

TWO TAUBERIAN THEOREMS IN THE THEORY OF FOURIER SERIES

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1. Let $f(x)$ be a function which is integrable in the sense of Lebesgue and periodic with period 2π . We consider the Fourier series of $f(x)$ and write

$$(1.1) \quad f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

$$A_0 = a_0/2, \quad A_n = A_n(x) = a_n \cos nx + b_n \sin nx \quad (n > 0),$$

$$(1.2) \quad \sigma_n^\alpha = \frac{S_n^\alpha}{C_{n+\alpha, n}} = \frac{1}{C_{n+\alpha, n}} \sum_{\nu=0}^n C_{n+\alpha-\nu, n-\nu} A_\nu \quad (\alpha > -1),$$

$$\phi(t) = \phi_0(t) = \{f(x+t) + f(x-t) - 2s\}/2,$$

$$(1.3) \quad \phi_p(t) = \frac{p}{t^p} \int_0^t (t-u)^{p-1} \phi(u) du \quad (p > 0).$$

It is a theorem of Paley⁽¹⁾ that if $\alpha \geq 0$ and $\sigma_n^\alpha \rightarrow s$, then $\phi_{1+\alpha+\delta} \rightarrow 0$ for every positive δ . The result is best possible of its kind⁽²⁾. That is to say, we cannot replace δ by 0 in the conclusion of the above theorem. We are interested in such a problem, whether we can replace δ by 0, whenever we emphasize the hypothesis a little. This has been done by Hardy and Littlewood⁽³⁾ in the case $\alpha=0$. They proved that if (i) $A_n = O(n^{-\delta})$ for some positive δ and (ii) $\sigma_n^0 - s = o(1/\log n)$, then $\phi_1(t) \rightarrow 0$. The object of this paper is to investigate the analogous problems for the case $\alpha > 0$. We prove that for $\alpha > 0$ a single condition corresponding to condition (ii) in Hardy and Littlewood's theorem is sufficient to deduce $\phi_{1+\alpha}(t) \rightarrow 0$. Our theorem runs as follows.

THEOREM 1. *If $\alpha > 0$, and*

$$(1.4) \quad \sigma_n^\alpha - s = o(1/\log n),$$

then

$$(1.5) \quad \phi_{1+\alpha}(t) \rightarrow 0.$$

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(¹) R. E. A. C. Paley, *On the Cesàro summability of Fourier series and allied series*, Proc. Cambridge Philos. Soc. vol. 26 (1930).

(²) *Ibid.*

(³) G. H. Hardy and J. E. Littlewood, *Some new convergence criteria for Fourier series*, Annali Scuola Normale Superiore, Pisa, vol. 3 (1934).

It is natural to ask whether the more stringent condition $\sigma_n^\alpha - s = O(n^{-\epsilon})$ ($1 > \epsilon > 0$) combined with an order condition on the coefficients, $A_n = O(n^{-\delta})$, is sufficient to deduce $\phi_\alpha(t) \rightarrow 0$. We prove that this is true when α is any positive integer, and also require a certain restriction on δ .

THEOREM 2. *If α is any positive integer such that*

$$(1.6) \quad \sigma_n^\alpha - s = O(n^{-\epsilon}) \quad (1 > \epsilon > 0),$$

$$(1.7) \quad A_n = O(n^{-\delta}) \quad (\delta > 1 - \epsilon),$$

then

$$(1.8) \quad \phi_\alpha(t) \rightarrow 0.$$

2. We begin by making the usual standard simplification of data, and discuss some properties of the function

$$(2.1) \quad \gamma_p(t) = \int_0^1 (1-u)^{p-1} \cos tu \, du \quad (p > 0).$$

We suppose throughout this paper that $a_0 = 0, s = 0$, then

$$(2.2) \quad \phi(t) = \{f(x+t) + f(x-t)\}/2 \sim \sum_{n=1}^{\infty} A_n \cos nt,$$

and if $p > 0$,

$$(2.3) \quad \begin{aligned} \frac{1}{p} \phi_p(t) &= \frac{1}{t^p} \int_0^t (t-u)^{p-1} \phi(u) \, du = \int_0^1 (1-u)^{p-1} \phi(tu) \, du \\ &= \sum_{n=1}^{\infty} A_n \int_0^1 (1-u)^{p-1} \cos ntu \, du = \sum_{n=1}^{\infty} A_n \gamma_p(nt). \end{aligned}$$

If $j \geq 1$, we denote by $\gamma_p^j(t)$ the j th derivative of $\gamma_p(t)$, while $\gamma_p^0(t)$ is defined as $\gamma_p(t)$. Then we have the following lemma.

LEMMA 1. $\gamma_p^j(t) = O(1)$ ($j = 0, 1, 2, \dots$) for all t , and if $t \geq 1$, then

$$(2.4) \quad \gamma_p^j(t) = O(1/t^{j+1+\gamma}) \quad (j = 0, 1, 2, \dots)$$

where $\gamma = \min(1, p - (j+1))$.

This is well known⁽⁴⁾.

LEMMA 2⁽⁵⁾. If $p > 0, q > p > 0$, then

⁽⁴⁾ E. W. Hobson, *The theory of functions of a real variable*, vol. 2, 2nd ed.

⁽⁵⁾ S. Verblunsky, *Note on the sum of an oscillating series*, Proc. Cambridge Philos. Soc. vol. 26 (1930).

$$\begin{aligned}
 \Delta^q \gamma_p(vt) &= \sum_{n=p}^{\infty} (-1)^{(n-p)} C_{q,n-p} \gamma_p(nt) \\
 (2.5) \quad &= \int_0^1 (1-u)^{p-1} (2 \sin(tu/2))^q \cos \left[\left(vt + \frac{q}{2} t \right) u - \frac{q}{2} \pi \right] du.
 \end{aligned}$$

We have

$$\begin{aligned}
 \sum_{n=p}^{\infty} (-1)^{(n-p)} C_{q,n-p} \gamma_p(nt) \\
 (2.6) \quad &= \int_0^1 (1-u)^{p-1} \left(\sum_{n=p}^{\infty} (-1)^{(n-p)} C_{q,n-p} \cos ntu \right) du,
 \end{aligned}$$

while

$$\begin{aligned}
 \sum_{n=p}^{\infty} (-1)^{(n-p)} C_{q,n-p} \cos ntu &= \cos vtu - C_{q,1} \cos(v+1)tu + C_{q,2} \cos(v+2)tu - \dots + \dots \\
 &= [e^{ivtu} - C_{q,1} e^{i(v+1)tu} + \dots - \dots] / 2 \\
 &\quad + [e^{-ivtu} - C_{q,1} e^{-i(v+1)tu} + \dots + \dots] / 2 \\
 &= e^{ivtu} (1 - e^{itu})^q / 2 + e^{-ivtu} (1 - e^{-itu})^q / 2 \\
 &= e^{ivtu} (2 \sin(tu/2))^q e^{iq(tu-\pi)/2} / 2 \\
 &\quad + e^{-ivtu} (2 \sin(tu/2))^q e^{-iq(tu-\pi)/2} / 2 \\
 &= (2 \sin(tu/2))^q \cos \left[(vt + (q/2)t)u - (q/2)\pi \right].
 \end{aligned}$$

Substituting into the right-hand side of (2.6), we get the required result (2.5).

LEMMA 3. If $p > 0$, $q > s > 0$, then

$$(2.7) \quad \sum_{n=p}^{\infty} (-1)^{(n-p)} C_{q-s,n-p} \Delta^s \gamma_p(nt) = \Delta^q \gamma_p(vt).$$

We have by Lemma 2,

$$\begin{aligned}
 \sum_{n=p}^{\infty} (-1)^{(n-p)} C_{q-s,n-p} \Delta^s \gamma_p(nt) \\
 (2.8) \quad &= \int_0^1 (1-u)^{p-1} \left(2 \sin \frac{tu}{2} \right)^s \sum_{n=p}^{\infty} (-1)^{(n-p)} C_{q-s,n-p} \\
 &\quad \cdot \cos \left[\left(nt + \frac{s}{2} t \right) u - \frac{s}{2} \pi \right] du,
 \end{aligned}$$

and using the same method as in the proof of Lemma 2 we can deduce that

$$\begin{aligned}
& \sum_{n=p}^{\infty} (-1)^{(n-p)} C_{q-s, n-p} \cos \left[\left(nt + \frac{s}{2} t \right) u - \frac{s}{2} \pi \right] \\
&= \left(2 \sin \frac{tu}{2} \right)^{q-s} \cos \left[\left(\nu t + \frac{s}{2} t + \frac{q-s}{2} t \right) u - \frac{s}{2} \pi - \frac{q-s}{2} \pi \right] \\
&= \left(2 \sin \frac{tu}{2} \right)^{q-s} \cos \left[\left(\nu t + \frac{q}{2} t \right) u - \frac{q}{2} \pi \right].
\end{aligned}$$

Hence

$$\begin{aligned}
& \sum_{n=p}^{\infty} (-1)^{(n-p)} C_{q-s, n-p} \Delta^s \gamma_p(nl) \\
&= \int_0^1 (1-u)^{p-1} \left(2 \sin \frac{tu}{2} \right)^q \cos \left[\left(\nu t + \frac{q}{2} t \right) u - \frac{q}{2} \pi \right] du.
\end{aligned}$$

Our result follows from Lemma 2.

3. To prove Theorem 1, we begin by choosing r such that

$$\begin{aligned}
r &> (1+\alpha)/\alpha \quad \text{if } \alpha \leq 1, \\
r &> 2 \quad \quad \quad \text{if } \alpha > 1,
\end{aligned}$$

and then write

$$(3.1) \quad l = [t^{-1}], \quad k = [t^{-r}].$$

We have by (2.3) that

$$(3.2) \quad \frac{1}{1+\alpha} \phi_{1+\alpha}(t) = \sum_{n=1}^{\infty} A_n \gamma_{1+\alpha}(nt) = \sum_{n=1}^k + \sum_{n=k+1}^{\infty} = \chi_1(t) + \chi_2(t),$$

say. Since the Fourier coefficient of any integrable function tends to zero, we notice that

$$(3.3) \quad \gamma_{1+\alpha}(nt) = \begin{cases} O((nt)^{-1-\alpha}), & 0 < \alpha \leq 1, \\ O((nt)^{-2}), & \alpha > 1, \end{cases}$$

in virtue of Lemma 1. Hence we have

$$(3.4) \quad \chi_2(t) = O\left(\frac{1}{t^{1+\alpha}} \sum_{k+1}^{\infty} n^{-1-\alpha}\right) = O\left(\frac{1}{t^{1+\alpha} k^{\alpha}}\right) = O(t^{r\alpha-1-\alpha}) = o(1),$$

when $\alpha \leq 1$, and

$$(3.5) \quad \chi_2(t) = O\left(\frac{1}{t^2} \sum_{k+1}^{\infty} n^{-2}\right) = O\left(\frac{1}{t^2 k}\right) = O(t^{r-2}) = o(1),$$

when $\alpha > 1$.

So it is enough to prove that $\chi_1(t) = o(1)$. Since

$$A_n = \sum_{r=0}^n (-1)^{(n-r)} C_{\alpha+1, n-r} S_r^\alpha$$

holds for any $\alpha > -1$, hence

$$\begin{aligned} \chi_1(t) &= \sum_{n=1}^k A_n \gamma_{1+\alpha}(nt) = \sum_{n=1}^k \gamma_{1+\alpha}(nt) \sum_{r=0}^n (-1)^{(n-r)} C_{\alpha+1, n-r} S_r^\alpha \\ (3.6) \quad &= \sum_{r=1}^k S_r^\alpha \sum_{n=r}^k (-1)^{(n-r)} C_{\alpha+1, n-r} \gamma_{1+\alpha}(nt). \end{aligned}$$

In order to establish $\chi_1(t) = o(1)$, we require some further lemmas.

LEMMA 4.

$$\begin{aligned} &\sum_{n=r}^{\infty} (-1)^{(n-r)} C_{\alpha+1, n-r} \gamma_{1+\alpha}(nt) \\ &= t^{1+\alpha} (1 + \zeta_t) \int_0^1 (1-u)^{\alpha} u^{1+\alpha} \cos \left[\left(vt + \frac{\alpha+1}{2} t \right) u - \frac{\alpha+1}{2} \pi \right] du \end{aligned}$$

where $\zeta_t \rightarrow 0$ as $t \rightarrow 0$, uniformly in n .

This is an immediate corollary of Lemma 2. From this lemma we easily see the following result.

LEMMA 5. For all positive t ,

$$(3.7) \quad \sum_{n=r}^{\infty} (-1)^{(n-r)} C_{\alpha+1, n-r} \gamma_{1+\alpha}(nt) = O(t^{1+\alpha}).$$

LEMMA 6. If $vt > 1$, then

$$(3.8) \quad \sum_{n=r}^{\infty} (-1)^{(n-r)} C_{\alpha+1, n-r} \gamma_{1+\alpha}(nt) = O\left(\frac{1}{v^{1+\alpha}}\right).$$

Putting, in Lemma 4, $vt = X$, $(\alpha+1)(ut-\pi)/2 = \beta$, we are going to prove

$$(3.9) \quad \int_0^1 (1-u)^{\alpha} u^{1+\alpha} \cos(Xu + \beta) du = O\left(\frac{1}{X^{1+\alpha}}\right).$$

If α is an integer, we have by successive integration by parts

$$\begin{aligned} &\left| \int_0^1 (1-u)^{\alpha} u^{1+\alpha} \cos(Xu + \beta) du \right| \\ &= \left| \frac{1}{X^{\alpha}} \int_0^1 \cos \left[Xu + \beta + (\alpha+1) \frac{\pi}{2} \right] \sum_{j=0}^{\alpha} A_j (1-u)^{\alpha-j} u^{1+j} du \right|, \end{aligned}$$

where the A 's are numbers which depend on α . Each term can be integrated by parts again, whence our result follows.

If α is not an integer, let $[\alpha]$ denote the greatest integer which does not exceed α . Integrating by parts gives

$$\left| \int_0^1 (1-u)^{\alpha} u^{1+\alpha} \cos(Xu + \beta) du \right| \\ = \left| \frac{1}{X^{[\alpha]+1}} \int_0^1 \cos \left[Xu + \beta + ([\alpha] + 1) \frac{\pi}{2} \right] \sum_{i=0}^{[\alpha]+1} A_i (1-u)^{\alpha-i} u^{\alpha-[\alpha]+i} du \right|.$$

Each of the first $[\alpha]+1$ terms can be integrated by parts again and is thus seen to be numerically less than

$$K/X^{[\alpha]+2},$$

which is the form required, since $X > 1$, and $[\alpha]+2 > \alpha+1$. The numerical value of the last term, on taking $x(1-u)$ as a new variable under the integrand, is seen to be

$$\left| \frac{A_{[\alpha]+1}}{X^{1+\alpha}} \int_0^X \cos \left[u - X - \beta - ([\alpha] + 1) \frac{\pi}{2} \right] \left(1 - \frac{u}{x} \right)^{1+\alpha} u^{\alpha-[\alpha]-1} du \right|,$$

which is again of order $O(1/X^{1+\alpha})$, since the last integral converges.

4. We are in a position to prove $\chi_1(t) = o(1)$. First, we shall confine our proof to the case $0 < \alpha \leq 1$. We have by (3.6)

$$(4.1) \quad \begin{aligned} \chi_1(t) &= \sum_{\nu=1}^k S_{\nu}^{\alpha} \sum_{n=\nu}^k (-1)^{(n-\nu)} C_{\alpha+1, n-\nu} \gamma_{1+\alpha}(nt) \\ &= \sum_{\nu=1}^k S_{\nu}^{\alpha} \sum_{n=\nu}^{\infty} (-1)^{(n-\nu)} C_{\alpha+1, n-\nu} \gamma_{1+\alpha}(nt) - \sum_{\nu=1}^k S_{\nu}^{\alpha} \sum_{n=k+1}^{\infty} . \end{aligned}$$

We put

$$(4.2) \quad \sum_{\nu=1}^k S_{\nu}^{\alpha} \sum_{n=\nu}^{\infty} (-1)^{(n-\nu)} C_{\alpha+1, n-\nu} \gamma_{1+\alpha}(nt) = \sum_{\nu=1}^l + \sum_{l+1}^k = \chi_3(t) + \chi_4(t)$$

say. In $\chi_3(t)$, $nt \leq 1$, we have by Lemma 5,

$$(4.3) \quad \chi_3(t) = t^{1+\alpha} \sum_{\nu=1}^l o(\nu^{\alpha}) = o(t^{1+\alpha} l^{1+\alpha}) = o(1).$$

On the other hand, in $\chi_4(t)$, $nt \geq 1$, we have by Lemma 6,

$$(4.4) \quad \begin{aligned} \chi_4(t) &= o \left(\sum_{l+1}^k \frac{\nu^{\alpha}}{\nu^{\alpha+1} \log n} \right) = o \left(\sum_{l+1}^k \frac{1}{n \log n} \right) \\ &= o(\log \log t^{\nu} - \log \log t^{-1}) = o(1). \end{aligned}$$

It remains to prove

$$(4.5) \quad \sum_{\nu=1}^k S_{\nu}^{\alpha} \sum_{n=k+1}^{\infty} (-1)^{(n-\nu)} C_{\alpha+1, n-\nu} \gamma_{1+\alpha}(nt) = o(1).$$

We have $\gamma_{1+\alpha}(nt) = O(1/(nt)^{1+\alpha})$ for $0 < \alpha \leq 1$, hence the left-hand side of (4.5) is

$$\begin{aligned} O\left(\frac{1}{t^{1+\alpha}} \sum_{\nu=1}^k \nu^{\alpha} \sum_{n=k+1}^{\infty} \frac{1}{n^{1+\alpha}(n-\nu)^{\alpha+2}}\right) &= O\left(\frac{1}{t^{1+\alpha}k} \sum_{\nu=1}^k \sum_{n=k+1}^{\infty} \frac{1}{(n-\nu)^{\alpha+2}}\right) \\ &= O\left(\frac{1}{t^{1+\alpha}k} \sum_{\nu=1}^k \frac{1}{(k+1-\nu)^{1+\alpha}}\right) = O\left(\frac{1}{t^{1+\alpha}k}\right) = O(t^{r-\alpha-1}) = o(1), \end{aligned}$$

for $r > (\alpha+1)/\alpha \geq \alpha+1$, when $0 < \alpha \leq 1$.

Collecting our results from (4.1), (4.2), (4.3), (4.4), and (4.5), we get $\chi_1(t) = o(1)$.

We notice that (4.3) and (4.4) are established for all $\alpha > 0$, while (4.5) holds in general only for $0 < \alpha \leq 1$; this is certainly the sole reason for us to treat the case $0 < \alpha \leq 1$ first. As for $\alpha > 1$, by repeated use of Abel's transformation $[\alpha]$ times, we have

$$\begin{aligned} (4.6) \quad \chi_1(t) &= \sum_{n=1}^k A_n \gamma_{1+\alpha}(nt) = \sum_{n=1}^{k-[\alpha]} S_n^{[\alpha]-1} \Delta^{[\alpha]} \gamma_{1+\alpha}(nt) \\ &\quad + \sum_{j=0}^{[\alpha]-1} S_{k-j}^j \Delta^j \gamma_{1+\alpha}((k-j)t). \end{aligned}$$

Also^(*),

$$\begin{aligned} \Delta^1 \gamma_{1+\alpha}(mt) &= \gamma_{1+\alpha}(mt) - \gamma_{1+\alpha}((m+1)t) \\ &= -t \gamma'_{1+\alpha}((m+\theta_1)t) \quad (0 < \theta_1 < 1), \\ \Delta^2 \gamma_{1+\alpha}(mt) &= \Delta^1 \gamma_{1+\alpha}(mt) - \Delta^1 \gamma_{1+\alpha}((m+1)t) \\ &= t^2 \gamma''_{1+\alpha}((m+\theta_1+\theta_2)t) \quad (0 < \theta_2 < 1), \end{aligned}$$

in general,

$$\begin{aligned} (4.7) \quad \Delta^j \gamma_{1+\alpha}(mt) &= (-1)^j t^j \gamma_{1+\alpha}^{(j)}((m+\theta_1+\theta_2+\cdots+\theta_j)t) \\ &= (-1)^j t^j \gamma_{1+\alpha}^{(j)}((m+\Theta)t) \end{aligned}$$

where $0 < \theta_i < 1$ ($i=1, 2, \dots, j$).

We have by Lemma 1 that

$$\begin{aligned} (4.8) \quad \gamma_{1+\alpha}^{(j)}((m+\Theta)t) &= O(1/(mt)^{j+2}) \quad (j=0, 1, 2, \dots, [\alpha]-1), \\ \gamma_{1+\alpha}^{[\alpha]}(mt) &= O(1/(mt)^{1+\alpha}), \end{aligned}$$

and by (4.7),

(*) Hereafter $\gamma^j(mt)$ denotes differentiation with respect to the argument.

$$(4.9) \quad \Delta^j \gamma_{1+\alpha}(mt) = O(t^j/(mt)^{j+2}) = O(1/m^{j+2}t^2) \\ (j = 0, 1, 2, \dots, [\alpha] - 1),$$

$$(4.10) \quad \Delta^{[\alpha]} \gamma_{1+\alpha}(mt) = O(t^{[\alpha]}/(mt)^{1+\alpha}) = O(1/m^{1+\alpha}t^{\alpha-[\alpha]+1}).$$

We notice that $A_n = o(1)$, hence $S_n^j = o(n^{j+1})$. Thus

$$(4.11) \quad S_{k-j}^j \Delta^j \gamma_{1+\alpha}((k-j)t) = O((k-j)^{j+1}/(k-j)^{j+2}t^2) \\ = O(1/kt^2) = O(t^{r-2}) = o(1) \\ (j = 0, 1, \dots, [\alpha] - 1)$$

since $r > 2$. That is to say, each term in the second sum of the right-hand side of (4.6) is $o(1)$. We thus obtain

$$(4.12) \quad \chi_1(t) = \sum_{n=1}^{k-[\alpha]} S_n^{[\alpha]-1} \Delta^{[\alpha]} \gamma_{1+\alpha}(nt) + o(1).$$

We notice that if $p > q > -1$, $S_n^q = \sum_{\nu=0}^n (-1)^{(n-\nu)} C_{p-q, n-\nu} S_\nu^p$. Hence

$$(4.13) \quad \sum_{n=1}^{k-[\alpha]} S_n^{[\alpha]-1} \Delta^{[\alpha]} \gamma_{1+\alpha}(nt) \\ = \sum_{\nu=1}^{k-[\alpha]} \Delta^{[\alpha]} \gamma_{1+\alpha}(nt) \sum_{\mu=1}^n (-1)^{(n-\mu)} C_{\alpha-[\alpha]+1, n-\mu} S_\mu^\alpha \\ = \sum_{\nu=1}^{k-[\alpha]} S_\nu^\alpha \sum_{\eta=\nu}^{k-[\alpha]} (-1)^{(n-\nu)} C_{\alpha-[\alpha]+1, n-\nu} \Delta^{[\alpha]} \gamma_{1+\alpha}(nt) \\ = \sum_{\nu=1}^{k-[\alpha]} S_\nu^\alpha \sum_{\eta=\nu}^{\infty} (-1)^{(n-\nu)} C_{\alpha-[\alpha]+1, n-\nu} \Delta^{[\alpha]} \gamma_{1+\alpha}(nt) + o(1).$$

The last formula is justified by

$$(4.14) \quad \sum_{\nu=1}^{k-[\alpha]} S_\nu^\alpha \sum_{n=k-[\alpha]+1}^{\infty} (-1)^{(n-\nu)} C_{\alpha-[\alpha]+1, n-\nu} \Delta^{[\alpha]} \gamma_{1+\alpha}(nt) = o(1),$$

which can be easily deduced by noticing that $S_\nu^\alpha = o(\nu^\alpha)$ and (4.10). Hence the left-hand side of (4.14) is

$$O\left(\frac{1}{t^{\alpha-[\alpha]+1}} \sum_{\nu=1}^{k-[\alpha]} \nu^\alpha \sum_{n=k-[\alpha]+1}^{\infty} \frac{1}{(n-\nu)^{\alpha-[\alpha]+2} n^{\alpha+1}}\right) \\ = O\left(\frac{1}{t^{\alpha-[\alpha]+1} k} \sum_{\nu=1}^{k-[\alpha]} \sum_{n=k-[\alpha]+1}^{\infty} \frac{1}{(n-\nu)^{\alpha-[\alpha]+2}}\right) \\ = O\left(\frac{1}{t^{\alpha-[\alpha]+1} k} \sum_{\nu=1}^{k-[\alpha]} \frac{1}{(n-\nu)^{\alpha-[\alpha]+1}}\right) \\ = O(1/t^{\alpha-[\alpha]+1} k) = O(t^{r-(\alpha-[\alpha]+1)}) = o(1),$$

since $r > 2 > \alpha - [\alpha] + 1$. Combined with (4.12) and (4.13) it follows that

$$(4.15) \quad \chi_1(t) = \sum_{n=1}^{k-[\alpha]} S_r^\alpha \sum_{n=r}^{\infty} (-1)^{(n-r)} C_{\alpha-[\alpha]+1, n-r} \Delta^{[\alpha]} \gamma_{1+\alpha}(nt) + o(1).$$

Moreover, by applying Lemma 3 to the above sum we obtain

$$(4.16) \quad \chi_1(t) = \sum_{n=1}^{k-[\alpha]} S_r^\alpha \sum_{n=r}^{\infty} (-1)^{(n-r)} C_{\alpha+1, n-r} \gamma_{1+\alpha}(nt) + o(1) = o(1),$$

in virtue of (4.2), (4.3) and (4.4). Theorem 1 is thus completely proved.

5. In the proof of Theorem 2, we may suppose $0 < \delta < 1$. We write

$$(5.1) \quad \frac{1}{\alpha} \phi_\alpha(t) = \sum_{n=1}^{\infty} A_n \gamma_\alpha(nt) = \sum_{n=1}^k + \sum_{n=k+1}^{\infty} = \chi_1(t) + \chi_2(t).$$

Notice that $A_n = O(n^{-\delta})$, and that

$$\gamma_\alpha(nt) = \begin{cases} O\left(\frac{1}{(nt)^\alpha}\right), & 0 < \alpha \leq 2, \\ O\left(\frac{1}{(nt)^2}\right), & \alpha > 2, \end{cases}$$

and write

$$(5.2) \quad l = [t^{-1}], \quad k = [t^r],$$

where r is so chosen that

$$(5.3) \quad 1/(1 - \epsilon) > r > 1/\delta.$$

Thus we have for $\alpha = 1$,

$$(5.4) \quad \begin{aligned} \chi_2(t) &= \sum_{n=k+1}^{\infty} A_n \gamma_\alpha(nt) = O\left(\frac{1}{t} \sum_{k+1}^{\infty} n^{-1-\delta}\right) = O\left(\frac{1}{k^\delta t}\right) \\ &= O(t^{r\delta-1}) = o(1), \end{aligned}$$

and for $\alpha = 2, 3, \dots$ we have

$$(5.5) \quad \begin{aligned} \chi_2(t) &= O\left(\frac{1}{t^2} \sum_{k+1}^{\infty} n^{-2-\delta}\right) = O\left(\frac{1}{k^{1+\delta} t^2}\right) \\ &= O(t^{(1+\delta)r-2}) = o(1), \end{aligned}$$

since $r > 1/\delta > 2/(1+\delta)$.

It is enough to prove $\chi_1(t) = o(1)$. By repeated use of Abel's transformation, we obtain

$$(5.6) \quad \chi_1(t) = \sum_{n=1}^k A_n \gamma_\alpha(nt) = \sum_{n=1}^{k-\alpha-1} S_n^\alpha \Delta^{\alpha+1} \gamma_\alpha(nt) + \sum_{j=0}^{\alpha} S_{k-j}^j \Delta^j \gamma_\alpha((k-j)t).$$

Making use of Lemma 1, we have

$$(5.7) \quad \gamma_{\alpha}^j(mt) = O(1)$$

for all t , and

$$(5.8) \quad \gamma_{\alpha}^j(mt) = \begin{cases} O(1/(mt)^{j+2}) & (j = 0, 1, \dots, \alpha - 2), \\ O(1/(mt)^2) & (j = \alpha - 1, \alpha, \alpha + 1), \end{cases}$$

when $mt > 1$.

From (5.7), (5.8) and

$$(5.9) \quad \Delta^j \gamma_{\alpha}(mt) = O(t^j |\gamma_{\alpha}^j(mt)|),$$

we easily deduce that

$$(5.10) \quad \Delta^i \gamma_{\alpha}(mt) = O(t^i)$$

for all positive t , and if $mt > 1$

$$(5.11) \quad \begin{aligned} \Delta^j \gamma_{\alpha}(mt) &= O(1/m^{j+2}t^2) \quad (j = 0, 1, \dots, (\alpha - 2)), \\ \Delta^{\alpha-1} \gamma_{\alpha}(mt) &= O(1/m^{\alpha}t), \quad \Delta^{\alpha} \gamma_{\alpha}(mt) = O(1/m^{\alpha}), \quad \Delta^{\alpha+1} \gamma_{\alpha}(mt) = O(t/m^{\alpha}). \end{aligned}$$

All of the following estimates depend upon (5.10) or (5.11).

Since $A_n = O(n^{-\delta})$, so that $S_n^j = O(n^{j+1-\delta})$, we have for $j = 0, 1, 2, \dots, (\alpha - 2)$,

$$(5.12) \quad \begin{aligned} S_{k-j}^j \Delta^j \gamma_{\alpha}((k-j)t) &= O((k-j)^{j+1-\delta} 1/(k-j)^{j+2} t^2) \\ &= O(1/k^{1+\delta} t^2) = O(t^{(1+\delta)r-2}) = o(1) \end{aligned}$$

since $r > 1/\delta > 2/(1+\delta)$; and for $j = \alpha - 1$ we have

$$(5.13) \quad \begin{aligned} S_{k-(\alpha-1)}^{\alpha-1} \Delta^{(\alpha-1)} \gamma_{\alpha}((k-(\alpha-1))t) &= O(k^{\alpha-\delta}/k^{\alpha} t) \\ &= O(1/k^{\delta} t) = O(t^{\gamma\delta-1}) = o(1). \end{aligned}$$

Moreover, since $S_n^{\alpha} = o(n^{\alpha-\epsilon})$, because of our hypothesis we have

$$(5.14) \quad S_{k-\alpha}^{\alpha} \Delta^{\alpha} \gamma_{\alpha}((k-\alpha)t) = O\left(k^{\alpha-\epsilon} \frac{1}{k^{\alpha}}\right) = O(k^{-\epsilon}) = o(1).$$

It follows that each term in the second sum of the right hand side of (5.6) is $o(1)$.

Lastly, we write

$$(5.15) \quad \sum_{n=1}^{k-\alpha-1} S_n^{\alpha} \Delta^{\alpha+1} \gamma_{\alpha}(nt) = \sum_{n=1}^l + \sum_{l+1}^{k-\alpha-1} = \chi_3(t) + \chi_4(t)$$

say; we note that in $\chi_3(t)$, $nt < 1$,

$$\begin{aligned}
 (5.16) \quad \chi_3(t) &= \sum_{n=1}^l S_n^\alpha \Delta^{\alpha+1} \gamma_\alpha(nt) = O\left(t^{\alpha+1} \sum_{n=1}^l n^{\alpha-\epsilon}\right) = O(l^{\alpha+1-\epsilon} t^{\alpha+1}) \\
 &= O(t^{\epsilon-\alpha-1+\alpha+1}) = O(t^\epsilon) = o(1),
 \end{aligned}$$

while in $\chi_4(t)$, $nt > 1$,

$$\begin{aligned}
 (5.17) \quad \chi_4(t) &= O\left(t \sum_{n=1}^{k-\alpha-1} n^{\alpha-\epsilon} \frac{1}{n^\alpha}\right) = O\left(t \sum_{n=1}^{k-\alpha-1} n^{-\epsilon}\right) = O(k^{1-\epsilon} t) \\
 &= O(t^{\epsilon-(\epsilon-1)+1}) = o(1),
 \end{aligned}$$

since $r < 1/(1-\epsilon)$. Collecting our results from (5.6), (5.12), (5.13), (5.14), (5.15), (5.16), (5.17) we get $\chi_1(t) = o(1)$. Theorem 2 is thus proved.

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