Z''_2$  to  $Z'_0$ . Then  $Z''_2 = Z'_0 \oplus \bar{Z}''$ , and the zero-subspace  $Z'_3$  of  $T_3$  is  $Z'_2 \oplus Z'_0$ . The projection  $P$  may be the projection through  $Z'_0$  into  $Z'_2 \oplus \bar{Z}''$ , or through  $Z'_0 \oplus$  (any subspace  $Z'_0$  of  $Z'_2$ ) into  $\bar{Z}'' \oplus$  (complementary subspace  $\bar{Z}'$  in  $Z'_2$  to  $Z'_0$ ).

6.7. THEOREM. *Let  $T_2$  be a distributive extension. Let  $P$  be a projection on*

$Z$  into  $X_1$  through  $\bar{X}_1$ , and let  $T_3 = T_2P$ . Then in order that there exist a projection  $Q$  such that  $T_3 = QT_2$ , it is necessary and sufficient that  $Z'_2 \cdot X_1$  and  $Z'_2 \cdot \bar{X}_1$  be complementary subspaces in  $Z'_2$ . If  $P$  does not satisfy this condition,  $Z'_3 \not\supset Z'_2$ . If  $Q$  exists, it is determined uniquely by  $T_2$  and  $P$ .

Since  $T_3 = T_2P$ , the zero subspace of  $T_3$ ,  $Z'_3 = (Z'_2 \cdot X_1) \oplus \bar{X}_1$ . For if  $z = x_1 + \bar{x}_1$ ,  $T_2Pz = T_2x_1 = 0$  if and only if  $x_1 \in Z'_2$ . If  $Q$  exists,  $Z'_3$  must contain  $Z'_2$  as in Theorem 6.6. Thus  $(Z'_2 \cdot X_1) \oplus \bar{X}_1 \supset Z'_2$ . Therefore any  $z'_2 \in Z'_2$  may be expressed in the form  $z'_2 = x_1 + \bar{x}_1$ ,  $x_1 \in Z'_2 \cdot X_1$ . Then  $\bar{x}_1 = z'_2 - x_1 \in Z'_2$ , that is,  $\bar{x}_1 \in Z'_2 \cdot \bar{X}_1$ , and the condition is necessary.

The condition is sufficient. For let  $X'_1$  and  $\bar{X}'_1$  be respectively complementary subspaces in  $X_1$  to  $Z'_2 \cdot X_1$  and in  $\bar{X}_1$  to  $Z'_2 \cdot \bar{X}_1$ . Then  $X'_1 \oplus \bar{X}'_1$  is a space  $Z'_2'$ , and  $Z'_2' \cdot X_1$  and  $Z'_2' \cdot \bar{X}_1$  are complementary subspaces in  $Z'_2'$ . The projection  $Q$  is the projection in  $W_2$  determined, through the isomorphism of  $W_2$  and  $Z'_2'$ , by the projection on  $Z'_2'$  into  $Z'_2' \cdot X_1 = X'_1$  through  $Z'_2' \cdot \bar{X}_1 = \bar{X}'_1$ . It is unique since it is  $T_2PT_2^{-1}$  where  $T_2$  is considered only on  $Z'_2'$ .

**6.8. THEOREM.** *If  $T_2$  and  $T_3$  are any two distributive extensions, there exists a unique maximal extension  $T_2 \cap T_3$  of which both  $T_2$  and  $T_3$  are collapsions, and a unique minimal extension  $T_2 \cup T_3$  which is a collassion of both  $T_2$  and  $T_3$ .*

The proof is by similar considerations to those already given, and is left to the reader.

**7. Classification of linear transformations.** In this section we present a classification of linear transformations into four fundamental types.

Let  $U$  be any l.t. on a n.l. space  $X$  into a n.l. space  $L \subset Y = L^c$ . By Theorem 3.1, we may assume without loss of generality that  $X$  is a Banach space. The range  $L$  may be complete or incomplete; accordingly we shall say that  $U$  is respectively of type II or of type I. (By Theorem 3, p. 38 of [1], if  $L$  is incomplete it is of category I in  $L^c$ .)

**7.1. LEMMA.** *Let  $X'$  be any closed linear subspace of a Banach space  $X$ . Let  $\mathcal{L}$  be the quotient space  $X/X'$ . With norm defined by  $\|x + X'\| = \inf_{x' \in X'} \|x + x'\|$ ,  $\mathcal{L}$  is a Banach space.*

As remarked by Alaoglu and Birkhoff [2, p. 298], this lemma may be shown by a straightforward procedure. It is proved, assuming that  $X$  is reflexive, by Murray (see [7, p. 78 and Lemma 6, p. 87]). (The assumption that  $X$  is reflexive is unnecessary.)

Now let  $U$  as above be defined on a Banach space  $X$ , and let  $X'$  be the closed linear subspace of all  $x' \in X$  such that  $Ux' = 0$ . Without loss of generality, evidently  $U$  may be assumed to be of norm 1; let this assumption be made. Denote by  $T$  the transformation on  $X$  into  $\mathcal{L}$  with the Banach norm of Lemma 7.1, defined by the correspondence of each  $x \in X$  to the hyperplane of  $\mathcal{L} = X/X'$  in which it is included. Define  $f(x) = \|Tx\|$ , and  $g(x) = \|Ux\|$ . Then

as in §4,  $f(x_1+x')$  and  $g(x_1+x')$  are constant over  $x' \in X'$ . Let the norm of Lemma 7.1 in  $\mathcal{L}$  be denoted by  $\|x_1+X'\|_f = f(x_1+x') = \|T(x_1+x')\| = \inf_{x' \in X'} \|x_1+x'\|$ , and define a second norm in  $\mathcal{L}$  by  $\|x_1+X'\|_g = g(x_1+x') = \|U(x_1+x')\| \leq \inf_{x' \in X'} \|x_1+x'\|$ . Then we have  $\|x_1+X'\|_g \leq \|x_1+X'\|_f$ , and, by §4,  $L$  is equivalent to  $\mathcal{L}$  with the norm  $\|x_1+X'\|_g$ .

Any complementary subspace  $X''$  to  $X'$  is (algebraically) isomorphic with  $\mathcal{L}$ ; each hyperplane  $x+X'$  has one and only one representative element in  $X''$  (Theorem 6.2). This isomorphism induces a third norm  $\|x+X'\|_{x''}$  on  $\mathcal{L}$ —the norm of  $X$  on the linear subspace  $X''$ . Then  $\|x+X'\|_f = \inf_{x' \in X'} \|x+x'\| \leq \|x+X'\|_{x''}$ , so that we have

$$\|x+X'\|_g \leq \|x+X'\|_f \leq \|x+X'\|_{x''}.$$

If  $U$  is of type II, then by Theorem 5, p. 41 of [1], the norm  $\|x+X'\|_g$  on  $\mathcal{L}$  is isomorphic<sup>(19)</sup> with the norm  $\|x+X'\|_f$ , and  $U$  is essentially the same as the operation  $T$  defined above. If  $U$  is of type I, the norm  $\|x+X'\|_g$  is incomplete on  $\mathcal{L}$ .

We shall say that  $U$  is of type (A) if there is a bounded projection on  $X$  into the subspace  $X'$  of the zeros of  $U$ ; otherwise we shall say that  $U$  is of type (B). If  $U$  is of type (A), there is a closed complementary subspace  $X''$ , and  $\|x+X'\|_f$  and  $\|x+X'\|_{x''}$  are isomorphic. If  $U$  is of type (B),  $\|x+X'\|_{x''}$  is incomplete on  $\mathcal{L}$ , for every choice of  $X''$ .

Following is a list of the possible types of  $U$ , with remarks concerning each type. Let  $\mathcal{L}$  with the respective norms be denoted by  $(\mathcal{L})_f$ ,  $(\mathcal{L})_g$ ,  $(\mathcal{L})_{x''}$ .

II(A) All three norms are isomorphic.  $(\mathcal{L})_f \cong (\mathcal{L})_g \cong (\mathcal{L})_{x''}$ . In this case  $U$  is essentially the bounded projection on  $X$  into  $X''$  through  $X'$ .

II(B) Every complementary subspace  $X''$  to  $X'$  is incomplete.  $U$  is one-to-one on incomplete  $X'' \cong (\mathcal{L})_{x''}$  to complete  $(\mathcal{L})_f \cong (\mathcal{L})_g \cong L$ .

I(A) There is a closed complementary subspace  $X''$ ,  $X'' \cong (\mathcal{L})_f$ .  $U$  is one-to-one on complete  $X''$  to incomplete  $L \cong (\mathcal{L})_g$ .

I(B) Any complementary subspace  $X''$  to  $X'$  is incomplete.  $U$  is one-to-one on incomplete  $X''$  to incomplete  $L \cong (\mathcal{L})_g$ . No two of the norms are isomorphic;  $\|x+X'\|_f$  is a Banach norm, while  $\|x+X'\|_g$  and  $\|x+X'\|_{x''}$  are both incomplete. Since  $U$  is defined on a Banach  $X$  into  $L$ , by Theorem 3.1 it is possible to have only one (continuous) extension on  $(X'')^c$  of  $U$  on  $X''$ . This extension necessarily coincides with  $U$  and does not enlarge  $L$  (so it is of course not one-to-one).

By Corollary 3.2, any  $U$  on a n.l.  $X$  corresponds uniquely to a  $U$  on a Banach  $X$  in all features used in the above classification, so essentially any  $U$  on a n.l.  $X$  into a n.l.  $L$  is of one of the above types.

The general form of II(A) is  $U=VP$ , where  $P$  is a bounded projection in the Banach space  $X$ ,  $V$  an isomorphism of the subspace  $X''$  of  $P$  into the

<sup>(19)</sup> Two norms  $\|x\|_1$  and  $\|x\|_2$  on the same linear space are called *isomorphic* if there exist constants  $k > 0$ ,  $K > 0$  such that  $k \cdot \|x\|_1 \leq \|x\|_2 \leq K \cdot \|x\|_1$  for all  $x$ .

range  $L$ . If  $X'$  is any closed subspace of a Banach space  $X$  for which there is no bounded projection, the transformation  $T$  on  $X$  into the quotient space  $(\mathcal{L})_f$  is of type II(B); the general form of II(B) is  $U=VT$ , where  $V$  is the isomorphism on  $(\mathcal{L})_f$  into  $L$ . The general form of I(B) is  $U=VT$ , where  $V$  is one-to-one of type I(A) on  $(\mathcal{L})_f$  into  $L$ .

*Examples of I(A):* The identity  $V$  on  $l_p$  to  $l_q$ , where  $1 \leq p < q \leq \infty$  [ $l_\infty = c_0$ ], is one-to-one of this type. Let  $X''=l_p$ , let  $X'$  be any Banach space, and let  $X$  be  $X' \oplus X''$  with norm  $\|(x', x'')\| = \|x'\| + \|x''\|$ <sup>(20)</sup>, where  $x' \in X'$ ,  $x'' \in X''$ . Let  $P$  be the bounded projection on  $X$  into  $X''$  through  $X'$ . Then  $U=VP$  is of type I(A), and not one-to-one. Or if  $X$  is a Hilbert space,  $l_2$  a proper closed linear subspace,  $P$  the orthogonal projection on  $X$  into  $l_2$ ,  $V$  the identity on  $l_2$  to  $l_q$ ,  $q > 2$ , then  $U=VP$  is of type I(A).

*Example of I(B):* We shall show that the example in Remarks 3 and 4 of §2 is of this type. In the notation of this section,  $X=l_p$ ,  $X'=l' \subset l_p$ ,  $L=l$  with norm  $\|x\|_2$ ,  $U$  is the unique continuous extension on  $X=l_p$  to  $L^c$  of the transformation on  $l \oplus l'$  into  $L$ .

To verify the above statements, in the first place the subspace  $X'$  of zeros of  $U$  is not larger than  $l'$ , since as is shown in the example at the end of §3,  $l'$  is closed in  $l_p$  with the  $l_2$  norm, and  $U$  is essentially the section on  $l_p \subset l_2$  of the orthogonal projection  $P$  on  $l_2$  into  $(l)^c$  (in  $l_2$ ) through  $(l')^c$  (in  $l_2$ ). There exist elements  $y \in l_p$  such that  $Py \notin l_p$  and  $(I-P)y \in l_p$ . To show that  $U$  is not of type II(B) (and thus that the linear correspondence on the quotient space  $l_p/l'$  to the Hilbert space  $L^c$  is *not* an isomorphism), we have to show that  $P(l_p)$  is a proper subspace of  $(l)^c = L^c$ , that is, that there exists an element  $x \in (l^c - l)$ , such that  $(x+x') \notin l_p$  for any  $x' \in [(l')^c - l']$ .

To show this, let  $y = (1, 0, 2^{-1/p}, 0, 0, 0, 3^{-1/p}, 0, \dots, \nu^{-1/p}, 0, \dots)$ . (For each  $\nu$ ,  $\nu^{-1/p}$  is followed by  $2^\nu - 1$  zeros.) For any  $z \in l_2$ , let  $\{z\}_\nu$  and  $\{z'\}_\nu$  denote the components in  $l_\nu$  and in  $l'_\nu$  respectively. Then by [9, pp. 81 and 83],  $x = 2Py = (1 + u_{11}^{(1)}, u_{21}^{(1)}, 2^{-1/p}(1 + u_{11}^{(2)}), 2^{-1/p}u_{21}^{(2)}, 2^{-1/p}u_{31}^{(2)}, 2^{-1/p}u_{41}^{(2)}, \dots)$ , and  $\nu^{-1/p}(n_\nu^{1/p-1/2} - 1) \leq \|\{x\}_\nu\|_p$ , where  $n_\nu = 2^\nu$ . By [9, pp. 85-86], for any  $x' \in (l')^c$ ,  $\|\{x\}_\nu\|_p \leq (n_\nu^{1/p-1/2} + 1)\|\{x+x'\}_\nu\|_p$ . Thus, essentially,  $\|\{x+x'\}_\nu\|_p \geq \nu^{-1/p}$ ,  $\|x+x'\|_p = \infty$ , and  $(x+x') \notin l_p$  for any  $x' \in (l')^c$ .

If there is no bounded projection on a closed subspace  $X'$  of a Banach space  $X$ , there are three logical possibilities regarding complementary subspaces  $X''$ : (1) For some  $X''$ ,  $(X'')^c$  is all of  $X$  or a subspace for which there is a bounded projection, and for other  $X''$ , there is no bounded project on  $X$  into  $(X'')^c$ . (2) For all  $X''$ ,  $(X'')^c$  is all of  $X$ , or a subspace for which there is a bounded projection. (3) For every  $X''$ , there is no bounded projection on  $X$  into  $(X'')^c$ . These give rise to a corresponding subclassification of

<sup>(20)</sup> The norm in  $X' \oplus X''$  equally well may be defined by  $\|(x', x'')\| = \|\|x'\|, \|x''\|\|_0$ , where  $\|\|_0$  is any norm on the two-dimensional space of real number pairs  $(x_1, x_2)$ , such that  $\|\|(x_1, x_2)\|_0 = \|\|(|x_1|, |x_2|)\|_0$ . If  $X$  is a Banach space,  $X'$  a subspace for which there is a complementary closed subspace  $X''$ , any such norm on  $X = X' \oplus X''$  is isomorphic with the original norm of  $X$ .

linear transformations of type (B).

Let  $Z$  be any Banach space with a closed subspace  $Z'$  for which there is no bounded projection, that is, no closed complement [5, p. 138, Lemma 1.1.1]. Let  $X''$  be a complementary subspace to  $Z'$ , and let  $X = (X'')^c$ ,  $X' = (X'')^c \cdot Z'$ . Then  $X'$  is closed, and  $X'$  and  $X''$  are complementary subspaces in  $X$ , such that  $(X'')^c = X$ . For if  $x \in X$ , and  $x = x'' + z'$ ,  $x'' \in X''$ ,  $z' \in Z'$ , then  $z' = (x - x'') \in X$ , and so  $z' \in X'$ . There can be no bounded projection on  $X$  into  $X'$ ; for if this were the case, there would be a closed complement  $X_2''$  to  $X'$  in  $X$ , and every  $z \in Z$  could be expressed in the form  $z = x'' + z' = x_2'' + (x' + z')$ ,  $x_2'' \in X_2''$ ,  $(x' + z') \in Z'$ , so that  $X_2''$  would be a closed complement to  $Z'$ , contrary to hypothesis. If  $l$  is any subspace of  $X'$ , there is a complement  $W$  to  $l$  in  $X$ , and by the algebraic relations of complements,  $W$  must contain a complement  $X_1''$  to  $X'$  in  $X$ . If  $W$  is closed,  $(X_1'')^c \subset W$ . (In particular, there is always a closed  $W$  if  $l$  is finite dimensional.) Thus this  $X$  and  $X'$  must be an example of either situation (1) or situation (2) of the above possibilities. Whether there exist realizations of all three possibilities is a question which I hope to study in another paper.

**8. One-to-one extensions.** For a one-to-one linear transformation, one may ask whether there is an extension preserving the norm and the one-to-one character of the transformation. This question is answered in the following three theorems.

**8.1. THEOREM.** *Given any n.l. space  $X$  which is isomorphic with a linear subspace  $Y$  of a n.l. space  $W$ , let  $U$  be the isomorphism. Then there always exists a n.l. space  $Z \supset X$  which is isomorphic with  $W$  through an isomorphism  $U_1$  which coincides with  $U$  on the subspace  $X$ , and  $|U_1| = |U|$ ,  $|U_1^{-1}| = |U^{-1}|$ .*

**Proof.** On  $Y$ , define a second norm  $\|y\|_X = \|U^{-1}y\|$ . Then by hypothesis

$$C_1 \cdot \|y\| \leq \|y\|_X \leq C_2 \cdot \|y\|$$

where  $C_2 = |U^{-1}|$ ,  $C_1 = 1/|U|$ . Let  $\{\bar{y}\}$  be any set of functionals on  $Y$  which are of norm 1 for  $\|y\|_X$  and such that  $\sup_{\bar{y} \in \{\bar{y}\}} |\bar{y}(y)| = \|y\|_X$  for each  $y \in Y$ . The functionals  $\bar{y}$  are of norm not greater than 1 for the norm  $C_2 \cdot \|y\|$ . Extend each  $\bar{y}$  to be a functional  $\bar{w}$  on  $W$  of norm not greater than 1 for the norm  $C_2 \cdot \|w\|$ . Define  $\|w\|' = \sup_{\bar{w} \in \{\bar{w}\}} [\bar{w}(w), C_1 \cdot \|w\|]$ . Then  $\|w\|'$  is an extension to  $W$  of  $\|y\|_X$ , and  $C_1 \cdot \|w\|' \leq \|w\|' \leq C_2 \cdot \|w\|$ . Let  $Z \supset X$  be a copy of  $W$  with norm  $\|w\|'$ . Then  $Z$  obviously satisfies the requirements of the theorem.

By Theorem 8.1, an isomorphism may always be extended as a larger isomorphism, with preservation of the norm in both directions.

**8.2. THEOREM.** *Given a one-to-one l.t.  $U$  on any n.l. space  $X$  into a n.l. space  $Y$ , then for any n.l. space  $W \supset Y$ , there is a n.l. space  $Z \supset X$  and a one-to-one extension  $U_1$  on  $Z$  into  $W$ , with  $|U_1| = |U|$ .*

**Proof.** Without loss of generality we may assume that  $|U| = 1$ . Choose

a complementary subspace  $Y'$  to  $Y$  in  $W$ . Let  $\|w\| = \|w\|_W$  be the original norm in  $W$ , and define a second norm  $\|w\|'$  on  $W = Y \oplus Y'$  by  $\|w\|' = \|y\|_X + \|y'\|_W$ , where  $w = y + y'$ . Define  $\|w\|'' = \max [\|w\|', \|w\|]$ . Then  $\|w\|''$  coincides with  $\|y\|_X$  on  $Y$  and  $\|w\| \leq \|w\|''$  for all  $w \in W$ . Thus if we take  $Z \supset X$  to be a copy of  $W$  with norm  $\|w\|''$ ,  $Z$  has the desired properties.

**8.3. THEOREM.** *Given a one-to-one linear transformation  $U$  on any n.l. space  $X$  into a n.l. space  $Y$ , then for any n.l. space  $Z \supset X$ , such that  $X$  is closed in  $Z$ , there is a n.l. space  $W \supset Y$  and a one-to-one extension  $U_1$  on  $Z$  into  $W$ , with  $|U_1| = |U|$ .*

**Proof.** Without loss of generality, we may assume  $|U| = 1$ . On  $X$ , define a second norm  $\|x\|_Y = \|Ux\|$ . Consider the set  $\Omega$  of all functionals  $\bar{z}$  on  $Z$  having both of the following properties:

- (1) Considered only on  $X$ ,  $|\bar{z}| \leq 1$  with respect to  $\|x\|_Y$ .
- (2)  $|\bar{z}| \leq 1$  with respect to  $\|z\|$ .

The set  $\Omega$  does not consist of only the functional  $\bar{z} \equiv 0$ . For  $\|x\|_Y = \|Ux\| \leq \|x\|$ , so that any functional  $\bar{x}$  of norm 1 with respect to  $\|x\|_Y$  is of norm not greater than 1 with respect to  $\|z\|$  on  $X$ , and by the Hahn-Banach theorem  $\bar{x}$  may be extended to be  $\bar{z}$  on  $Z$  with preservation of the norm; then  $\bar{z} \in \Omega$ . Define  $\|z\|' = \sup |\bar{z}(z)|$ ,  $\bar{z} \in \Omega$ . Then  $\|z\|'$  is a pseudo-norm, coincides with  $\|z\|_Y$  for  $z \in X$ , and  $\|z\|' \leq \|z\|$ .

For each  $z \notin X$ , in consequence of the hypothesis that  $X$  is closed in  $Z$ , there is a functional  $\bar{z}$  such that  $\bar{z}(z) \neq 0$  and  $\bar{z}(x) = 0$  for all  $x \in X$  [1, p. 57, lemma]. Since any multiple of  $\bar{z}$  has the same properties, we may assume  $|\bar{z}| \leq 1$ . By definition the set  $\Omega$  contains  $\bar{z}$  ( $|\bar{z}| = 0 \leq 1$  on  $X$ ). This shows that the convex functional  $\|z\|'$  is a norm. Let  $W$  be a copy of  $Z$  with the new norm  $\|z\|'$ , and  $U_1$  the identity.  $U_1$  and  $W$  then have the properties required for the theorem.

The assumption in the above theorem that  $X$  is closed in  $Z$  is essential, because of the required continuity of  $U_1$ : if  $U$  is a one-to-one l.t. on an incomplete n.l.  $X$  into a complete n.l.  $Y$ , the continuous extension  $U_1$  on  $Z = X^c$  is unique and cannot be one-to-one (Theorem 3.1).

Combination of the idea of maximal extension in §6, of the relation  $\|x + X'\|_o \leq \|x + X'\|_f$  of §7, and of Theorem 8.3 yields the following theorem.

**8.4. THEOREM.** *Given any l.t.  $U$  on a closed subspace  $X$  of a Banach space  $Z$ , into a n.l.  $L$ , then  $L$  is equivalent to  $X/X'$  with norm  $\|x + X'\|_o$ . On  $Z/X'$ , there exists an extension  $\|z + X'\|_o$  of  $\|x + X'\|_o$ ,  $\|z + X'\|_o \leq \|z + X'\|_f$ , where  $\|\cdot\|_f$  is the Banach quotient space norm on  $Z/X'$ . The maximal extension  $T_1$  of  $U$  into  $Z/X'$  with the norm  $\|z + X'\|_o$  is bounded, and  $|T_1| = |U|$ .*

Let  $U$  be normalized so that  $|U| = 1$ , as in §7. By Theorem 8.3 the norm  $\|x + X'\|_o$  on  $L = X/X' \subset Z/X'$  may be extended to  $Z/X'$ , with  $\|z + X'\|_o \leq \|z + X'\|_f$ , since  $L$  with norm  $\|x + X'\|_f$  is a Banach space and hence closed

in  $Z/X'$  with norm  $\|z+X'\|_f$ . Then  $\|T_1(z+x')\| = \|z+X'\|_o \leq \|z+X'\|_f$ ,  $= \inf_{x' \in X'} \|z+x'\| \leq \|z+x'\|$ , and  $|T_1| = 1 = |U|$ .

The proof of the following theorem is similar to that of Theorem 8.2.

**8.5. THEOREM.** *Any linear transformation of type I may be extended to be of type II.*

Let  $L$  be the range of the l.t. of type I, and let  $Y'$  be a copy of a complementary subspace to  $L$  in  $L^c = W$ . Define a norm on  $X \oplus Y'$  by  $\|x+y'\|' = \|x\| + \|y'\|_W$ , where  $\|\cdot\|_W$  is the norm in  $W = L^c$ . (Or  $\|x+y'\|'$  may be any norm on  $X \oplus Y'$  coinciding with  $\|x\|$  on  $X$ .) Define  $U(x+y') = Ux+y'$ , and  $\|x+y'\|_1 = \max[\|x+y'\|', \|U(x+y')\|_W]$ . Complete  $X \oplus Y'$  with  $\|x+y'\|_1$ , as in §3. Then the unique extension  $U^c$  on  $(X \oplus Y')^c$  of  $U$  (Theorem 3.1) is an extension of type II of  $U$ .

*Remark.* If  $U$  is of type IA, the projection  $P$  through  $(X'' \oplus Y')^c$  into the zero subspace  $X'$  of  $U$  is bounded. For  $\|x'\| \leq \|x'+x''\| + \|y'\|_W \leq \|x+y'\|_1$ . By Theorem 3.3, the space  $X' \oplus (X'' \oplus Y')^c$  is closed in  $(X \oplus Y')^c$ . Since this space contains  $X \oplus Y'$ , it is therefore the whole of  $(X \oplus Y')^c$ .

**9. Problems.** The extension  $U^c$  of type II in Theorem 8.5, considered only on  $X'' = X'' \oplus Y'$ , is one-to-one, but is not an isomorphism;  $X''$  is incomplete. Still  $U^c$  might be of type IIA, if there were a complementary subspace  $X_1''$ ,  $X_1'' \subset (X \oplus Y')^c$ , to the zero subspace  $X' \supset X'$  of  $U^c$ , such that  $U^c$  is an isomorphism on  $X_1''$  into  $L^c$ . Then necessarily  $X_1'' \cdot X$  would be either the zero element or such that  $U$  on  $X_1'' \cdot X$  is an isomorphism into only part of  $L$ . So far we have not answered the questions whether, if a transformation  $U$  is of type IA (type IB), it may always be regarded as a section of both, or of only one specifically, of transformations of type IIA and of type IIB.

When  $X'$  and  $X''$  are complementary closed subspaces corresponding to a bounded projection  $P$  in a Banach space  $X$ , can there ever exist a closed subspace  $R$ , such that  $R$  intersects  $X'$  and  $X''$  only in the origin, and  $PR$  is an incomplete subspace? If so, we would have a transformation of type IA as a section of a transformation of type IIA:  $P$  is of type IIA, while  $P$  considered only on  $R$  would be a one-to-one transformation of type IA.

As already noted in §3, if  $P$  is a bounded projection through an infinite dimensional  $X'$ , it is easy to construct a not-closed subspace  $R$  such that  $P$  considered only on  $R$  is one-to-one, and  $PR$  is closed. Is it possible to find such an  $R$  so that also  $P$  considered only on  $R^c$  is of type IIB? This would require that all other complementary subspaces in  $R^c$  to  $X' \cdot R^c$ , as well as  $R$ , be incomplete. If this situation could be realized, we would have a transformation of type IIB as a section of a transformation of type IIA. Or it may be that  $R^c$  must always contain a closed complementary subspace to  $X' \cdot R^c$ , so that  $P$  only on  $R^c$  is also of type IIA.



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