ARITHMETIC OF ORDERED SYSTEMS

BY

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This paper was first conceived as a short note in which two operations—called ordered addition and ordered multiplication—were to be defined for ordered systems and shown to include all but the sixth of the assorted operations of ordinal and cardinal addition, multiplication, and exponentiation discussed by G. Birkhoff in [1](1). These facts are still in the paper but are completely overshadowed by far more important considerations, mostly arising from the rather unexpected properties of the operation of ordered multiplication. The general purpose of this paper is easily explained. We define these operations of ordered addition and multiplication of families of systems and define certain unary operations called transitization and contraction, which are applied to single systems. We wish to discuss, first, the properties of these operations singly and in combination, and, second, the nature of the ordered systems which arise when these are applied to systems with assigned properties. Examples of the first type of theorem are the general associative laws satisfied by ordered addition and multiplication; a sample of the second type is Theorem 5.14 which shows that while the product of transitive systems need not be transitive it has a property (defined below) which is closely allied to transitivity.

The systems (called numbers) studied by Birkhoff have the two properties of transitivity (if $a \geqq b \geqq c$, then $a \geqq c$) and antisymmetry (if $a \geqq b \geqq a$, then $a = b$). It is noted in [1] that the ordinal power of such systems need not be antisymmetric; that transitivity also fails is easily seen by an example (see §3 below) in which the base is a two-element well-ordered system and the exponent is the system of integers ordered by magnitude. It can be seen from the systems used in this example that any restriction on base and exponent so great that the ordinal power is transitive must be very strong indeed. (For example, we show in §4 that when base and exponent are both numbers, the ordinal power is a number if and only if the base is a cardinal number or the exponent satisfies the ascending chain condition.)

In this paper an ordered system $\mathfrak{R} = (R, \geqq)$ will be a set R in which a reflexive binary relation $\geqq$ holds between some pairs of elements of R. The preceding paragraph shows why no further restriction is placed on the systems involved; apparently no reasonable subclass is closed under ordinal exponentiation. Even the ordinal power of countable ordinals leads to non-transitive systems! Since we often prefer transitive systems or numbers, this

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(1) Numbers in brackets refer to the Bibliography at the end of the paper.

makes useful the simple methods (see §1) by which we can associate to each ordered system $\mathfrak{S}$ a unique transitive system tr($\mathfrak{S}$) with a minimal set of extra related pairs and to each transitive system $\mathfrak{S}$ a unique number $c(\mathfrak{S})$ defined by identifying all elements in each subset of $\mathfrak{S}$ in which each element follows every other. We shall be interested in the relationship of these operations with those of ordered multiplication and addition.

In this connection it soon appears that there are degrees of intransitivity. To make this more precise, note that the condition of transitivity says that if $r_1 \geqq r_2 \geqq \cdots \geqq r_n$, then $r_1 \geqq r_n$; that is, if r_1 and r_n can be connected by a finite chain of r_i such that $r_i \geqq r_{i+1}$, then the chain can be shortened until no middle links are left. Even if this is not possible in a system $\mathfrak{R}$ it may still be possible to shorten any such chain to some definite length; this suggests the following definition: $\mathfrak{R}$ is called k-transitive if the conditions $r_1 \geqq r_2 \geqq \cdots \geqq r_n$ imply the existence of k elements $r_1', \cdots, r_k'$ such that $r_1 \geqq r_1' \geqq \cdots \geqq r_k' \geqq r_n$. That such a definition is not without fruitful content follows from the fact proved in §5 that the example mentioned in the second paragraph of this introduction is 1-transitive.

Briefly the contents of the various sections are as follows: In §1 properties and processes for single systems are discussed; it is here that transitization, contraction, and k-transitivity are carefully defined and relations given between these properties and various order and equivalence relations among ordered systems. In §2 an ordered sum, $\sum_{(R,\geqq)}\mathfrak{S}_r$, is defined for ordered systems $\mathfrak{R}$ and $\mathfrak{S}_r$, $r \in R$, and conditions are given under which the sum is transitive; transitized and contracted sums are also studied. A relation is given between iterated sums and sums over sums which is shown to be the generalization to ordered addition of the ordinary associative law.

The remaining sections are devoted to the ordered product $\prod_{(R,\geqq)}\mathfrak{S}_r$. The definition is given in §3 along with some preliminary but important properties; the ones used most in succeeding sections are 3.7—if all $\mathfrak{S}_r$ are k-transitive and $\prod_{(R,\geqq)} \text{tr}(\mathfrak{S}_r)$ is m-transitive, then $\prod_{(R,\geqq)} \mathfrak{S}_r$ is $(2m+k)$-transitive—and 3.8—if all $\mathfrak{S}_r$ are k-transitive, then $\text{tr}(\prod_{(R,\geqq)} \text{tr}(\mathfrak{S}_r))$ is isomorphic to $\text{tr}(\prod_{(R,\geqq)} \mathfrak{S}_r)$; these facts enable us to avoid many computations with intransitive factors. §4 deals with transitivity of the product; it leads up to Theorem 4.12 which gives a set of conditions on $\mathfrak{R}$ and $\mathfrak{S}_r$ necessary and sufficient that the ordered product $\prod_{(R,\geqq)} \mathfrak{S}_r$ be transitive.

§5 studies k-transitivity of products over numbers and over transitive systems, completing a sequence of theorems of which the important ones are 4.2, 5.5, 5.7 and 5.14; for example, the last of these says that if $\mathfrak{R}$ and all $\mathfrak{S}_r$ are transitive, then $\prod_{(R,\geqq)} \mathfrak{S}_r$ is 2-transitive. Rather simple examples show that the values of the transitivity numbers given in these theorems are best possible. §6 deals with certain homomorphisms and isomorphisms between iterated products and products over sums; these are in the nature of associative laws for the ordered product.

§7 is concerned with the problems on ordinal exponentiation of numbers which were studied by Birkhoff in [1]. We study in this section not only the ordered power ${}^{(R,\geqq)}\mathfrak{S}$, defined as $\prod_{(R,\geqq)}\mathfrak{S}_r$ where $\mathfrak{S}_r=\mathfrak{S}$, but also $\mathrm{tr}({}^{(R,\geqq)}\mathfrak{S})$ and $\mathrm{ctr}({}^{(R,\geqq)}\mathfrak{S})$; for example, Theorems 7.3 and 7.10 give conditions under which these assorted systems are lattices; we also discuss when these are simply ordered or well-ordered or complete lattices or other types of numbers. §8 is a brief appendix discussing the relationship of the product and sum defined here to those defined by Whitehead and Russell in *Principia Mathematica*.

Notation for point set operations will be as usual; that is, $\in$, $\subset$, $\cup$, $\cap$ will have the usual meanings of element of, is contained in, union, and intersection, respectively; $\{p|Q\}$ means the set of all p having the property Q. Due to difficulties in printing the following convention has been adopted to avoid subscripts attached to subscripts and superscripts: If N is a symbol to which it is desired to attach as a subscript a compound symbol i_j, the compound symbol shall be rewritten as $i(j)$ when it is actually used as a subscript so the symbol with subscript appears as $N_{i(j)}$. For another example the sum over $(R, \geqq_i)$ will appear as $\sum_{(R,\geqq(i))}$.

1. **The elementary operations and relations.** If R is a non-empty set and $\geqq$ a reflexive binary relation that holds between some pairs of its elements, we shall call the combination of R and $\geqq$ an *ordered system* and use the symbols $\mathfrak{R}$ and $(R, \geqq)$ for this system; $r>r'$ will mean that $r\geqq r'$ but $r\neq r'$. The fundamental equivalence relation to be used is isomorphism; $\mathfrak{R}$ and $\mathfrak{S}$ are called *isomorphic* (symbol: $\mathfrak{R}\sim\mathfrak{S}$) if there is a one-to-one function h from $\mathfrak{R}$ onto $\mathfrak{S}$ such that $r\geqq r'$ if and only if $h(r)\geqq h(r')$. We shall consider two order relations between ordered systems. $\mathfrak{S}$ is a *homomorphic image* of $\mathfrak{R}$ (symbol: $\mathfrak{R}\succ\mathfrak{S}$) if there is a function h from $\mathfrak{R}$ onto $\mathfrak{S}$ such that $h(r)\geqq h(r')$ if $r\geqq r'$; that is, if and only if there is a monotonic function h defined on $\mathfrak{R}$ whose values fill up $\mathfrak{S}$. A *subsystem* $\mathfrak{R}'=(R', \geqq)$ of $\mathfrak{R}=(R, \geqq)$ is a subset R' of R with the order relation in $\mathfrak{R}'$ imposed by that in $\mathfrak{R}$; that is, for r and r' in R', $r\geqq r'$ in $\mathfrak{R}'$ if and only if $r\geqq r'$ in $\mathfrak{R}$. Say that $\mathfrak{S}$ *follows* $\mathfrak{R}$ (symbol: $\mathfrak{S}>\mathfrak{R}$) if $\mathfrak{R}$ is isomorphic to a subsystem of $\mathfrak{S}$.

There are certain common ordered systems that will be used frequently. If R is any non-empty set, the system $(R, =)$ will be called a *cardinal number*. If $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are both cardinal numbers, obviously $\mathfrak{R}_1\sim\mathfrak{R}_2$ if and only if there is a one-to-one mapping of R_1 onto R_2; also, for any relation $\geqq$ in R, $(R, =)\succ(R, \geqq)$. Systems which are well-ordered (that is, in which every subset has a first element) will be called *ordinal numbers*; in particular, the system (N, w) of all positive integers ordered by magnitude is an ordinal and $\mathfrak{R}>(N, w)$ if $\mathfrak{R}$ is an infinite ordinal. If n is any positive integer, let (N_n, w) be the subsystem containing the first n elements of (N, w); then the systems (N_n, w) are finite ordinals; the systems $(N_n, =)$ are, of course, finite cardinals. If R is a non-empty set, the *universal relation* u in R is that in which every element follows every other; that is, $r\ u\ r'$ for every r, r' in R. Note that $(N_1, w)=(N_1, =)=(N_1, u)$.

If $\geqq$ and $\geqq'$ are two order relations in one set R, say that $\geqq$ *includes* $\geqq'$ if $r\geqq' r'$ implies that $r\geqq r'$; clearly if $\geqq$ includes $\geqq'$, then $(R, \geqq')\succ(R, \geqq)$. If $\mathfrak{R}=(R, \geqq)$ is an ordered system, define tr$(\mathfrak{R})$, the *transitization* of $\mathfrak{R}$, to be the ordered system $(R, \geqq_t)$ where $\geqq_t$ is the least transitive relation including $\geqq$; that is, $r_0\geqq_t r$ means that there exist $r_1, \cdots, r_n$, $n\geqq 0$, such that $r_0\geqq r_1\geqq \cdots \geqq r_n\geqq r$. Clearly $\mathfrak{R}\succ$ tr$(\mathfrak{R})$ for every $\mathfrak{R}$; $\mathfrak{R}$ is called *transitive* if $\mathfrak{R}=$ tr$(\mathfrak{R})$. Since transitivity is clearly preserved under isomorphism, and since tr(tr$(\mathfrak{R})$)$=$tr$(\mathfrak{R})$, $\mathfrak{R}$ is transitive if and only if $\mathfrak{R}\sim$tr$(\mathfrak{R})$ and if and only if there exists $\mathfrak{S}$ such that $\mathfrak{R}\sim$tr$(\mathfrak{S})$. Also $\mathfrak{R}$ is transitive if and only if the conditions $r_0\geqq r_1\geqq \cdots \geqq r_n$ imply $r_0\geqq r_n$.

A second operation can be applied most profitably to transitive systems. If $\mathfrak{R}$ is transitive and $r\in R$, let $c(r)=\{r' \mid r\geqq r'\geqq r\}$; that is, $(c(r), \geqq)$ as a subsystem of $(R, \geqq)$ has the universal order relation u and $c(r)$ is the largest such set containing r. Let $c(\mathfrak{R})$, the *contraction* of $\mathfrak{R}$, be the system whose elements are these sets $c(r)$, where $c(r)\geqq c(r')$ if and only if $r\geqq r'$. It is easily verified that there is no contradiction in defining the order relation in this way; it is also clear that if R' is a subset of R such that $R'\cap c(r)$ contains just one point for each r, then $(R', \geqq)\sim c(\mathfrak{R})$ so $\mathfrak{R}>c(\mathfrak{R})$; since the contraction mapping is a homomorphism, $\mathfrak{R}\succ c(\mathfrak{R})$ also holds. As in [1] $\mathfrak{R}$ will be called a *number* if and only if the natural homomorphism $r\rightarrow c(r)$ of $\mathfrak{R}$ onto c(tr$(\mathfrak{R})$) is an isomorphism; that is, if and only if $\mathfrak{R}$ is transitive and has no pairs of distinct points r and r' for which $r>r'>r$. From one point of view a number may be regarded as an extremely transitive system; precisely, $(R, \geqq)$ is a number if and only if the relation $>$ is transitive.

1.1 LEMMA. tr *and* c *are invariant under isomorphism and monotone under homomorphism; that is, if* $\mathfrak{R}\sim\mathfrak{S}$, *then* tr$(\mathfrak{R})\sim$tr$(\mathfrak{S})$ *and* c(tr$(\mathfrak{R})$)$\sim c$(tr$(\mathfrak{S})$) *while if* $\mathfrak{S}\succ\mathfrak{R}$, *then* tr$(\mathfrak{S})\succ$tr$(\mathfrak{R})$ *and* c(tr$(\mathfrak{S})$)$\succ c$(tr$(\mathfrak{R})$). *If* $\mathfrak{S}$ *is transitive and* $\mathfrak{S}>\mathfrak{R}$, *then* $\mathfrak{R}$ *is transitive and* $c(\mathfrak{S})>c(\mathfrak{R})$.

These properties can be verified directly from the appropriate definitions. Note that $\mathfrak{S}>\mathfrak{R}$ need not imply that tr$(\mathfrak{S})>$tr$(\mathfrak{R})$; for example, let $\mathfrak{S}$ contain elements s_1, s_2, s_3 where, besides equality, all the relations that hold are $s_1>s_2$, $s_2>s_1$, $s_1>s_3$, $s_3>s_1$; let $\mathfrak{R}$ be the subsystem of $\mathfrak{S}$ containing only s_2 and s_3; then tr$(\mathfrak{S})=(S, u)$ while tr$(\mathfrak{R})=\mathfrak{R}=(R, =)$, so tr$(\mathfrak{R})$ is not isomorphic to any subsystem of tr$(\mathfrak{S})$.

We give next a theorem on factorization of homomorphisms.

1.2 THEOREM. *If* $\mathfrak{S}$ *is a number and* $\mathfrak{R}=(R, \geqq)$, *then* $\mathfrak{R}\succ\mathfrak{S}$ *if and only if there exists a transitive relation* $\geqq'$ *including* $\geqq$ *in* R *such that* $c(R, \geqq')\sim\mathfrak{S}$. *If* h *and* H *are, respectively, the homomorphism and isomorphism involved, one can be calculated from the other by the relation* $H(c(r))=h(r)$.

If h is given so that $h(r) \geqq h(r')$ if $r \geqq r'$, define $r \geqq' r'$ to mean that $h(r) \geqq h(r')$; since $\mathfrak{S}$ is transitive, $\geqq'$ is transitive; since h is monotone, $\geqq'$ includes $\geqq$. Then in $(R, \geqq')$ we have $c(r) = h^{-1}h(r)$, so the mapping H defined by the equation above is one-to-one between $c(R, \geqq')$ and $\mathfrak{S}$. $c(r) \geqq c(r')$ means $r \geqq' r'$; that is, $h(r) \geqq h(r')$ or $H(c(r)) \geqq H(c(r'))$, so H is an isomorphism. If $\geqq'$ and H are given, define h by the above equation; since the identity is a homomorphism of $(R, \geqq)$ onto $(R, \geqq')$, since contraction is a homomorphism of $(R, \geqq')$ onto $c(R, \geqq')$, and since H is an isomorphism of $c(R, \geqq')$ onto $\mathfrak{S}$, h is a homomorphism of $(R, \geqq)$ onto $\mathfrak{S}$.

1.3 Corollary. *If h is a homomorphism of an ordered system $\mathfrak{R}$ onto the number $\mathfrak{S}$, then h can be factored into three pieces, $h = HcI$, where I is the identity mapping of $(R, \geqq)$ onto $(R, \geqq')$ and $\geqq'$ is a transitive relation including $\geqq$, c is the contraction of $(R, \geqq')$ onto $c(R, \geqq')$, and H is an isomorphism of $c(R, \geqq')$ onto $\mathfrak{S}$.*

Note that if $\mathfrak{S}$ is only transitive but not a number, a similar factoring, $h = Hc'I$, is possible with first and third factors of the same nature as before, but the middle factor c' is only a partial contraction of $(R, \geqq')$. If $\mathfrak{S}$ is not even transitive, then factoring is still possible but $\geqq'$ need not be transitive.

We have already defined transitivity of ordered systems, but the lumping together of all intransitive systems into one class is too crude a procedure for some parts of this paper. We shall define a property of k-transitivity of ordered systems, k a non-negative integer, in such a way that ordinary transitivity is the special case for which $k=0$. The system $(S, \geqq)$ or the relation $\geqq$ is called *k-transitive* if and only if the existence of a chain $s_1 \geqq s_2 \geqq \cdots \geqq s_n$ connecting s_1 with s_n implies the existence of $s_1', \cdots, s_{k'}'$ in S such that $s_1 \geqq s_1' \geqq s_2' \geqq \cdots \geqq s_k' \geqq s_n$. Note that no assertion is made about the distinctness of the s_i' nor do the s_i' have to be among the original s_j; hence if $\mathfrak{S}$ is k-transitive it is also n-transitive for every integer $n \geqq k$; this property has the disadvantage that $\mathfrak{R} > \mathfrak{S}$ and $\mathfrak{R}$ k-transitive do not imply $\mathfrak{S}$ n-transitive for any n (except where $k=0$). This is unfortunate but not fatal; after all, even transitivity is not preserved under the relation $\succ$.

Another formulation of this property may add some clarity. If E is any subset of the ordered system $\mathfrak{S} = (S, \geqq)$, define $E^U = \{s \mid \text{there exists } s' \text{ in } E \text{ for which } s \geqq s'\}$; similarly, let $E^D = \{s \mid \text{there exists } s' \text{ in } E \text{ for which } s' \geqq s\}$. Then the operation $E \to E^U$ (or $E \to E^D$) is a closure function in S in the sense of my earlier paper [3]; it has several elementary properties: $E^U \supset E$; if $E_1 \subset E_2$, then $E_1^U \subset E_2^U$; much more than this is true; this operation is additive for an arbitrary number of terms; that is, if R is a set and if for each r in R, $E_r \subset S$, then $(\bigcup_{r \in R} E_r)^U = \bigcup_{r \in R} (E_r^U)$; $E^U = E$ for every finite set E if and only if $E^U = E$ for every $E \subset S$; that is, if and only if $\mathfrak{S}$ is $(S, =)$. The various transitivity properties of $\mathfrak{S}$ can be simply expressed in terms of this operation; $\mathfrak{S}$ is transitive if and only if $E^{UU} = E^U$ for every $E \subset S$; that is, if and only if

^{U}(or, dually, ^{D}) is idempotent. Similarly, $\mathfrak{S}$ is k-transitive if and only if $E^{(k+2)U}=E^{(k+1)U}$ where E^{kU} means to apply ^{U} k times in succession.

For future reference we give here two definitions. A *star* in $\mathfrak{S}$ is a subset E of $\mathfrak{S}$ such that $E^{U}=E$; that is, E is a star if it contains every successor of each of its elements. Clearly the stars in $\mathfrak{S}$ and in tr($\mathfrak{S}$) are the same subsets of S. A set E is *cofinal* [*coinitial*] in $\mathfrak{S}$ if and only if $E^{D}=S$ [$E^{U}=S$].

A *terminal element* of an ordered system $\mathfrak{R}$ is an element r_0 with no successors (different from itself); that is, r_0 is a terminal element of $\mathfrak{R}$ if the condition $r>r_0$ is not satisfied by any r in R. (This is not quite the usage of [2] but agrees with it for numbers.) r_0 is a terminal element of $\mathfrak{R}$ if and only if it is a terminal element of tr($\mathfrak{R}$); if r_0 is a terminal element of a transitive $\mathfrak{R}$, then $c(r_0)$ is a terminal element of $c(\mathfrak{R})$, but not necessarily conversely. If $E\subset R$, let $E^{(1)}$ be the set of terminal elements of the subsystem $(E, \geqq)$ of $(R, \geqq)$. For any ordinal number $\alpha>1$ define $E^{(\alpha)}$ to be the set $(E-\bigcup_{\lambda<\alpha}E^{(\lambda)})^{(1)}$. Then there must be a smallest ordinal $\lambda_0\geqq 1$ such that $E^{(\lambda(0))}$ is empty; if $\lambda>\lambda_0$, $E^{(\lambda)}$ is also empty; define $E'=\bigcup_{\lambda<\lambda(0)}E^{(\lambda)}$.

1.4 LEMMA. *For any $\mathfrak{R}$ and any α, $(\bigcup_{\lambda<\alpha}R^{(\lambda)})^{U}=\bigcup_{\lambda<\alpha}R^{(\lambda)}$; that is, $\bigcup_{\lambda<\alpha}R^{(\lambda)}$ is a star in $\mathfrak{R}$; hence R' is a star in $\mathfrak{R}$.*

If $r\in R^{(\lambda(1))}$ for some λ_1, then r is a terminal element of $R-\bigcup_{\lambda<\lambda(1)}R^{(\lambda)}$; hence every successor of r is in $\bigcup_{\lambda<\lambda(1)}R^{(\lambda)}$; that is, $(\bigcup_{\lambda<\alpha}R^{(\lambda)})^{U}=\bigcup_{\lambda<\alpha}R^{(\lambda)}$.

1.5 COROLLARY. *If $\mathfrak{R}$ is transitive and $R'=R$, then $\mathfrak{R}$ is a number.*

Let $\lambda(r)$ be the ordinal less than λ_0 such that $r\in R^{(\lambda)}$; then $\lambda(r)$ is defined for every r in R and is strictly decreasing; that is, $r>r'$ implies $\lambda(r)<\lambda(r')$. Hence $r>r'>r$ would imply $\lambda(r)<\lambda(r')<\lambda(r)$ and this is impossible under the relation among ordinals.

1.6 THEOREM. *If $(R, \geqq)$ is transitive, $R=R'$ if and only if $E\subset E^{(1)D}$ for every $E\subset R$.*

If $R\neq R'$, let $E=R-R'$; then $E^{(1)}$ is empty so $E^{(1)D}\not\supset E$. If $R'=R$ and $E\subset R$, let $r\in E$; then there is a smallest ordinal λ_0 in the set of ordinals $\{\lambda(r')\mid r'\geqq r \text{ and } r'\in E\}$. If r_0 is a successor of r in E such that $\lambda(r_0)=\lambda_0$, then no successor of r_0 can lie in E, so $r_0\in E^{(1)}$ and $r_0\geqq r$; that is, $E\subset E^{(1)D}$.

1.7 THEOREM. *$R=R'$ if and only if ascending chains in $\mathfrak{R}$ are finite; that is, if and only if the conditions $r_1\leqq r_2\leqq\cdots\leqq r_n\leqq\cdots$ imply that $r_n=r_{n(0)}$ for all $n\geqq$ some n_0.*

If $R=R'$ and $r_1\leqq r_2\leqq\cdots\leqq r_n\leqq\cdots$, then $\lambda(r_1)\geqq\lambda(r_2)\geqq\cdots\geqq\lambda(r_n)\geqq\cdots$; the set $\{\lambda(r_n)\mid n\in N\}$ contains a smallest element, say $\lambda(r_{n(0)})$, so $\lambda(r_n)=\lambda(r_{n(0)})$ if $n\geqq n_0$. Hence $r_n=r_{n(0)}$ if $n\geqq n_0$. If $R'\neq R$, every element of $R-R'$ has a successor different from itself, so there exists an infinite chain $r_1<r_2<\cdots<r_n<\cdots$.

Note that if the transitive system $\mathfrak{R}$ is finite, $R'=R$ if and only if $\mathfrak{R}$ is a number.

For §4 we shall need the next two lemmas.

1.8 LEMMA. *If A and B are subsets of $(R, \geqq)$, then $(A\cup B)^{(1)}\subset A^{(1)}\cup B^{(1)}$; in fact, $(A\cup B)^{(1)}=(A^{(1)}-B^D)\cup(B^{(1)}-A^D)\cup(A^{(1)}\cap B^{(1)})$.*

If $r\in(A\cup B)^{(1)}$, if $r'\geqq r$ and $r'\in A$, then $r'\in(A\cup B)$ and $r'\geqq r$ so $r'=r$; hence if $r\in(A\cup B)^{(1)}\cap A^D$, $r\in A^{(1)}$. Similarly if $r\in(A\cup B)^{(1)}\cap B^D$, $r\in B^{(1)}$; since every r in $(A\cup B)^{(1)}$ is in $A^D\cup B^D$, we have the three distinct possibilities $r\in A^{(1)}-B^D$ or $B^{(1)}-A^D$ or $A^{(1)}\cap B^{(1)}$.

1.9 LEMMA. *If $\mathfrak{R}$ is a number with no terminal elements, there exists three disjoint cofinal subsets of $\mathfrak{R}$.*

Well-order the elements of $\mathfrak{R}$ and let r_1 be the first element in this ordering, let r_2 be the first that follows r_1, and so on as long as possible; that is, let r_α be the first element of R that follows all r_λ, $\lambda<\alpha$, as long as such an r_α exists. Then there must be a first α_0 such that the set $\{r_\alpha|\alpha<\alpha_0\}$ has no common successors; since $\mathfrak{R}$ has no terminal elements such an α_0 must be a limit ordinal. At such a point define $r_{\alpha(0)}$ to be the first element of R which does not precede any r_α, $\alpha<\alpha_0$, and proceed from $r_{\alpha(0)}$ as from r_1 until stuck again at α_1. Repetition of this process defines limit ordinals α_λ, $\lambda<\lambda_0$, and points r_α, $\alpha<\alpha_{\lambda(0)}$, such that $r_\alpha<r_{\alpha'}$ if $\alpha_\lambda\leqq\alpha<\alpha'<\alpha_{\lambda+1}$ while $r_{\alpha'}$ does not precede or equal r_α if $\alpha'>\alpha$; moreover, for each r in R there is an $\alpha<\alpha_{\lambda(0)}$ such that $r_\alpha\geqq r$. Let $E_i=\{r_\alpha|\alpha$ a limit ordinal plus $i-1$ plus a multiple of three, $\alpha<\alpha_{\lambda(0)}\}$; then the sets E_i have the desired properties, since $E_1\cup E_2\cup E_3$ is cofinal in R, and if $r_\alpha\in E_i$, then $r_{\alpha+1}\in E_{i+1(\text{mod } 3)}$.

2. **Ordered addition.** For disjoint systems a notion of ordered addition should, to fit our intuitive notions, have something to do with an ordering of the point-set union of the systems. It should include the notions of cardinal and ordinal addition used in [4] and [1], and, if all the terms in the sum are alike, should specialize to some kind of multiplication, in this case the ordinal multiplication of [4] and [1]. It is easy to give a rough description of the sum over $\mathfrak{R}$ of the systems $\mathfrak{S}_r$; the sum is obtained by putting each $\mathfrak{S}_r$ in place of r in the system $\mathfrak{R}$. This will be made more precise in the next paragraph, but it should be mentioned that this definition of sum need not give transitive sums even when all systems concerned are transitive. Since it is often useful to construct transitive systems or numbers from systems of the same sort, we shall also define modified sums obtained by following the operation of addition by tr or ctr. (It will be seen by those who have read [2] that the operation of ordered addition used there is the one called $\sum^t$ here.) In [4] a definition of ordered sum of relations is given which translates into the definition of sum used here; see §8 for further remarks on this subject.

We give now the precise definition of sum. If $\mathfrak{R}$ is an ordered system and if, for each r in $\mathfrak{R}$, $\mathfrak{S}_r$ is an ordered system, define $\sum_{(R,\geqq)} \mathfrak{S}_r$, the *ordered sum* over $\mathfrak{R}$ of the systems $\mathfrak{S}_r$, to be the system $\mathfrak{P}=(P, \geqq)$ where the elements of P are the ordered pairs (r, s) with r in R and s in S_r, and $(r, s) \geqq (r', s')$ means that $r>r'$ or else $r=r'$ and $s \geqq s'$; $\mathfrak{R}$ will be called the *index system* and the $\mathfrak{S}_r$ the *terms* of the sum. (Note that if the $\mathfrak{S}_r$ are disjoint, there is a natural one-to-one correspondence between P and $\bigcup_{r \in R} S_r$.) Define $\sum^t_{(R,\geqq)} \mathfrak{S}_r$ to be $\text{tr}(\sum_{(R,\geqq)} \mathfrak{S}_r)$ and $\sum^c_{(R,\geqq)} \mathfrak{S}_r=\text{ctr}(\sum_{(R,\geqq)} \mathfrak{S}_r)$; these are, respectively, the *transitive* and *contracted* sums over $\mathfrak{R}$ of the $\mathfrak{S}_r$; we shall use $\geqq^t$ and $\geqq^c$ for the order relations in these systems.

2.1 THEOREM. *If $\mathfrak{R}$ and all $\mathfrak{S}_r$ are transitive, $(r, s) \geqq^t (r', s')$ if and only if* (a) $(r, s) \geqq (r', s')$ *or* (b) $r=r'$ *but there exists r'' such that $r>r''>r$; hence $\sum_{(R,\geqq)} \mathfrak{S}_r$ is 1-transitive if $\mathfrak{R}$ and $\mathfrak{S}_r$ are transitive.*

Obviously $(r, s) \geqq^t (r', s')$ if (a) or (b) holds. $(r, s) \geqq^t (r', s')$ means that there exist points (r_i, s_i) such that $(r, s) \geqq (r_1, s_1) \geqq (r_2, s_2) \geqq \cdots \geqq (r_n, s_n) \geqq (r', s')$. Hence $r \geqq r_1 \geqq r_2 \geqq \cdots \geqq r_n \geqq r'$. If equality holds all down this chain, then $s \geqq s_1 \geqq s_2 \geqq \cdots \geqq s_n \geqq s'$, so $r=r'$ and $s \geqq s'$ by transitivity in $\mathfrak{S}_r$; that is, $(r, s) \geqq (r', s')$. If at least one of these is not an equality, by transitivity in $\mathfrak{R}$ either $r>r'$ (implying (a)) or (b) holds.

2.2 LEMMA. *The index system and the terms of a sum are isomorphic to subsystems of the sum; that is, $\mathfrak{R}<\sum_{(R,\geqq)} \mathfrak{S}_r$ and $\mathfrak{S}_{r(0)}<\sum_{(R,\geqq)} \mathfrak{S}_r$ for every choice of $\mathfrak{R}$, $\mathfrak{S}_r$, and r_0.*

If s_r is any point of S_r, the function $h(r)=(r, s_r)$ is an isomorphism of $\mathfrak{R}$ into $\sum_{(R,\geqq)} \mathfrak{S}_r$; the function g from $\mathfrak{S}_{r(0)}$ into $\sum_{(R,\geqq)} \mathfrak{S}_r$ defined by $g(s) =(r_0, s)$ is also an isomorphism. Note that if $r_0>r_1>r_0$, $\mathfrak{S}_{r(0)}$ need not be isomorphic to a subsystem of $\sum^t_{(R,\geqq)} \mathfrak{S}_r$.

2.3 COROLLARY. *$\sum_{(R,\geqq)} \mathfrak{S}_r$ is transitive if and only if* (a) *$\mathfrak{R}$ and every $\mathfrak{S}_r$ are transitive, and* (b) *if $r>r'>r$, then $\mathfrak{S}_r$ has the universal order relation.*

(a) is necessary by 2.2 since every subsystem of a transitive system is transitive. If $r>r'>r$, let $s, s'' \in S_r$ and $s' \in S_{r'}$; then $(r, s)>(r', s')>(r, s'')$ so, by transitivity, $(r, s) \geqq (r, s'')$; hence $s \geqq s''$ for every pair of points in $\mathfrak{S}_r$; that is, $\mathfrak{S}_r$ is (S_r, u). If (a) and (b) hold, by 2.1 $\sum_{(R,\geqq)} \mathfrak{S}_r=\sum^t_{(R,\geqq)} \mathfrak{S}_r$ so the first system is transitive.

2.4 COROLLARY. *If $\mathfrak{R}$ is a number, $\sum_{(R,\geqq)} \mathfrak{S}_r$ is transitive if and only if all $\mathfrak{S}_r$ are transitive and $\sum_{(R,\geqq)} \mathfrak{S}_r$ is a number if and only if all $\mathfrak{S}_r$ are numbers.*

The first of these statements follows immediately from 2.1 and 2.3 since neither case (b) can occur if $\mathfrak{R}$ is a number. For the second result we can use the first; if all $\mathfrak{S}_r$ are numbers and $(r, s) \geqq (r', s') \geqq (r, s)$, then $r \geqq r' \geqq r$ so $r=r'$ and $s \geqq s' \geqq s$; hence $s=s'$. If $\sum_{(R,\geqq)} \mathfrak{S}_r$ is a number, 2.2 implies that $\mathfrak{R}$ and $\mathfrak{S}_r$ are numbers.

2.5 LEMMA. *If* $\geqq_1$ *includes* $\geqq_2$ *in* R *and if* $\geqq^i$ *is the order relation in* $\sum_{(R,\geqq(i))}\mathfrak{S}_r$, *then* $\geqq^1$ *includes* $\geqq^2$ *and a similar relation holds for the transitive sums; hence* $\sum_{(R,\geqq(1))}\mathfrak{S}_r \prec \sum_{(R,\geqq(2))}\mathfrak{S}_r$, $\sum^t_{(R,\leqq(1))}\mathfrak{S}_r \prec \sum^t_{(R,\geqq(2))}\mathfrak{S}_r$ *and* $\sum^c_{(R,\geqq(1))}\mathfrak{S}_r \prec \sum^c_{(R,\geqq(2))}\mathfrak{S}_r$.

If $(r, s)\geqq^2(r', s')$, then $r\geqq_2 r'$ or $r=r'$ and $s\geqq s'$; hence $r\geqq_1 r'$ or $r=r'$ and $s\geqq s'$, so $(r, s)\geqq^1(r', s')$; that is, $\geqq^1$ includes $\geqq^2$. If $\geqq^{it}$ is the order relation in $\operatorname{tr}(\sum_{(R,\geqq(i)},\mathfrak{S}_r)$, it follows immediately that $\geqq^{1t}$ includes $\geqq^{2t}$. The first two homomorphisms are immediate consequences of these two inclusions; the third follows from the second and 1.1.

2.6 LEMMA. *If in* $\sum_{(R,\geqq)}\mathfrak{S}_r$ *a chain* $(r_1, s_1)\geqq(r_2, s_2)\geqq \cdots \geqq(r_n, s_n)$ *is given, either* (1) $r_i=r_{i+1}$ *and* $s_i\geqq s_{i+1}$ *for all* i *or* (2) *there exists a shortest subchain* $(r_{i(1)}, s_{i(1)})> \cdots >(r_{i(k)}, s_{i(k)})$ *such that* $r_{i(j)}>r_{i(j+1)}$ *for all* $j<k$, *while* $i_1=1$ *and* $r_{i(k)}=r_n$.

If all r_i are equal, then we have $s_i\geqq s_{i+1}$. If some r_i differ, we can find integers i_j such that $r_1=r_2= \cdots =r_{i(2)-1}>r_{i(2)}= \cdots =r_{i(3)-1}>r_{i(3)} \cdots r_{i(k)-1} >r_{i(k)}= \cdots =r_n$; these r_i satisfy the given conditions.

This result has two useful consequences, 2.7 and 2.8.

2.7 THEOREM. *If* $\mathfrak{R}$ *is* m-*transitive and all* $\mathfrak{S}_r$ *are* k-*transitive, then* $\sum_{(R,\geqq)}\mathfrak{S}_r$ *is* n-*transitive, where* $n=\sup(m+1, k)$.

If $(r_1, s_1)\geqq \cdots \geqq(r_p, s_p)$ and all r_i are equal, $s_1\geqq s_2\geqq \cdots \geqq s_p$ in $\mathfrak{S}_r$; since $\mathfrak{S}_r$ is k-transitive, this chain can be replaced by $s_1\geqq s_1' \geqq \cdots \geqq s_k' \geqq s_p$ so $(r_1, s_1)\geqq(r_1, s_1')\geqq \cdots \geqq(r_1, s_k')\geqq(r_p, s_p)$ and a chain with not more than k middle links connects the ends. If some $r_i\neq r_{i+1}$ we can use the second chain of 2.6, $\{(r_{i(j)}, s_{i(j)})\}$. If $r_1\neq r_p$, shorten the chain down by m-transitivity in $\mathfrak{R}$ till there is a chain with not more than m middle links r_i' such that $r_1>r_1' > \cdots >r_q' >r_p$; then for any s_i' in $S_{r'(i)}$, $(r_1, s_1)>(r_1', s_1')> \cdots >(r_q', s_q')>(r_p, s_p)$ so there is a chain with not more than m middle links connecting (r_1, s_1) with (r_p, s_p). If $r_1=r_p$, take the last $r_{i(j)}$ before the chain first closes up to r_1 again; then there exist r_i' such that $r_1>r_1' > \cdots >r_q' >r_{i(j)}>r_1$; using the argument just above we see that there is a chain with not more than $m+1$ middle links connecting (r_1, s_1) and (r_p, s_p). This shows that any chain can be shortened until it contains not more than n middle links; that is, $\sum_{(R,\geqq)}\mathfrak{S}_r$ is n-transitive.

The next theorem shows that we can alter the order relations in $\mathfrak{R}$ and the $\mathfrak{S}_r$ to some extent without altering the order relation in the transitive sum.

2.8 THEOREM. *For every choice of the systems* $\mathfrak{R}$ *and* $\mathfrak{S}_r$ *we have*

$$\sum_{(R,\geqq)}{}^t\mathfrak{S}_r = \sum_{\operatorname{tr}(R,\geqq)}{}^t \operatorname{tr}(\mathfrak{S}_r).$$

Let $\geqq^t$ and $\geqq^T$ be the order relations in the left and right-hand systems respectively, and use $\geqq_t$ for the relations in $\operatorname{tr}(\mathfrak{R})$ and $\operatorname{tr}(\mathfrak{S}_r)$; clearly $\geqq^T$ includes $\geqq^t$. $(r, s) \geqq^T (r', s')$ means, by 2.1, that either (1) $r >_t r'$ or (2) $r = r'$ but $s >_t s'$ or (3) $r >_t r'' >_t r = r'$. In case (1) either $r > r'$ or there exist r_i such that $r > r_1 > \cdots > r_n > r'$; in either case $(r, s) \geqq^t (r', s')$. In case (2) there exist s_i such that $s \geqq s_1 \geqq \cdots \geqq s_n \geqq s'$ so $(r, s) \geqq (r, s_1) \geqq \cdots \geqq (r, s_n) \geqq (r', s')$ so again $(r, s) \geqq^t (r', s')$. In case (3) take any s'' in $S_{r''}$; then as in case (1), $(r, s) \geqq^t (r'', s'') \geqq^t (r', s')$ so $(r, s) \geqq^t (r', s')$. Hence $\geqq^t$ and $\geqq^T$ are equal.

2.9 COROLLARY. *If* $\operatorname{tr}(R, \geqq_1) = \operatorname{tr}(R, \geqq_2)$ *and* $\operatorname{tr}(S_r, \geqq_{1r}) = \operatorname{tr}(S_r, \geqq_{2r})$ *for each* r, *then* $\sum^t_{(R,\geqq(1))} (S_r, \geqq_{1r}) = \sum^t_{(R,\geqq(2))} (S_r, \geqq_{2r})$; *in particular, both the transitive sums of* 2.8 *are equal to* $\sum^t_{(R,\geqq)} \operatorname{tr}(\mathfrak{S}_r) = \sum^t_{\operatorname{tr}(R,\geqq)} \mathfrak{S}_r$.

The relations between $\sum$, $\succ$ and $>$ are not surprising.

2.10 THEOREM. *If* $\mathfrak{S}_r \succ \mathfrak{P}_r$, *then* $\sum_{(R,\geqq)} \mathfrak{S}_r \succ \sum_{(R,\geqq)} \mathfrak{P}_r$; *if* $\mathfrak{S}_r < \mathfrak{P}_r$, *then* $\sum_{(R,\geqq)} \mathfrak{S}_r < \sum_{(R,\geqq)} \mathfrak{P}_r$. *The first relation holds for* $\sum^t$ *and* $\sum^c$; *the second holds for* $\sum^t$ *and* $\sum^c$ *if* $\operatorname{tr}(\mathfrak{S}_r) < \operatorname{tr}(\mathfrak{P}_r)$.

If h_r is a homomorphism of $\mathfrak{S}_r$ onto $\mathfrak{P}_r$, define $h(r, s) = (r, h_r(s))$; it is easily verified that h is a homomorphism of $\sum_{(R,\geqq)} \mathfrak{S}_r$ onto $\sum_{(R,\geqq)} \mathfrak{P}_r$. If instead h_r is an isomorphism of $\mathfrak{S}_r$ into $\mathfrak{P}_r$, clearly $h(r, s) \geqq h(r', s')$ if $(r, s) \geqq (r', s')$. If $h(r, s) \geqq h(r', s')$, either $r > r'$ or $r = r'$ and $h_r(s) \geqq h_r(s')$; in the first case $(r, s) > (r', s')$; in the second $s \geqq s'$ as h_r is an isomorphism, so, again, $(r, s) \geqq (r', s')$; that is, h is an isomorphism of $\sum_{(R,\geqq)} \mathfrak{S}_r$ and a subsystem of $\sum_{(R,\geqq)} \mathfrak{P}_r$.

1.1 now implies the relation $\succ$ for $\sum^t$ and $\sum^c$. To prove that $\sum^t_{(R,\geqq)} \mathfrak{S}_r < \sum^t_{(R,\geqq)} \mathfrak{P}_r$, we see by 2.8 that this holds if and only if it holds when all systems are transitive; that is, we need only prove the special case:

2.11. If $\mathfrak{R}$ and $\mathfrak{P}_r$ are transitive and $\mathfrak{S}_r < \mathfrak{P}_r$, then $\sum^t_{(R,\geqq)} \mathfrak{S}_r < \sum^t_{(R,\geqq)} \mathfrak{P}_r$.

Let h be the function previously defined; then $(r, s) \geqq^t (r', s')$ means by 2.1 that $r > r'$ or $r = r'$, $s \geqq s'$ or $r > r'' > r' = r$. One of these cases occurs if and only if the same relation holds for $(r, h_r(s))$ and $(r', h_r'(s'))$, so h is again an isomorphism. 1.1 gives the same relation for $\sum^c$.

Another special case is:

2.12 COROLLARY. *If the systems* $\mathfrak{P}_r$ *are transitive and* $\mathfrak{P}_r > \mathfrak{S}_r$, *then* $\sum^t_{(R,\geqq)} \mathfrak{P}_r > \sum^t_{(R,\geqq)} \mathfrak{S}_r$ *and* $\sum^c_{(R,\geqq)} \mathfrak{P}_r > \sum^c_{(R,\geqq)} \mathfrak{S}_r$.

The relation 2.8 between $\sum$ and tr is simpler than the corresponding relation between $\sum$ and c.

2.13 COROLLARY. $\sum^c_{(R,\geqq)} \mathfrak{S}_r \sim \sum^c_{\operatorname{tr}(R,\geqq)} \operatorname{ctr}(\mathfrak{S}_r)$.

Since $\sum^t_{(R,\geqq)} \mathfrak{S}_r = \sum^t_{\operatorname{tr}(R,\geqq)} \operatorname{tr}(\mathfrak{S}_r)$, by 1.1 $\sum^c_{(R,\geqq)} \mathfrak{S}_r \sim \sum^c_{\operatorname{tr}(R,\geqq)} \operatorname{tr}(\mathfrak{S}_r)$

so we need only prove $\sum^c_{\mathrm{tr}(R,\geqq)} \mathrm{tr}(\mathfrak{S}_r) \sim \sum^c_{\mathrm{tr}(R,\geqq)} \mathrm{ctr}(\mathfrak{S}_r)$; that is, we need only prove the following special case of 2.13.

2.14. If $\mathfrak{R}$ and all $\mathfrak{S}_r$ are transitive, $\sum^c_{(R,\geqq)} \mathfrak{S}_r \sim \sum^c_{(R,\geqq)} c(\mathfrak{S}_r)$.

Since the contraction c of $\mathfrak{S}_r$ onto $c(\mathfrak{S}_r)$ is a homomorphism, by the proof of 2.10 the function h defined by $h(r, s) = (r, c(s))$ is a homomorphism of $\sum^t_{(R,\geqq)} \mathfrak{S}_r$ onto $\sum^t_{(R,\geqq)} c(\mathfrak{S}_r)$; hence the function H defined from $c(\sum^t_{(R,\geqq)} \mathfrak{S}_r)$ onto $c(\sum^t_{(R,\geqq)} c(\mathfrak{S}_r))$ by $H(c(r, s)) = c(h(r, s))$ is also a homomorphism. Suppose $H(c(r', s')) \leqq H(c(r, s))$; that is, $c(h(r, s)) \geqq c(h(r', s'))$; this is equivalent to $h(r, s) \geqq^t h(r', s')$ or $(r, c(s)) \geqq^t (r', c(s'))$. Then by 2.1, $r > r'$ or $r > r'' > r' = r$ or $r = r'$, $c(s) \geqq c(s')$; that is, $r > r'$ or $r > r'' > r' = r$ or $r = r'$, $s \geqq s'$; hence $(r, s) \geqq^t (r', s')$ if $H(c(r, s)) \geqq H(c(r', s'))$. From this it follows that $c(r, s) \geqq c(r', s')$ if $H(c(r, s)) \geqq H(c(r', s'))$, so H is one-one and an isomorphism of $\sum^c_{(R,\geqq)} \mathfrak{S}_r$ onto $\sum^c_{(R,\geqq)} c(\mathfrak{S}_r)$.

We have an elementary relation between addition and contraction which will be most useful in §5.

2.15 THEOREM. *If $\mathfrak{S}$ is transitive, there is a number $\mathfrak{R}$ ($c(\mathfrak{S})$ will do) and subsystems $\mathfrak{S}_r = (S_r, u)$ of $\mathfrak{S}$ such that $\mathfrak{S} \sim \sum_{(R,\geqq)} \mathfrak{S}_r$.*

We need only let $\mathfrak{R} = c(\mathfrak{S})$ and let $S_r = c^{-1}(r)$, where c is the contraction mapping of $\mathfrak{S}$ onto $\mathfrak{R}$.

We give next a theorem on subsystems.

2.16 THEOREM. *If $\mathfrak{R}_1 = (R_1, \geqq)$ is a subsystem of $\mathfrak{R} = (R, \geqq)$ and if $\mathfrak{P}_r$ is a subsystem of $\mathfrak{S}_r$ for each r in R_1, then* (a) *$\sum_{(R(1),\geqq)} \mathfrak{P}_r$ is a subsystem of $\sum_{(R,\geqq)} \mathfrak{S}_r$;* (b) *$\sum_{\mathrm{tr}(R(1),\geqq)} \mathfrak{P}_r$ is a subsystem of $\sum_{\mathrm{tr}(R,\geqq)} \mathfrak{S}_r$ if and only if $\mathrm{tr}(R_1, \geqq)$ is a subsystem of $\mathrm{tr}(R, \geqq)$;* (c) *$\sum^t_{(R(1),\geqq)} \mathfrak{P}_r$ is a subsystem of $\sum^t_{(R,\geqq)} \mathfrak{S}_r$ if and only if* (1) *$\mathrm{tr}(R_1, \geqq)$ is a subsystem of $\mathrm{tr}(R, \geqq)$,* (2) *if r in R_1 is such that an r' in R exists for which $r >_t r' >_t r$ in $\mathrm{tr}(R, \geqq)$, either there is an r_1 in R_1 such that $r >_{1t} r_1 >_{1t} r$ in $\mathrm{tr}(R_1, \geqq)$ or $\mathrm{tr}(\mathfrak{P}_r) = (P_r, u)$, and* (3) *$\mathrm{tr}(\mathfrak{P}_r)$ is a subsystem of $\mathrm{tr}(\mathfrak{S}_r)$ if r is not such a point of R_1.*

(a) follows immediately from the definitions. If $\mathrm{tr}(R_1, \geqq)$ is a subsystem of $\mathrm{tr}(R, \geqq)$ it follows from (a) that $\sum_{\mathrm{tr}(R(1),\geqq)} \mathfrak{P}_r$ is a subsystem of $\sum_{\mathrm{tr}(R,\geqq)} \mathfrak{S}_r$. If $\sum_{\mathrm{tr}(R(1),\geqq)} \mathfrak{P}_r$ is a subsystem of $\sum_{\mathrm{tr}(R,\geqq)} \mathfrak{S}_r$, if r and r' are in R_1 and if $r >_t r'$ in $\mathrm{tr}(R, \geqq)$, then for any p in P_r and any p' in $P_{r'}$, $(r, p) \geqq (r', p')$ in $\sum_{\mathrm{tr}(R,\geqq)} \mathfrak{S}_r$; hence $(r, p) \geqq (r', p')$ in $\sum_{\mathrm{tr}(R(1),\geqq)} \mathfrak{P}_r$. Therefore $r >_t r'$ in $\mathrm{tr}(R_1, \geqq)$ and this latter system is a subsystem of $\mathrm{tr}(R, \geqq)$. This proves (b).

For (c) we apply 2.8 and see that $\sum^t_{(R(1),\geqq)} \mathfrak{P}_r$ is a subsystem of $\sum^t_{(R,\geqq)} \mathfrak{S}_r$ if and only if $\sum^t_{\mathrm{tr}(R(1),\geqq)} \mathrm{tr}(\mathfrak{P}_r)$ is a subsystem of $\sum^t_{\mathrm{tr}(R,\geqq)} \mathrm{tr}(\mathfrak{S}_r)$. Let $\geqq^1$ and $\geqq^2$, respectively, be the order relations in these latter systems. By 2.1, $(r, p) \geqq^1 (r', p')$ if and only if $r >_{1t} r'$ or $r >_{1t} r_1 >_{1t} r' = r$ for some r_1 in R_1 or $r = r'$ and $p \geqq_{1t} p'$ in $\mathrm{tr}(\mathfrak{P}_r)$; $(r, p) \geqq^2 (r', p')$ if and only if $r >_t r'$ or

$r >_t r'' >_t r' = r$ for some r'' in R or $r = r'$, $p \geqq_t p'$ in $\mathrm{tr}(\mathfrak{S}_r)$. If (1), (2) and (3) hold, (1) implies that $\geqq_{1t}$ in $\mathrm{tr}(R_1, \geqq)$ is the same as $\geqq_t$, so the first condition is the same for $\geqq^1$ as for $\geqq^2$. The middle terms are the same by (2) and the last from (3). If $\geqq^1$ and $\geqq^2$ agree and $r >_t r_1$, $(r, p) >^2 (r_1, p_1)$ for any p and p_1, so $(r, p) >^1 (r_1, p_1)$ and $r >_{1t} r_1$; that is, (1) holds. If $r >_t r' >_t r$, if p and p'' are in $\mathfrak{P}_r$ and if s' is in $\mathfrak{S}_{r'}$, then $(r, p) >^2 (r', s') >^2 (r, p'')$; hence $(r, p) \geqq^1 (r, p'')$ so either there is an r_1 in R_1 such that $r >_{1t} r_1 >_{1t} r$ or else $p \geqq_{1t} p''$ in $\mathrm{tr}(\mathfrak{P}_r)$ for every p, p'' in P_r. If r is not such a point and $p \geqq_t p'$ in $\mathrm{tr}(\mathfrak{S}_r)$, then $(r, p) \geqq^2 (r, p')$ so $(r, p) \geqq^1 (r, p')$ or $p \geqq_{1t} p'$ in $\mathrm{tr}(\mathfrak{P}_r)$.

An arithmetical property of these sums is contained in the next theorem. As we shall show, this has a number of special cases such as the associative laws of cardinal and ordinal addition and of ordinal multiplication.

2.17 THEOREM. *If $\mathfrak{R}$, $\mathfrak{S}_r$ and $\mathfrak{P}_{rs}$ are ordered systems, let $(T, \geqq) = \sum_{(R,\geqq)} \mathfrak{S}_r$; then the following associative laws hold for the various types of sums:*

$$\sum_{(R,\geqq)} \sum_{(S_r,\geqq)} \mathfrak{P}_{rs} \sim \sum_{(T,\geqq)} \mathfrak{P}_{rs},$$

$$\sum^t_{(R,\geqq)} \sum^t_{(S_r,\geqq)} \mathfrak{P}_{rs} \sim \sum^t_{\mathrm{tr}(T,\geqq)} \mathfrak{P}_{rs},$$

and

$$\sum^c_{(R,\geqq)} \sum^c_{(S_r,\geqq)} \mathfrak{P}_{rs} \sim \sum^c_{\mathrm{tr}(T,\geqq)} \mathfrak{P}_{rs}.$$

In the first isomorphism the elements of the left-hand system are the points $(r, (s, p))$ with r in R, s in S_r, and p in P_{rs}; $(r, (s, p)) \geqq (r', (s', p'))$ means that $r > r'$ or $r = r'$, $s > s'$ or $r = r'$, $s = s'$, $p \geqq p'$. The elements of the right-hand system are the points $((r, s), p)$ with r in R, s in S_r and p in P_{rs}; $((r, s), p) \geqq ((r', s'), p')$ means that $r > r'$ or $r = r'$, $s > s'$ or $r = r'$, $s = s'$, $p \geqq p'$. Hence the natural correspondence $(r, (s, p)) \rightleftarrows ((r, s), p)$ is an isomorphism of the two sums. The second isomorphism follows from the first, 2.8, and 2.9 for

$$\sum^t_{(R,\geqq)} \sum^t_{(S(r),\geqq)} \mathfrak{P}_{rs} = \mathrm{tr}\left(\sum_{(R,\geqq)} \mathrm{tr}\left(\sum_{(S(r),\geqq)} \mathfrak{P}_{rs} \right)\right)$$

$$= \mathrm{tr}\left(\sum_{(R,\geqq)} \sum_{(S(r),\geqq)} \mathfrak{P}_{rs} \right) \sim \mathrm{tr}\left(\sum_{(T,\geqq)} \mathfrak{P}_{rs} \right) = \mathrm{tr}\left(\sum_{\mathrm{tr}(T,\geqq)} \mathfrak{P}_{rs} \right).$$

For the last isomorphism we note from these results and 2.15 that

$$\sum^c_{(R,\geqq)} \sum^c_{(S(r),\geqq)} \mathfrak{P}_{rs} = \sum^c_{(R,\geqq)} c\left(\sum^t_{(S(r),\geqq)} \mathfrak{P}_{rs} \right) \sim \sum^c_{(R,\geqq)} \sum^t_{(S(r),\geqq)} \mathfrak{P}_{rs}$$

$$= c\left(\sum^t_{(R,\geqq)} \sum^t_{(S(r),\geqq)} \mathfrak{P}_{rs} \right) \sim c\left(\sum^t_{\mathrm{tr}(T,\geqq)} \mathfrak{P}_{rs} \right) = \sum^t_{\mathrm{tr}(T,\geqq)} \mathfrak{P}_{rs}.$$

Note that by 2.8 in the last two isomorphisms of the theorem the operator tr can be applied or left off at will anywhere after the first $\sum^t$ or $\sum^c$ on each side.

For the last general results of ordered addition we give two decomposition theorems.

2.18 THEOREM. *$\mathfrak{S} \succ \sum_{(R,\geqq)} \mathfrak{P}_r$ if and only if there exist disjoint subsystems $\mathfrak{S}_r$ of $\mathfrak{S}$ such that $\mathfrak{S}_r \succ \mathfrak{P}_r$ for each r and $\mathfrak{S} \succ \sum_{(R,\geqq)} \mathfrak{S}_r$; if $\mathfrak{R}$ is a cardinal, $\mathfrak{S} \sim \sum_{(R,\geqq)} \mathfrak{S}_r$.*

If $\mathfrak{S} \succ \sum_{(R,\geqq)} \mathfrak{S}_r$ and $\mathfrak{S}_r \succ \mathfrak{P}_r$, by 2.10 and the transitivity of $\succ$ we can conclude that $\mathfrak{S} \succ \sum_{(R,\geqq)} \mathfrak{P}_r$. If h is a homomorphism of $\mathfrak{S}$ onto $\sum_{(R,\geqq)} \mathfrak{P}_r$ let $S_r = \{s \mid \text{there exists } p \text{ for which } h(s) = (r, p)\}$; on S_r define h_r by $h_r(s) = p$ if $h(s) = (r, p)$. If $s \geqq s'$ in $\mathfrak{S}_r$, $s \geqq s'$ in $\mathfrak{S}$ so $h(s) \geqq h(s')$ in $\sum_{(R,\geqq)} \mathfrak{P}_r$; that is, $(r, h_r(s)) \geqq (r, h_r(s'))$ so $h_r(s) \geqq h_r(s')$ and h_r is a homomorphism of $\mathfrak{S}_r$ onto $\mathfrak{P}_r$. If $g(s) = (r, s)$ whenever $s \in S_r$, g is clearly a homomorphism of $\mathfrak{S}$ onto $\sum_{(R,\geqq)} \mathfrak{S}_r$. If $\mathfrak{R}$ is a cardinal number, $(R, =)$, and if $(r, s) = g(s) \geqq g(s') = (r', s')$, then $r \geqq r'$ so $r = r'$; hence $s \geqq s'$ and g is an isomorphism.

Note that if $\mathfrak{R}$ is a number, the relations given above for $\sum$ will also hold for $\sum^t$ although $\sum$ and $\sum^t$ need not be the same unless in addition all the $\mathfrak{S}_r$ are transitive.

2.19 THEOREM. *$\mathfrak{S} < \sum_{(R,\geqq)} \mathfrak{P}_r$ if and only if there exists a subsystem $\mathfrak{R}_1$ of $\mathfrak{R}$ and disjoint subsystems $\mathfrak{S}_r$ of $\mathfrak{S}$ for r in R_1 such that $\mathfrak{S}_r < \mathfrak{P}_r$ and $\mathfrak{S} \sim \sum_{(R,\geqq)} \mathfrak{S}_r$.*

Sufficiency of the condition is clear. For necessity let h be an isomorphism of $\mathfrak{S}$ into $\sum_{(R,\geqq)} \mathfrak{P}_r$ and let R_1, the projection of $h(\mathfrak{S})$ into R, be $\{r \mid \text{there exist } s \text{ and } p \text{ such that } h(s) = (r, p)\}$. Let $S_r = \{s \mid \text{there exist } p \text{ such that } h(s) = (r, p)\}$ and let h_r be defined from $\mathfrak{S}_r$ into $\mathfrak{P}_r$ by $h_r(s) = p$ if $h(s) = (r, p)$. Then $s \geqq s'$ means $h(s) \geqq h(s')$; that is, $(r, h_r(s)) \geqq (r', h_{r'}(s'))$; hence if s and s' are in one S_r, $r = r'$ and $s \geqq s'$ if and only if $h_r(s) \geqq h_r(s')$; that is, h_r is an isomorphism of $\mathfrak{S}_r$ into $\mathfrak{P}_r$. Let g be the mapping of $\mathfrak{S}$ onto $\sum_{(R,\geqq)} \mathfrak{S}_r$ defined by $g(s) = (r, s)$ if $s \in S_r$; then $s \geqq s'$ means $(r, h_r(s)) \geqq (r', h_{r'}(s'))$ which means that either $r > r'$ or $r = r'$ and $h_r(s) \geqq h_r(s')$; since h_r is an isomorphism, this condition means that either $r > r'$ or $r = r'$ and $s \geqq s'$; that is, $s \geqq s'$ is equivalent to $g(s) \geqq g(s')$, so g is an isomorphism of $\mathfrak{S}$ and $\sum_{(R,\geqq)} \mathfrak{S}_r$.

Finally we give an elementary result on terminal elements in sums; we do not give a condition for initial elements since it is dual to this.

2.20. (r, s) is a terminal element of $\sum_{(R,\geqq)} \mathfrak{S}_r$ (or of $\sum^t_{(R,\geqq)} \mathfrak{S}_r$) if and only if r is a terminal element of $\mathfrak{R}$ and s is a terminal element of $\mathfrak{S}_r$. $c(r, s)$ is a terminal element of $\sum^c_{(R,\geqq)} \mathfrak{S}_r$ if and only if (1) r is a terminal element of $\mathfrak{R}$ and $c(s)$ is a terminal element of $\operatorname{ctr}(\mathfrak{S}_r)$ or (2) $c(r)$ is a terminal element of $\operatorname{ctr}(\mathfrak{R})$ which contains at least one element in addition to r.

If $(r, s) \in \sum_{(R,\geqq)} \mathfrak{S}_r$ and $r' > r$, then $(r', s') > (r, s)$ so (r, s) can not be a terminal element of $\sum_{(R,\geqq)} \mathfrak{S}_r$ unless r is a terminal element of $\mathfrak{R}$. Similarly $s'>s$ is impossible if (r, s) is a terminal element, so s must be a terminal element. On the other hand if r and s are terminal elements and $(r', s')\geqq(r, s)$, $r'\geqq r$ so $r'=r$; hence $s'\geqq s$ so $s'=s$ so (r, s) is a terminal element. The same is true for $\sum^t$ since a system and its transitization have the same terminal elements. The proof of the last statement is of the same nature and can be left to the reader with the remark that for any ordered $\mathfrak{S}$, $c(s_0)$ is a terminal element of ctr($\mathfrak{S}$) if and only if s_0 is an element of $\mathfrak{S}$ such that $s\geqq_t s_0$ implies that $s_0\geqq_t s$ in tr($\mathfrak{S}$).

Certain special cases of this addition operation have been given before, for instance in [1]; most of these writers have considered more restricted systems. $\sum_{(N(k),w)} \mathfrak{S}_n$ will be called the *ordinal sum* $\mathfrak{S}_1\oplus\mathfrak{S}_2\oplus\cdots\oplus\mathfrak{S}_k$ of the $\mathfrak{S}_n$. $\sum_{(R,=)} \mathfrak{S}_r$ will be called the *cardinal sum* of the $\mathfrak{S}_r$; if $R=N_k$, we write $\mathfrak{S}_1+\mathfrak{S}_2+\cdots+\mathfrak{S}_k$ for this. Since $(N_3, =)\sim(N_2, =)+(N_1, =)\sim(N_1, =)+(N_2, =)$ and $(N_3, w)\sim(N_1, w)\oplus(N_2, w)\sim(N_2, w)\oplus(N_1, w)$, after a renumbering of the systems the relation 2.17 specializes to $\mathfrak{S}_1\oplus(\mathfrak{S}_2\oplus\mathfrak{S}_3)\sim\mathfrak{S}_1\oplus\mathfrak{S}_2\oplus\mathfrak{S}_3\sim(\mathfrak{S}_1\oplus\mathfrak{S}_2)\oplus\mathfrak{S}_3$ and $\mathfrak{S}_1+(\mathfrak{S}_2+\mathfrak{S}_3)\sim\mathfrak{S}_1+\mathfrak{S}_2+\mathfrak{S}_3\sim(\mathfrak{S}_1+\mathfrak{S}_2)+\mathfrak{S}_3$; these rules are slightly stronger than the associative law for cardinal and ordinal addition. A check of the properties (10)–(18) given in [1, §3] for ordinal and cardinal addition shows that they are special cases of various formulas in this section.

Ordinal multiplication is another special case of ordered addition. If all $\mathfrak{S}_r=\mathfrak{S}$, we write $\mathfrak{R}\circ\mathfrak{S}$ for $\sum_{(R,\geqq)} \mathfrak{S}_r$; explicitly, $\mathfrak{R}\circ\mathfrak{S}$ is the set of ordered pairs (r, s) with r in R, s in S, where $(r, s)\geqq(r', s')$ means $r>r'$ or $r=r'$, $s\geqq s'$. Taking all $\mathfrak{S}_r=\mathfrak{S}$ and all $\mathfrak{P}_{rs}=\mathfrak{P}$, we derive from 2.17 the associative law of ordinal multiplication $\mathfrak{R}\circ(\mathfrak{S}\circ\mathfrak{P})\sim(\mathfrak{R}\circ\mathfrak{S})\circ\mathfrak{P}$. Similarly, if $\mathfrak{R}=(N_2, w)$ and all $\mathfrak{P}_{rs}=\mathfrak{P}$, we have from 2.17 that $(\mathfrak{S}_1\circ\mathfrak{P})\oplus(\mathfrak{S}_2\circ\mathfrak{P})\sim(\mathfrak{S}_1\oplus\mathfrak{S}_2)\circ\mathfrak{P}$, a one-sided distributive law for ordinal multiplication; similarly, the other rules (30)–(35) given in [1, §6] for ordinal multiplication can be found among the preceding theorems.

3. **Ordered multiplication.** The ordered product that is to be defined includes not only the ordinal and cardinal products of [4] and [1] but also the ordinal exponentiation of [1]. As with addition we shall give three multiplication operations, an ordered product $\prod_{(R,\geqq)} \mathfrak{S}_r$, and the related transitive and contracted products. The definition is essentially that of ordinal exponentiation but it does not have the virtues claimed for it in [1, §§8, 9]; even when all the systems involved are ordinal numbers, the ordered product need not be transitive. We shall show, however, that the ordered product is often almost transitive; for example, if $\mathfrak{R}$ is a number and all $\mathfrak{S}_r$ are transitive, 5.7 shows that $\prod_{(R,\geqq)} \mathfrak{S}_r$ is 1-transitive.

If $\mathfrak{R}=(R, \geqq)$ and $\mathfrak{S}_r$ are ordered systems, define $\prod_{(R,\geqq)} \mathfrak{S}_r$, the *ordered product over* $\mathfrak{R}$ of the $\mathfrak{S}_r$, to be the system $(P, \geqq)$ where the elements of P are the functions f defined on R such that $f(r)\in S_r$, while $f\geqq f'$ means that if

$f(r) \neq f'(r)$ there exists $r' \geqq r$ such that $f(r') \geqq f'(r')$; that is, $\{r \mid f(r) > f'(r)\}$ is cofinal in $\{r \mid f(r) \neq f'(r)\}$. In the case where $\mathfrak{R}$ is a finite ordinal it is easily seen that this becomes the lexicographic ordering of the "words" in P except that spelling is backwards. Let $\prod^t_{(R,\geqq)} \mathfrak{S}_r = \mathrm{tr}(\prod_{(R,\geqq)} \mathfrak{S}_r)$ and $\prod^c_{(R,\geqq)} \mathfrak{S}_r = c(\prod^t_{(R,\geqq)} \mathfrak{S}_r)$; these are, respectively, the *transitive* and *contracted* products.

The special cases mentioned above are easily defined. $\prod_{(R,=)} \mathfrak{S}_r$ is the ordinary *direct* or *cardinal* product of the systems $\mathfrak{S}_r$ in which $f \geqq f'$ if and only if $f(r) \geqq f'(r)$ for every r in R. If $\mathfrak{R} = (N_2, w)$, $\prod_{(N(2),w)} \mathfrak{S}_n$ reduces to $\mathfrak{S}_2 \circ \mathfrak{S}_1$, the ordinal product of the systems in reverse order. If all $\mathfrak{S}_r = \mathfrak{S}$, $\prod_{(R,\geqq)} \mathfrak{S}_r$ reduces to the *ordinal power* ${}^{(R,\geqq)}\mathfrak{S}$.

Let us give first the example mentioned in the introduction. Recall that (N, w) is the system of integers ordered by magnitude and that (N_2, w) is the subsystem containing the first two elements of N; if $\mathfrak{R} = (N, w)$ and $\mathfrak{S}_r = (N_2, w)$ for every r, then $\prod_{(R,\geqq)} \mathfrak{S}_r = {}^{(N,w)}(N_2, w)$ is not transitive. To prove this we construct three functions such that $f_1 > f_2 > f_3$ but f_1 does not follow or equal f_3; for every n let

$$f_1(2n) = 2, \qquad f_2(2n) = 1, \qquad f_3(2n) = f_3(4n + 1) = 2,$$
$$f_1(2n + 1) = 1, \qquad f_2(2n + 1) = 2, \qquad f_3(4n + 3) = 1.$$

Then $f_1(2n) > f_2(2n)$ for every n so $f_1 > f_2$; $f_2(4n+3) > f_3(4n+3)$ for every n so $f_2 > f_3$; however $f_3(n) \geqq f_1(n)$ for every n and equality does not always hold so f_1 does not follow or equal f_3.

We proceed with a discussion of properties of this product.

3.1 LEMMA. *If* $(R_0, \geqq)$ *is a subsystem of* $(R, \geqq)$ *and if, for each* r *in* R_0, $\mathfrak{P}_r$ *is a subsystem of* $\mathfrak{S}_r$, *then* $\prod_{(R(0),\geqq)} \mathfrak{P}_r < \prod_{(R,\geqq)} \mathfrak{S}_r$.

Let s_r be a fixed point in S_r for each r not in R_0 and define h from $\prod_{(R(0),\geqq)} \mathfrak{P}_r$ into $\prod_{(R,\geqq)} \mathfrak{S}_r$ by $hf_0 = f$ if and only if $f(r) = f_0(r)$ if $r \in R_0$, $f(r) = s_r$ if $r \notin R_0$. h is clearly an isomorphism of the left-hand system into the right.

3.2 LEMMA. *If* $\geqq_1$ *includes* $\geqq_2$ *in* R *and* $\geqq^i$ *is the order relation in* $\prod_{(R,\geqq(i))} \mathfrak{S}_r$, *then* $\geqq^1$ *includes* $\geqq^2$.

$f \geqq^2 f'$ means that if $f(r) \neq f'(r)$ there exists $r' \geqq_2 r$ such that $f(r') > f'(r')$; such an $r' \geqq_1 r$ so $f \geqq^1 f'$.

3.3 LEMMA. *If* $\geqq_{1r}$ *includes* $\geqq_{2r}$ *in* S_r *for each* r *and if* $\geqq^i$ *is the order relation in* $\prod_{(R,\geqq)} (S_r, \geqq_{ir})$, *then* $\geqq^1$ *includes* $\geqq^2$.

$f \geqq^2 f'$ means that if $f(r) \neq f'(r)$ there exists $r' \geqq r$ such that $f(r') >_{2r} f'(r')$; hence $f(r') >_{1r} f'(r')$ so $f \geqq^1 f'$.

It might be hoped that some relation such as $\mathfrak{S}_r \succ \mathfrak{P}_r$ for all r would imply that $\prod_{(R,\geqq)} \mathfrak{S}_r \succ \prod_{(R,\geqq)} \mathfrak{P}_r$. That this is false is easily seen by letting $\mathfrak{S} = (N_2, w) \circ (N_2, u)$ and $\mathfrak{P} = (N_2, w)$; then ${}^{(N,w)}\mathfrak{S} \succ {}^{(N,w)}\mathfrak{P}$ would imply that $\mathrm{tr}({}^{(N,w)}\mathfrak{S}) \succ \mathrm{tr}({}^{(N,w)}\mathfrak{P})$; this is false since in $\mathrm{tr}({}^{(N,w)}\mathfrak{S})$ every element follows every other while this is not the case in $\mathrm{tr}({}^{(N,w)}\mathfrak{P})$. Note that $\mathfrak{S}>\mathfrak{P}$ so, by 3.1, ${}^{(N,w)}\mathfrak{S}>{}^{(N,w)}\mathfrak{P}$. Note that the factorization theorem 1.2 for homomorphisms suggests that some such trouble might occur. If h_r is a homomorphism of $\mathfrak{S}_r$ onto $\mathfrak{P}_r$, then h_r factors into $H_rc_rI_r$, where H_r is an isomorphism of $\mathfrak{P}_r$ and $c_r(S_r, \geqq_r')$, $\geqq_r'$ includes $\geqq_r$ in S_r, and c_r is a partial or total contraction of $(S_r, \geqq_r')$. $\prod_{(R,\geqq)}(S_r, \geqq_r)>\prod_{(R,\geqq)}(S_r, \geqq_r')$ by 3.3, but $\mathfrak{S}_r'>c(\mathfrak{S}_r')$ is the only fact we are able to use about contractions, so we get the relations $\prod_{(R,\geqq)}\mathfrak{S}_r>\prod_{(R,\geqq)}\mathfrak{S}_r'>\prod_{(R,\geqq)}\mathfrak{P}_r$.

From 3.2 and 3.3 we derive relations between $\prod$ and tr; these are not as simple as the corresponding relations for $\sum$ and tr.

3.4 THEOREM. $\prod^t_{(R,\geqq)}\mathfrak{S}_r>\prod^t_{(R,\geqq)}\mathrm{tr}(\mathfrak{S}_r)>\prod^t_{tr(R,\geqq)}\mathrm{tr}(\mathfrak{S}_r)$ *and* $\prod^t_{(R,\geqq)}\mathfrak{S}_r>\prod^t_{tr(R,\geqq)}\mathfrak{S}_r>\prod^t_{tr(R,\geqq)}\mathrm{tr}(\mathfrak{S}_r)$.

We give two examples to show that these homomorphisms need not be isomorphisms. Note that the fourth homomorphism is a special case of the first and the second is a special case of the third.

3.5a. Let $\mathfrak{S}=(N, \geqq)$ where $j>k$ means that $j=k+1$; then $\mathrm{tr}({}^{(N,-)}\mathfrak{S})$ is not isomorphic to $\mathrm{tr}({}^{(N,-)}\mathrm{tr}(\mathfrak{S}))$.

Clearly $\mathrm{tr}(\mathfrak{S})=(N, w)$ so $f\geqq^1 f'$ in ${}^{(N,-)}\mathrm{tr}(\mathfrak{S})$ means that $f(n)wf'(n)$ for every n; it is easily seen that this system is already transitive and is even a lattice; in particular, every pair of elements has an upper bound. $\mathrm{tr}({}^{(N,-)}\mathfrak{S})$ does not have this property. $f>^2f'$ means that there exist $f_1, \cdots, f_n$ such that $f=f_1\geqq f_2\geqq\cdots\geqq f_n=f'$ in ${}^{(N,\geqq)}\mathfrak{S}$; that is, such that $f_i(n)=f_{i+1}(n)$ or $f_i(n)=f_{i+1}(n)+1$ for every n; that is, $f\geqq^2 f'$ means that $f(n)wf'(n)$ for all n and that the difference of f and f' is a bounded function. Hence two functions whose difference is unbounded have no common successor in $\mathrm{tr}({}^{(N,-)}\mathfrak{S})$ so this system is not isomorphic to the other.

3.5b. Let $\mathfrak{R}=(N, \geqq)$ (the $\mathfrak{S}$ of the example above) and let $\mathfrak{S}=(N_2, w)$; then $\mathrm{tr}({}^{(N,w)}\mathfrak{S})$ is not isomorphic to $\mathrm{tr}({}^{(N,\geqq)}\mathfrak{S})$.

$\geqq^1$ in this first system is easily described; clearly each f for which $f(n)=2$ for an infinite number of values of $n\geqq^1$ each f' for which $f'(n)=1$ for an infinite number of values of n. Hence the system falls into three parts: (1) If $f>^1f'$ and f is ultimately equal to 1, then there exists n_0 such that $f(n_0)=2>f'(n_0)$ while $f(n)=f'(n)=1$ for all n beyond n_0. This relation well-orders those functions which are ultimately equal to 1; there are only a countable number of them. (2) Above all these lie those functions which are not ultimately constant; each of these follows every other and also follows every element of the first set. (3) Above these lie the functions which are ultimately equal to 2; these are well-ordered in reverse and all follow all the elements of the first two classes. From this we see that $\mathrm{tr}({}^{(N,w)}\mathfrak{S})$ is isomorphic to $(N, w)\oplus(D, u)\oplus(N, w^*)$, where D is of the power of the continuum and w^* is the relation w turned end for end; that is, w^* is the usual ordering of the negative integers by magnitude.

The relation $\geqq^2$ of $\text{tr}({}^{(N,\geqq)}\mathfrak{S})$ is harder to describe; however, to show that this system is not isomorphic to the other it suffices to show that there exist two points neither of which follows the other. For this let $\geqq_2$ be the relation in ${}^{(N,\geqq)}\mathfrak{S}$ and note that $f\geqq_2 f'$ means that if $f(n)<f'(n)$, then $f(n+1)>f'(n+1)$. Then $f\geqq^2 f'$ in $\text{tr}({}^{(N,\geqq)}\mathfrak{S})$ means that there exist $f_1, \cdots, f_k$ such that $f\geqq_2 f_1 \geqq_2 \cdots \geqq_2 f_k \geqq_2 f'$. Define f and f' by $f(4^n)=2, f(j)=1$ if $j\neq 4^n$ for some n, $f'(2\cdot 4^n)=2, f'(j)=1$ if $j\neq 2\cdot 4^n$ for some n. Then $f\geqq_2 f_1$ means that $f_1(j)=1$ if $j\neq 4^n$ or 4^n-1 for some n while for each n either $f_1(4^n-1)=1$ or $f_1(4^n)=1$. That is, if $f\geqq_2 f_1$, the set of points where f_1 is equal to 2 can not be shifted back more than one unit from the set of points where f is equal to 2; clearly the same is true for $f_1\geqq_2 f_2$ and so on, so no finite chain can connect these two functions f and f'. Such incomparability does not occur in $\text{tr}({}^{(N,w)}\mathfrak{S})$ so these systems are not isomorphic.

3.8 shows that on certain occasions the first homomorphism of 3.4 is an isomorphism; the proof uses the following lemma which has also the important consequence 3.7.

3.6 LEMMA. *Let* $\geqq^1$ *be the relation in* $\prod_{(R,\geqq)} \mathfrak{S}_r$ *and* $\geqq^2$ *be that in* $\prod_{(R,\geqq)} \text{tr}(\mathfrak{S}_r)$. *If* $f>^2 f'$ *and if all* $\mathfrak{S}_r$ *are k-transitive, there exist* $f_1, \cdots, f_k$ *such that* $f>^1 f_1$ *and* $f_1(r)\geqq f_2(r)\geqq \cdots \geqq f_k(r)\geqq f'(r)$ *in* $\mathfrak{S}_r$ *for every* r *in* R (*so* $f_1\geqq^1 f_2 \geqq^1 \cdots \geqq^1 f_k \geqq^1 f'$).

Let $E=\{r \mid f(r)>_t f'(r) \text{ in } \text{tr}(\mathfrak{S}_r)\}$; by k-transitivity of $\mathfrak{S}_r$ there exist points $s_{r1}, \cdots, s_{rk}$ in $\mathfrak{S}_r$ such that $f(r)>s_{r1}\geqq s_{r2}\geqq \cdots \geqq s_{rk}\geqq f'(r)$ in $\mathfrak{S}_r$. Define $f_k(r)=s_{rk}$ if $r\in E$, $f_k(r)=f'(r)$ if not; then $f_1(r)\geqq \cdots \geqq f'(r)$ for all r and $\{r \mid f(r)>f_1(r) \text{ in } \mathfrak{S}_r\}=E$ so $f>^1 f_1$.

3.7 THEOREM. *If all* $\mathfrak{S}_r$ *are k-transitive and if* $\prod_{(R,\geqq)} \text{tr}(\mathfrak{S}_r)$ *is m-transitive, then* $\prod_{(R,\geqq)} \mathfrak{S}_r$ *is* $(k+2m)$-*transitive*; *if, in addition,* $\text{tr}(\mathfrak{S}_r)$ *is a number for every* r, $\prod_{(R,\geqq)} \mathfrak{S}_r$ *is* $(k+m)$-*transitive.*

If $f_0\geqq^1 f_1 \geqq^1 \cdots \geqq^1 f_n$ in $\prod_{(R,\geqq)} \mathfrak{S}_r$, then $f_0\geqq^2 f_1\geqq^2 \cdots \geqq^2 f_n$ in $\prod_{(R,\geqq)} \text{tr}(\mathfrak{S}_r)$; hence there exist $f_1', \cdots, f_m'$ such that $f_0 \geqq^2 f_1' \geqq^2 \cdots \geqq^2 f_m' \geqq^2 f_n$. If $f_1'=f_0$, let $f_{11}=f_1'$; if not, $f_0>^2 f_1'$ so by 3.6 there is a chain whose top element f_{11} has the properties $f_0>^1 f_{11}$ and $f_{11}(r)\geqq_t f_1'(r)$ for all r. Let $E_1=\{r \mid f_1'(r)>_t f_2'(r)=f_{11}(r)\}$; if E_1 is empty, then $f_{11}>^2 f_2'$ and we let $f_{12}=f_{11}$; if E_1 is not empty, define f_{12} so that $f_{11}(r)>f_{12}(r)\geqq_t f_1'(r)$ if $r\in E_1$, $f_{12}(r)=f_{11}(r)$ if $r\notin E_1$. Then $f_{11}(r)\geqq f_{12}(r)\geqq_t f_{11}(r)$ for every r and $\{r \mid f_{12}(r)>_t f_2'(r)\} \supset \{r \mid f_1'(r)>_t f_2'(r)\}$ so $f_{12}\geqq^2 f_2'$. Repeating this process we see that there exist f_{i1} and f_{i2} such that $f_0\geqq^1 f_{11}\geqq^1 f_{12}\geqq^1 f_{21}\geqq^1 f_{22}\geqq^1 \cdots \geqq^1 f_{m1}\geqq^1 f_{m2}\geqq^2 f_n$. Insert k elements in this last gap by 3.6; then the resulting chain connecting f_0 with f_n has $2m+k$ middle links, so $\prod_{(R,\geqq)} \mathfrak{S}_r$ is $(2m+k)$-transitive. If $\text{tr}(\mathfrak{S}_r)$ is a number for every r, then $f_{i1}=f_{i2}$ for every i so $\prod_{(R,\geqq)} \mathfrak{S}_r$ is $(k+m)$-transitive.

3.8 THEOREM. *If all* $\mathfrak{S}_r$ *are k-transitive, then*

$$\prod^{t}_{(R,\geqq)} \mathfrak{S}_r = \prod^{t}_{(R,\geqq)} \operatorname{tr}(\mathfrak{S}_r).$$

We have already shown in 3.3 that the relation on the left-hand side is included in that on the right. $f \geqq^T f'$ in the right-hand system means that there exist $f_1, \cdots, f_n$ such that $f \geqq^2 f_1 \geqq^2 \cdots \geqq^2 f_n \geqq^2 f'$ in $\prod_{(R,\geqq)} \operatorname{tr}(\mathfrak{S}_r)$; by 3.6 there exist $f_1', \cdots, f_j'$ such that $f \geqq^1 f_1' \geqq^1 \cdots \geqq^1 f_j' \geqq^1 f'$ in $\prod_{(R,\geqq)} \mathfrak{S}_r$; that is, $f >^t f'$ in $\prod^t_{(R,\geqq)} \mathfrak{S}_r$ and the relations in the two systems are the same.

Related to this is

3.8′. If R is a finite set, $\prod^t_{(R,\geqq)} \mathfrak{S}_r = \prod^t_{(R,\geqq)} \operatorname{tr}(\mathfrak{S}_r)$.

As in the proof of 3.6 when $f >^2 f'$ in $\prod_{(R,\geqq)} \operatorname{tr}(\mathfrak{S}_r)$, let $E = \{r \mid f(r) >_t f'(r)\}$; then for each r in E there are points $s_{r1}, \cdots, s_{rk(r)}$ such that $f(r) > s_{r1} \geqq \cdots \geqq s_{rk(r)} \geqq f'(r)$ in $\mathfrak{S}_r$; since $\sup_{r \in E} k_r$ must be finite, we can continue the argument of 3.6 and then of 3.8.

The ordered product contains subsystems isomorphic to the factors and to part of the index system; this need not be true of transitive and contracted products.

3.9 LEMMA. $\prod_{(R,\geqq)} \mathfrak{S}_r > \mathfrak{S}_{r(0)}$ *for every choice of* $\mathfrak{R}$, $\mathfrak{S}_r$, *and* r_0.

This follows immediately from 3.1 since $\mathfrak{S} \sim^{(N(1),=)} \mathfrak{S}$.

Note that $\operatorname{tr}(^{(N,w)}[(N, w^*) \oplus (N, w)])$ has the universal order relation so it contains no subsystems not of the same sort; hence no such conclusion as 3.9 holds for $\prod^t$ or $\prod^c$.

3.10 LEMMA. *If* $R_1 = \{r \mid \operatorname{ctr}(\mathfrak{S}_r)$ *is not a cardinal number*$\}$ *is not empty, then* $(R_1, \geqq) < \prod_{(R,\geqq)} \mathfrak{S}_r$ *and* $(R_1, \leqq) < \prod_{(R,\geqq)} \mathfrak{S}_r$.

If $r \in R_1$, there exist points s_{ri} in S_r such that $s_{r2} > s_{r1}$ but s_{r1} does not follow or equal s_{r2}; for r not in R_1 let s_r be any point of S_r. Define h from R_1 into $\prod_{(R,\geqq)} \mathfrak{S}_r$ by $h_{r(0)} = f$ if $f(r) = s_r$ for r not in R_1, $f(r) = s_{r1}$ if $r \in R_1$ but $r \neq r_0$, $f(r_0) = s_{r2}$. Then $r_2 > r_1$ implies that $h_{r(2)} > h_{r(1)}$ for $h_{r(2)}(r) = h_{r(1)}(r)$ if $r \neq r_2$ or r_1, and $h_{r(2)}(r_2) > h_{r(1)}(r_2)$. If $h_{r(2)} > h_{r(1)}$, $h_{r(1)}(r_1) > h_{r(2)}(r_1)$ and r_2 is the only point at which $h_{r(2)}(r_2) > h_{r(1)}(r_2)$ so $r_2 > r_1$; that is, h is an isomorphism. Interchanging the roles of s_{r1} and s_{r2} gives the last relation.

To see that no bigger subsystem of $\mathfrak{R}$ need be isomorphic to a subsystem of the product, let $\mathfrak{R} = \mathfrak{R}_1 + \mathfrak{R}_2$, where $\mathfrak{R}_2 \not< \prod_{(R(1),\geqq)} \mathfrak{S}_r$ and $\mathfrak{R}_2$ is a lattice; let $\mathfrak{S}_r$ be a cardinal number if $r \in R_2$. Then (see 6.5) $\prod_{(R(1),\geqq)+(R(2),\geqq)} \mathfrak{S}_r$ is isomorphic to $(\prod_{(R(1),\geqq)} \mathfrak{S}_r) \cdot (\prod_{(R(2),\geqq)} \mathfrak{S}_r)$. This latter factor is a cardinal number also, so, if $\mathfrak{R}_1 + \mathfrak{R}_2 < \prod_{(R(1),\geqq)+(R(2),\geqq)} \mathfrak{S}_r$, the lattice character of $\mathfrak{R}_2$ would force a system isomorphic to $\mathfrak{R}_2$ to be a subsystem of a part of the product isomorphic to $\prod_{(R(1),\geqq)} \mathfrak{S}_r$; we chose $\mathfrak{R}_2$ so this could not happen. A simple special case of this is given by $\mathfrak{R}_1 = \mathfrak{S}_r = (N_2, w)$ if $r \in R_1$, $\mathfrak{R}_2 = (N, w)$ and $\mathfrak{S}_r = (N, =)$ if $r \in R_2$.

The example before 3.10 shows also that nothing like 3.10 can be expected from the transitive product; however, the reader who considers the proofs of 7.12 and 7.12′ will see that certain subsystems of $(R, \geqq)$ and $(R, \leqq)$ can often be embedded in the transitive or contracted product.

4. **Transitivity of the ordered product.** This section gives conditions under which the ordered product, $\prod_{(R,\geqq)} \mathfrak{S}_r$, is already transitive. We begin with a simple special case which in some ways suggests the principal part of what may happen in general. Recall the definition of R' from §1.

4.1 THEOREM. *If $\mathfrak{R}$ and all $\mathfrak{S}_r$ are numbers and $R'=R$, then $\prod_{(R,\geqq)} \mathfrak{S}_r$ is transitive.*

If $f_1 \geqq f_2 \geqq f_3$, let $E_i = \{r \mid f_i(r) \neq f_{i+1}(r)\}$, $i=1, 2$, and let $E = \{r \mid f_1(r) \neq f_3(r)\}$; then $E \subset E_1 \cup E_2$, and $f_i(r) > f_{i+1}(r)$ if $r \in E_i^{(1)}$. We shall show that if $r \in (E_1 \cup E_2)^{(1)}$, then $f_1(r) > f_3(r)$; since $E \subset E_1 \cup E_2 \subset (E_1 \cup E_2)^{(1)D}$ (by 1.5) this will prove $f_1 \geqq f_3$. By 1.7, $(E_1 \cup E_2)^{(1)} = (E_1^{(1)} - E_2^D) \cup (E_2^{(1)} - E_1^D) \cup (E_1^{(1)} \cap E_2^{(1)})$. If r is in the first of these sets, $f_1(r) > f_2(r) = f_3(r)$; if r is in the second, $f_1(r) = f_2(r) > f_3(r)$; if in the third, $f_1(r) > f_2(r) > f_3(r)$; since $>$ is transitive in a number, $f_1(r) > f_3(r)$ in all these cases, so $\{r \mid f_1(r) > f_3(r)\}$ is cofinal in E and $f_1 \geqq f_3$.

This can easily be extended slightly.

4.2 THEOREM. *If $\mathfrak{R}$ and all $\mathfrak{S}_r$ are numbers and $E = \{r \mid \mathfrak{S}_r$ is not a cardinal number$\}$, then $\prod_{(R,\geqq)} \mathfrak{S}_r$ is transitive if and only if $E = E'$.*

If $E \neq E'$, $E - E'$ is not empty and has no terminal elements; by 1.9 there exist three disjoint cofinal subsets E_1, E_2 and E_3 of $E - E'$. If $r \in E$, there exist points s_{ri}, $i=1, 2$, in $\mathfrak{S}_r$ such that $s_{r2} > s_{r1}$ but s_{r1} does not follow or equal s_{r2}. The example at the beginning of §3 now suggests the proper procedure; define

$$f_1(r) = f_2(r) = f_3(r) \text{ to be any point of } \mathfrak{S}_r \text{ if } r \in R - E_1 - E_2 - E_3,$$

$$f_1(r) = s_{r2}, \quad f_2(r) = s_{r1}, \quad f_3(r) = s_{r2} \quad \text{if} \quad r \in E_1,$$

$$f_1(r) = s_{r1}, \quad f_2(r) = s_{r2}, \quad f_3(r) = s_{r1} \quad \text{if} \quad r \in E_2,$$

$$f_1(r) = s_{r1}, \quad f_2(r) = s_{r2}, \quad f_3(r) = s_{r2} \quad \text{if} \quad r \in E_3.$$

Then $f_1 > f_2 > f_3$, but f_1 does not follow or equal f_3.

If $E = E'$, the argument of the previous theorem is easily applied to show that $\prod_{(R,\geqq)} \mathfrak{S}_r$ is transitive.

4.3 COROLLARY. *If $\mathfrak{R}$ and all $\mathfrak{S}_r$ are numbers, $\prod_{(R,\geqq)} \mathfrak{S}_r$ is a number if and only if $E = E'$.*

If the product is a number, it is transitive, so, by the theorem, $E = E'$. If $E = E'$, the product is transitive; to prove it antisymmetric let $f_1 \geqq f_2 \geqq f_1$ and suppose that there exists r such that $f_1(r) \neq f_2(r)$. Then $(E_1 \cup E_2)^{(1)}$ is not empty and for r in $(E_1 \cup E_2)^{(1)}$, $f_1(r) > f_3(r)$ so $f_1 > f_3$ if $f_1 > f_2$; that is, $f_1 \geqq f_2 \geqq f_1$ implies $f_1 = f_2$.

4.4 COROLLARY. *If $\mathfrak{R}$ is a finite number and all $\mathfrak{S}_r$ are numbers, $\prod_{(R,\geqq)} \mathfrak{S}_r$ is a number.*

From 1.7, $R=R'$ if R is finite.

The next sequence of lemmas gives a number of necessary conditions for transitivity of the product; all together they will turn out to be sufficient (Theorem 4.12).

4.5 LEMMA. If $\prod_{(R,\geqq)} \mathfrak{S}_r$ *is transitive, then every* $\mathfrak{S}_r$ *is transitive.*

This follows from 3.9 and the obvious fact that transitivity is preserved by $>$.

4.6 LEMMA. *If $\prod_{(R,\geqq)} \mathfrak{S}_r$ is transitive and $R_1 = \{r \mid$ ctr$(\mathfrak{S}_r)$ is not a cardinal number$\}$, then the subsystem $\mathfrak{R}_1$ of $\mathfrak{R}$ is transitive.*

This follows from 3.10 as 4.5 did from 3.9.

4.7 LEMMA. *Under the hypothesis of* 4.6, *if $r_1 \geqq r_2 \geqq r_3$, if r_1 and r_2 are elements of R_1 and if $\mathfrak{S}_{r(3)} \neq (S_{r(3)}, u)$, then $r_1 \geqq r_3$.*

If $r_1=r_2$ or $r_2=r_3$, there is no more to prove; if $r_1>r_2>r_3$, define s_{ij}, $i=1, 2, j=1, 2, 3$, in $\mathfrak{S}_{r(j)}$ so that $s_{2j}>s_{1j}$ and s_{1j} does not follow or equal s_{2j} for $j=1, 2$ while s_{13} does not follow or equal s_{23}; define f_i to be equal except on the r_i and define

$$f_1(r_1) = s_{21}, \quad f_2(r_1) = s_{21}, \quad f_3(r_1) = s_{11},$$
$$f_1(r_2) = s_{22}, \quad f_2(r_2) = s_{12}, \quad f_3(r_2) = s_{22},$$
$$f_1(r_3) = s_{13}, \quad f_2(r_3) = s_{23}, \quad f_3(r_3) = s_{23}.$$

Then $f_1>f_2>f_3$, so $f_1>f_3$; since $f_1(r_2)=f_3(r_2)$ and $f_1(r_3) \not> f_3(r_3)$, $r_1>r_3$.

4.8 LEMMA. *If $\prod_{(R,\geqq)} \mathfrak{S}_r$ is transitive and $\mathfrak{S}_{r(0)}$ is not a number, then $\mathfrak{S}_r=(S_r, u)$ if $r<r_0$.*

If $\mathfrak{S}_{r(0)}$ is not a number, since it is transitive by 4.5, there exist two points s_i in $\mathfrak{S}_{r(0)}$ such that $s_1>s_2>s_1$. Let r_1 be a point such that $r_1<r_0$; define f_i so that $f_1(r)=f_2(r)=f_3(r)$ if $r\neq r_0$ or r_1, $f_1(r_0)=f_3(r_0)=s_1$, $f_2(r_0)=s_2$, and let $f_i(r_1)$ be any point of $\mathfrak{S}_{r(1)}$. Then $f_1>f_2>f_3$ so $f_1\geqq f_3$; since $f_1(r_0)=f_3(r_0)$, it follows that $f_1(r_1)\geqq f_3(r_1)$; since these were any two points of $\mathfrak{S}_{r(1)}$, it follows that $\mathfrak{S}_{r(1)}$ has the universal relation.

Though we have shown that $\mathfrak{R}_1$ is transitive, it need not be a number. An example is the system ${}^{(N(2),u)}(N_2, w)$; this has four elements which may be represented by (i, j), $i, j \in N_2$. $(1, 1)<$any other element and $(2, 2)>$any other element, $(2, 1)>(1, 2)>(2, 1)$; these are all the relations that hold in the system so transitivity can easily be verified. Here $R_1=R$ and R' is empty. We are going to show in 4.9 and 4.11 that this example is really typical instead of very special, for $R_1 \cap c(r)$ can contain not more than two points if the product is transitive and the corresponding factors must be of this form (N_2, w).

4.9 LEMMA. *Under the hypotheses of* 4.6, $(R_1, \geqq)$ *has no subsystem* (R_2, u) *such that* R_2 *contains more than two elements.*

Any one-point set in (R_2, u) is a cofinal subset of (R_2, u); if R_2 contains more than two points, (R_2, u) contains three disjoint cofinal subsets and the construction of 4.2 shows that $\prod_{(R,\geqq)} \mathfrak{S}_r$ can not be transitive.

4.10 LEMMA. *Under the hypotheses of* 4.6, *if* $r_2 > r_1 > r_2$ *in* R_1, *if* $r < r_2$ *and if* $r < r_1$, *then* $\mathfrak{S}_r = (S_r, u)$.

Take points s_{ij}, $i, j = 1, 2$, in $\mathfrak{S}_{r(j)}$ so that $s_{2j} > s_{1j}$ but s_{1j} does not follow or equal s_{2j}. Define

$$\begin{aligned} f_1(r') &= f_2(r') = f_3(r') \quad \text{if} \quad r' \neq r_2, r_1, \text{ or } r, \\ f_1(r_1) &= f_3(r_1) = s_{21}, \qquad f_2(r_1) = s_{11}, \\ f_1(r_2) &= f_3(r_2) = s_{12}, \qquad f_2(r_2) = s_{22}, \\ f_i(r) &\text{ any point of } \mathfrak{S}_r. \end{aligned}$$

Then $f_1(r_1) > f_2(r_1)$ so $f_1 > f_2$; $f_2(r_2) > f_3(r_2)$ so $f_2 > f_3$; hence $f_1 \geqq f_3$; that is, every point of $\mathfrak{S}_r$ follows every other.

4.11 LEMMA. *Under the hypotheses of* 4.6, *if* $r_0 > r_1 > r_0$ *in* R_1, *then* $\mathfrak{S}_{r(0)} \sim (N_2, w)$.

Since $c(\mathfrak{S}_{r(0)})$ is not a cardinal number, $\mathfrak{S}_{r(0)}$ contains a subsystem isomorphic to (N_2, w); that is, there exist s_i, $i = 1, 2$, such that $s_1 > s_2$ but s_2 does not follow or equal s_1. Hence by 4.8, $\mathfrak{S}_{r(0)}$ is a number (since $r_0 < r_1$ and $\mathfrak{S}_{r(1)}$ has this same property). If $\mathfrak{S}_{r(0)}$ contains one more point, then $\mathfrak{S}_{r(0)}$ contains a subsystem isomorphic to one of the systems (N_3, w), $(N_1, =) + (N_2, w)$, $(N_1, =) \oplus (N_2, =)$ or $(N_2, =) \oplus (N_1, =)$. If $\mathfrak{P}_1$ is one of these four systems and $\mathfrak{P}_2 = (N_2, w)$, we know that $\mathfrak{S}_{r(1)} > \mathfrak{P}_2$; if $\mathfrak{S}_{r(0)} > \mathfrak{P}_1$ then $\prod_{(R,\geqq)} \mathfrak{S}_r > \prod_{(N(2),u)} \mathfrak{P}_n$; we show that for no one of these choices of $\mathfrak{P}_1$ is this latter system transitive, so transitivity of $\prod_{(R,\geqq)} \mathfrak{S}_r$ implies that $\mathfrak{S}_{r(0)} \not> \mathfrak{P}_1$ for any of these choices of $\mathfrak{P}_1$.

If $\mathfrak{P}_1 = (N_3, w)$, the elements of the product are pairs (i, j), $i = 1, 2, 3$, $j = 1, 2$; $(i, j) > (i', j')$ if $i > i'$ or $j > j'$; hence $(2, 1) > (1, 2) > (3, 1)$ but $(2, 1)$ does not follow or equal $(3, 1)$.

If $\mathfrak{P}_1 = (N_1, =) + (N_2, w)$, this system may be represented by a system of three points a, b and c where $b > c$ is the only relation besides equality. $\prod_{(N(2),u)} \mathfrak{P}_n$ is the system of pairs (p, i), $p = a$, b or c, $i = 1, 2$, where $(q, 2) > (p, 1)$ for every p and q in $\mathfrak{P}_1$, and $(b, i) > (c, i)$ for $i = 1$ or 2; then $(b, 1) > (c, 2) > (a, 1)$ but $(b, 1)$ does not follow or equal $(a, 1)$.

If $\mathfrak{P}_1 = (N_1, =) \oplus (N_2, =)$, the only difference between this case and the preceding is that an extra relation $a > c$ has been added; the same example holds here. If $\mathfrak{P}_1=(N_2,\ =)\oplus(N_1,\ =)$ the extra relation $b>a$ is added in the system of the preceding paragraph; then $(c, 2)>(b, 1)>(a, 2)$, but $(c, 2)$ does not follow or equal $(a, 2)$.

Hence $\prod_{(N(2),u)}\mathfrak{P}_n$ is not transitive in any one of these cases; it follows that $\mathfrak{P}_1<\mathfrak{S}_{r(0)}$ is false; hence $\mathfrak{S}_{r(0)}\sim(N_2, w)$.

These conditions can now be collected into one big set of necessary and sufficient conditions for transitivity of the product.

4.12 THEOREM. *$\prod_{(R,\geqq)}\mathfrak{S}_r$ is transitive if and only if all of the following conditions hold:*

(1) *Every $\mathfrak{S}_r$ is transitive.*

(2) *If $R_0=\{r\mid \mathfrak{S}_r$ is not a number$\}$ and $r\in R_0^D-R_0^{(1)}$, then $\mathfrak{S}_r=(S_r, u)$.*

(3) *If $R_1=\{r\mid c(\mathfrak{S}_r)$ is not a cardinal number$\}$, $(R_1,\ \geqq)$ is a transitive subsystem of $(R,\ \geqq)$.*

(4) *If $r_1\geqq r_2\geqq r_3$, if r_1 and $r_2\in R_1$, and $\mathfrak{S}_{r(3)}\neq(S_{r(3)}, u)$, then $r_1\geqq r_3$.*

(5) *$(R_1,\ \geqq)$ contains no subsystem (A, u) where A contains more than two points.*

(6) *If $R_2=\{r\mid r\in R_1$ and there exists r_1 in R_1 such that $r>r_1>r\}$ and r is a point which is less than both elements in such a pair r', r_1 of R_1, then $\mathfrak{S}_r=(S_r, u)$.*

(7) *If $r\in R_2$, then $\mathfrak{S}_r\sim(N_2, w)$ (so R_2 is disjoint from R_0).*

(8) *If $E=R_1-R_0^D-R_2$, then $E=E'$.*

Suppose that $\prod_{(R,\geqq)}\mathfrak{S}_r$ is transitive; 4.5 gives (1); 4.8 gives (2); 4.6 gives (3). 4.7 gives (4); 4.9 gives (5); 4.10 gives (6); 4.11 gives (7). $\prod_{(E,\geqq)}\mathfrak{S}_r<\prod_{(R,\geqq)}\mathfrak{S}_r$ by 3.1; since E is a number and $\mathfrak{S}_r$ is a number if $r\in E$, 4.2 gives (8).

If the conditions (1)–(8) hold, suppose that $f_1>f_2>f_3$ and that r is a point for which $f_1(r)\neq f_3(r)$; we produce a point $r'\geqq r$ such that $f_1(r')>f_3(r')$ by considering various cases. (a) If $r\in R_0^D-R_0^{(1)}$ or if $r\in R_2^D-R_2$, by (2) or (6) $f_1(r)>f_3(r)$. (b) If $r\in R_2$, either (b_1) $f_1(r')=f_2(r')=f_3(r')$ for every $r'>r$ but not in R_2 or (b_2) not. In case (b_1) if r_1 is the element of R_2 such that $r>r_1>r$, (7) and the example before 4.9 assure that $f_1(r)>f_3(r)$ or $f_1(r_1)>f_3(r_1)$. In case (b_2) if $f_1(r)\not>f_3(r)$, there exists $r'>r$ and not in R_2 such that $f_1(r')\neq f_2(r')$ or $f_2(r')\neq f_3(r')$; r' must lie in R_1-R_2 so is in E. Since $f_1>f_2>f_3$ in E, and the product is transitive there (by (8) and 4.2), there exist $r''>r'$ such that $f_1(r'')>f_3(r'')$; by (3), $r''>r$. (c) If $r\in R-R_1-R_0^D$ and $f_1(r)\neq f_3(r)$, either $f_1(r)\neq f_2(r)$ or $f_2(r)\neq f_3(r)$ (or both) so there exists $r'>r$ such that $f_i(r')>f_{i+1}(r')$ for $i=1$ or 2; this r' must be in E; as before it follows that there exists r'' in E with $r''>r'$ and $f_1(r'')>f_3(r'')$. By (4) this $r''>r$. (d) If $r\in E$, (8) and 4.2 are all that are needed. If $r\in R_0^{(1)}-R_2^D$, the usual argument provides r' in E such that $r'>r$ and $f_i(r')>f_{i+1}(r')$ for $i=1$ or 2; as before, r'' can be found in E and $r''>r$ by (3) or (4).

A corollary of this which extends 4.3 is:

4.13 THEOREM. *$\prod_{(R,\geqq)}\mathfrak{S}_r$ is a number if and only if* (1) *if $R_1=\{r\mid \mathfrak{S}_r$ is* *not a cardinal number*$\}$, $R_1'=R_1$, (2) $(R_1, \geqq)$ *and all* $\mathfrak{S}_r$ *are numbers, and* (3) *if* $r_1 \geqq r_2 \geqq r_3$, *if* r_1 *and* $r_2 \in R_1$, *and* $\mathfrak{S}_{r(3)} \neq (N_1, =)$, *then* $r_1 \geqq r_3$.

If these conditions hold, the sets R_0 and R_2 of 4.12 are empty, so $R_1 = E$. It is easily verified that the conditions (1)–(8) of that theorem hold so $\prod_{(R,\geqq)} \mathfrak{S}_r$ is transitive. The relations $f > f' > f$ in $\prod_{(R,\geqq)} \mathfrak{S}_r$ would imply the same relation for the functions equal to these but defined only over R_1; 4.1 and the condition (2) prevent this so the product is a number.

Suppose that $\prod_{(R,\geqq)} \mathfrak{S}_r$ is a number; then conditions (1)–(8) of 4.12 hold for $\mathfrak{R}$ and the $\mathfrak{S}_r$. Since $\mathfrak{S}_{r(0)} < \prod_{(R,\geqq)} \mathfrak{S}_r$, every $\mathfrak{S}_{r(0)}$ is a number. Hence R_0 is empty; if R_2 were not empty, $\prod_{(R,\geqq)} \mathfrak{S}_r$ would contain a subsystem isomorphic to ${}^{(N(2),w)}(N_2, w)$; the discussion of this system before 4.9 shows that it is not a number, so R_2 must be empty. Hence R_1 is a number and $R_1 = E$ so $R' = R$. (3) follows immediately from (4) of 4.12.

Also from 4.12 and 3.7 (with $m=0$) we have the following corollary.

4.14 Corollary. *If* $\mathfrak{R}$ *and* $\operatorname{tr}(\mathfrak{S}_r)$ *satisfy the conditions* (1)–(8) *and all* $\mathfrak{S}_r$ *are k-transitive, then* $\prod_{(R,\geqq)} \mathfrak{S}_r$ *is k-transitive.*

In the special cases of cardinal and ordinal multiplication most of the conditions of 4.12 are satisfied automatically. We state this as another corollary:

4.15 Corollary. $\prod_{(R,=)} \mathfrak{S}_r$ *is transitive (a number) if and only if all* $\mathfrak{S}_r$ *are transitive (numbers).* $\mathfrak{S}_2 \circ \mathfrak{S}_1 = \prod_{(N(2),w)} \mathfrak{S}_n$ *is transitive if and only if* (a) *both* $\mathfrak{S}_n$ *are transitive and* (b) $\mathfrak{S}_2$ *is a number or* $\mathfrak{S}_1 = (S_1, u)$; $\mathfrak{S}_2 \circ \mathfrak{S}_1$ *is a number if and only if both the factors are numbers.*

(2) through (8) of 4.12 hold if $\mathfrak{R} = (R, =)$ so (1) is necessary and sufficient for transitivity of the cardinal product. (3) through (8) hold for $\mathfrak{R} = (N_2, w)$; (1) and (2) are then equivalent to (a) and (b). The last condition follows from 4.13.

The condition for transitivity of the ordinal power is not quite as complicated as that for the product in general.

4.16 Corollary. ${}^{(R,\geqq)}\mathfrak{S}$ *is transitive if and only if one of the following conditions holds*: (1) $\mathfrak{S}$ *is a cardinal number.* (2) $\mathfrak{S} = (S, u)$. (3) $\mathfrak{S}$ *is transitive and* $\mathfrak{R}$ *is a cardinal number.* (4) $\mathfrak{R}$ *is transitive and* $\mathfrak{S}$ *is a number; each* $c(r)$ *contains not more than two points; if* $R_2 = \{r \mid c(r)$ *contains two points*$\}$, *then* $R_2^D = R_2$; *if* R_2 *is not empty,* $\mathfrak{S}$ *is isomorphic to* (N_2, w); $R' = R - R_2$.

When all $\mathfrak{S}_r = \mathfrak{S}$, the sets R_0 and R_1 of 4.12 can only be empty or equal to R; the various combinations of these possibilities give the four cases of this corollary.

4.17 Corollary. ${}^{(R,\geqq)}\mathfrak{S}$ *is a number if and only if* $\mathfrak{S}$ *is a cardinal number or* $\mathfrak{R}$ *and* $\mathfrak{S}$ *are numbers and* $R = R'$.

It is to be noted that this condition says that the ordinal power of numbers is a number if and only if it is transitive; 4.13 said the same for ordered products in which factors and index system were both numbers.

5. **k-transitivity of the ordered product.** We have in this section a chain of principal theorems with successively weaker hypotheses and conclusions. The first of these (5.5) shows that if $\mathfrak{R}$ and all $\mathfrak{S}_r$ are numbers, then $\prod_{(R,\geqq)} \mathfrak{S}_r$ is 1-transitive; 5.7 improves on this by showing that if all $\mathfrak{S}_r$ are transitive and $\mathfrak{R}$ is a number, then the product is still 1-transitive. 5.9 asserts that if $\{r \mid \mathfrak{S}_r$ is not a cardinal number$\}$ contains more than one point, then $\prod_{(R,u)} \mathfrak{S}_r$ is 2-transitive. At this level a simple computation with the preceding results shows that if $\mathfrak{R}$ and $\mathfrak{S}_r$ are transitive, then $\prod_{(R,\geqq)} \mathfrak{S}_r$ is 4-transitive; however, a refinement of the proofs of 5.5 and 5.7 allows us to prove the stronger result 5.14 that if $\mathfrak{R}$ and all $\mathfrak{S}_r$ are transitive, then $\prod_{(R,\geqq)} \mathfrak{S}_r$ is 2-transitive; this result is best possible as is shown by an example before 5.9.

For convenience in the calculations to follow, if $E \subset R$, define $f_1 \geqq f_2$ *over* E to mean that the functions are related as in $\prod_{(E,\geqq)} \mathfrak{S}_r$; that is, $f_1 \geqq f_2$ over E means that if r is a point of E such that $f_1(r) \neq f_2(r)$, there is a point r' in E such that $r' \geqq r$ and $f_1(r') > f_2(r')$. Recall that E is a star in $\mathfrak{R}$ if $E^U = E$.

5.1 LEMMA. *If E is a star in $\mathfrak{R}$ and $f_1 \geqq f_2$ over R, then $f_1 \geqq f_2$ over E.*

Since a star contains all successors of each of its elements, if r exists in E such that $f_1(r) \neq f_2(r)$, the r' in R which exists since $f_1 \geqq f_2$ over R must lie in E.

5.2 LEMMA. *If $f_1(r) > f_2(r)$ for every r in a set $E \subset R$, then $f_1 > f_2$ over E^D no matter how the f_i are defined on $E^D - E$.*

Every point r' of E^D has a successor r in E; for such an r, $f_1(r) > f_2(r)$.

5.3 LEMMA. *If E_p, $p \in P$, are subsets of R and $f_1 \geqq f_2$ over each E_p, then $f_1 \geqq f_2$ over $\bigcup_{p \in P} E_p$.*

This is clear from the definitions.

5.4 THEOREM. *If $\mathfrak{R}$ and all $\mathfrak{S}_r$ are numbers and if $f_1 \geqq f_2 \geqq f_3 \geqq f_4$ over R, then there exists f such that $f_1 \geqq f \geqq f_4$ over R.*

The proof gives a stepwise construction of f and is rather dull reading; however, it is the fundamental construction of this section; modifications of it are used to prove 5.7 and 5.14. Let $E_i = \{r \mid f_i(r) \neq f_{i+1}(r)\}$, $i = 1, 2, 3$, and let $F_i = \{r \mid f_i(r) > f_{i+1}(r)\}$; then the given condition that $f_i \geqq f_{i+1}$ over R means that F_i is a cofinal subset of E_i; that is, that $F_i^D \supset E_i$.

Consider now the first three functions, f_1, f_2, f_3; we can not generally conclude that $f_1 \geqq f_3$ over all of R, but there is usually a large subset of R over which $f_1 \geqq f_3$; we begin by constructing such a subset whose complement has certain properties. If $R_0 = E_1 \cup E_2$, then $f_1(r) = f_2(r) = f_3(r)$ if r is not in R_0. If $r \in R_0^{(1)}$, then by 1.8 and the fact that F_i is cofinal in E_i either $f_1(r) > f_2(r)$

$=f_3(r)$ or $f_1(r)=f_2(r)>f_3(r)$ or $f_1(r)>f_2(r)>f_3(r)$. By transitivity of $>$ (not only of $\geqq$) in the number $\mathfrak{S}_r$ we see that $f_1(r)>f_3(r)$. We now know by 5.2 and 5.3 that

(A) If $R_1=R_0-R_0^{(1)D}$, then $f_1\geqq f_3$ over $R-R_1$.

If $r\in E_1-E_2^D$, then $f_1(r)\neq f_2(r)$ so there exists $r'\geqq r$ in F_1; this r' can not be in E_2^D so $f_1(r')>f_2(r')=f_3(r')$; by 5.2, $f_1>f_3$ over $(E_1-E_2^D)^D$. Similarly $f_1>f_3$ over $(E_2-E_1^D)^D$; by 5.3 we have

(B) If $R_2=R_1-(E_1-E_2^D)^D-(E_2-E_1^D)^D$, then $f_1\geqq f_3$ over $R-R_2$.

Since no element of R_2 has a successor in R_0-R_2, it is clear that

(C) If $f(r)=f_1(r)$ and $f'(r)=f_3(r)$ for every r in $R-R_2$, then $f\geqq f'$ over R if and only if $f\geqq f'$ over R_2.

We next prove

(D) $R_2^{(1)}$ is empty, $R_2\subset E_1\cup E_2$ and $R_2\cap F_i$ is cofinal in R_2 for $i=1, 2$.

Since R_2 is a star in $(R_1, \geqq)$ and $R_1^{(1)}$ is empty, $R_2^{(1)}$ is empty. $R_2\subset R_1\subset R_0 =E_1\cup E_2$. If $r\in R_2=R_1-(E_1-E_2^D)^D-(E_2-E_1^D)^D$, if $r_1>r$ and if $r_1\in E_i$, there exists r_2 in E_{3-i} such that $r_2>r_1>r$; therefore $E_i\cap R_2$ is cofinal in R_2 for each i; since R_2 is a star in $(R_0, \geqq)$, $F_i\cap R_2$ is cofinal in $E_i\cap R_2$ and hence in R_2.

We now include the extra function f_4 and, as seems reasonable from (B) and (C), define f on $R-R_2$ by $f(r)=f_3(r)$ if $r\in R-R_2$. Let $R_3=R_2-(F_3-R_2)^D$. Then the values of f in R_2-R_3 will have no effect on whether $f\geqq f_4$, so we can define $f(r)=f_1(r)$ if $r\in R_2-R_3$; then by 5.2 and 5:3 we have

(E) $f_1\geqq f\geqq f_4$ over $R-R_3$.

Since R_3 is a star in R_2 we have

(F) $F_i\cap R_3$ is cofinal in R_3 if $i=1, 2$; $R_3^{(1)}$ is empty.

We prove next that

(G) If $E=\{r\mid f_4(r)$ is not a terminal element of $\mathfrak{S}_r\}$, then $E\cap R_3$ is cofinal in R_3.

If $r\in R_3$, either $f_3(r')=f_4(r')$ for each $r'\geqq r$ in R_3 or not; if the former, there exists $r'\geqq r$ in $F_2\cap R_3$, so $f_2(r')>f_3(r')=f_4(r')$; if the latter, there exists $r'\geqq r$ in R_3 such that $f_3(r')>f_4(r')$; in either case $r'\in E\cap R_3$.

Since $F_1\cap R_3$ and $E\cap R_3$ are both cofinal subsets of $(R_3, \geqq)$ and since $R_3^{(1)}$ is empty, there exist disjoint cofinal subsets G and H of $(R_3, \geqq)$ such that $G\subset F_1\cap R_3$ and $H\subset E\cap R_3$. Define $f(r)>f_4(r)$ if $r\in H, f(r)=f_2(r)$ if $r\in R_3-H$. Then it is clear that $f_1\geqq f\geqq f_4$ over R if and only if $f_1\geqq f\geqq f_4$ over R_3; however, $\{r\mid f_1(r)>f(r)\}\supset G$ while $\{r\mid f(r)>f_4(r)\}\supset H$ so $f_1\geqq f\geqq f_4$ does hold over R_3 and therefore over R.

5.5 THEOREM. *If $\mathfrak{R}$ and all $\mathfrak{S}_r$ are numbers, then $\prod_{(R,\geqq)}\mathfrak{S}_r$ is 1-transitive; that is, if $f_1\geqq f_2\geqq\cdots\geqq f_n$, there exists f such that $f_1\geqq f\geqq f_n$.*

This follows from 5.4 and an obvious induction on n.

We wish now to improve 5.4 by weakening the hypothesis on the $\mathfrak{S}_r$.

5.6 LEMMA. *If $\mathfrak{R}$ is a number and all $\mathfrak{S}_r$ are transitive and if $f_1\geqq f_2\geqq f_3\geqq f_4$ over R, there exists f such that $f_1\geqq f\geqq f_4$ over R.*

Since we used transitivity of $>$ in $\mathfrak{S}_r$ in the proof that $f_1 \geqq f_3$ over $R_0^{(1)D}$ we must modify that construction somewhat. (See step (3) below.) As before let $R_0 = E_1 \cup E_2$; then

(1) Define $f(r) = f_1(r) = f_3(r)$ on $R - R_0$; clearly $f_1 \geqq f \geqq f_4$ over $R - R_0$.

(2) Let $f(r) = f_1(r)$ if $r \in (F_3 - R_0^D)^D$. If $A_0 = R_0 - (F_3 - R_0^D)^D$, $f_1 \geqq f \geqq f_4$ over $R - A_0$.

(3) Let $A = A_0^{(1)} \cap \{r \mid f_1(r) > f_2(r) > f_3(r) = f_1(r)\}$; well-order A in a transfinite sequence $\{r_\alpha\}$ and let $B_\alpha = A_0 \cap \{r \mid r_\alpha \geqq r$ but r_λ does not follow or equal r if $\lambda < \alpha\}$. If $r \in B_\alpha$, let $f(r) = f_1(r)$ if $f_4(r_\alpha) \neq f_1(r_\alpha)$; let $f(r) = f_2(r)$ if $f_4(r_\alpha) = f_1(r_\alpha)$. In the first case $f_1 = f$ over B_α while $f(r_\alpha) = f_3(r_\alpha) \neq f_4(r_\alpha)$; since no $r' > r_\alpha$ exists for which $f_3(r') > f_4(r')$, it follows that $f(r_\alpha) > f_4(r_\alpha)$. In the second case $f_1(r_\alpha) > f(r_\alpha) > f_4(r_\alpha)$; in both cases $f_1 \geqq f \geqq f_4$ over B_α. Hence by 5.3 we have, setting $A_1 = A_0 - A^D$, that $f_1 \geqq f \geqq f_4$ over $R - A_1$.

(4) For r in $A_1^{(1)} = A_0^{(1)} - A$, either $f_1(r) > f_2(r) = f_3(r)$ or $f_1(r) = f_2(r) > f_3(r)$ or $f_1(r) > f_2(r) > f_3(r) \neq f_1(r)$; in all these cases $f_1(r) > f_3(r)$ so we define $f = f_3$ in $A_1 \cap A_1^{(1)D}$. If $A_2 = A_1 - A_1^{(1)D}$, then $f_1 \geqq f \geqq f_4$ over $R - A_2$.

(5) Let $A_3 = A_2 - (E_1 - E_2^D)^D - (E_2 - E_1^D)^D$; then precisely as in (B) of 5.4, f can be taken equal to f_3 in $A_2 - A_3$ and $f_1 \geqq f \geqq f_4$ over $R - A_3$.

(6) As before with R_3 it is easily seen that A_3 has no terminal elements, that $F_1 \cap A_3$ is cofinal in $(A_3, \geqq)$ and that if $E = \{r \mid f_4(r)$ is not a terminal element of $\mathfrak{S}_r\}$, $E \cap A_3$ is also cofinal in $(A_3, \geqq)$; these properties were used in defining f in R_3; the same technique gives f in A_3 in such a way that $f_1 \geqq f \geqq f_4$ over R.

From 5.6 we derive the following theorem by induction.

5.7 THEOREM. *If $\mathfrak{R}$ is a number and all $\mathfrak{S}_r$ are transitive, then $\prod_{(R,\geqq)} \mathfrak{S}_r$ is 1-transitive.*

From this and 3.7 we derive the following corollary.

5.8 COROLLARY. *If $\mathfrak{R}$ is a number and all $\mathfrak{S}_r$ are k-transitive, then $\prod_{(R,\geqq)} \mathfrak{S}_r$ is $(k+2)$-transitive.*

We wish next to relax the conditions on $\mathfrak{R}$; that this can not be done without penalty can be seen by a simple example, ${}^{(N(2),u)}[(N_2, w) + (N_2, w)]$, where the exponent is the simplest possible transitive system that is not a number. The base is isomorphic to the subsystem of four points 1, 2, i, $2i$ of the complex plane where $a + bi \geqq a' + b'i$ if $a \geqq a'$ and $b \geqq b'$. Define $f_1, \cdots, f_4$ on (N_2, u) by

$$f_1(1) = 1, \qquad f_2(1) = 2, \qquad f_3(1) = 1, \qquad f_4(1) = 2i,$$

$$f_1(2) = 2, \qquad f_2(2) = 1, \qquad f_3(2) = 2i, \qquad f_4(2) = i.$$

Then $f_1 > f_2 > f_3 > f_4$ but $f_1 > f$ implies $f(2) = 1$ and $f > f_4$ implies $f(2) = 2i$; since these conditions can not be satisfied simultaneously, $f_1 > f > f_4$ is impossible.

Clearly f_1 does not follow or equal f_4 so this system is at best 2-transitive: 5.9 shows that it is actually 2-transitive.

5.9 THEOREM. *If* $E=\{r \mid \mathfrak{S}_r$ *is not a cardinal number*$\}$ *is empty*, $\prod_{(R,u)} \mathfrak{S}_r$ *is a cardinal number; if E contains only one point* r_0, $\prod_{(R,u)} \mathfrak{S}_r \sim \prod_{(R-E(D),u)} \mathfrak{S}_r \cdot (\mathfrak{S}_{r(0)} \circ \prod_{(E(D)-E,u)} \mathfrak{S}_r)$, *where the first and third factors are cardinal numbers; if E contains more than one point*, $\prod_{(R,u)} \mathfrak{S}_r$ *is 2-transitive.*

$\prod_{(R,\geqq)} \mathfrak{S}_r$ is always a cardinal number if all $\mathfrak{S}_r$ are cardinal numbers. If r_0 is the only point of E, $f(r_0)>f'(r_0)$ if and only if $f>f'$; hence the mapping $f\rightleftarrows(h, s, g)$ if $h(r)=f(r)$ if $r\in R-E^D$, $f(r_0)=s$, and $g(r)=f(r)$ if $r\in E^D-E$ is an isomorphism of the given systems. In the last case suppose $f_1\geqq f_2\geqq f_3\geqq f_4\geqq f_5$; if one equality holds the chain can be shortened to four members. If no equality holds, then for each $i\leqq 4$ there exists r_i such that $f_i(r_i)>f_{i+1}(r_i)$. If for $i=1$ or 2, $r_i\neq r_{i+2}$, define $f(r_i)<f_i(r_1)$, $f(r_{i+2})>f_{i+3}(r_{i+2})$, $f(r)$ arbitrary if $r\neq r_i$ or r_{i+2}; then $f_i>f>f_{i+3}$ so the chain can be shortened one link. If $r_1\neq r_4$, the same device shows that the chain can be shortened. If none of these things happens, $r_1=r_3=r_4=r_2$ and for every other r in R, $f_i(r)\leqq f_{i+1}(r)$ for every i. By hypothesis E must contain at least one point $r_0\neq r_1$; choose $f(r_1)<f_1(r_1)$, $f(r_0)$ not an initial element of $\mathfrak{S}_{r(0)}$, f arbitrary elsewhere, and define $f'(r_1)>f_5(r_1)$, $f'(r_0)<f(r_0)$, f' arbitrary elsewhere; then $f_1>f>f'>f_5$. Hence $\prod_{(R,u)} \mathfrak{S}_r$ is 2-transitive since every chain with more than four elements can be shortened repeatedly until there are only two middle links.

The next lemma is closely related to 6.1–6.5 but we use it here to prove a relation between products over a general transitive system and over numbers.

5.10 LEMMA. *If* $\mathfrak{R}$ *is a number, if* $\mathfrak{S}_r=(S_r, u)$ *for each* r, *and if* $(T, \geqq) =\sum_{(R,\geqq)} \mathfrak{S}_r$, *then*

$$\prod_{(R,\geqq)} \prod_{(S(r),u)} \mathfrak{P}_{rs} \sim \prod_{(T,\geqq)} \mathfrak{P}_{rs}.$$

The elements of the left-hand system are functions F defined on R with F_r in $\prod_{(S(r),u)} \mathfrak{P}_{rs}$ for each r; $F\geqq F'$ means that if $F_r\neq F'_r$ there exists $r'\geqq r$ and s' in $\mathfrak{S}_r$ such that $F_{r'}(s')>F'_{r'}(s')$. The elements of the right-hand system are functions f defined on $\sum_{(R,\geqq)} (S_r, u)$ with $f(r, s)$ in $\mathfrak{P}_{rs}$; $f\geqq f'$ means that if $f(r, s)\neq f'(r, s)$ there exists $(r', s')\geqq(r, s)$ such that $f(r', s')>f'(r', s')$. Since $(r', s')\geqq(r, s)$ means here that $r\geqq r'$, $f\geqq f'$ means that if $f(r, s)\neq f'(r, s)$, there exists (r', s') with $r'\geqq r$ such that $f(r', s')>f'(r', s')$. Under the one-to-one mapping $F\rightleftarrows f$ if $F_r(s)=f(r, s)$ for all r, s, these conditions are equivalent so the systems are isomorphic.

From the three preceding results we derive a fact which we improve in 5.14.

5.11 COROLLARY. *If* $\mathfrak{R}$ *and all* $\mathfrak{S}_r$ *are transitive*, $\prod_{(R,\geqq)} \mathfrak{S}_r$ *is 4-transitive.*

If $\mathfrak{P}=c(\mathfrak{R})$ and $\mathfrak{V}_p=(c^{-1}(p), u)$, then $(V, \geqq)=\sum_{(P,\geqq)} \mathfrak{V}_p\sim\mathfrak{R}$; if h is this isomorphism, let $\mathfrak{S}_{pv}=\mathfrak{S}_{h(p,v)}$; then by 5.10

$$\prod_{(R,\geqq)}\mathfrak{S}_r\sim\prod_{(V,\geqq)}\mathfrak{S}_{pv}\sim\prod_{(P,\geqq)}\ \prod_{(V(p),u)}\mathfrak{S}_{pv}.$$

By 5.9, $\prod_{(V(p),u)}\mathfrak{S}_{pv}$ is 2-transitive for every p, for if $\mathfrak{S}_{pv}$ is a cardinal for every v in V_p then the product is a cardinal and is transitive; if V_p contains just one element for which $\mathfrak{S}_{pv}$ is not a cardinal it is easily seen that $\prod_{(V(p),u)}\mathfrak{S}_{pv}$ is at worst 1-transitive when all $\mathfrak{S}_{pv}$ are transitive; if V_p contains two elements such that $\mathfrak{S}_{pv}$ is not a cardinal, 5.9 asserts directly that the product is 2-transitive. By 5.8 the right-hand system is 4-transitive so the same is true of $\prod_{(R,\geqq)}\mathfrak{S}_r$.

One last squeeze on the proof of 5.4–5.6 gives us an even better estimate of the transitivity number of the product over a transitive system of transitive systems. We need some additional lemmas.

5.12 LEMMA. *If $\mathfrak{P}_v$ are transitive and if $f_1>^t f_2>^t f_1$ in* $\text{tr}(\prod_{(V,u)}\mathfrak{P}_v)$, *then there exist f and f' such that $f_1>f\geqq f'>f_1$ in $\prod_{(V,u)}\mathfrak{P}_v$.*

By 5.9 there exist f_1', f_1'' such that $f_1>f_1'\geqq f_1''\geqq f_2$ in $\prod_{(V,u)}\mathfrak{P}_v$. Since $f_1'>^t f_1$, again by 2-transitivity there exist f_2', f_3' such that $f_1>f_1'>f_2'\geqq f_3'\geqq f_1$. If equality holds once here, f_1' and f_2' can be used for f and f'; if equality holds twice, $f=f'=f_1'$ will do; if neither of these equalities holds, then there exist points v_i such that $f_1(v_1)>f_1'(v_1)$, $f_1'(v_2)>f_2'(v_2)$, $f_2'(v_3)>f_3'(v_3)$, and $f_3'(v_4)>f_1(v_4)$. If v_1 and v_4 can be chosen unequal, define f so that $f(v_1)<f_1(v_1)$, $f(v_4)>f_1(v_4)$ and f is defined arbitrarily elsewhere; then $f_1>f>f_1$ and the conditions are satisfied with $f'=f$. If v_1 must be chosen equal to v_4, there are several cases: (1) $f_1'(v_1)>f_1(v_1)$; then $f=f'=f_1'$ will satisfy the given conditions. (2) $f_1'(v_1)$ does not follow or equal $f_1(v_1)$, $v_2\neq v_1$; take $f=f_1'$ and define f' so that $f'(v_2)<f(v_2)$, $f'(v_1)>f_1(v_1)$; then $f_1>f>f'>f_1$. (3) $f_1'(v_1)$ does not follow or equal $f_1(v_1)$, $v_2=v_1$; then $f_1(v_1)>f_1'(v_1)>f_2'(v_1)$ so $f_1(v_1)\geqq f_2'(v_1)$; equality is impossible here since $f_1'(v_1)$ does not follow or equal $f_1(v_1)$ so $f_1(v_1)>f_2'(v_1)$; hence we may take $f=f_2'$ and $f'=f_3'$.

5.13 LEMMA. *If $\mathfrak{R}$ is a number, if $\mathfrak{S}_r=\prod_{(V(r),u)}\mathfrak{P}_{rv}$ and if $f_1\geqq^2 f_2\geqq^2 f_3$ in $\prod_{(R,\geqq)}$* $\text{tr}(\mathfrak{S}_r)$, *there exist f and f' such that $f_1\geqq^1 f\geqq^1 f'\geqq^1 f_3$ in $\prod_{(R,\geqq)}\mathfrak{S}_r$.*

Let $E_i=\{r\mid f_i(r)\neq f_{i+1}(r)\}$, $F_i=\{r\mid f_i(r)>^t f_{i+1}(r)$ in $\text{tr}(\mathfrak{S}_r)\}$ and let $R_0=E_1\cup E_2$.

In $R-R_0$ define $f=f'=f_1=f_3$. As in the proof of 5.6 let $A=\{r\mid r\in R_0^{(1)}$ and $f_1(r)>^t f_2(r)>^t f_3(r)=f_1(r)$ in $\text{tr}(\mathfrak{S}_r)\}$; by 5.12 define $f(r)$ and $f'(r)$ for r in A so that $f_1(r)>f(r)\geqq f'(r)>f_1(r)$ in $\mathfrak{S}_r$. For r in A^D-A define $f(r)=f'(r)$ in any way; then if $B=R_0-A^D$, $f_1\geqq^1 f\geqq^1 f'\geqq^1 f_3$ over $R-B$. If $r\in B^{(1)}=R_0^{(1)}-A$, then $f_1(r)>^t f_3(r)$ so, by 2-transitivity, $f(r)$ and $f'(r)$ can be chosen so that $f_1(r)>f(r)\geqq f'(r)\geqq f_3(r)$ in $\mathfrak{S}_r$; if $r\in B^{(1)D}-B^{(1)}$, let $f(r)=f'(r)=f_3(r)$. If $B_1=B-B^{(1)D}$, then $f_1\geqq^1 f\geqq^1 f'\geqq^1 f_3$ over $R-B_1$. As before we see that for r

in $(F_1-E_2^D)\cap B_1$, $f_1(r)>^t f_2(r)=f_3(r)$ and in $(F_2-E_1^D)\cap B_1$, $f_1(r)=f_2(r)>^t f_3(r)$; again by 2-transitivity we can define $f_1(r)>f(r)\geqq f'(r)\geqq f_3(r)$ in either of these two sets and define $f=f'=f_3$ for all other r in $[(F_1-E_2^D)^D\cup(F_2-E_1^D)^D]\cap B_1$. If $B_2=B_1$ minus this last set, then $f_1\geqq^1 f\geqq^1 f'\geqq^1 f_3$ over $R-B_2$; B_2 is precisely the subsystem R_2 of 5.4 (B) and (D). From this we define $f=f'$ in B_2 just as we defined f in R_3 in that proof to be below f_1 on one set cofinal in B_2 and above f_3 on another set cofinal in B_2; then $f_1\geqq^1 f\geqq^1 f'\geqq^1 f_3$ everywhere over R.

From this lemma and 5.7 we can improve 5.11; the example before 5.9 shows that this result is the best possible.

5.14 THEOREM. *If $\mathfrak{R}$ and all $\mathfrak{S}_r$ are transitive, then $\prod_{(R,\geqq)}\mathfrak{S}_r$ is 2-transitive.*

Use the notation of 5.11; then $\mathfrak{R}\sim(V,\geqq)=\sum_{(P,\geqq)}(V_p,u)$ and $\prod_{(R,\geqq)}\mathfrak{S}_r\sim\prod_{(V,\geqq)}\mathfrak{S}_{pv}\sim\prod_{(P,\geqq)}\prod_{(V(p),u)}\mathfrak{S}_{pv}$. If $f_1\geqq^1 f_2\geqq^1\cdots\geqq^1 f_n$ in this last system, then $f_1\geqq^2\cdots\geqq^2 f_n$ in $\prod_{(P,\geqq)}\operatorname{tr}(\prod_{(V(p),u)}\mathfrak{S}_{pv})$, so, by 5.6, there exists f'' such that $f_1\geqq^2 f''\geqq^2 f_n$ in $\prod_{(P,\geqq)}\operatorname{tr}(\prod_{(V(p),u)}\mathfrak{S}_{pv})$. By 5.13 there exist f and f' such that $f_1\geqq^1 f\geqq^1 f'\geqq^1 f_n$ in $\prod_{(P,\geqq)}\prod_{(V(p),u)}\mathfrak{S}_{pv}$; hence this latter system is 2-transitive so the original product is also 2-transitive.

5.15. COROLLARY. *If $\mathfrak{R}$ is transitive and all $\mathfrak{S}_r$ are k-transitive, then $\prod_{(R,\geqq)}\mathfrak{S}_r$ is $(k+4)$-transitive.*

This follows from 5.14 and 3.7.

A problem which is still unsettled is to determine whether k-transitivity of $\mathfrak{R}$ and all $\mathfrak{S}_r$ implies that $\prod_{(R,\geqq)}\mathfrak{S}_r$ is m-transitive for some m.

6. **Some properties of ordered products.** In a special case 5.10 asserts that if $(T,\geqq)=\sum_{(R,\geqq)}\mathfrak{S}_r$, then $\prod_{(R,\geqq)}\prod_{(S(r),\geqq)}\mathfrak{P}_{rs}\sim\prod_{(T,\geqq)}\mathfrak{P}_{rs}$; this is not true in general but a homomorphism one way always holds and a similar relation is true if $\prod^t$ is used in place of $\prod$ in the first place on each side. We also give some conditions under which the isomorphism above does hold. The elements of the left-hand side we represent by functions F for which $F_r\in\prod_{(S(r),\geqq)}\mathfrak{P}_{rs}$ and $F\geqq F'$ means that if $F_r\neq F_r'$, there exists $r'\geqq r$ such that $F_{r'}>F_{r'}'$ in $\prod_{(S(r'),\geqq)}\mathfrak{P}_{r's}$. The elements of the right-hand system are functions f for which $f(r,s)\in\mathfrak{P}_{rs}$ and $f\geqq f'$ means that if $f(r,s)\neq f'(r,s)$ there exists $(r',s')\geqq(r,s)$ such that $f(r',s')>f'(r',s')$. There is an obvious one-to-one correspondence h between these systems, defined by $hF=f$ if $F_r(s)=f(r,s)$ for all r and s.

6.1 THEOREM. *The function h just defined is a homomorphism so*

$$\prod_{(R,\geqq)}\prod_{(S(r),\geqq)}\mathfrak{P}_{rs}\succ\prod_{(T,\geqq)}\mathfrak{P}_{rs}\quad\text{where}\quad(T,\geqq)=\sum_{(R,\geqq)}\mathfrak{S}_r.$$

If $F>F'$, if $hF=f$ and $hF'=f'$, let (r,s) be a point such that $f(r,s)\neq f'(r,s)$; then $F_r(s)\neq F_r'(s)$ so $F_r\neq F_r'$. Hence there exists $r'\geqq r$ such that $F_{r'}>F_{r'}'$; if $r'>r$, take s' to be any point of $\mathfrak{S}_{r'}$ such that $F_{r'}(s')>F_{r'}'(s')$; if $r'=r$, take

$s' \geqq s$ such that $F_r(s') > F'_r(s')$. Then $(r', s') \geqq (r, s)$ and $f(r', s') > f'(r', s')$; hence $f \geqq f'$ if $F \geqq F'$ and h is a homomorphism.

It is easy to give an example of systems where this homomorphism is not an isomorphism. Let $\mathfrak{R} = (N, w)$, $\mathfrak{S}_r = \mathfrak{P}_{rs} = (N_2, w)$; then define f and f' by $f(r, 1) = f'(r, 2) = 2$, $f(r, 2) = f'(r, 1) = 1$ for all r. Then $(r+1, 1) > (r, s)$ for all r, and $2 = f(r+1, 1) > f'(r+1, 1) = 1$, so $f > f'$; since $F_r(2) < F'_r(2)$ for every r, $F_r \not> F'_r$ no matter how r is chosen, so F does not follow or equal F' and the homomorphism is not an isomorphism.

The next result has two useful consequences.

6.2 LEMMA. *If $f \geqq f'$, there exists F'' such that $F \geqq F'' \geqq F'$.*

If $f > f'$, let $R_1 = \{r \mid$ there is an s in $\mathfrak{S}_r$ such that $f(r, s) > f'(r, s)\}$. If $r \in R_1^{(1)}$ and $F_r(s) \neq F'_r(s)$, there exists $(r', s') \geqq (r, s)$ in $\sum_{(R, \geqq)} \mathfrak{S}_r$ such that $f(r', s') > f'(r', s')$; hence $r' \geqq r$ and $r' \in R$ so $r = r'$ and $s' \geqq s$; that is, if $r \in R_1^{(1)}$ and $F_r(s) \neq F'_r(s)$ there exists $s' \geqq s$ such that $F_r(s') > F'_r(s')$; hence $F_r > F'_r$ if $r \in R_1^{(1)}$ and $F > F'$ over $R_1^{(1)D}$. Let $R_2 = R_1 - R_1^{(1)D}$ and let $R_3 = \{r \mid c(r)$ is a terminal element of $c(R_2, \geqq)\}$. Then if $r \in R_3$, $c(r)$ contains at least one r' different from r, while in $R_2 - R_3$ every element r has a successor not in $c(r)$; hence there are two disjoint cofinal subsets A_1 and A_2 of $(R_2, \geqq)$. If $r \in A_1$ there exists s_0 in S_r such that $f(r, s_0) > f'(r, s_0)$; define F'_r by $F''_r(s) = F_r(s)$ if $s \neq s_0$, $F''_r(s_0) = F'_r(s_0)$. Then $F_r > F''_r$ if $r \in A_1$. Similarly, if $r \in A_2$, choose s_0 so that $f(r, s_0) > f'(r, s_0)$; then let F'' be defined by $F''_r(s) = F'_r(s)$ if $s \neq s_0$, $F''_r(s_0) = F_r(s_0)$; then $F''_r > F'_r$ for r in A_2. Let $F''_r = F_r$ for every r where F''_r has not been defined; then it can easily be verified that $F \geqq F'' \geqq F'$.

This lemma is used in the proof of two important results.

6.3 THEOREM. *Let $(T, \geqq) = \sum_{(R, \geqq)} \mathfrak{S}_r$. If $\prod_{(T, \geqq)} \mathfrak{P}_{rs}$ is k-transitive, then $\prod_{(R, \geqq)} \prod_{(S(r), \geqq)} \mathfrak{P}_{rs}$ is $(2k+1)$-transitive. If $\prod_{(R, \geqq)} \prod_{(S(r), \geqq)} \mathfrak{P}_{rs}$ is k-transitive, so is $\prod_{(T, \geqq)} \mathfrak{P}_{rs}$; that is, the homomorphism h of* 6.1 *does not increase the transitivity number.*

If $F_1 \geqq F_2 \geqq \cdots \geqq F_n$, $f_1 \geqq f_2 \geqq \cdots \geqq f_n$ by 5.1; hence there exist $f'_1, \cdots, f'_k$ such that $f_1 \geqq f'_1 \geqq \cdots \geqq f'_k \geqq f_n$. By 5.2 there exist $F''_1, \cdots, F''_{k+1}$ such that $F_1 > F''_1 > F'_1 > \cdots > F'_k > F''_{k+1} > F_n$; that is, the left-hand system is $(2k+1)$-transitive. The proof in the other direction is of the same sort but easier.

6.4 THEOREM. *If $(T, \geqq) = \sum_{(R, \geqq)} \mathfrak{S}_r$, then $\prod^t_{(R, \geqq)} \prod_{(S(r), \geqq)} \mathfrak{P}_{rs} \sim \prod^t_{(T, \geqq)} \mathfrak{P}_{rs}$ under the natural mapping h.*

h is a homomorphism of the left-hand system on the right by 6.1 and 1.1. By 6.2, h^{-1} is a homomorphism of $\prod_{(T, \geqq)} \mathfrak{P}_{rs}$ onto $\prod^t_{(R, \geqq)} \prod_{(S(r), \leqq)} \mathfrak{P}_{rs}$; hence by 1.1, h^{-1} is a homomorphism of $\prod^t_{(T, \geqq)} \mathfrak{P}_{rs}$ onto $\prod^t_{(R, \geqq)} \prod_{(S(r), \leqq)} \mathfrak{P}_{rs}$.

6.5 THEOREM. *Let $(T, \geqq) = \sum_{(R, \geqq)} \mathfrak{S}_r$. If* (a) *$\mathfrak{R}$ and all $\mathfrak{S}_r$ are transitive* *and all* $\mathfrak{P}_{rs}$ *are k-transitive, or* (b) *if R is a finite set, then the mapping h is an isomorphism of* $\prod^t_{(T,\geqq)} \mathfrak{P}_{rs}$ *and* $\prod^t_{(R,\geqq)} \prod^t_{(S(r),\geqq)} \mathfrak{P}_{rs}$.

Under the first set of hypotheses by 5.15, $\prod_{(S(r),\geqq)} \mathfrak{P}_{rs}$ is $(k+4)$-transitive for every r; by 3.8 the right-hand system here is equal to the left-hand system in the isomorphism of 6.4; this implies the desired isomorphism. If R is finite, 3.8′ can be used in place of 3.8 to give the same conclusion.

Under sufficient restriction h is an isomorphism of $\prod\prod$ and $\prod_\Sigma$.

6.6 THEOREM. *If* $\mathfrak{R}$ *is a number such that* $R=R'$ *and if* $(T, \geqq)=\sum_{(R,\geqq)} \mathfrak{S}_r$, *then*

$$\prod_{(R,\geqq)} \prod_{(S(r),>)} \mathfrak{P}_{rs} \sim \prod_{(T,\geqq)} \mathfrak{P}_{rs}.$$

The proof that h^{-1} is a homomorphism makes use of 1.6 and follows about half way down the proof of 6.2. Using the notation of that proof we showed that if $r\in R_1^{(1)}$ then $F_r>F_r'$; by 1.6, $R_1^{(1)D}\supset R_1$ so $F>F'$.

To deal with contracted products we need the following lemma.

6.7 LEMMA. *If all* $\mathfrak{S}_r$ *are transitive,* $\prod_{(R,\geqq)} c(\mathfrak{S}_r) \succ \prod^c_{(R,\geqq)} \mathfrak{S}_r$.

Let $\mathfrak{P}_r=c(\mathfrak{S}_r)$ and define ϕ from $\prod_{(R,\geqq)} \mathfrak{S}_r$ onto $\prod_{(R,\geqq)} c(\mathfrak{S}_r)$ by $\phi f=g$ if $g(r)=c(f(r))$, where c is the contraction mapping of $\mathfrak{S}_r$ onto $\mathfrak{P}_r$. If $\phi f_1=\phi f_2$, $f_1(r)\geqq f_2(r)\geqq f_1(r)$ for every r so that $f_1\geqq f_2\geqq f_1$ in $\prod_{(R,\geqq)} \mathfrak{S}_r$; that is, $\phi f_1=\phi f_2$ implies that $c(f_1)=c(f_2)$ in $\prod^c_{(R,\geqq)} \mathfrak{S}_r$. Define Φ from $\prod_{(R,\geqq)} \mathfrak{P}_r$ by letting Φg be the element of $\prod^c_{(R,\geqq)} \mathfrak{S}_r$ which contains $\phi^{-1}g$; clearly Φ maps $\prod_{(R,\geqq)} \mathfrak{P}_r$ onto $\prod^c_{(R,\geqq)} \mathfrak{S}_r$ so we need only prove Φ monotone. If $g_1>g_2$ in $\prod_{(R,\geqq)} \mathfrak{P}_r$, take f_i in $\phi^{-1}(g_i)$ and let r be a point such that $f_1(r)\neq f_2(r)$; then either $g_1(r)=g_2(r)$ so $f_1(r)>f_2(r)>f_1(r)$ or $g_1(r)\neq g_2(r)$ so there exists $r'\geqq r$ such that $g_1(r')>g_2(r')$; this implies that $f_1(r')>f_2(r')$ so $f_1>f_2$ in $\prod_{(R,\geqq)} \mathfrak{S}_r$; hence by the definition of order in a contraction, $\Phi g_1\geqq \Phi g_2$ in $\prod^c_{(R,\geqq)} \mathfrak{S}_r$ if $g_1>g_2$ in $\prod_{(R,\geqq)} c(\mathfrak{S}_r)$.

6.8 COROLLARY. $\prod^t_{(R,\geqq)} c(\mathfrak{S}_r) \succ \prod^c_{(R,\geqq)} c(\mathfrak{S}_r) \succ \prod^c_{(R,\geqq)} \mathfrak{S}_r$.

Since $\prod^c$ is already transitive and contracted, 1.1 and 6.7 imply that $\prod^t_{(R,\geqq)} c(\mathfrak{S}_r) \succ \prod^c_{(R,\geqq)} \mathfrak{S}_r$ and this in turn implies that $\prod^c_{(R,\geqq)} c(\mathfrak{S}_r) \succ \prod^c_{(R,\geqq)} \mathfrak{S}_r$, so $\prod^t_{(R,\geqq)} c(\mathfrak{S}_r) \succ \prod^c_{(R,\geqq)} c(\mathfrak{S}_r) \succ \prod^c_{(R,\geqq)} \mathfrak{S}_r$.

None of these homomorphisms need be isomorphisms; note that if all $\mathfrak{S}_r$ are numbers these conclusions are tautologies.

6.9 THEOREM. *Let* $(T, \geqq)=\sum_{(R,\geqq)} \mathfrak{S}_r$. *Then* $\prod^c_{(R,\geqq)} \prod_{(S(r),\geqq)} \mathfrak{P}_{rs} \sim \prod^c_{(T,\geqq)} \mathfrak{P}_{rs}$. *If R is finite or if* $\mathfrak{R}$ *and all* $\mathfrak{S}_r$ *are transitive and* $\mathfrak{P}_{rs}$ *are k-transitive,*

$$\prod^c_{(R,\geqq)} \prod^t_{(S(r),\geqq)} \mathfrak{P}_{rs} \sim \prod^c_{(T,\geqq)} \mathfrak{P}_{rs}$$

so

$$\prod^{c}_{(R,\geqq)} \prod^{c}_{(S(r),\geqq)} \mathfrak{P}_{rs} > \prod^{c}_{(T,\geqq)} \mathfrak{P}_{rs}.$$

The first isomorphism follows from 6.1 and 1.1; the second follows in the same way from 6.5, and the homomorphism from this and 6.8.

The relations between products over $\mathfrak{R}$ and over subsystems of $\mathfrak{R}$ are not very satisfactory.

6.10. If R_1 is a subset of $\mathfrak{R}$, $(\prod_{(R(1),\geqq)} \mathfrak{S}_r)\cdot(\prod_{(R-R(1),\geqq)} \mathfrak{S}_r) > \prod_{(R,\geqq)} \mathfrak{S}_r$.

This follows from 5.3.

6.11. If R_1 is a star in $\mathfrak{R}$, $\prod_{(R,\geqq)} \mathfrak{S}_r > \prod_{(R(1),\geqq)} \mathfrak{S}_r$ and $\prod_{(R,\geqq)} \mathfrak{S}_r > (\prod_{(R(1),\leqq)} \mathfrak{S}_r) \circ (\prod_{(R-R(1),\geqq)} \mathfrak{S}_r)$.

The function Φ such that $\Phi f(r) = f(r)$ in R_1 is a homomorphism of $\prod_{(R,\geqq)} \mathfrak{S}_r$ onto $\prod_{(R(1),\geqq)} \mathfrak{S}_r$ by 5.1. If ϕ is defined by $\phi f(r) = f(r)$ in $R - R_1$, then the map $f \rightleftarrows (\Phi f, \phi f)$ is a one-to-one correspondence between $\prod_{(R,\geqq)} \mathfrak{S}_r$ and $(\prod_{(R(1),\geqq)} \mathfrak{S}_r) \circ (\prod_{(R-R(1),\geqq)} \mathfrak{S}_r)$. By the first homomorphism stated, $\Phi f \geqq \Phi f'$ if $f \geqq f'$; clearly if $f \geqq f'$ and $\Phi f = \Phi f'$, then $\phi f \geqq \phi f'$, so this mapping is a homomorphism.

Note that, by 6.6, if $(R, \geqq) = (R_1, \geqq) \oplus (R - R_1, \geqq)$ this last homomorphism is an isomorphism.

A number of special cases of these isomorphisms and homomorphisms are very familiar. If we take $\mathfrak{R} = (N_2, w)$, $\mathfrak{S}_1 = (N_1, w)$, $\mathfrak{S}_2 = (N_2, w)$, $\mathfrak{S}_1' = (N_2, w)$, and $\mathfrak{S}_2' = (N_1, w)$, then $\sum_{(R,\geqq)} \mathfrak{S}_r \sim \sum_{(R,\geqq)} \mathfrak{S}_r' \sim (N_3, w)$. This fact, 6.6 and a renumbering of the systems $\mathfrak{P}_{rs}$ give the associative law of ordinal multiplication.

6.12. $\mathfrak{P}_1 \circ (\mathfrak{P}_2 \circ \mathfrak{P}_3) \sim (\mathfrak{P}_1 \circ \mathfrak{P}_2) \circ \mathfrak{P}_3$.

In a similar way the associative law of cardinal multiplication is a special case of 6.6.

Recall that ${}^{(R,\geqq)}\mathfrak{S}$ is the product $\prod_{(R,\geqq)} \mathfrak{S}_r$ in which all $\mathfrak{S}_r = \mathfrak{S}$; we shall define $\mathfrak{S}^{(R,\geqq)}$ to be $\mathrm{tr}({}^{(R,\geqq)}\mathfrak{S})$. Taking $\mathfrak{R} = (N_2, w)$ and all $\mathfrak{P}_{rs} = \mathfrak{P}$, we have from 6.6,

6.13. ${}^{(S(2),\geqq)}\mathfrak{P} \circ {}^{(S(1),\geqq)}\mathfrak{P} \sim {}^{(S(1),\geqq)\oplus(S(2),\geqq)}\mathfrak{P}$ and ${}^{(S(2),\geqq)}\mathfrak{P} \cdot {}^{(S(1),\geqq)}\mathfrak{P} \sim {}^{(S(1),\geqq)+(S(2),\geqq)}\mathfrak{P}$.

These are given for numbers in [1] but the factors on the left-hand side of the first isomorphism are reversed.

From 6.5 we derive in the same way

6.14. $\mathfrak{P}^{(S(2),\geqq)} \circ \mathfrak{P}^{(S(1),\geqq)} \sim \mathfrak{P}^{(S(1),\geqq)\oplus(S(2),\geqq)}$ and $\mathfrak{P}^{(S(2),\geqq)} \cdot \mathfrak{P}^{(S(1),\geqq)} \sim \mathfrak{P}^{(S(2),\geqq)+(S(1),\geqq)}$.

If all $\mathfrak{S}_r = \mathfrak{S}$ and all $\mathfrak{P}_{rs} = \mathfrak{P}$, 6.1 and 6.4 give

6.15. ${}^{(R,\geqq)}({}^{(S,\geqq)}\mathfrak{P}) > {}^{(R,\geqq)\circ(S,\geqq)}\mathfrak{P}$ and $({}^{(S,\geqq)}\mathfrak{P})^{(R,\geqq)} \sim \mathfrak{P}^{(R,\geqq)\circ(S,\geqq)}$.

If $\mathfrak{R}$ is a number for which $R = R'$, 6.6 gives

6.16. ${}^{(R,\geqq)}({}^{(S,\geqq)}\mathfrak{P}) \sim {}^{(R,\geqq)\circ(S,\geqq)}\mathfrak{P}$.

From 6.5 we also see

6.17. If $\mathfrak{R}$ is finite or if $\mathfrak{R}$ and $\mathfrak{S}$ are transitive and $\mathfrak{P}$ is k-transitive, then $(\mathfrak{P}^{(S,\geqq)})^{(R,\geqq)} \sim \mathfrak{P}^{(R,\geqq)\circ(S,\geqq)}$ and $c([c(\mathfrak{P}^{(S,\geqq)})]^{(R,\geqq)}) \succ (\mathfrak{P}^{(R,\geqq)\circ(S,\geqq)})$.

7. **Ordinal exponentiation with numbers.** Since many of the results of [1] are proved under the inaccurate assumption that ordinal powers of numbers are transitive, this section discusses the various statements made there. These include a list of properties of ordinal exponentiation [1, §7], a set of conditions necessary and sufficient that ${}^{(R,\geqq)}\mathfrak{S}$ be a lattice [1, Theorem 12] and a list of closure properties [1, §13]. We shall test most of these statements giving correct hypotheses and conclusions for three kinds of ordinal exponentiation: ${}^{(R,\geqq)}\mathfrak{S}$ which we have defined to mean the ordered product $\prod_{(R,\geqq)} \mathfrak{S}_r$ when all $\mathfrak{S}_r = \mathfrak{S}$; $\mathfrak{S}^{(R,\geqq)} = \operatorname{tr}({}^{(R,\geqq)}\mathfrak{S})$ and $\mathfrak{S}^{[(R,\geqq)]}$ which will mean $c(\mathfrak{S}^{(R,\geqq)}) = \operatorname{ctr}({}^{(R,\geqq)}\mathfrak{S})$.

§7 of [1] gives a list of properties of ordinal exponentiation with numbers; we shall give properties of ${}^{(R,\geqq)}\mathfrak{S}$ in the notation of this paper but with the numbering scheme used in [1].

(45) $\mathfrak{S}_1 \sim \mathfrak{S}_2$ implies ${}^{(S(1),\geqq)}\mathfrak{R} \sim {}^{(S(2),\geqq)}\mathfrak{R}$ and ${}^{(R,\geqq)}\mathfrak{S}_1 \sim {}^{(R,\geqq)}\mathfrak{S}_2$.

(46) $\mathfrak{S}_1 < \mathfrak{S}_2$ implies ${}^{(R,\geqq)}\mathfrak{S}_1 < {}^{(R,\geqq)}\mathfrak{S}_2$.

(47) ${}^{(R(1),\geqq)+(R(2),\geqq)}\mathfrak{S} \sim ({}^{(R(1),\geqq)}\mathfrak{S}) \cdot ({}^{(R(2),\geqq)}\mathfrak{S})$.

(48) ${}^{(R(1),\geqq)\oplus(R(2),\geqq)}\mathfrak{S} \sim ({}^{(R(2),\geqq)}\mathfrak{S}) \circ ({}^{(R(1),\geqq)}\mathfrak{S})$.

(49) ${}^{(R,\geqq)}({}^{(S,\geqq)}\mathfrak{P}) \succ {}^{(R,\geqq)\circ(S,\geqq)}\mathfrak{P}$ and isomorphism holds if $R = R'$.

(50) ${}^{(N(1),=)}\mathfrak{S} \sim \mathfrak{S}$ and ${}^{(S,\geqq)}(N_1, =) \sim (N_1, =)$.

To these we add

(46′) $\mathfrak{S}_1 < \mathfrak{S}_2$ implies that ${}^{(S(1),\geqq)}\mathfrak{R} < {}^{(S(2),\geqq)}\mathfrak{R}$.

(48) is misstated in [1]; the interchange of order in the exponents is an accidental result of the particular phrasing (used both here and in [1]) of the definitions of ordinal multiplication and exponentiation. [1] claims isomorphism in (49) only in case both $\mathfrak{R}$ and $\mathfrak{S}$ satisfy the ascending chain condition. There is a typographical error in the last half of (50) in [1].

7.1 THEOREM. *These properties hold for* ${}^{(R,\geqq)}\mathfrak{S}$. *The analogues of* (46) *and* (46′) *fail for* $\mathfrak{S}^{(R,\geqq)}$, (49) *becomes an isomorphism for all* $\mathfrak{R}$, $\mathfrak{S}$ *and* $\mathfrak{P}$; *the others hold. The analogues of* (46), (46′) *and* (48) *fail for* $\mathfrak{S}^{[(R,\geqq)]}$; *the homomorphism holds in* (49); (45), (47) *and* (50) *hold.*

For ${}^{(R,\geqq)}\mathfrak{S}$, (45) and (50) are obvious and (46) and (46′) are special cases of 3.1; (47) and (48) come from 6.13; (49) comes from the first half of 6.15 and from 6.16.

For $\mathfrak{S}^{(R,\geqq)}$, (45) and (50) are obvious again (since $\mathfrak{S}$ is transitive). That (46) fails is clear; let $\mathfrak{S} = (N, w^*) \oplus (N, w)$; then $\mathfrak{S}^{(N,w)}$ has the universal ordering while $(N, w)^{(N,w)}$ does not. That (46′) fails follows from the systems $\mathfrak{S}^{(N,w)}$ and $\mathfrak{S}^{(N(1),=)}$ with the same $\mathfrak{S}$ as before. (47) and (48) follow from 6.14; (49) follows from 6.17.

For $\mathfrak{S}^{[(R,\geqq)]}$, (45) and (50) are still obvious (since $\mathfrak{S}$ is a number) and the same examples show that (46) and (46′) fail to hold. To prove that (48) fails to hold it suffices to show that $(N_2, w)^{[(N,w)]} \circ (N_2, w)^{[(N(1),=)]}$ is not isomorphic to $(N_2, w)^{[(N(1),=)\oplus(N,w)]}$. To show this note that the left-hand side is an ordinal product of numbers; the system $(N_2, w)^{(N,w)}$ was discussed in 3.5b from which we can see that $(N_2, w)^{[(N,w)]} \sim (N, w) \oplus (N_1, =) \oplus (N, w^*)$. From the one-sided distributive law at the end of §2 we see that $(N_2, w)^{[(N,w)]} \circ (N_2, w)^{[(N(1),=)]} \sim \{(N, w) \circ \mathfrak{S}\} \oplus \mathfrak{S} + \{(N, w^*) \circ \mathfrak{S}\}$ where $\mathfrak{S} = (N_2, w)^{[(N(1),=)]} \sim (N_2, w)$; since $(N, w) \circ (N_2, w) \sim (N, w)$ and $(N, w^*) \circ (N_2, w) \sim (N, w^*)$, we see that the first system is isomorphic to $(N, w) + \mathfrak{S} + (N, w^*)$. Since $(N_1, =) \oplus (N, w) \sim (N, w)$, the second system is isomorphic to $(N, w) \oplus (N_1, =) \oplus (N, w^*)$; this system is not isomorphic to the first.

Some remarks may be made; many other systems could have been used in place of $(N_1, =)$ in this example and the proof that the two sides are not isomorphic could then be carried through in much the same way. Also it is rather simple to show that if $R_2 = R_2'$, then $\mathfrak{S}^{[(R(1),\geqq)\oplus(R(2),\geqq)]} \sim (\mathfrak{S}^{[(R(2),\geqq)]}) \circ (\mathfrak{S}^{[(R(1),\geqq)]})$; that is, (48) holds with this extra hypothesis. (47) for $\mathfrak{S}^{[(R,\geqq)]}$ follows from the same result for $\mathfrak{S}^{(R,\geqq)}$ and from the following lemma.

7.2 Lemma. *If* $\mathfrak{R}$ *and* $\mathfrak{S}$ *are transitive,* $c(\mathfrak{R} \cdot \mathfrak{S}) \sim c(\mathfrak{R}) \cdot c(\mathfrak{S})$.

The mapping $c(r, s) \rightleftarrows (c(r), c(s))$ is an isomorphism.

Let us turn next to conditions under which the ordinal power is a lattice; recall that a lattice is a number in which every pair of elements s_1 and s_2 have a least upper bound $s_1 \vee s_2$ and a greatest lower bound $s_1 \wedge s_2$. The conditions of [1, Theorem 12] are not sufficient without the extra hypothesis $R = R'$ but they are necessary.

In the terminology of [1] a *chain* is a simply ordered number; that is, a number in which every two elements are comparable. A *semi-root* is a number in which the set of successors of every element is a chain. The conditions (1)–(3) below are those of [1, Theorem 12]; that they are not sufficient is shown by the example ${}^{(N,w)}(N_2, w)$ which satisfies both (1) and (3) but not $N = N'$. The sufficiency proofs of [1] can be carried through with the extra hypothesis $R = R'$ but are meaningless without it since "critical values" need not exist.

7.3 Theorem. ${}^{(R,\geqq)}\mathfrak{S}$ *is a lattice if and only if* (a) $\mathfrak{S} = (N_1, =)$ *or* (b) $R = R'$ *and one of the following conditions holds*: (1) $\mathfrak{S}$ *is a bounded lattice*; (2) $\mathfrak{S}$ *is a lattice and* $\mathfrak{R}$ *is a cardinal*; (3) $\mathfrak{S}$ *is a chain and* $\mathfrak{R}$ *a semi-root.*

If ${}^{(R,\geqq)}\mathfrak{S}$ is a lattice, it is a number; by 4.17, $\mathfrak{S}$ is a cardinal number or $R = R'$. If $\mathfrak{S}$ is a cardinal, it can not have two distinct elements or there would be two elements of ${}^{(R,\geqq)}\mathfrak{S}$ with no upper bound so $\mathfrak{S} \sim (N_1, =)$ if $\mathfrak{S}$ is a cardinal. If $R = R'$, suppose that $\mathfrak{R}$ is a cardinal and that $f = f_1 \vee f_2$; for each r in $\mathfrak{R}$ it is clear that $f(r)$ is an upper bound of $f_1(r)$ and $f_2(r)$; it is a least upper bound because any other upper bound can be used to define a function f' which is also an upper bound of the f_i. Therefore least upper bounds exist in $\mathfrak{S}$ and by a dual argument we see that $\mathfrak{S}$ is a lattice. If $R=R'$ and $\mathfrak{S}$ is not a chain, there exist two incomparable elements s_1 and s_2 in $\mathfrak{S}$. Let $r_0 \in R^{(1)}$ and define $f_1(r)=f_2(r)$ if $r \neq r_0$, $f_i(r_0)=s_i$; if $f=f_1 \vee f_2$, $f(r_0)$ is clearly an upper bound of the s_i. As before we see that it must be a least upper bound so $s_1 \vee s_2$ exists if the s_i are incomparable; however if one follows the other, then the larger one is a least upper bound of the two so $s_1 \vee s_2$ always exists. A dual argument shows that $\mathfrak{S}$ must be a lattice. Moreover if r is a terminal element of the set of predecessors of r_1 and if $f_i(r_1)$ are incomparable, $f(r)$ must be smaller than any other element of $\mathfrak{S}$ if $f=f_1 \vee f_2$; hence if $\mathfrak{S}$ is not a chain and $\mathfrak{R}$ is not a cardinal, $\mathfrak{S}$ has a smallest element 0; dually there is a largest element 1 so $\mathfrak{S}$ is a bounded lattice if it is not a chain and $\mathfrak{R}$ is not a cardinal number. If $\mathfrak{S}$ is a chain but not a bounded lattice, $\mathfrak{R}$ must be a semi-root, for let r_1 and r_2 be incomparable elements of $\mathfrak{R}$ and let $s_1 > s_2$; define $f_1(r_1)=f_2(r_2)=s_2$ and $f_1(r_2)=f_2(r_1)=s_1$, $f_1(r)=f_2(r)$ for all other r in R. If r_0 is a terminal element in $\{r \mid r<r_1 \text{ and } r<r_2\}$ and $f=f_1 \vee f_2$, then $f(r)$ would have to be the smallest element of $\mathfrak{S}$. Dually $\mathfrak{S}$ would have a largest element contrary to the assumption that $\mathfrak{S}$ is not a bounded lattice. Hence the set of common predecessors of r_1 and r_2 has no terminal elements; since $R=R'$, the set is empty, so any two successors of a given element must be comparable.

If (a) holds, ${}^{(R,\geqq)}\mathfrak{S} \sim (N_1, =)$. if (a) is false and (b) holds, take f_i in ${}^{(R,\geqq)}\mathfrak{S}$; then $\mathfrak{S}$ is a lattice in all cases in (b) so $g(r)=f_1(r) \vee f_2(r)$ is defined for each r in R. Let $E_i=\{r \mid \text{there exists } r' \geqq r \text{ for which } g(r')>f_i(r')\}$. Then in $R-E_1-E_2$ let $f(r)=f_1(r)=f_2(r)$; if $r \in (E_1^{(1)}-E_2) \cup (E_2^{(1)}-E_1) \cup (E_1^{(1)} \cap E_2^{(1)})$, let $f(r)=g(r)$. If r is a cardinal, this defines f everywhere and it is clear that f is a least upper bound of the f_i. If $\mathfrak{S}$ is a bounded lattice, define $f(r)=0$ for every r where it is not yet given; then $f>f_i$ for $f=f_1=f_2$ in $R-E_1-E_2$, $f>f_1$ in $E_1^{(1)}-E_2$, $f>f_2$ in $E_2^{(1)}-E_1$ and $f>$both f_i in $E_1^{(1)} \cap E_2^{(1)}$. Since $(E_1 \cup E_2)^{(1)} = (E_1^{(1)}-E_2) \cup (E_2^{(1)}-E_1) \cup (E_1^{(1)} \cap E_2^{(1)})$, every point where $f(r)$ was defined to be 0 is below a point r' where $f(r')>f_i(r')$, $i=1, 2$; hence $f \geqq f_i$. If $f' \geqq f_i$ it can now be shown that $f' \geqq f$ so $f=f_1 \vee f_2$; dually we can provide $f_1 \wedge f_2$ so ${}^{(R,\geqq)}\mathfrak{S}$ is a lattice if $\mathfrak{S}$ is a bounded lattice. If $\mathfrak{S}$ is a chain and $\mathfrak{R}$ a semi-root each r in $E_1 \cup E_2$ has a *unique* successor r' in $(E_1 \cup E_2)^{(1)}$. $E_1^{(1)} \cap E_2^{(1)}$ is empty, for $f_1(r')>f_2(r')>f_1(r')$ is impossible in a chain. If $r' \in E_1^{(1)}-E_2$ and $r \leqq r'$, let $f(r)=f_2(r)$; if $r' \in E_2^{(1)}-E_1$ and $r \leqq r'$, let $f(r)=f_1(r)$. This defines f everywhere and it is easily seen to be a least upper bound for the f_i. A dual argument gives $f_1 \wedge f_2$.

7.4 COROLLARY. *$\mathfrak{S}^{(R,\geqq)}$ is a lattice if and only if ${}^{(R,\geqq)}\mathfrak{S}$ is a lattice.*

By 4.17, $\mathfrak{S}^{(R,\geqq)}$ is a number if and only if $\mathfrak{S}$ is a cardinal or $R=R'$; in either of these cases $\mathfrak{S}^{(R,\geqq)}={}^{(R,\geqq)}\mathfrak{S}$. Hence $\mathfrak{S}^{(R,\geqq)}$ is a lattice if and only if ${}^{(R,\geqq)}\mathfrak{S}$ is a lattice.

Conditions under which $\mathfrak{S}^{[(R,\geqq)]}$ is a lattice are more difficult to prove; we give a sequence of lemmas containing various necessary or sufficient conditions and then combine them in 7.10 into one set of conditions both necessary and sufficient that $\mathfrak{S}^{[(R,\geqq)]}$ be a lattice.

7.5 LEMMA. *If $\mathfrak{S}$ has no terminal or initial elements and if $\mathfrak{R}$ has no terminal elements, then $\mathfrak{S}^{[(R,\geqq)]}$ is a lattice isomorphic to $(N_1, =)$.*

If f_1 and f_2 are any two elements of ${}^{(R,\geqq)}\mathfrak{S}$ it is easy to construct f so that $f_1>f>f_2$, for there must exist two disjoint cofinal subsets E_1 and E_2 in $\mathfrak{R}$ and we need only take $f(r)<f_1(r)$ if $r\in E_1$ and $f(r)>f_2(r)$ if $r\in E_2$. Hence $f_1>{}'f_2$ and by a dual argument $f_2>{}'f_1$ so $c(f_1)=c(f_2)$ in $\mathfrak{S}^{[(R,\geqq)]}$; that is, $\mathfrak{S}^{[(R,\geqq)]}$ contains only one element.

7.6 LEMMA. *If $\mathfrak{S}$ has neither terminal nor initial elements and $\mathfrak{R}$ has a terminal element (that is, if R' is not empty), then $\mathfrak{S}^{[(R,\geqq)]}\sim{}^{(R',\geqq)}\mathfrak{S}\sim\mathfrak{S}^{[(R',\geqq)]}$.*

As in 6.11 define Φ from ${}^{(R,\geqq)}\mathfrak{S}$ onto ${}^{(R',\geqq)}\mathfrak{S}$ by $\Phi f=f'$ if $f'(r)=f(r)$ for r in R', f' not defined elsewhere; 6.11 asserts that Φ is a homomorphism. 4.17 shows that ${}^{(R',\geqq)}\mathfrak{S}$ is a number so ${}^{(R',\geqq)}\mathfrak{S}\sim\mathfrak{S}^{[(R',\geqq)]}$. If $\Phi f_1\geqq\Phi f_2$, as in 7.5 there exists f such that $f_1>f>f_2$ over $R-R'$; defining $f=f_1$ in R' we have $f_1\geqq f\geqq f_2$; that is, $f_1\geqq{}'f_2$ if $\Phi f_1\geqq\Phi f_2$. Hence $c(f_1)=c(f_2)$ if $\Phi f_1=\Phi f_2$ and $c(f_1)\geqq c(f_2)$ if $\Phi f_1\geqq\Phi f_2$. Let $\Psi(c(f))=\Phi f$; by 1.1, Ψ is a homomorphism of $\mathfrak{S}^{[(R,\geqq)]}$ onto ${}^{(R',\geqq)}\mathfrak{S}$; we have just shown that Ψ^{-1} exists and is a homomorphism so the lemma is proved.

7.7 COROLLARY. *If* (a) *$\mathfrak{S}$ is a lattice with neither initial nor terminal elements and $(R', \geqq)$ is a cardinal or* (b) *if $\mathfrak{S}$ is a chain with neither initial nor terminal elements and $(R', \geqq)$ is a semi-root, then $\mathfrak{S}^{[(R,\geqq)]}$ is a lattice.*

This follows from 7.6 and 7.3.

7.8 LEMMA. *If $\mathfrak{S}^{[(R,\geqq)]}$ is a lattice and if $\mathfrak{R}$ has a terminal element or $\mathfrak{S}$ has a terminal or initial element, then $\mathfrak{S}$ is a lattice; moreover, if $\mathfrak{S}$ has either a terminal or initial element, then* (a) *$\mathfrak{S}$ is a bounded lattice or* (b) *$\mathfrak{S}$ is a chain and $\{r\mid r<r_1$ and $r<r_2\}$ has no terminal elements when r_1 and r_2 are incomparable, or* (c) *$\{r\mid r<r_1$ and $r<r_2\}$ has no terminal elements for any choice of r_1, r_2 in R.*

If $r\in R^{(1)}$ and $c(f)=c(f_1)\vee c(f_2)$, it is clear that $f(r)$ must be a least upper bound of $f_1(r)$ and $f_2(r)$; that is, $s_1\vee s_2$ must exist for every pair of points in $\mathfrak{S}$. Dually $s_1\wedge s_2$ exists so $\mathfrak{S}$ is a lattice.

By duality we may assume without loss of generality that $\mathfrak{S}$ contains an initial element s_0. If s_1 and s_2 are comparable in $\mathfrak{S}$, then $s_1\vee s_2$ and $s_1\wedge s_2$ obviously exist; if the s_i are not comparable, take r_0 in $\mathfrak{R}$ and for $i=1, 2$ define $f_i(r_0)=s_i$, $f_i(r)=s_0$ if $r\neq r_0$. Then if $c(f)=c(f_1)\wedge c(f_2)$, $f\leqq f_i$ so $f(r)=s_0$ if r does not precede or equal r_0. Hence $f(r_0)<f_i(r_0)$ and the points s_i have a common lower bound. If $f_s(r_0)=s$ and $f_s(r)=s_0$ if $r\neq r_0$, then $f_s\leqq f_i$ whenever s is a lower

bound of the s_i; hence $f(r_0)>s$ if s is a lower bound of the s_i so $f(r_0)$ is a greatest lower bound of the s_i and $s_1\wedge s_2$ exists in $\mathfrak{S}$.

We show next that if there exist s_1 and s_2 with no common upper bound and if $r_0\in\mathfrak{R}$, there exists an element $r_1>r_0$ such that $r\geqq r_1$ if $r>r_0$. Define f_i as before and let $c(g)=c(f_1)\vee c(f_2)$; then $g\geqq f_i$ and $g(r_0)$ can not follow both $f_1(r_0)$ and $f_2(r_0)$ so there exists $r_1>r_0$ such that $g(r_1)>s_0$. If $r>r_0$, let $f_r(r)>s_0$ and $f_r(r')=s_0$ if $r'\neq r$; then f_r is an upper bound for the f_i so $f_r>'g$; hence if r' does not precede or equal $r>r_0$, then $g(r')=s_0$. Since $g(r_1)>s_0$, it follows that $r_1\leqq r$ if $r>r_0$; that is, there is a first proper successor r_1 of r.

Now for $i=1, 2$ define $f_i'(r_1)=s_i$ and $f_i'(r)=s_0$ if $r\neq r_1$; let $c(f')=c(f_1')\wedge c(f_2')$; then by the second paragraph of this proof $f'(r)=s_0$ if r does not precede or equal r_1, while $f'(r_1)=s_1\wedge s_2$ so $f'(r_1)<s_i$, $i=1, 2$. r_0 is a terminal element in the set of predecessors of r_1; hence it is easily seen that $f(r_0)$ must follow every other element of $\mathfrak{S}$, for if $f_s'(r_0)=s$ and $f_s'(r_1)=s_1\wedge s_2$ and $f_s'(r)=s_0$ for all other r, f_s' is a lower bound of the f_i' so $f'\geqq' f_s'$ for every s in $\mathfrak{S}$. Since $f_s'(r)=f'(r)$ if $r>r_0$ and since $f'(r)$ is an initial element of $\mathfrak{S}$ if $r>r_1$, we see that $f'(r_0)\geqq f_s'(r_0)=s$ if $s\in\mathfrak{S}$. This proves that the assumption that s_1 and s_2 have no upper bound eventually enables us to construct an upper bound; this contradiction proves that every pair of elements of $\mathfrak{S}$ has an upper bound.

Knowing this we return to the functions f_i defined in the second paragraph of this proof and let $c(g)=c(f_1)\vee c(f_2)$; then define $f_s(r_0)=s$ and $f_s(r)=s_0$ if $r\neq r_0$; $f_s\geqq f_i$ if $s>s_i$ so, as before, $g(s)\leqq s$ if s is an upper bound of the s_i; hence $g(r_0)=s_1\vee s_2$ and we have proved that $\mathfrak{S}$ is a lattice. Incidentally we have also shown that if $\mathfrak{S}$ is not a chain $\{r \mid r<r_0\}$ can not have a terminal element unless $\mathfrak{S}$ has a largest element. Since s_0 is an initial element of the lattice $\mathfrak{S}$, s_0 is a smallest element in $\mathfrak{S}$, so in this case $\mathfrak{S}$ is a bounded lattice.

Similarly, if r_1 and r_2 are incomparable and $\mathfrak{S}$ is not isomorphic to $(N_1, =)$ but $\{r \mid r<r_1 \text{ and } r<r_2\}$ has a terminal element, then there exist $s_2>s_1$ in $\mathfrak{S}$; define $f_1(r_1)=f_2(r_2)=s_2$, $f_1(r_2)=f_2(r_1)=s_1$ and $f_i(r)=s_0$ elsewhere. Then $c(f)=c(f_1)\vee c(f_2)$ implies that $f(r_i)>f_{3-i}(r_i)$ so $f(r_0)$ must be a smallest element of $\mathfrak{S}$; the dual proof with $f_1\wedge f_2$ shows that there must also be a largest element in $\mathfrak{S}$; that is, that $\mathfrak{S}$ is a bounded lattice.

Note that in a semi-root two incomparable elements r_1 and r_2 have no common predecessors so $\{r \mid r<r_1 \text{ and } r<r_2\}$ inevitably has no terminal elements; if $\mathfrak{R}=(N, w)\oplus(N_2, =)$, $\mathfrak{R}$ is the simplest system with this last property which is not a semi-root.

7.9 LEMMA. *If $\mathfrak{S}$ is a bounded lattice, then $\mathfrak{S}^{[(R,\geqq)]}$ is a bounded lattice.*

To prove this we shall work in $\mathfrak{S}^{(R,\geqq)}$ instead and shall there construct for a given pair of functions f_1 and f_2 an f such that $f\geqq f_i$ while if $f'\geqq' f_i$, $i=1$ and 2, then $f'\geqq' f$ also; then $c(f)$ will be a least upper bound of $c(f_i)$. Let $g(r)=f_1(r)\vee f_2(r)$ and let $E_i=\{r \mid g(r')=f_i(r') \text{ for every } r'\geqq r\}$; then $E_i=E_i^U$.

Let $E=R-(E_1-E_2)^D-(E_2-E_1)^D-(E_1\cap E_2)$ and suppose $r\in E^{(1)}$; then if r has any successor r' in $\mathfrak{R}$, $r'\in E_1\cap E_2$ so $g(r')=f_1(r')=f_2(r')$; define $f(r)=g(r)$ in $E^{(1)}$ and let $f(r)=0$ in $E^{(1)D}-E^{(1)}$. Let $f(r)=f_i(r)$ if $r\in E_i$ or if $r\in(E_i-E_{3-i})^D-(E_{3-i}-E_i)^D$ and let $f(r)=0$ if $r\in(E_1-E_2)^D\cap(E_2-E_1)^D$. Let $A=E-E^{(1)D}$; then A has no terminal elements and there exist cofinal subsets A_i of A such that $g(r)>f_i(r)$ for each r in A_i. Hence $f_{3-i}(r)>0$ and $f_i(r)<1$ for every r in A_i; since each A_i has two disjoint cofinal subsets it follows that f can be defined in A so that $f_1>f>f_2>f>f_1$ over A. Hence $f>f_i$ over R.

If $f'\geqq' f_i$ for $i=1$ and 2, since $f=f_i$ over E_i, $f'>'f$ over $E_1\cup E_2$; a similar argument shows that $f'\geqq' f$ over $(E_i-E_{3-i})^D$. Clearly $f'\geqq f$ over $E^{(1)}$ and hence over $E^{(1)D}$; in A, $f'\geqq' f_1\geqq' f$ so $f'\geqq' f$ everywhere. Passing to the contraction this shows that $c(f)=c(f_1)\vee c(f_2)$; a dual construction would produce $c(f_1)\wedge c(f_2)$.

7.10 THEOREM. $\mathfrak{S}^{[(R,\geqq)]}$ *is a lattice if and only if one of the following cases holds*: (1) $\mathfrak{S}$ *is a bounded lattice.* (2) $\mathfrak{S}$ *has neither terminal nor initial elements and either* (a) $\mathfrak{R}$ *has no terminal elements or* (b) R' *is not empty and* ${}^{(R',\geqq)}\mathfrak{S}$ *is a lattice* (*see* 7.3 *for these conditions*). (3) $\mathfrak{S}$ *is a lattice with a terminal or initial element but not both and either* (c) $\mathfrak{S}$ *is a chain and the set* $\{r \mid r<r_1 \text{ and } r<r_2\}$ *has no terminal elements if* r_1 *and* r_2 *are incomparable, or* (d) *the set defined in* (c) *has no terminal elements no matter how* r_1 *and* r_2 *are related in* $\mathfrak{R}$.

Necessity of these conditions follows easily from the preceding lemmas; assume $\mathfrak{S}^{[(R,\geqq)]}$ a lattice. If $\mathfrak{R}$ has a terminal element $\mathfrak{S}$ is a lattice by 7.8; this proves the necessity of (2) if $\mathfrak{S}$ has neither terminal nor initial elements. If $\mathfrak{S}$ has one or the other, 7.8 again gives (1) or (3).

7.9 gives the sufficiency of (1) and 7.5 and 7.6 prove sufficiency of (2). In case (3) by duality we may consider only the case in which $\mathfrak{S}$ has an initial element. If $f_i\in\mathfrak{S}^{(R,\geqq)}$, f can be constructed just as in 7.9 since $\mathfrak{S}$ has a zero element; the proof that $c(f)=c(f_1)\vee c(f_2)$ can be repeated in the same way. The extra hypotheses (c) or (d) are needed to construct a greatest lower bound for the f_i. For this we set $h(r)=f_1(r)\wedge f_2(r)$ and let $F_i=\{r \mid f_i(r')=h(r') \text{ for every } r'\geqq r\}$ and in F_i define $f(r)=h(r)=f_i(r)$. As in the previous construction let $f=R-(F_1-F_2)^D-(F_2-F_1)^D-(F_1\cap F_2)$ and define $f(r)=h(r)=f_1(r)\wedge f_2(r)$ if $r\in F^{(1)}$; define $f(r)=f_i(r)$ if $r\in(F_i-F_{3-i})^D-(F_{3-i}-F_i)^D$. In $F-F^{(1)D}$ we can intertwine f, f_1 and f_2 as we did in the set A of the preceding proof. This defines f everywhere except in $F^{(1)D}-F^{(1)}$ and $(F_1-F_2)^D\cap(F_2-F_1)^D$; if $\mathfrak{S}$ is a chain, $F^{(1)}$ is empty; if $\mathfrak{S}$ is not a chain $F^{(1)D}-F^{(1)}$ has no terminal elements, for r a terminal element of this set would imply that there exists r' in $F^{(1)}$ such that r is a terminal element of the set of predecessors of r'; this is prevented by (d) with $r_1=r_2=r'$. In either case (c) or (d) $(F_1-F_2)^D\cap(F_2-F_1)^D$ has no terminal elements for if r_0 is in this set there exists r_1 in F_1-F_2 and r_2 in F_2-F_1 such that $r_0\in\{r \mid r<r_1 \text{ and } r<r_2\}$; since this set has no terminal elements r_0 can not be a terminal element of the origi-

nal set. In $(F^{(1)D}-F^{(1)})\cup[(F_1-F_2)^D\cap(F_2-F_1)^D]$ define $f(r)$ in any way so long as it is not zero or one; then for every f' such that $f'\leqq' f_i$ we have $f'\leqq' f$ over this set; if $f'\leqq' f_i$ it is then clear that $f'\leqq' f$ over R, so $c(f)=c(f_1)\wedge c(f_2)$. This proves that $\mathfrak{S}^{[(R,>)]}$ is a lattice.

All that remains is to check the list of closure properties given in [1, p. 294]. Recalling that $\mathfrak{S}^{(R,\geqq)}$ is a number if and only if $\mathfrak{S}^{(R,\geqq)}={}^{(R,\geqq)}\mathfrak{S}$ we can deal with the first two cases together; the result are collected in the following table.

Property of ${}^{(R,\geqq)}\mathfrak{S}$ $(=\mathfrak{S}^{(R,\geqq)})$	Necessary and sufficient condition on $\mathfrak{R}$ and $\mathfrak{S}$
(1) Cardinal	$\mathfrak{S}$ a cardinal.
(2) Chain	$\mathfrak{S}\sim(N_1,\ =)$ or $\mathfrak{R}$ and $\mathfrak{S}$ chains and $R=R'$.
(3) Ordinal	$\mathfrak{S}\sim(N_1,\ =)$ or $\mathfrak{S}$ an ordinal and $\mathfrak{R}$ a finite ordinal.
(4) Bounded number	$\mathfrak{S}\sim(N_1,\ =)$ or $\mathfrak{S}$ bounded and $R=R'$.
(5) Finite number	$\mathfrak{S}\sim(N_1,\ =)$ or $\mathfrak{S}$ and $\mathfrak{R}$ finite.
(6) Lattice	See 7.3.
(7) Complete lattice	$\mathfrak{S}\sim(N_1,\ =)$ or $\mathfrak{S}$ a complete lattice and $R=R'$.

The proofs can safely be left to the reader, but we wish to use the sufficiency proof of (7) again so we give it in the next paragraph. The necessity proofs follow almost immediately from 3.9 and 3.10. [1] also mentions "striated" numbers; the condition for them is probably all right since it includes finiteness of $\mathfrak{R}$ which implies $R=R'$. In [1] no condition is given for ordinals and the condition given for lattices is a much worse approximation to the truth on this subject than is [1, Theorem 12].

If $R=R'$ and $\mathfrak{S}$ is a complete lattice, let f_p, $p\in P$, be any set of elements of ${}^{(R,\geqq)}\mathfrak{S}$; then for r in $R^{(1)}$ define $f(r)=\bigvee_{p\in P}f_p(r)$. Then for any r such that $f(r)$ is defined over the set E_r of all proper successors of r and $f\geqq f_p$ over E_r for all p, define $P_r=\{p\mid f_p(r')=f(r')$ for all $r'>r\}$; let $f(r)=\bigvee_{p\in P(r)}f_p(r)$ if P_r is not empty, let $f(r)=0$ if P_r is empty. Then $f\geqq f_p$ over $(r)^U$; since $R=R'=\bigcup_{\lambda<\lambda(0)}R^{(\lambda)}$, this process defines f on all of R by transfinite induction so that $f\geqq f_p$ over R for every p. If $f'\geqq$ all f_p and r is a point where $f'(r)$ does not follow or equal $f(r)$, P_r can not be empty since $f(r)\neq 0$; then $f'\geqq f_p=f$ over E_r so that either there exists $r'>r$ with $f'(r')>f(r')$ or $f'(r')=f_p(r')$ for all p in P_r and r' in E_r; in the second case $f'(r)\geqq\bigvee_{p\in P(r)}f_p(r)=f(r)$. Hence $f'\geqq f$ over R and $f=\bigvee_{p\in P}f_p$; a dual argument would produce $\bigwedge_{p\in P}f_p$ so ${}^{(R,\geqq)}\mathfrak{S}$ is a complete lattice. The necessity proof uses the usual embedding argument.

The corresponding conditions for $\mathfrak{S}^{[(R,\geqq)]}$ are vaguely reminiscent of these but more complicated; for example, compare 7.3 and 7.10. In the proofs certain elementary facts related to 3.9 and 3.10 are quite useful. Recall that in this section $\mathfrak{R}$ and $\mathfrak{S}$ are assumed to be numbers.

7.11. If $\mathfrak{R}$ has a terminal element, $\mathfrak{S}<\mathfrak{S}^{[(R,\geqq)]}$.

Choose r_0 in $R^{(1)}$ and s_0 in $\mathfrak{S}$; as in 3.1 define $f_s(r_0)=s$, $f_s(r)=s_0$ if $r\neq r_0$. Then the map Φ such that $\Phi(s)=c(f_s)$ is an isomorphism of $\mathfrak{S}$ into $\mathfrak{S}^{[(R,\geqq)]}$.

7.11′. If $\mathfrak{S}$ has a terminal or initial element, $\mathfrak{S}<\mathfrak{S}^{[(R,\geqq)]}$.

Choose r_0 in R and s_0 an initial or terminal element of $\mathfrak{S}$ and define f_s and Φ as in 7.11; then $f_s \geqq {}^t f^{s'}$ if and only if $s\geqq s'$ so Φ is an isomorphism.

7.12. If there exists an initial element s_0 in $\mathfrak{S}$ and a point $s_1>s_0$, then $\mathfrak{R}<\mathfrak{S}^{[(R,\geqq)]}$.

Define $f_r(r)=s_1$, $f_r(r')=s_0$ if $r'\neq r$; then $f_{r(1)}\geqq {}^t f_{r(2)}$ if and only if $r_1\geqq r_2$; defining $\phi(r)=c(f_r)$ we see that ϕ is an isomorphism of $\mathfrak{R}$ into $\mathfrak{S}^{[(R,\geqq)]}$.

7.12′. If $\mathfrak{S}$ has a terminal element s_1 and an element $s_0<s_1$ and if $\mathfrak{R}^*=(R,\leqq)$, then $\mathfrak{R}^*<\mathfrak{S}^{[(R,\geqq)]}$.

We define ϕ in the same way with this choice of the s_i.

By means of these facts we are able to prove the conditions given in the table below; we sketch the proofs since they are generally worse than the corresponding proofs for ${}^{(R,\geqq)}\mathfrak{S}$. In the table below (E) stands for the condition that $\mathfrak{S}$ has neither initial nor terminal elements and $\mathfrak{R}$ has no terminal elements; (I) stands for the condition that $\mathfrak{S}\sim(N_1, =)$.

Property of $\mathfrak{S}^{[(R,\geqq)]}$	Necessary and sufficient conditions on $\mathfrak{S}$ and $\mathfrak{R}$
(1) Cardinal	(E) *or* $\mathfrak{S}$ is a cardinal.
(2) Ordinal	(E) *or* (I) *or* $\mathfrak{S}$ and $\mathfrak{R}$ ordinals such that either $\mathfrak{S}$ is unbounded or $\mathfrak{R}$ is finite.
(3) Chain	(E) *or* (I) *or* $\mathfrak{S}$ a chain without terminal or initial elements, $\mathfrak{R}'$ a chain *or* $\mathfrak{R}$ and $\mathfrak{S}$ chains.
(4) Bounded	(E) *or* (I) *or* $\mathfrak{S}$ bounded *or* R' empty and any initial element of $\mathfrak{S}$ precedes all other elements of $\mathfrak{S}$ and any terminal element of $\mathfrak{S}$ follows all other elements of $\mathfrak{S}$.
(5) Finite	(E) *or* (I) *or* $\mathfrak{R}$ and $\mathfrak{S}$ finite.
(6) Lattice	See 7.10.
(7) Complete lattice	(E) *or* (I) *or* $\mathfrak{S}$ a complete lattice *or* $\mathfrak{S}$ or $\mathfrak{S}^*$ is an ordinal while every element of $\mathfrak{R}$ has a unique smallest proper successor.

(1) $\rightarrow$ If $\mathfrak{R}$ or $\mathfrak{S}$ has a terminal element, $\mathfrak{S}<\mathfrak{S}^{[(R,\geqq)]}$ so $\mathfrak{S}$ is a cardinal. $\leftarrow$ is obvious.

(2) $\rightarrow$ If $\mathfrak{R}$ or $\mathfrak{S}$ has a terminal element, $\mathfrak{S}<\mathfrak{S}^{[(R,\geqq)]}$; hence $\mathfrak{S}$ is an ordinal and has an initial element so $\mathfrak{R}<\mathfrak{S}^{(R,\geqq)}$ if $\mathfrak{S}$ is not isomorphic to $(N_1, =)$. If in addition $\mathfrak{S}$ has a terminal element, $\mathfrak{R}^*<\mathfrak{S}^{[(R,\geqq)]}$ so $\mathfrak{R}$ is a finite ordinal. $\leftarrow$ is straightforward although the proof in the next to the last case is rather long.

(3) $\rightarrow$ If $\mathfrak{R}$ has a terminal element, $\mathfrak{S}<\mathfrak{S}^{[(R,\geqq)]}$ so $\mathfrak{S}$ is a chain; if $\mathfrak{S}$ has neither terminal nor initial elements and (E) fails to hold, then $\mathfrak{S}^{[(R,\geqq)]}\sim^{(R',\geqq)}\mathfrak{S}$ so $\mathfrak{S}$ and $\mathfrak{R}'$ are chains by case (2) of the discussion of ${}^{(R,\geqq)}\mathfrak{S}$. If

$\mathfrak{S}$ has either a terminal or initial element and $\mathfrak{S}$ is not isomorphic to $(N_1, =)$, then $\mathfrak{R}$ or $\mathfrak{R}^* < \mathfrak{S}^{[(R,\geqq)]}$ so $\mathfrak{S}$ and $\mathfrak{R}$ are chains. $\leftarrow$ is obvious for all but the last condition and there it follows from the fact that any two elements of $\mathfrak{S}^{(R,\geqq)}$ are comparable.

(4) $\rightarrow$ If $\mathfrak{R}$ has a terminal element r_0, let $c(f_1)$ and $c(f_2)$ be the smallest and largest elements of $\mathfrak{S}^{[(R,\geqq)]}$; using the notation of the proof of 7.11 we have for every s that $c(f_1) \geqq \Phi(s) \geqq c(f_2)$ or $f_1 \geqq {}^t f_s \geqq {}^t f_2$ so $f_1(r_0) \geqq s \geqq f_2(r_0)$ for every s. Hence $\mathfrak{S}$ is bounded. If $\mathfrak{S}$ has an initial element s_0, then the function f_0 for which $f_0(r) = s_0$ for all r is an initial element of $\mathfrak{S}^{(R,\geqq)}$ so $c(f_0)$ must be the smallest element of $\mathfrak{S}^{[(R,\geqq)]}$; hence s_0 is the smallest element of $\mathfrak{S}$. Dually any terminal element of $\mathfrak{S}$ must be the largest element of $\mathfrak{S}$.

$\leftarrow$ If $\mathfrak{S}$ is bounded, the obvious functions define upper and lower bounds in $\mathfrak{S}^{[(R,\geqq)]}$. If $\mathfrak{S}$ and $\mathfrak{R}$ have no terminal elements and if there is a smallest element s_0 such that $s \geqq s_0$ for all s in $\mathfrak{S}$, let $f(r)$ be chosen greater than s_0 for every r; then $c(f)$ is the largest element in $\mathfrak{S}^{[(R,\geqq)]}$ and $c(f_0)$ is the smallest element so $\mathfrak{S}^{[(R,\geqq)]}$ is bounded. Dually if $\mathfrak{S}$ has a largest element but no initial element, $\mathfrak{S}^{[(R,\geqq)]}$ is again bounded.

(5) $\rightarrow$ If R' is not empty or if $\mathfrak{S}$ has an initial or terminal element, $\mathfrak{S} < \mathfrak{S}^{[(R,\geqq)]}$ so $\mathfrak{S}$ is finite. If $\mathfrak{S}$ is a cardinal number, $\mathfrak{S}^{[(R,\geqq)]} \sim {}^{(R,\geqq)}\mathfrak{S}$ which is finite if and only if $\mathfrak{R}$ is also finite; if $\mathfrak{S}$ is not a cardinal number, 7.12 shows that $\mathfrak{R} < \mathfrak{S}^{[(R,\geqq)]}$ so $\mathfrak{R}$ is finite. $\leftarrow$ is obvious.

(6) has been given in 7.10.

(7) $\rightarrow$ If R' is not empty, the embedding Φ of 7.11 can be used to show that $\mathfrak{S}$ is a complete lattice. If R' is empty, and (E) and (I) are false, suppose that $\mathfrak{S}$ has an initial element s_0; by the argument in (4), s_0 must be the smallest element of $\mathfrak{S}$ and an extension of an argument used in 7.10 shows that $\bigwedge_{p \in P} s_p$ must exist for all choices of s_p in $\mathfrak{S}$. If $\mathfrak{S}$ is not bounded the same argument shows that every element r_0 of $\mathfrak{R}$ has a unique smallest proper successor $r_1 > r_0$; as in 7.10 we see that $\mathfrak{S}$ must be a chain in this case. If s_1 is any element of $\mathfrak{S}$, let $f_s(r_0) = s$, $f_s(r_1) = s_1$, and $f_s(r) = s_0$ for all other r; if $\mathfrak{S}$ is unbounded and $f \geqq^t$ all f_s, either $f(r_1) > s_1$ or $f(r) > s_0$ for some $r > r_1$. If $f_s'(r_1) = s$ and $f_s'(r) = s_0$ for all other r, $f_s' > f_{s'}$ for every s' and every $s > s_1$. If $c(f) = \bigvee_{s \in S} c(f_s)$, then $f(r) = s_0$ if r does not precede or equal r_1, and $f(r_1) \leqq s$ if $s > s_1$; that is, s_1 has a unique smallest proper successor if $\mathfrak{S}$ is unbounded. Since $\mathfrak{S}$ is also a chain in which every set has a greatest lower bound, we see that $\mathfrak{S}$ is an (unbounded) ordinal. A dual argument starting from the assumption that $\mathfrak{S}$ has a terminal element but no initial element gives instead the conclusion that $\mathfrak{S}^*$ is an ordinal.

$\leftarrow$ (E) or (I) implies that $\mathfrak{S}^{[(R,\geqq)]} \sim (N_1, =)$, a complete lattice. If $\mathfrak{S}$ is a complete lattice let 0 and 1 be its smallest and largest elements and take f_p, $p \in P$, in $\mathfrak{S}^{(R,\geqq)}$. We shall construct in a stepwise fashion an f in $\mathfrak{S}^{(R,\geqq)}$ such that $f \geqq^t$ all f_p while $f' \geqq^t$ all f_p implies that $f' \geqq^t f$. Let $E_{1p} = \{r \mid f_p(r') = 1 \text{ for every } r' \geqq r\}$, let $E_1 = \{r \mid f_p(r') = 0 \text{ for every } p \text{ in } P$

and every $r' \geqq r\}$; define $f(r)=1$ in $\bigcup_{p\in P} E_{1p}$, $f(r)=0$ in E_1. Let $A_1=R-E_1-\bigcup_{p\in P} E_{1p}$ and in A_1' define f as in the proof of (7) of the preceding table; that is, if $f(r')$ is defined for all $r'>r$, let $P_r=\{p \mid f_p(r')=f(r')$ for all $r'>r\}$ and let $f(r)=\bigvee_{p\in P(r)} f_p(r)$ if P_r is not empty, let $f(r)=0$ if P_r is empty. Let $A_1-A_1'=R_2$ and in R_2 repeat the same process; that is, if R_λ and A_λ, $\lambda<\alpha$, have been defined, let $R_\alpha=\bigcap_{\lambda<\alpha}(A_\lambda-A_\lambda')$; if f was defined in $R-R_\alpha$, let $E_\alpha=\{r \mid r\in R_\alpha$, and for every p either there exists $r'>r$, $r'\notin R_\alpha$, for which $f(r')>f_p(r')$ or $f_p(r')=f(r')$ for every successor r' of R which lies in $R-R_\alpha$ and $f_p(r')=0$ for every $r'\geqq r$ which lies in $R_\alpha\}$; let $E_{\alpha p}=\{r \mid r\in R_\alpha$ and $f_p(r')=1$ if $r'\geqq r$ and $r'\in R_\alpha$ and $f_p(r')=f(r')$ if $r'>r$ and $r'\in R-R_\alpha\}$. Define $f(r)=1$ in $\bigcup_{p\in P} E_{\alpha p}$, $f(r)=0$ in E_α; let $A_\alpha=R_\alpha-E_\alpha-\bigcup_{p\in P} E_{\alpha p}$ and define $f(r)$ in A_α' as in A_1'. This construction yields a decreasing sequence (probably transfinite) of subsets R_α of R such that $f\geqq f_p$ over $R-R_\alpha$ for each p and α. There will exist a smallest α_0 such that $R_{\alpha(0)}=R_{\alpha(0)+1}$ so $R_{\alpha(0)}=R_\alpha$ if $\alpha\geqq\alpha_0$ and $f\geqq f_p$ for every p over $R-\bigcap_{\alpha\leqq\alpha(0)} R_\alpha=R-R_{\alpha(0)}$.

$R_{\alpha(0)}$ may be empty; if it is not empty it has certain useful properties. $E_{\alpha(0)}$ is empty; hence (a) if $r\in R_{\alpha(0)}$, there exists p such that $f(r')=f_p(r')$ for each r' in $(r)^U-R_{\alpha(0)}$ and there exists r' in $(r)^U\cap R_{\alpha(0)}$ such that $f_p(r')>0$. Each $E_{\alpha(0)p}$ is empty; hence (b) for every p and every r in $R_{\alpha(0)}$ such that $f_p(r')=f(r')$ for every r' in $(r)^U-R_{\alpha(0)}$ there exists r' in $(r)^U\cap R_{\alpha(0)}$ such that $f_p(r')<1$; hence the set $(r)^U\cap R_{\alpha(0)}\cap\{r' \mid f_p(r')<1\}$ is cofinal in $(r)^U\cap R_{\alpha(0)}$. $A'_{\alpha(0)}$ is empty; hence (c) $R_{\alpha(0)}$ has no terminal elements.

f was so defined that $f\geqq$ all f_p over $R-R_{\alpha(0)}$. Let B and $R_{\alpha(0)}-B$ be cofinal subsets of $(R_{\alpha(0)},\geqq)$ and define $f(r)>0$ on B, $f(r)<1$ on $R_{\alpha(0)}-B$. Then (b) and (c) can be used to show that $f\geqq^t$ all f_p over all of R. This shows that $c(f)$ is an upper bound for the $c(f_p)$; to prove it a least upper bound we must show that $f'\geqq^t$ all f_p implies that $f'\geqq^t f$.

To carry this through requires two induction arguments (which we shall omit) to prove the following statements: (d) If $f'\geqq^t$ all f_p over $R-R_{\alpha(0)}$, then $f'\geqq f$ over $R-R_{\alpha(0)}$. (e) If E is a star in $R-R_{\alpha(0)}$, if $f'\geqq f''\geqq f$ over E, and if $f''(r)>f(r)$ for some r in E, then there is an $r'\geqq r$ such that $f'(r')>f(r')$. Now if $F=\{r \mid f'(r)>f(r)\}-R_{\alpha(0)}$ and if $G=R_{\alpha(0)}-F^D$, we see that $f'\geqq f$ over $R-G$. If $r\in G$, by (a) there is a p such that $f=f_p$ in $(r)^U-G$ and $f_p(r')>0$ for some r' in $(r)^U\cap G$. Now $f'\geqq^t f_p$ so by 1-transitivity there exists f'' for which $f'\geqq f''\geqq f_p$. In $(r)^U-G$ we have $f'=f_p=f$ so $f=f'\geqq f''\geqq f$ over $(r)^U-G$. By (e), $f'=f''=f$ over $(r)^U-G$. Hence there must exist r_1 in $(r')^U\cap G$ such that $f''(r_1)>0$, and therefore there exists r_2 in $(r_1)^U\cap G$ such that $f'(r_2)>0$. This proves that $\{r \mid f'(r)>0\}$ is cofinal in G so, by (c), $f'\geqq^t f$ over G. Hence $f'\geqq^t f$ over all R.

From this we see that $c(f)=\bigvee_{p\in P} c(f_p)$; the usual dual argument would give $\bigwedge_{p\in P} c(f_p)$ so $\mathfrak{S}^{[(R,\geqq)]}$ is a complete lattice.

In case $\mathfrak{S}$ is an unbounded ordinal with smallest element 0 and every element r_0 of $\mathfrak{R}$ has a unique smallest proper successor, this construction for f

can be used with minor modifications noting that every $E_{\alpha p}$ must be empty; a useful fact in the proof that $\bigvee_{p\in P} c(f)$ can be constructed is that in such an $\mathfrak{R}$, $E' = \bigcup_{n\in N} E^{(n)}$ for every $E\subset R$.

8. **Appendix on the calculus of relations.** In their monumental work *Principia mathematica*, Whitehead and Russell devote several sections to properties of certain addition and multiplication operations among relations. Since they take the very natural attitude that a function is not defined unless its class of arguments is defined, and since a relation is a yes-or-no valued function of two variables, in their notation they speak not of the ordered system (R, P), where R is a set and P a binary relation in R, but merely of the relation P itself. Since almost no one ever reads the *Principia* for its mathematics and since the discussion on the arithmetic of relations begins about the middle of vol. 2, we include in this section a sketchy outline of the definitions given there together with a comparison with the operations of this paper; however this will be translated to the notation of this paper.

Their definition of sum given in §162 is isomorphic under this translation to that of §2; their Theorem 162.34 is their form of the general associative law 2.17. In §172 a product of relations is defined which is not equivalent to that used here. For ordered systems their definition is equivalent to a different ordering of those same functions which are the elements of $\prod_{(R,\geqq)} \mathfrak{S}_r$; define $\geqq'$ in $\prod'_{(R,\geqq)} \mathfrak{S}_r$ by $f_1 \geqq' f_2$ if and only if there exists r_0 such that $f_1(r_0) > f_2(r_0)$ in $\mathfrak{S}_{r(0)}$ while $f_1(r) = f_2(r)$ if $r > r_0$. If $\mathfrak{R}$ is a chain, then $\geqq'$ is included in $>$; if $\mathfrak{R}$ is not a chain, $\geqq'$ and $\geqq$ need not be related; in fact, note that $\prod'_{(R,=)} \mathfrak{S}_r = \prod_{(R,u)} \mathfrak{S}_r$ so $\prod'$ and $\prod$ may be exceedingly unlike when $(R, \geqq)$ is not a chain.

It may be noted that the discussion in the *Principia* is pointed toward relations which are there called "series"; the corresponding ordered systems are chains. For such systems $\prod'$ is a number but need not be a chain; in fact, it generally turns out to be a cardinal sum of chains. In contrast to this property of $\prod'$, $\prod_{(R,\geqq)} \mathfrak{S}_r$ need be neither transitive nor antisymmetric, but if $\mathfrak{R}$ and $\mathfrak{S}_r$ are chains, then $\prod^c_{(R,\geqq)} \mathfrak{S}_r$ is always a chain. If $\mathfrak{R}$ is not a chain, $\prod'$ is usually intransitive as is $\prod$.

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