

NOTE ON THE STRONG SUMMABILITY OF FOURIER SERIES

BY
CHING-TSÜN LOO

1. Let $S_n(x)$ ($n=0, 1, 2, \dots$) be the partial sums of the Fourier series of a function $f(x) \in L^p(-\pi, \pi)$. Put

$$(1.1) \quad 2\phi(t) = f(x+t) + f(x-t) - 2S,$$

and let

$$(1.2) \quad \Phi(t) = \int_0^t |\phi(u)|^p du = o(t).$$

Hardy and Littlewood have proved the following two theorems:

A⁽¹⁾. If $p > 1$, then (1.2) gives

$$(1.3) \quad \sum_{m=0}^n |S_m - S|^2 = o(n).$$

B⁽²⁾. If $p = 1$, then (1.2) gives

$$(1.4) \quad \sum_{m=0}^n |S_m - S|^2 = o(n \log n).$$

In my previous paper⁽³⁾, I have replaced S_m in (1.4) by S_{mk} ($k=2, 3, \dots$). The object of the present paper is to show that the S_m in (1.3) can also be replaced by the lacunary partial sums S_{mk} , if (1.2) holds with $p > k$. In other words:

If p is greater than a positive integer k , and

$$(1.5) \quad \int_0^t |\phi(u)|^p du = o(t),$$

then

Presented to the Society, November 25, 1944; received by the editors, March 16, 1944.

⁽¹⁾ G. H. Hardy and J. E. Littlewood, *Notes on the theory of series*. IV. *On the strong summability of Fourier series*, Proc. London Math. Soc. (2) vol. 26 (1926) pp. 273-286.

⁽²⁾ G. H. Hardy and J. E. Littlewood, *The strong summability of Fourier series*, Fund. Math. vol. 25 (1935) pp. 162-189.

⁽³⁾ C. T. Loo, *Note on the strong summability of Fourier series*, Science Record Academia Sinica vol. 1 (1942) pp. 76-83.

$$(1.6) \quad \sum_{m=0}^n |S_{m^k} - S|^2 = o(n).$$

Without loss of generality, we may suppose that

$$f(t) \sim \sum_1^{\infty} a_n \cos nt$$

is an even function with zero mean value, and that $x=0$, $S=0$, so that

$$\phi(t) = f(t), \quad S_m = a_1 + a_2 + \cdots + a_m.$$

We write

$$\begin{aligned} \pi S_{m^k} &= \int_0^{\pi} \frac{\sin(m^k + 1/2)t}{\sin(t/2)} f(t) dt \\ (1.7) \quad &= \int_0^{\pi} \cos m^k t f(t) dt + \int_0^{1/n^k} \sin m^k t \cot(t/2) f(t) dt \\ &\quad + \int_{1/n^k}^{1/n} \sin m^k t \cot(t/2) f(t) dt + \int_{1/n}^{\pi} \sin m^k t \cot(t/2) f(t) dt \\ &= \alpha_m + \beta_m + \gamma_m + \delta_m. \end{aligned}$$

It suffices to prove the sum $\sum_{m=0}^n (\alpha_m^2 + \beta_m^2 + \gamma_m^2 + \delta_m^2)$ is equal to $o(n)$.

2. It is plain that $\alpha_m = o(1)$, so that

$$(2.1) \quad \sum_0^n \alpha_m^2 = o(n).$$

Hereafter, we shall write briefly

$$(2.2) \quad \xi = 1/n, \quad \eta = \pi/kn^{k-1}, \quad \zeta = 1/n^k.$$

If

$$(2.3) \quad F(t) = \int_0^t |f(u)|^p du = o(t) \quad (p > 1),$$

then the same relation holds true when p is replaced by any positive small index. Hence we have

$$(2.4) \quad |\beta_m| = \left| \int_0^{1/n^k} \sin m^k t \cot(t/2) f(t) dt \right| \leq m^k \int_0^{1/n^k} |f(t)| dt = o(1),$$

and

$$(2.5) \quad \sum_0^n \beta_m^2 = o(n).$$

Assuming

$$r > 2, \quad r' = r/(r-1) < p,$$

we are going to prove

$$(2.6) \quad \sum_0^n |\delta_m|^r = o(n).$$

This is stronger than

$$(2.7) \quad \sum_1^n \delta_m^2 = o(n).$$

We denote by $C_n(r)$ the n th Fourier sine coefficient of the odd function $\chi(t)$, which is equal to $f(t)$ in $(0, \tau)$ and to zero in (τ, π) . We have then

$$(2.8) \quad \begin{aligned} \delta_m &= \int_{\xi}^{\pi} \sin m^k t \cot(t/2) f(t) dt \\ &= \int_{\xi}^{\pi} \cot(t/2) \left(\frac{d}{dt} \int_0^t \sin m^k u f(u) du \right) dt \\ &= -(\pi/2) \cot(\xi/2) C_{mk}(\xi) + (\pi/4) \int_{\xi}^{\pi} \csc^2(t/2) C_{mk}(t) dt. \end{aligned}$$

From $r' < p$, it follows by Hausdorff's inequality that

$$(2.9) \quad \begin{aligned} \left(\sum_1^n |C_{mk}(t)|^r \right)^{1/r} &\leq \left(\frac{1}{\pi} \int_{-\pi}^{\pi} |\chi(u)|^{r'} du \right)^{1/r'} \\ &= \left(\frac{1}{\pi} \int_{-t}^t |f(u)|^{r'} du \right)^{1/r'} = o(t^{1/r'}). \end{aligned}$$

Hence

$$(2.10) \quad \begin{aligned} \left(\sum_1^n |\delta_m|^r \right)^{1/r} &= o\left(\frac{1}{\xi} \xi^{1/r'}\right) + o\left(\int_{\xi}^{\pi} \frac{1}{t^2} t^{1/r'} dt\right) \\ &= o(\xi^{-1/r}) = o(n^{1/r}). \end{aligned}$$

This establishes (2.6).

3. By the theorem of Riemann and Lebesgue, we have

$$(3.1) \quad \begin{aligned} \gamma_m &= \int_{\xi}^{\pi} \sin m^k t \cot(t/2) f(t) dt \\ &= 2 \int_{\xi}^{\pi} \frac{\sin m^k t}{t} f(t) dt + o(1). \end{aligned}$$

Hence

$$\begin{aligned}
 \sum_1^n \gamma_m^2 &= 4 \int_{\zeta}^{\xi} \frac{f(u)}{u} du \int_{\zeta}^{\xi} \frac{f(v)}{v} \sum_1^n \sin m^k u \sin m^k v dv + o(n) \\
 (3.2) \quad &= 2 \int_{\zeta}^{\xi} \frac{f(u)}{u} du \int_{\zeta}^{\xi} \frac{f(v)}{v} \sum_1^n \cos m^k(u-v) dv \\
 &\quad - 2 \int_{\zeta}^{\xi} \frac{f(u)}{u} du \int_{\zeta}^{\xi} \frac{f(v)}{v} \sum_1^n \cos m^k(u+v) dv + o(n) \\
 &= I_1 + I_2 + o(n).
 \end{aligned}$$

We shall use the fact that if

$$J_n(t) = \sum_{m=0}^n e^{i m^k t},$$

then

$$(3.3) \quad \left| \sum_1^n \cos m^k t \right| \leq |J_n(t)|.$$

We also require the following lemmas:

LEMMA 1. Let a, b and k be three integers such that $a < b, k \geq 2$. Let $g(x)$ be a real function having differential coefficients of the first k orders. If $Rg^{(k)}(x) \geq 1$ (or ≤ -1) throughout the interval (a, b) , write

$$(3.4) \quad K = 2^k, \quad P = R |g^{(k-1)}(b) - g^{(k-1)}(a)|,$$

where R is a positive number. Then

$$(3.5) \quad \left| \sum_{m=a}^b e^{2\pi i g(m)} \right| \leq 100P \{ R^{-1/(K-2)} + R^{2/K} P^{-2k/K} + P^{-2/K} \}.$$

This theorem is due to Van der Corput⁽⁴⁾.

LEMMA 2. If $t > 0$, then

$$(3.6) \quad J_n(t) = O(nt^{1/(K-2)}) + O(n^{1-2k/K} t^{-2/K}) + O(n^{1-2/K}).$$

Put $a=0, b=n, g(x) = x^k t / 2\pi$ ($t > 0$), $R^{-1} = k! t / 2\pi$. We have

$$g^{(k-1)}(x) = k! x t / 2\pi, \quad g^{(k)}(x) = k! t / 2\pi = 1/R > 0, \quad P = n.$$

Lemma 2 is therefore an immediate corollary of Lemma 1.

LEMMA 3. Let $g(x)$ be a real differentiable function in the interval (a, b) , $g'(x)$ be monotonic in this interval, and $|g'(x)| < 1/2$. Then

⁽⁴⁾ J. G. Van der Corput, *Zahlentheoretische Abschätzungen mit Anwendung auf Gitterpunktprobleme*, Math. Zeit. vol. 28 (1928) p. 303.

$$(3.7) \quad \sum_{a \leq n \leq b} e^{2\pi i g(n)} = \int_a^b e^{2\pi i g(x)} dx + O(1).$$

This theorem is due to Titchmarsh⁽⁵⁾.

LEMMA 4. If $0 < t \leq \pi/kn^{k-1} = \eta$, then

$$(3.8) \quad J_n(t) = O(1/t^{1/k}).$$

Put $a=0$, $b=n$, $g(x) = x^k t / 2\pi$ ($t > 0$). Evidently, $g'(x) = kx^{k-1}t/2\pi$ is monotone in the interval $(0, \pi)$ and $0 < g'(x) < 1/2$. Hence, by Lemma 3,

$$\begin{aligned} J_n(t) &= \sum_0^n e^{imkt} = \int_0^n e^{ix^k t} dx + O(1) \\ &= \frac{1}{kt^{1/k}} \int_0^{n^k t} e^{iu} u^{1/k-1} du + O(1) = O\left(\frac{1}{t^{1/k}}\right). \end{aligned}$$

4. We are now in a position to estimate $\sum_1^n \gamma_m^2$. Firstly, remembering $\xi = 1/n$, $\zeta = 1/n^k$, we have

$$\begin{aligned} I_{21} &= n \int_{\zeta}^{\xi} \frac{|f(u)|}{u} du \int_{\zeta}^{\xi} \frac{|f(v)|}{v} (u+v)^{1/(K-2)} dv \\ &\leq n \int_{\zeta}^{\xi} u^{-1+1/(K-2)} |f(u)| du \int_{\zeta}^{\xi} \frac{|f(v)|}{v} dv \\ &\quad + n \int_{\zeta}^{\xi} \frac{|f(u)|}{u} du \int_{\zeta}^{\xi} v^{-1+1/(K-2)} |f(v)| dv \\ &= o(n^{1-1/(K-2)} \log n) = o(n), \\ I_{22} &= n^{1-2k/K} \int_{\zeta}^{\xi} \frac{|f(u)|}{u} du \int_{\zeta}^{\xi} \frac{|f(v)|}{v} (u+v)^{-2/K} dv \\ &\leq n^{1-2k/K} \int_{\zeta}^{\xi} u^{-1-1/K} |f(u)| du \int_{\zeta}^{\xi} v^{-1-1/K} |f(v)| dv \\ &= o(n^{1-2k/K} \cdot n^{k/K} \cdot n^{k/K}) = o(n), \\ I_{23} &= n^{1-2/K} \int_{\zeta}^{\xi} \frac{|f(u)|}{u} du \int_{\zeta}^{\xi} \frac{|f(v)|}{v} dv \\ &= o(n^{1-2/K} \cdot (\log n)^2) = o(n). \end{aligned}$$

Observing the definition of I_2 in (3.2) it follows from (3.3) and (3.6) that

⁽⁵⁾ E. C. Titchmarsh, *On Van der Corput's method and the zeta-function of Riemann*, Quart. J. Math. vol. 2 (1931) pp. 161-173.

$$(4.1) \quad I_2 = O(I_{21} + I_{22} + I_{23}) = o(n).$$

Secondly, observing (3.2) and (3.3) we have

$$(4.2) \quad \begin{aligned} \left| \frac{I_1}{4} \right| &\leq \int_{\zeta}^{\xi} \frac{|f(u)|}{u} du \int_{\zeta}^u \frac{|f(v)|}{v} \left| \sum_0^n \cos m^k(u-v) \right| dv \\ &\leq \int_{\zeta}^{2\zeta} \frac{|f(u)|}{u} du \int_{\zeta}^u \frac{|f(v)|}{v} |J_n(u-v)| dv \\ &\quad + \int_{2\zeta}^{\xi} \frac{|f(u)|}{u} du \int_{u-\zeta}^u \frac{|f(v)|}{v} |J_n(u-v)| dv \\ &\quad + \int_{2\zeta}^{\eta+\zeta} \frac{|f(u)|}{u} du \int_{\zeta}^{u-\zeta} \frac{|f(v)|}{v} |J_n(u-v)| dv \\ &\quad + \int_{\eta+\zeta}^{\xi} \frac{|f(u)|}{u} du \int_{u-\eta}^{u-\zeta} \frac{|f(v)|}{v} |J_n(u-v)| dv \\ &\quad + \int_{\eta+\zeta}^{\xi} \frac{|f(u)|}{u} du \int_{\zeta}^{u-\eta} \frac{|f(v)|}{v} |J_n(u-v)| dv \\ &= I_{11} + I_{12} + I_{13} + I_{14} + I_{15}. \end{aligned}$$

Writing

$$\begin{aligned} I_{11} &= \int_{\zeta}^{2\zeta} \frac{|f(u)|}{u^2} du \int_{\zeta}^u |f(v)| |J_n(u-v)| dv \\ &\quad + \int_{\zeta}^{2\zeta} \frac{|f(u)|}{u^2} du \int_{\zeta}^u \frac{|f(v)|}{v} (u-v) |J_n(u-v)| dv \\ &= I_{111} + I_{112}, \end{aligned}$$

we have

$$I_{111} \leq n \int_{\zeta}^{2\zeta} \frac{|f(u)|}{u} du \int_{\zeta}^u |f(v)| dv,$$

and

$$\begin{aligned} \int_{\zeta}^u |f(v)| dv &\leq \left(\int_{\zeta}^u |f(v)|^p dv \right)^{1/p} \left(\int_{\zeta}^u dv \right)^{1/q} \\ &= o(u^{1/p}(u-\zeta)^{1/q}) = o(u^{1/p}\zeta^{1/q}) = o(n^{-k/q}u^{1/p}), \end{aligned}$$

since $u \leq 2\zeta$. It follows that

$$\begin{aligned} I_{111} &\leq n^{1-k/q} \int_{\zeta}^{2\zeta} u^{-2+1/p} |f(u)| du = n^{1-k/q} \int_{\zeta}^{2\zeta} u^{-1-1/q} |f(u)| du \\ &= o(n^{1-k/q}\zeta^{-1/q}) = o(n^{1-k/q}n^{k/q}) = o(n). \end{aligned}$$

Again, from

$$I_{112} \leq n^{1-k} \int_{\zeta}^{2\zeta} \frac{|f(u)|}{u} du \int_{\zeta}^u \frac{|f(v)|}{v} dv,$$

and

$$\int_{\zeta}^u \frac{|f(v)|}{v} dv = o\left(\log \frac{u}{\zeta}\right) = o(\log 2) = o(1),$$

we obtain

$$I_{112} \leq n^{1-k} \int_{\zeta}^{2\zeta} \frac{|f(u)|}{u^2} du = o(n^{1-k}n^k) = o(n).$$

Therefore

$$(4.3) \quad I_{11} = O(I_{111} + I_{112}) = o(n).$$

The same method of estimation can be applied to I_{12} , by assuming $2\zeta \leq u$. Thus we obtain

$$(4.4) \quad I_{12} = o(n).$$

Apply Lemma 2 to I_{15} . We have

$$\begin{aligned} I_{151} &= n \int_{\eta+\zeta}^{\xi} \frac{|f(u)|}{u} du \int_{\zeta}^{u-\eta} \frac{|f(v)|}{v} (u-v)^{1/(K-2)} dv \\ &\leq n \int_{\eta+\zeta}^{\xi} u^{-1+1/(K-2)} |f(u)| du \int_{\zeta}^{u-\eta} \frac{|f(v)|}{v} dv \\ &= o(n^{1-1/(K-2)} \log n) = o(n), \\ I_{152} &= n^{1-2k/K} \int_{\eta+\zeta}^{\xi} \frac{|f(u)|}{u} du \int_{\zeta}^{u-\eta} \frac{|f(v)|}{v} (u-v)^{-2/K} dv \\ &\leq n^{1-2k/K} \eta^{-2/K} \int_{\eta+\zeta}^{\xi} \frac{|f(u)|}{u} du \int_{\zeta}^{u-\eta} \frac{|f(v)|}{v} dv \\ &= o(n^{1-2k/K} \cdot n^{(k-1)2/K} \cdot (\log n)^2) \\ &= o(n^{1-2/K} (\log n)^2) = o(n), \\ I_{153} &= n^{1-2/K} \int_{\eta+\zeta}^{\xi} \frac{|f(u)|}{u} du \int_{\zeta}^{u-\eta} \frac{|f(v)|}{v} dv \\ &= o(n^{1-2/K} (\log n)^2) = o(n). \end{aligned}$$

It follows that

$$(4.5) \quad I_{15} = O(I_{151} + I_{152} + I_{153}) = o(n).$$

Observe that in I_{13}

$$u \leq \eta + \zeta \quad \text{and} \quad \zeta \leq v \leq u - \zeta,$$

or

$$0 < \zeta \leq u - v \leq u - \zeta \leq \eta.$$

Using Lemma 4, we have

$$\begin{aligned} I_{13} &\leq \int_{2\zeta}^{\eta+\zeta} \frac{|f(u)|}{u} du \int_{\zeta}^{u-\zeta} \frac{|f(v)|}{v} (u-v)^{-1/k} dv \\ &= \int_{2\zeta}^{\eta+\zeta} \frac{|f(u)|}{u^2} du \int_{\zeta}^{u-\zeta} \frac{|f(v)|}{v} (u-v)^{-1/k} dv \\ &\quad + \int_{2\zeta}^{\eta+\zeta} \frac{|f(u)|}{u^2} du \int_{\zeta}^{u-\zeta} |f(v)| (u-v)^{-1/k} dv \\ &= I_{131} + I_{132}. \end{aligned}$$

From

$$I_{131} \leq \int_{2\zeta}^{\eta+\zeta} u^{-1-1/k} |f(u)| du \int_{\zeta}^{u-\zeta} \frac{|f(v)|}{v} dv,$$

and

$$\begin{aligned} \int_{\zeta}^{u-\zeta} \frac{|f(v)|}{v} dv &\leq \left(\int_{\zeta}^{u-\zeta} |f(v)|^p dv \right)^{1/p} \left(\int_{\zeta}^{u-\zeta} v^{-q} dv \right)^{1/q} \\ &= o(u^{1/p} \zeta^{-1+1/q}) = o(u^{1/p} n^{k-q/q}) = o(u^{1/p} n^{k/p}), \end{aligned}$$

we obtain

$$\begin{aligned} I_{131} &\leq n^{k/p} \int_{2\zeta}^{\eta+\zeta} u^{-1-1/k+1/p} |f(u)| du \\ &= o(n^{k/p} \zeta^{-1/k+1/p}) = o(n^{k/p} n^{1-k/p}) = o(n). \end{aligned}$$

Again, from

$$I_{132} = \int_{2\zeta}^{\eta+\zeta} \frac{|f(u)|}{u^2} du \int_{\zeta}^{u-\zeta} |f(v)| (u-v)^{-1/k} dv$$

and

$$\begin{aligned} \int_{\zeta}^{u-\zeta} |f(v)| (u-v)^{-1/k} dv &\leq \left(\int_{\zeta}^{u-\zeta} |f(v)|^p dv \right)^{1/p} \left(\int_{\zeta}^{u-\zeta} (u-v)^{-q/k} dv \right)^{1/q} \\ &= o(u^{1/p} (u-\zeta)^{-1/k+1/q}) = o(u^{1-1/k}), \end{aligned}$$

since $-1/k+1/q > 0$, we obtain

$$\begin{aligned} I_{132} &\leq \int_{2\zeta}^{\eta+\zeta} u^{-1-1/k} |f(u)| du \\ &= o(\zeta^{-1/k}) = o(n). \end{aligned}$$

Therefore

$$(4.6) \quad I_{13} = O(I_{131} + I_{132}) = o(n).$$

Finally, by applying Lemma 4 to I_{14} we easily deduce that

$$(4.7) \quad I_{14} = o(n).$$

Combining (4.2), (4.3), (4.4), (4.5), (4.6) and (4.7) we get

$$(4.8) \quad I_1 = O(I_{11} + I_{12} + I_{13} + I_{14} + I_{15}) = o(n).$$

It follows from (3.2), (4.1) and (4.8) that

$$(4.9) \quad \sum_1^n \gamma_m^2 = o(n).$$

Summarizing (2.1), (2.5), (2.7) and (4.9) we obtain

$$\sum_1^n (\alpha_m^2 + \beta_m^2 + \gamma_m^2 + \delta_m^2) = o(n).$$

The theorem is thus proved.

NATIONAL UNIVERSITY OF CHEKIANG,
KWEICHOW, CHINA.