CLEFT RINGS

BY
BERNARD VINOGRADE

A non-nilpotent associative ring R with minimum condition on left ideals (ML) has a nilpotent radical and is semisimple modulo the radical, that is, the minimum condition insures that R be semi-primary(1). In this paper the assumption is made that R is an associative ring with ML, and the main purpose is to launch an investigation of R when it is a *cleft* ring. R will be called *cleft* when it contains a subring $R^* \cong R/N$ such that $R = R^* + N$, a group direct sum.

Perhaps the most widely known results pertaining to cleft rings are those concerning linear associative algebras. Wedderburn proved that every algebra over a field of characteristic zero is cleft [14, p. 158]. More recently this theorem has been generalized to the theorem that an algebra is cleft if it is separable modulo its radical [7, p. 24]. Cleft algebras also suggest themselves as having simpler multiplication tables than uncleft algebras [1, p. 172].

Not only does the study of cleft rings appear to be a natural step toward the study of general rings with ML, but the cleavage property (property of being cleft) also permits the exploitation of methods used to prove the general structure theorems on semisimple rings [2]. The cleavage of R is equivalent to the cleavage of the completely primary rings defined by the primitive idempotents of R. This property is the key to the theory developed here.

If R is a cleft ring with a unit e, then a commutative group $V = eV$ with R as left operator set and a finite admissible composition series is shown to be a sum of vector spaces (composite module) over certain division rings derived from R. In particular R furnishes a composite representation module for itself. This representation is developed and finally applied to the case of algebras.

Regarding the notation used: When R has a unit e and is a left operator set for the module $V = eV$, then the inverse R-homomorphism ring of V will be denoted by R', and the application of an element r' of R' to V will be indicated by a right multiplication Vr'. Then V is an R-left and R'-right

Presented to the Society, April 18, 1942, under the title *Split rings and their representation theory*; received by the editors March 12, 1944. This paper is substantially a thesis accepted by the University of Michigan in May, 1942. The author wishes to express his gratitude to Professors C. J. Nesbitt and R. M. Thrall for their generous aid.

(1) See [3] or [10]. Adopting the notation of many writers on ring theory (Artin, Kothe, Deuring, and so on), semi-primary will mean that R/N is semisimple; primary, that R/N is simple; completely primary, that R/N is a division ring. In the latter two cases it is assumed that R has a unit. For conditions equivalent to the semi-primary property see [12]. Numbers in brackets refer to the Bibliography at the end of the paper.

module, or briefly, an (R, R') module. R'' will denote the R'-homomorphism ring of V, and is applied to the left of V. Since it will be assumed that $rV=0$ only if $r=0$, R will be isomorphic to a subring of R''.

Part I is devoted mainly to the properties of R as an abstract ring. Part II deals with the representation theory of R when it has a unit, and without loss of generality only rings which are indecomposable two-sided ideals are considered.

I

1. R is a ring with ML. If R can be expressed as a group direct sum $R=R^*+N$, where R^* is a semisimple ring and N is the radical, then R is a cleft ring.

THEOREM I. *If R^* is any semisimple subring of R, then R^* is isomorphic to a subring of $\overline{R}\equiv R/N$ and defines a cleft subring R^*+N of R.*

Proof. Let r^* and s^* be distinct elements of R^*. Then r^*-s^* is in R^* and hence not in N. Therefore $\bar{r}^*$ and $\bar{s}^*$ are distinct elements of $\overline{R}$. If $\overline{R}^*$ is the subset of $\overline{R}$ which can be represented by R^*, then $\overline{R}^*\cong R^*$, and if P is the totality of elements of R which define the elements of $\overline{R}^*$ then $P=R^*+N$ is a cleft ring.

If $R^*\cong\overline{R}$, and if R is an algebra over a field F and R^* a subalgebra (over F), then $R=R^*+N$; the converse is obvious. If r is an element of R which has an inverse, then rRr^{-1} gives another cleavage of R, for $R=rRr^{-1}=rR^*r^{-1}+rNr^{-1}$ and $rNr^{-1}=N$.

LEMMA 1. *If e^* is the unit of R^* in the cleft ring R, then e^* is a principal idempotent of R. If R has a unit, then e^* is that unit.*

Proof. If there were an idempotent $e'=r^*+n\neq 0$ such that $e'e^*=e^*e'=0$ then $e^*e'=e^*r^*+e^*n=0$ implies $e^*r^*=r^*=0$. Therefore $e'=n$, which is impossible unless $n=0$. Hence $e'=0$. If R has a unit e, then $(e-e^*)e^*=e^*(e-e^*)=0$, hence the idempotent $e-e^*=0$, that is, $e=e^*$.

Let $R=\sum Re_{ij}+N_0$ be a decomposition of R where Re_{ij} is an indecomposable left ideal and N_0 is a subset of N [7, p. 13]. The e_{ij} are primitive orthogonal idempotents, $e=\sum e_{ij}$ is a principal idempotent, and the notation is chosen so that $\overline{R}=\sum_{i=1}^{s}\overline{R}^i=\sum_{i=1}^{s}\sum_{j=1}^{n_i}\overline{R}\bar{e}_{ij}$ is the corresponding decomposition of the semisimple ring $\overline{R}$ into simple rings and sets of isomorphic left ideals respectively.

LEMMA 2. *The number of primitive orthogonal idempotents into which any principal idempotent of R can be decomposed is constant.*

Proof. n_i and s are unique for $\overline{R}$. Hence $\sum_{i=1}^{s}n_i$ is constant.

Let C_{ij} denote the completely primary ring $e_{ij}Re_{ij}$. Let (C_{ij}) and (C'_{ij}) be the sets of completely primary rings corresponding to two arbitrary principal idempotents $e=\sum e_{ij}$ and $e'=\sum e'_{ij}$ of R.

LEMMA 3. *The rings in the sets (C_{ij}) and (C'_{ij}) are isomorphic, in the order for instance of the subscripts.*

Proof. From the uniqueness of $\bar{R}^i$ it follows that $\bar{R}^i=\sum_{k=1}^{n_i}\bar{R}\bar{e}_{ik}=\sum_{k=1}^{n_i}\bar{R}\bar{e}'_{ik}$, where $\bar{R}\bar{e}_{ik}\cong\bar{R}\bar{e}'_{il}$ for every k and l. Now, it is known that an isomorphism between two indecomposable left ideals Re_1 and Re_2 as R left spaces (e_1 and e_2 are primitive idempotents) is equivalent to the isomorphism of $\bar{R}\bar{e}_1$ and $\bar{R}\bar{e}_2$ as $\bar{R}$ left spaces [12, p. 182]. But $\bar{R}\bar{e}_{ik}\cong\bar{R}\bar{e}'_{il}$ not only as an $\bar{R}^i$ space but as an $\bar{R}$ space, because $\bar{R}^i\bar{R}^j=\delta_{ij}\bar{R}^i$. Hence $Re_{ik}\cong Re'_{il}$ as R left spaces. Further, the inverse homomorphism rings of these two ideals are isomorphic to C_{ik} and C'_{il} respectively; for if $r\in Re_{ik}$ and σ is a mapping in the inverse homomorphism ring, $\sigma(r)=r\sigma(e_{ij})=re_{ij}\sigma(e_{ij})e_{ij}$; hence $C_{ik}\cong C'_{il}$.

LEMMA 4. *If each C_{ij} of (C_{ij}) is cleft, then each C'_{ij} of (C'_{ij}) is cleft.*

Proof. The proof follows from Lemma 3.

According to a known decomposition theorem [7, p. 17] R can be written as $R=\sum_{i=1}^{s}P^i+N_0$ corresponding to the decomposition of $r\in R$ in the expression $r=ere+(r-ere)$, where e is a principal idempotent. $P^i\equiv e^iRe^i$, where $e^i=\sum_{j=1}^{n_i}e_{ij}$, is a primary ring such that $\bar{P}^i\equiv P^i/e^iNe^i\cong\bar{R}^i$. Since the elements of N_0 are of the form $\sum_{j\neq k}e^jre^k+r-ere$, then $e^iN_0e^i=e_{ij}N_0e_{ij}=0$.

Let e_i be an arbitrary idempotent of the set e_{ij}, $j=1, \cdots, n_i$; let C_i be the corresponding ring of the set (C_{ij}); let $N^i\equiv e^iNe^i$; let $(Q)_n$ be a total matric set of degree n over a set Q, ϵ_{ij} the matric units such that $\epsilon_{ij}\epsilon_{kl}=\delta_{jk}\epsilon_{il}$ and $q\epsilon_{ij}=\epsilon_{ij}q$ for $q\in Q$, hence $(Q)_n=\sum Q\epsilon_{kl}$.

LEMMA 5. *If R is cleft, then every C_{ij} is cleft.*

Proof. Choose, by Lemma 1, the unit of R^* for e and decompose it in R^*. Then from $R^*=R+N$ it follows that $C_i=e_iR^*e_i+e_iNe_i$. Now $e_iR^*e_i$ is a division ring $K_i\cong\bar{C}_i\equiv C_i/e_iNe_i$ and e_iNe_i is the radical of C_i. Hence C_i is cleft.

LEMMA 6. *The radical of $\sum P^i+N_0$ is $\sum N^i+N_0$.*

Proof. N_0 and the N^i are contained in N because of their definition. On the other hand, suppose $n\in N$, $n=\sum p^i+n_0$ where $p^i\in P^i$ and $n_0\in N_0$. Then $e^ine^i=p^i\in N^i$ for every i. Hence $n\in\sum N^i+N_0$.

LEMMA 7. $P^i\cong(C_i)_{n_i}$.

Proof. $P^i=\sum_{k=1}^{n_i}e_{ik}P^i$, where the $e_{ik}P^i$ are isomorphic indecomposable right ideals. P^i is its own homomorphism ring as a P^i right space, and C_{ik} is the homomorphism ring of $e_{ik}P^i$ as a P^i right space. Hence, by a known theorem [7, p. 18], $P^i\cong(C_i)_{n_i}$.

LEMMA 8. *$e_iNe_i\epsilon_{kl}$ is the intersection of $C_i\epsilon_{kl}$ and the radical of $(C_i)_{n_i}$.*

Proof. If $e_ire_i \in e_iNe_i$, then $e_ire_i = e_ine_i$ for some $n \in N$. Suppose there is no n such that $e_ire_i = e_ine_i$, then $e_ire_i \notin e_iNe_i$. Then there exists $\rho \in C_i$ such that $e_ire_i\rho$ is not nilpotent, and therefore $e_ire_i\epsilon_{kl}\rho\epsilon_{lk} = e_ire_i\rho\epsilon_{kk}$ is not nilpotent. Therefore $e_ire_i\epsilon_{kl}$ is not in the radical of $(C_i)_{n_i}$. On the other hand, $e_ine_i\epsilon_{kl}$ is properly nilpotent in $(C_i)_{n_i}$.

LEMMA 9. *$(e_iNe_i)_{n_i}$ is the radical of $(C_i)_{n_i}$.*

Proof. By Lemma 8, $(e_iNe_i)_{n_i}$ is contained in the radical of $(C_i)_{n_i}$. Now let n be an element of the radical of $(C_i)_{n_i}$. Then $n = \sum c_i^{kl}\epsilon_{kl}$ and so $e_i\epsilon_{kk}ne_i\epsilon_{ll} = c_i^{kl}\epsilon_{kl}$. Hence $c_i^{kl}\epsilon_{kl}$ is an element of the radical of $(C_i)_{n_i}$. But $c_i^{kl} \in C_i$, and therefore, by Lemma 8, c_i^{kl} is an element of e_iNe_i.

LEMMA 10. *If every C_{ij} is cleft, then R is cleft.*

Proof. Since $C_i = K_i + e_iNe_i$, then $(C_i)_{n_i} = (K_i)_{n_i} + (e_iNe_i)_{n_i}$. But it is known that $(K_i)_{n_i}$ is simple. Also, by Lemma 9, $(e_iNe_i)_{n_i}$ is the radical of $(C_i)_{n_i}$. And by Lemma 7, $P^i \cong (C_i)_{n_i}$, so P^i is cleft, that is, $P^i = S^i + N^i$ where $S^i \cong (K_i)_{n_i}$ and $N^i \cong (e^iNe^i)_{n_i}$. Then from $R = \sum P^i + N_0$ it follows that $R = \sum S^i + \sum N^i + N_0$. Since $P^iP^j = \delta_{ij}P^i$, then $R^* = \sum S^i$ is a semisimple ring. Moreover, by Lemma 6, $N = \sum N^i + N_0$. Hence R is a cleft ring $R^* + N$.

THEOREM II. *The condition that R be cleft is equivalent to the condition that every C_{ij} of an arbitrary set (C_{ij}) be cleft.*

Proof. The proof follows from Lemmas 5 and 10.

LEMMA 11. *If R contains a division ring K and any idempotent e such that $eK = Ke \neq 0$, then $eKe \cong K$.*

Proof. $eK = Ke \neq 0$ implies that $ek = k'e$ is an automorphism of K: $k \rightleftarrows k'$. It also implies that $ek = eke$. Similarly, $ek'' = ke$ implies that $ke = eke$. Hence $ke = ek$. Therefore $k \rightleftarrows eke$ is an isomorphic mapping of K onto eKe.

In particular, if $e = \sum e_{ij}$ is a principal idempotent such that $e_{ij}K = Ke_{ij} \neq 0$ for every i, j, then each $e_{ij}Ke_{ij} \cong K$.

THEOREM III. *If R is an algebra over a field F and contains division subalgebras (over F) G_i, $i = 1, \cdots, s$, such that $e_{ij}G_i = G_ie_{ij} \neq 0$ and $G_i \cong \overline{C}_{ij}$ for at least one j for every i, then R is cleft, and conversely.*

Proof. By Lemma 11, $C_i \supseteq e_iG_ie_i \cong G_i \cong \overline{C}_i$, $i = 1, \cdots, s$. Hence $C_i = e_iG_ie_i + e_iNe_i$ shows that C_i is cleft. Therefore every C_{ij} is cleft and, by Theorem II, R is cleft. On the other hand, if R is cleft, then take $G_i \equiv K_i$.

LEMMA 12. *Let A be an algebra over a field F and let $G_i \equiv Fe_i$ the set of scalar multiples of the idempotent e_i. Then $e_iG_ie_i = G_i \cong F$.*

COROLLARY 1. *If $\overline{A} \equiv A/N$ is a ring direct sum of total matric rings over F, then A is cleft* [1, p. 47].

Proof. $P^i \cong (C_i)_{n_i}$ by Lemma 7, and $\overline{P}^i \cong \overline{A}^i$. Hence $\overline{A}^i \cong (\overline{C}_i)_{n_i}$ and $\overline{C}_i \cong F$. Therefore, by Lemma 12, $\overline{C}_i \cong e_i F e_i \equiv G_i$. By Theorem III it now follows that A is cleft.

COROLLARY 2. *If K is a splitting field of $\overline{A}$ over F, then A_K is cleft. (If F is algebraically closed take $K \equiv F$.)*

Proof. $(\overline{A})_K$ is a ring direct sum of total matric rings over F, hence, by Corollary 1, A_K is cleft.

2. The following discussion about algebras is of interest in light of the known theorem [7, p. 24] that the separability of A/N implies that A is cleft. Let $Z(R)$ denote the center of R.

THEOREM IV. *The separability of A/N is equivalent to the separability of each C_{ij} in the set (C_{ij}).*

Proof. For any ring, $\overline{R}^i \cong (\overline{C}_i)_{n_i}$ and $Z((\overline{C}_i)_{n_i}) \cong Z(\overline{C}_i)$. Hence if R is an algebra A then the separability of $Z(\overline{A}^i)$ is equivalent to the separability of $Z(\overline{C}_i)$. Now, it is known that the separability of a semisimple algebra is equivalent to the separability of its center [1, p. 44]. Therefore the separability of A/N is equivalent to the separability of $Z(\overline{A}^i)$, and the separability of $\overline{C}_i$ is equivalent to the separability of $Z(\overline{C}_i)$. This implies the theorem.

If A is a simple algebra over F, then the separability of A is implied by the centrality of A [1, p. 43]. If A is a primary (or completely primary) algebra over F, then A/N is separable if F is isomorphic to the center of A/N, which implies (besides the fact that A is cleft) that A is an algebra over the center of A/N. On the other hand, if A is cleft, then A/N need not be separable.

3. The expression $R = \sum P^i + N_0$ yields at least two interesting decompositions of R in which the cleavage of P^i has a role. When R is not cleft, it may be said to be *partially cleft* if the R^* of the following theorem is not vacuous.

THEOREM V A. *$R = R^* + (R^{s,t} \cup N)$, where R^* is a semisimple ring (unique within isomorphisms) and $R^{s,t}$ is a ring with unit whose radical is $R^{s,t} \cap N$.*

Proof. Rearrange the P^i in $\sum P^i + N_0$ so that the cleft ones come first, that is, $P^i = S^i + N^i$ for $i = 1, \cdots, t$ only. Then $R^* = \sum_{i=1}^t S^i$ is semisimple and unique within isomorphism because the P^i are. Let $N^* = \sum_{i=1}^s N^i$ and $R^{s,t} = \sum_{i=t+1}^s P^i$. Then $R^{s,t} + N^* + N_0 = R^{s,t} \cup N$. Hence the theorem.

THEOREM V B. *The P^i for which $e^i N = N e^i = 0$ form a unique semisimple ring.*

Proof. From the properties of R it follows that $\overline{R}^i = \overline{P^i \cup N} = \sum_{j=1}^{n_i} \overline{R}\bar{e}_{ij} = \sum_{j=1}^{n_i} \overline{e_{ij} R}$, and that the isomorphism as R left spaces of the $\overline{R}\bar{e}_{ij}$ is equivalent to the isomorphism as left ideals of the Re_{ij} for $j=1, \cdots, n_i$, and similarly for the $\overline{e_{ij}R}$ and the $e_{ij}R$ (see proof of Lemma 3). Now consider e_{ik} and e_{il} for a fixed i. Suppose $Ne_{ik}=0$ and in an isomorphic mapping as R left spaces of Re_{ik} on Re_{il} that $re_{ik}\rightleftarrows e_{il}$. Then $nre_{ik}\rightleftarrows ne_{il}$ for $n\in N$. But $nr\in N$, hence $nre_{ik}=0$ and $ne_{il}=0$. Therefore $Ne_{il}=0$. Similarly, $e_{ik}N=0$ implies $e_{il}N=0$; and $e_{ik}N=Ne_{ik}=0$ implies $e_{il}N=Ne_{il}=0$. Now define the following rings:

$R^* =\sum P^i$, where $e^iN=Ne^i=0$,
$R_1 =\sum P^i$, where $e^iN=0\neq Ne^i$,
$R_2 =\sum P^i$, where $e^iN\neq 0=Ne^i$,
$R_3 =\sum P^i$, summed over the remaining P^i,
$P =R_1+R_2+R_3+N_0=R_1+R_2+(R_3\cup N)$.

Then $R=R^*+P$ and $R^*P=PR^*=0$. R_1 and R_2 are semisimple, R_3 has a unit, its radical is $R_3\cap N$, and $R_3\cup N$ is a ring, R_1, R_2, and R_3 are orthogonal. Indeed, so far this decomposition seems to be like that obtained by M. Hall [9]. To show that it actually is identical it is only necessary to prove that P is a bound ring, that is, the only elements of P which are two-sided annihilators of N are in N. Consider an element p of one of the P^i which appear in P such that $pN=Np=0$. P^ipP^i is a two-sided ideal of P^i and the elements in the classes of $(P^ipP^i\cup N^i)/N^i$ must coincide with the elements of the classes of a two-sided ideal of P^i/N^i. Therefore the non-radical elements of $P^ipP^i\cup N^i$ and therefore of P^ipP^i must include an idempotent e such that $eN=0$ and $Ne=0$ could not both be true, which contradicts the fact that the elements of P^ipP^i are two-sided annihilators of N. Therefore P^ipP^i has no non-radical elements, in particular $e^ipe^i=p\in N^i$. This shows that P is bound. The fact that the decomposition $R=R^*+P$ is unique can now be easily proved exactly as Hall did it [9, p. 393].

II

1. Although many of the results below can be proved in greater generality, the following conditions will be assumed throughout: R is a cleft ring with unit e; $V=eV=Vf$, where f is the unit of R', is an R left module with a finite composition series as an R left module and a finite composition series as an R' right module; $rV=0$ implies $r=0$; R' is cleft and has MR (minimum condition on right ideals); and finally, $R''=R$. These conditions are known to hold if R is a semisimple ring with ML and if $V=eV$ has minimum condition on R left sub-modules [11, p. 70].

Consider a particular cleavage R^*+N. By Lemma 1 the unit of R^* is the unit e of R. Hereafter e will be decomposed in R^*: $e=\sum e_{ij}$ with e_{ij} from $R^* =\sum R^*e_{ij}$. From the meaning of R' and R'' it follows that $e_{ij}V$ and Vf_{ij} are not zero for any i, j, where $f=\sum f_{ij}$.

Let $V\equiv V_1\supset V_2\supset \cdots \supset V_{n+1}=0$ be a composition series of V as an R left module. Hence $V_t/V_{t+1}\equiv\overline{V}_t$ is irreducible, that is, there exists an $\bar{x}\neq 0$

in $\overline{V}_t$ such that $R\bar{x}=\overline{V}_t$. Or, since $R=R^*+N$ and $N\overline{V}_t=0$, it follows that $R^*\bar{x}=\overline{V}_t$.

Let $W=RW$ be any irreducible R left module. Then $W=R^*W=\sum R^*e_{ij}W$. Each $R^*e_{ij}W=0$ or W. Hence $W\cong R^*e_{ij}$ as R^* left module for some i and for $j=1, \cdots, n_i$ by the correspondence $e_{ij}\rightleftarrows e_{ij}w_j$ where $w_j\in W$.

Suppose $\overline{V}_t$ is not isomorphic to R^*e_{ij}. Consider $e_{ij}V_t$. Since V_t properly contains V_{t+1}, then $e_{ij}V_t\supseteq e_{ij}V_{t+1}$. If $e_{ij}V_t$ properly contains $e_{ij}V_{t+1}$, then there exists a nonzero $\bar{x}=e_{ij}\bar{x}$ in $e_{ij}\overline{V}_t$, and $R^*e_{ij}\bar{x}=\overline{V}_t$. This implies that $\overline{V}_t\cong R^*e_{ij}$, contrary to assumption. So $e_{ij}V_t=e_{ij}V_{t+1}$. If necessary, treat V_{t+1} in the same way, continuing until a module V_{t+n} is finally reached such that $\overline{V}_{t+n}\cong R^*e_{ij}$ or $\overline{V}_{t+n}=V_{n+1}=0$.

Now, for each $V_t\neq 0$ there exists an i such that $\overline{V}_t\cong R^*e_{ij}$, $j=1, \cdots, n_i$ for a fixed i. Let σ_i equal the number of V_t for which this is true. $R^*e_{ij}\overline{V}_t=\overline{V}_t$, $j=1, \cdots, n_i$.

For different idempotents $e_{ij}V_t\neq e_{kl}V_t$ if $e_{ij}V_t\neq 0$. If $\overline{V}_t\cong R^*e_{ij}$ then $e_{ij}\overline{V}_t\neq 0$ and there exists a nonzero x_{ijt} in V_t such that $R^*e_{ij}\bar{x}_{ijt}=\overline{V}_t$, or $V_t=R^*e_{ij}x_{ijt}+V_{t+1}$. Hence $e_{ij}V_t=K_{ij}x_{ijt}+e_{ij}V_{t+1}$, where $K_{ij}\equiv e_{ij}R^*e_{ij}$. This proves the following lemma:

Lemma 13. *$e_{ij}V=\sum e_{ij}V_t=\sum_{t=1}^{\sigma_i}K_{ij}x_{ijt}$ (first summation over $\overline{V}_t\cong R^*e_{ij}$, $j=1, \cdots, n_i$). Similarly, $Vf_{ij}=\sum_{t=1}^{\tau_i}y_{ijt}L_{ij}$, where L_{ij} and y_{ijt} are the counterparts in R' and V of K_{ij} and x_{ijt}.*

Lemma 14. *$\sum_{t=1}^{\sigma_i}K_{ij}x_{ijt}$ is a group direct sum.*

Proof. Suppose $\sum_{t=1}^{\sigma_i}k_{ij}^t x_{ijt}=0$. Then if k_{ij}^r is the first nonzero coefficient, x_{ijr} is expressible as a linear combination of elements from sub-modules of the composition series for V which are below the sub-module from which $x_{ijr}\neq 0$ was chosen. This is impossible, hence $k_{ij}^r=0$.

Theorem VI. *$V=\sum_{ijt}e_{ij}V_t=\sum_{ijt}K_{ij}x_{ijt}$ is a group direct sum.*

Proof. $V=eV=\sum e_{ij}V$. Then the theorem follows by Lemmas 13 and 14.

Theorem VII. *$e_{ij}V$ is an indecomposable R' right module and $C_{ij}\equiv e_{ij}Re_{ij}$ is its homomorphism ring. Vf_{ij} is an indecomposable R left module and $C'_{ij}\equiv f_{ij}R'f_{ij}$ is its homomorphism ring* (see [8, p. 533]).

Proof. Any mapping σ of $e_{ij}V$ into itself as an R' right module is completely specified by $\sigma(e_{ij}v)=r''(e_{ij}v)=e_{ij}r''e_{ij}v$ for some $r''\in R''$. But $R=R''$ insures that $r''\in R$. Conversely every element of C_{ij} is easily seen to give a mapping σ.

Also, every mapping σ of Vf_{ij} into itself is by definition of R' given by some $r'\in R'$. Hence $\sigma(vf_{ij})=vf_{ij}r'=vf_{ij}r'f_{ij}$.

Now suppose $e_{ij}V=e_{ij}V_1+e_{ij}V_2$ is a decomposition of $e_{ij}V$. Then C_{ij}, the homomorphism ring of $e_{ij}V$, would contain two orthogonal idempotents, one which leaves $e_{ij}V_1$ undisturbed but sends $e_{ij}V_2$ into zero and one which leaves $e_{ij}V_2$ undisturbed but sends $e_{ij}V_l$ into zero. This is a contradiction. A similar argument holds for Vf_{ij}.

THEOREM VIII. *An isomorphism between $e_{ij}V$ and $e_{kl}V$ as R' right modules is equivalent to an isomorphism between $e_{ij}R$ and $e_{kl}R$ as right ideals of R. An isomorphism between Vf_{ij} and Vf_{kl} as R left modules is equivalent to an isomorphism between $R'f_{ij}$ and $R'f_{kl}$ as left ideals of R'.* (See [8, p. 530] *for a similar theorem for groups.*)

Proof. Any mapping $e_{ij}V \rightleftarrows e_{kl}V$ as R' right modules is given by an element of $e_{ij}Re_{kl}$. Suppose $e_{ij}V \cong e_{kl}V$. Then there exist elements r_{ik} and r_{ki} of $e_{ij}Re_{kl}$ and $e_{kl}Re_{ij}$ respectively such that $r_{ik}V = e_{ij}V$ and $r_{ik}r_{ki}V = e_{ij}V$ and $r_{ik}r_{ki} = e_{ij}$. Now consider the mapping $e_{ij}r \rightleftarrows r_{ki}r$, $r \in R$. $r_{ki}r = 0$ would imply $r_{ik}r_{ki}r = e_{ik}r = 0$. Hence the mapping defines an isomorphism between $e_{ij}R$ and $e_{kl}R$ as right ideals. On the other hand, suppose $e_{ij}R \cong e_{kl}R$. Then there exists an element r_{ki} which will effect the mapping, and an element r_{ik} such that $r_{ik}r_{ki} = e_{ij}$. Then the mapping $e_{ij}v \rightleftarrows r_{ki}v$, $v \in V$, maps $e_{ij}V$ onto $e_{kl}V$ isomorphically as R' modules.

COROLLARY. *For a particular decomposition of e, the number of isomorphic indecomposable modules $e_{ij}V$ equals the number of isomorphic right ideals $e_{ij}R$ or left ideals Re_{ij}, and hence equals the number of minimal right or left ideals in R^{*i}.*

Now any irreducible R^{*i} left module which is isomorphic to R^*e_{ij} has a K_{ij} right dimension equal to the number n_i of minimal left or right ideals of R^{*i}, and n_i is the dimension of the Wedderburn representation([2]) of R^{*i} [2].

THEOREM IX. *The results of this section can be summarized in a table as follows*:

Spaces of V	Space	Multiplicity	Dimension
Isomorphic indecomposable R left modules	Vf_{ij}	n_i'	τ_i over L_{ij}
Irreducible R' right modules	$\overline{V}_l' \cong f_{ij}R'^*$	τ_i	n_i' over K_{ij}
Isomorphic indecomposable R' right modules	$e_{ij}V$	n_i	σ_i over K_{ij}
Irreducible R left modules	$\overline{V}_l \cong R^*e_{ij}$	σ_i	n_i over L_{ij}

As an illustration, let R be a matric algebra over an algebraically closed field F, and suppose $R = R''$. Then R is cleft, as is its commuting algebra R'.

([2]) We refer to the representation of a simple ring R by a total matric set over a division ring K as the Wedderburn representation of R, and K is called a Wedderburn division ring for R.

Let V be a column vector space with coefficients from F. The modules of Theorem IX will reflect the indecomposable and irreducible constituents of R and R'. Furthermore F is the Wedderburn division ring of all the irreducible constituents. Hence we have the following relations: To every indecomposable constituent I_i of R corresponds an irreducible constituent G_i of R'; to every indecomposable constituent J_i of R' corresponds an irreducible constituent F_i of R. There are n_i' constituents I_i and n_i' is the degree of G_i; there are n_i constituents J_i and n_i is the degree of F_i(3).

2. Hereafter let the ring R be the module V. Assume that R has MLR(4); then it must have MaLR(5) [10, p. 726; 11, p. 71]. Hence it has a finite composition series of ideals, left or right [11, p. 71]. The elements of R, R', and R'' may be identified. Without loss of generality R will always be assumed an indecomposable two-sided ideal.

When speaking of a module let (K, R) denote a set of left operators K and right operators R.

$e_{ij}V$ and Vf_{ij} are now $e_{ij}R$ and Re_{ij} respectively.

Lemma 15. *$e_{ij}N$ and Ne_{ij} are the unique maximal sub-ideals of $e_{ij}R$ and Re_{ij} respectively* [13, p. 362].

By Theorems VI and VII, $e_{ij}R=\sum K_{ij}x_{ijt}$ is an indecomposable (K_{ij}, R) space. Because it is a right ideal of R it has a finite composition series of (K_{ij}, R) modules, and by Lemma 15 such a series will start out as $e_{ij}R\supset e_{ij}N \equiv V_2\supset \cdots$. Each composition factor module will contain a finite number of K_{ij} left independent elements in terms of which a residue system can be expressed as linear combinations over K_{ij}. In fact a set of residue systems exists which will serve in all composition series [6, p. 505]. Hence, if a column vector X_{ij}, formed from K_{ij} left independent elements of all the composition factor modules is right multiplied by R: $X_{ij}R=I_{ij}X_{ij}$, then the representation I_{ij} is similar to that due to any other composition series. Schematically,

$$I_{ij} = \begin{pmatrix} \overset{1}{F}_{ij} & & & & * \\ & \cdot & & & \\ & & \cdot & & \\ & & & \overset{p}{F}_{ij} & \\ & & & & \cdot \\ & & & & \cdot \\ 0 & & & & \cdot \end{pmatrix}, \text{ of degree } \sigma_i.$$

If X_{ij} is inverted then the zero and the star reverse.

Lemma 16. *For a suitable basis X_{ij}, I_{ij} contains*

(3) This proves a theorem due to Brauer [4].

(4) MLR ≡ minimum condition on left and right ideals; MaLR ≡ maximum condition on left and right ideals.

$$\begin{pmatrix} F^1_{ij} & & & & & 0 \\ & \cdot & & & & \\ & & \cdot & & & \\ & & & F^p_{ij} & & \\ & & & & \cdot & \\ 0 & & & & & \cdot \end{pmatrix}.$$

Proof. Form the composition series from a refinement of the Loewy series $e_{ij}R^*+e_{ij}N\supset e_{ij}N\supset e_{ij}N^2\supset \cdots$. Each factor module of this Loewy series is a (K_{ij}, R^*) module and is right annihilated by N, and a refinement of it to a composition series of (K_{ij}, R^*) modules will give the desired result.

COROLLARY. *For X_{ij} chosen this way, I_{ij} is a group direct sum of*

$$I_{ij}(R^*) = \begin{pmatrix} F^1_{ij} & & & & & 0 \\ & \cdot & & & & \\ & & \cdot & & & \\ & & & F^p_{ij} & & \\ & & & & \cdot & \\ 0 & & & & & \cdot \end{pmatrix} \quad \text{and} \quad I_{ij}(N) = \begin{pmatrix} 0 & & & * \\ & 0 & & \\ & & \cdot & \\ 0 & & & \cdot \end{pmatrix}.$$

LEMMA 17. *The irreducible constituent F^1_{ij} is a Wedderburn representation of R^{*i}, that is, $F^1_{ij}=(K_{ij})_{n_i}$.*

Proof. $e_{ij}R^*=e_{ij}R^{*i}$ is a K_{ij} left module of dimension n_i over K_{ij}, where n_i is the number of minimal left or right ideals of R^{*i}. In the composition series for $e_{ij}R$, choose a basis X^*_{ij} of $e_{ij}R^*$ as the residue system of $e_{ij}R/e_{ij}N$. Then $X^*_{ij}R^{*i}=F^1_{ij}X^*_{ij}$. But this gives a Wedderburn representation of R^{*i} [2].

COROLLARY. *For every $i=1, \cdots, s$ there exists an $F^1_{ij}\cong R^{*i}$.*

Proof. $e_{ij}R\neq 0$ for any i.

LEMMA 18. *F^p_{ij} is an irreducible representation over K_{ij} of R^{*k} for some k.*

Proof. Any irreducible (K_{ij}, R^*) module is annihilated by all except one of the R^{*i} and hence gives a representation of one of them.

Hereafter X_{ij} will be chosen from the composition series indicated in Lemma 16.

If the whole composite representation module $R=\sum e_{ij}R=\sum_{ijt}K_{ij}x_{ijt}$ be considered with X_{ij} for each $e_{ij}R$ chosen as agreed upon, then the first composite representation is generated:

$$\mathfrak{R} = \begin{pmatrix} I_{11} & & & 0 \\ & \cdot & & \\ & & \cdot & \\ 0 & & & I_{sn_s} \end{pmatrix}.$$

A second representation

$$\mathfrak{S} = \begin{pmatrix} J_{11} & & 0 \\ & \ddots & \\ 0 & & J_{sn_s} \end{pmatrix}$$

is generated by the ideals Re_{ij}: $RY_{ij} = Y_{ij}J_{ij}$, where

$$J_{ij} = \begin{pmatrix} G_{ij}^1 & & & & * \\ & \ddots & & & \\ & & G_{ij}^p & & \\ & & & \ddots & \\ 0 & & & & \cdot \end{pmatrix}, \text{ of degree } \tau_i.$$

Because of the choice of the columns X_{ij}, $\mathfrak{R}$ can be written as $\mathfrak{R}^* + \mathfrak{N}$ where $\mathfrak{R}^*$ is a diagonal representation of R^*. Similarly for $\mathfrak{S}$ because of the choice of the rows Y_{ij}.

THEOREM X. *The following table summarizes some of the properties of* $\mathfrak{R}$ *and* $\mathfrak{S}$.

	I_{ij}	F_{ij}^p	J_{ij}	G_{ij}^p
Number of distinct	s	s	s	s
Number of isomorphic	n_i	$\leqq \tau_i$	n_i	$\leqq \sigma_i$
Degree (for $p=1$)	σ_i	n_i	τ_i	n_i
Total of all kinds	$\sum n_i$	$\leqq \sum \tau_i$	$\sum n_i$	$\leqq \sum \sigma_i$

Remarks. (1) $e_{kl}Re_{ij}$ is a K_{ij} right module of finite dimension.

Proof. It is a K_{ij} sub-module of Re_{ij}.

(2) The number of times an irreducible representation of R^{*i} occurs in I_{kl} is less than or equal to the dimension of $e_{kl}Re_{ij}$ as a K_{ij} right module. Similarly, the representation of R^{*i} occurs in J_{kl} with multiplicity less than or equal to the dimension of $e_{ij}Re_{kl}$ as a K_{ij} left module [13, p. 364].

Proof. Apply the procedure of §1 but with $V = e_{kl}R$.

(3) Any two division rings K_{ij} and K_{kl} have representations over one another.

Proof. Every F_{kl}^p is isomorphic to some R^{*i} and hence to some F_{ij}^1. The existence of F_{kl}^p implies $e_{kl}Re_{ij} \neq 0$ by Remark 2. Also $e_{kl}Re_{ij}$ is a K_{ij} right and K_{kl} left module of finite dimension both ways by Remark 1. Hence it is a representation module for K_{ij} over K_{kl} and vice versa. Now, by use of a known theorem [13, p. 364] every K_{ij} and K_{kl} can be chained by a sequence

K_{ij}, K_{ab}, K_{cd}, $\cdots$, K_{st}, K_{kl} such that K_{ij} is represented over K_{ab}, K_{ab} over K_{cd}, $\cdots$, K_{st} over K_{kl}.

3. LEMMA 19. *$e_{ij}R^*e_{ik}$ does not vanish and is of left dimension one over K_{ij}.*

Proof. In the decomposition $R^{*i}=\sum_{r=1}^{n_i} e_{ir}R^*$, $e_{ij}R^*\cong e_{ik}R^*$ for every j, k; and in a mapping σ of the two ideals the image of e_{ik} is $\sigma(e_{ik})=\sigma(e_{ik})e_{ik}=e_{ij}\sigma(e_{ik})e_{ik}$, $\sigma(e_{ik})$ in R^{*i}, hence $e_{ij}R^*e_{ik}\neq 0$. Now consider the sum $e_{ij}R^*=\sum_{t=1}^{n_i} e_{ij}R^*e_{it}$. None of the summands vanishes and each is a K_{ij} left module of dimension one, for the dimension of $e_{ij}R^*$ over K_{ij} is exactly n_i.

Because of this lemma a basis X_{ij}^* of $e_{ij}R^*$ can be chosen as follows: $x_{ij1}e_{i1}$, $x_{ij2}e_{i2}$, $\cdots$, $x_{ijj-1}e_{ij-1}$, e_{ij}, $x_{ijj+1}e_{ij+1}$, $\cdots$, $x_{ijn_i}e_{in_i}$. Extend this to a basis X_{ij} of $e_{ij}R$ in accordance with the stipulations of §2. This is the basis that will be used hereafter.

Let ρ_{ij} denote the top n_i rows of I_{ij}.

LEMMA 20. *There exist elements of $\mathfrak{R}$ with arbitrary coefficients in ρ_{ij} for any i, j.*

Proof. The jth basis element in X_{ij} is e_{ij}. Hence the jth line of coefficients in I_{ij} is arbitrary over K_{ij}. Furthermore, F_{ij}^1 is complete (a total matric set) over K_{ij}. Therefore there is an element A in $\mathfrak{R}^*$ with zeros everywhere in F_{ij}^1 except for e_{ij} in the j, jth place. Let M be an element of $\mathfrak{R}$ with arbitrary coefficients in the jth line, then AM will have these same coefficients in the jth line again but zeros elsewhere in ρ_{ij}. Now, there is an element B in $\mathfrak{R}^*$ with zeros everywhere in F_{ij}^1 except for e_{ij} in the t, jth place. Then BAM will have zeros everywhere in ρ_{ij} except for arbitrary coefficients in the tth line.

LEMMA 21. *ρ_{ij} is independent of ρ_{kl} if $k\neq i$.*

Proof. $X_{ij}^*R=X_{ij}^*\sum_{t,s}e_{ts}R=X_{ij}^*\sum_s e_{is}R$, and $X_{kl}^*\sum_s e_{is}R=0$ *for* $k\neq i$.

LEMMA 22. *No nonzero element of $\mathfrak{R}$ vanishes in every ρ_{ij} simultaneously.*

Proof. If r vanished in all ρ_{ij} then $e_{ij}r=0$ for all i, j; hence $\sum_{i,j}e_{ij}r=er=r=0$.

COROLLARY. *Every element of $\mathfrak{R}$ has some nonzero coefficient in some ρ_{ij}.*

LEMMA 23. *If $e_{i1}R^*r=0$, where $r\in R$, then $e_{ij}R^*r=0$ for $j=1, \cdots, n_i$.*

Proof. $e_{i1}R^*r=0$ implies that $e_{i1}s^*r=0$ where s^* is any element of R^{*i}. Now $e_{i2}R\cong e_{i1}R$ as R right modules implies $e_{i2}\rightleftarrows e_{i1}t^*$ and hence $e_{i2}r\rightleftarrows e_{i1}t^*r$. Therefore $e_{i2}r=0$. Similarly $e_{i3}r=e_{i4}r=\cdots=0$. Now a glance at the structure of X_{ij}^* shows that $e_{ij}R^*r=0$, $j=1, \cdots, n_i$.

COROLLARY. *ρ_{ij}, $j=1, \cdots, n_i$, all vanish if any one does.*

The elements of $\mathfrak{R}$ can evidently be separated into s classes $\rho_1, \cdots, \rho_s$, where ρ_k consists of elements with zeros in every ρ_{ij} except those with $i=k$. (From the structure of X_{ij} it can be seen that ρ_k represents the right ideal e^kR.) If $E=\sum_{i=1}^{s}E_i$ is a decomposition of the unit of $\mathfrak{R}^*$, then $\mathfrak{R}=\sum\rho_i=E\sum\rho_i=\sum E_i\rho_i$, since $E_j\rho_i=0$ if $i\neq j$. So instead of ρ_k, $E_k\rho_k$ may be used for elements which vanish in all ρ_{ij} except for $i=k$. Thus the revised ρ_k has coefficients only in rows which intersect the diagonal in a representation of R^{*k}.

Define further as the set χ_{i1} all elements in $\mathfrak{R}^*$ (hence in ρ_i) which have $n_i\times K_{i1}$ in the F_{i1}^1 constituent. Obviously $\chi_{i1}\cong K_{i1}$.

Finally let T_{i1}^{uv} be the element of $\mathfrak{R}$ with e_{i1} in the u, v place of ρ_{i1}; $u=1, \cdots, n_i$; $v=1, \cdots, \sigma_i$.

THEOREM XI. $\rho_i=\sum_{u,v}\chi_{i1}T_{i1}^{uv}$.

Proof. If any element of ρ_i has no coefficient in ρ_{i1} then it is zero by the corollary to Lemma 22.

COROLLARY. $\mathfrak{R}=\sum_{i,u,v}\chi_{i1}T_{i1}^{uv}$.

Consider now the rings C_{ij} and P^i. The following results characterize all completely primary and primary rings which are cleft and have MLR.

First specializing to the case of C_{ij}: There will be just one indecomposable representation, call it I. And since $C_{ij}=K_{ij}+e_{ij}Ne_{ij}=\sum_{t=1}^{\sigma}K_{ij}x_{ijt}$, the basis X will start out with $X^*=e_{ij}$. Hence $\rho_{i1}\equiv\rho$ is a single line and is arbitrary over K_{ij} because $e_{ij}C_{ij}=C_{ij}$. Each $F_{ij}^p\equiv F^p$ is a representation of K_{ij} with coefficients from K_{ij}. Call the elements of $\rho_1\equiv\rho$: $T^1, \cdots, T^\sigma$. $\chi_{i1}\equiv\chi$ is just the representation of K_{ij} and has K_{ij} itself in the F^1 place.

THEOREM XII. *C_{ij} is isomorphic to the indecomposable matric set $I(C_{ij})=\sum_{i=1}^{\sigma}\chi T^i$ of degree σ, where $\chi\cong K_{ij}$. F^1 is K_{ij} and the top row is arbitrary over K_{ij}. Every element of I has a nonzero coefficient in the top row.*

THEOREM XIII. *P^i is isomorphic to an indecomposable matrix set $I(P^i)=\sum_{v=1,\cdots,\sigma}^{u=1,\cdots,n_i}\chi T^{uv}$ of degree $n_i\sigma$, where $\chi\cong K_{ij}$ has $n_i\times K_{ij}$ in the F^1 place. $F^1=(K_{ij})_{n_i}$ and the first n_i rows are arbitrary over K_{ij}. Every element has a nonzero coefficient in one of the first n_i rows.*

Proof. $P^i\cong(C_{ij})_{n_i}$ and the cleavage of P^i is equivalent to the cleavage of C_{ij}. Hence $I(C_{ij})$ may be substituted for C_{ij} in $(C_{ij})_{n_i}$. Then an easy coordinate adjustment gives the desired result.

4. When R is an algebra over a field F, the usual regular representation of R is generated from a composition series whose factor modules are just R right admissible rather than (K_{ij}, R) admissible. The Loewy series for $e_{ij}R$ is the same in both cases. But in general the two types of modules are not the same. For example consider the irreducible (K, F) space $V\equiv K$, where $K=\sum_{i=1}^{n}u_iF$. As an F right space, V breaks up into irreducible spaces $u_iF, \cdots, u_nF$.

Suppose R is an algebra over F. Express each K_{ij} over F: $K_{ij}=\sum_{i=1}^{h_i}Fu_i$. Then $e_{ij}R$ will have dimension $\sigma_i h_i$ over F. Now form a composition series of R right modules for $e_{ij}R$. Then I_{ij} will be of degree $\sigma_i h_i$. As an immediate consequence the multiplicity of all isomorphic G_{ij}^p will be exactly the degree of I_{ij} divided by the degree of K_{ij} over F, that is, equal to σ_i. This sharpens Theorem X(5).

THEOREM XIV. *Theorem X holds for the usual representations of an algebra over a field F if n_i, σ_i, and τ_i are replaced by n_ih_i, σ_ih_i, and τ_ih_i respectively, where h_i is the dimension of K_{ij} over F, and if the inequalities are replaced by equalities. A similar statement holds for Remark* 2.

If F is the Wedderburn division ring of every irreducible representation (for example if F is algebraically closed) then the elementary module theory can be derived from the considerations of §3(6).

BIBLIOGRAPHY

1. A. A. Albert, *Structure of algebras*, Amer. Math. Soc. Colloquium Publications, vol. 24, 1939.
2. E. Artin, C. J. Nesbitt, and R. M. Thrall, *Rings with minimum condition*, University of Michigan Press, 1944.
3. R. Brauer, *On the nilpotency of the radical of a ring*, Bull. Amer. Math. Soc. vol. 48 (1942) p. 752.
4. ———, *Darstellung von Gruppen in Galoisschen Feldern*, Exposés Mathématiques, vol. 7, 1935.
5. ———, *On modular and p-adic representations of algebras*, Proc. Nat. Acad. Sci. U.S.A. vol. 25 (1939) p. 252.
6. ———, *On sets of matrices with coefficients in a division ring*, Trans. Amer. Math. Soc. vol. 49 (1941) p. 502.
7. M. Deuring, *Algebren*, Ergebnisse der Mathematik, vol. 4, 1935.
8. H. Fitting, *Die Theorie der Automorphismenringe*, Math. Ann. vol. 107 (1932) p. 514.
9. M. Hall, *The position of the radical in an algebra*, Trans. Amer. Math. Soc. vol. 48 (1940) p. 391.
10. C. Hopkins, *Rings with minimal condition for left ideals*, Ann. of Math. (2) vol. 40 (1939) p. 712.
11. N. Jacobson, *The theory of rings*, Mathematical Surveys, vol. 2, 1943.
12. G. Kothe, *Die Struktur der Ringe, deren Restklassenring nach dem Radikal vollständig reduzibel ist*, Math. Zeit. vol. 32 (1930) p. 161.
13. T. Nakayama, *Some studies on regular, induced, and modular representations*, Ann. of Math. (2) vol. 39 (1938) p. 361.
14. J. L. Wedderburn, *Lectures on matrices*, Amer. Math. Soc. Colloquium Publications, vol. 17, 1934.

(5) Theorem XIV, which is derived for cleft algebras, corroborates properties summarized by Brauer [5].

(6) See W. M. Scott, *On matrix algebras over an algebraically closed field*, Ann. of Math. (2) vol. 43 (1942) pp. 147–160, for the elementary module theory.

UNIVERSITY OF WISCONSIN,
MADISON, WIS.