CONTRIBUTIONS TO THE THEORY OF SURFACES IN A 4-SPACE OF CONSTANT CURVATURE

BY

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1.1. **Introduction.** A Riemannian 4-space is a space of constant curvature and is denoted by S_4, if its Riemann tensor is of the form

$$R_{\kappa\lambda\mu}^{\rho} = -C(a_{\kappa\lambda}\delta_{\mu}^{\rho} - a_{\kappa\mu}\delta_{\lambda}^{\rho}) \quad (\rho, \kappa, \lambda, \mu = 1, \cdots, 4),$$

where $a_{\kappa\lambda}$ is the fundamental tensor, δ_{μ}^{ρ} the Kronecker delta and C a scalar, which is automatically a constant. A Euclidean 4-space, which we denote by R_4, is a special case of S_4. In this paper we study a few special surfaces V_2 in an S_4, including those in an R_4. The method used is invariant and is similar to those of Ricci[2] [14] and Graustein [9] for their studies of surfaces in a Euclidean 3-space. In essence, we first set up a suitable system of invariant fundamental equations for a V_2 in S_4 and then express the imbedding requirements of V_2 in S_4 in terms of the intrinsic properties of V_2. §1.2 contains some formulas in the intrinsic theory of V_2 which are useful for our later work. In §2, the curvature tensor $H_{cb}^{\cdot\cdot\kappa}$ and the curvature conic (G) for a V_2 in S_4 are introduced and discussed. Regarding $H_{cb}^{\cdot\cdot\kappa}$ as a pencil of tensors, we find a certain relationship between the nature of the pencil and that of the conic (G). Some formulas concerning (G) are then given, including one which leads to an interesting interpretation of the Gaussian curvature of V_2 in terms of (G). §3 contains the fundamental equations for V_2 in S_4 in invariant form and an outline of the method used for the study of some surfaces with preassigned curvature properties. §4 is devoted to the study of minimal V_2, and, in particular, the R-surface of Kommerell. In §5, ruled surfaces in S_4 are considered. §6 gives a special imbedding of R_2 in R_4. The paper ends in §7 with the determination of those V_2 in S_4 whose first fundamental form and one of whose second fundamental forms are respectively identical with the first and second fundamental forms of a V_2 in S_3.

1.2. **Some useful formulas.** Here are some intrinsic formulas for a V_2, which will be used in our later work and can be verified readily. Let g_{cb} $(a, b, c = 1, 2)$ be the fundamental tensor of a V_2 which is supposed to be positive definite, and let $i_{(1)}^{a}$, $i_{(2)}^{a}$ be two mutually orthogonal congruences (that

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[2] Numbers in brackets refer to the literature listed at the end of the paper.

is, unit vector fields) in V_2. Then

$$g_{cb} = \overset{.(1)}{i_c}\overset{.(1)}{i_b} + \overset{.(2)}{i_c}\overset{.(2)}{i_b}, \qquad \overset{.a}{i}_{(b)}\overset{.(c)}{i_a} = \delta^c_b, \qquad \overset{.(a)}{i_b}\overset{.c}{i}_{(a)} = \delta^c_b, \tag{1.1}$$

where $i^{\cdot(b)}_a = g_{ac} i^{\cdot c}_{(b)}$, and δ^c_b is the Kronecker delta.

The geodesic curvatures (to within sign) of the two congruences $i^a_{(1)}, i^a_{(2)}$ are, respectively,

$$\beta = - \overset{.b}{i}_{(2)}\overset{.c}{i}_{(1)}\overset{.(1)}{i}_{b,c}, \qquad \alpha = - \overset{.b}{i}_{(1)}\overset{.c}{i}_{(2)}\overset{.(2)}{i}_{b,c}, \tag{1.2}$$

where the comma denotes covariant differentiation with respect to g_{cb}.

Let us write

$$\delta_1 f = \overset{.c}{i}_{(1)} f_{,c}, \qquad \delta_2 f = \overset{.c}{i}_{(2)} f_{,c} \tag{1.3}$$

as the directional derivatives along the congruences $i^a_{(1)}$, $i^a_{(2)}$. Then the integrability condition for $\delta_1 f$, $\delta_2 f$ is

$$(\delta_2\delta_1 - \delta_1\delta_2 + \beta\delta_1 - \alpha\delta_2) f = 0, \tag{1.4}$$

that is, (1.4) is the necessary and sufficient condition that two scalars, which we denote by $\delta_1 f$, $\delta_2 f$, be the directional derivatives along $i^a_{(1)}$, $i^a_{(2)}$.

The first and second differential parameters for g_{cb} are, respectively,

$$\Delta_1 f = (\delta_1 f)^2 + (\delta_2 f)^2, \qquad \Delta_2 f = (\delta_1\delta_1 + \delta_2\delta_2 + \alpha\delta_1 + \beta\delta_2) f. \tag{1.5}$$

The Gaussian curvature K (that is, the scalar curvature) of V_2 is given by the Liouville formula:

$$K = -(\delta_1\alpha + \delta_2\beta + \alpha^2 + \beta^2). \tag{1.6}$$

(1.5) and (1.6) are of course independent of the choice of $i^a_{(1)}$, $i^a_{(2)}$.

The condition for $i^a_{(1)}$, $i^a_{(2)}$ to be an isometric net is

$$\delta_2\alpha - \delta_1\beta = 0. \tag{1.7}$$

If $\bar{i}^a_{(1)}$, $\bar{i}^a_{(2)}$ are two other mutually orthogonal congruences defined by

$$\overset{.a}{\bar{i}}_{(1)} = \overset{.a}{i}_{(1)} \cos\theta + \overset{.a}{i}_{(2)} \sin\theta, \quad \epsilon\overset{.a}{\bar{i}}_{(2)} = - \overset{.a}{i}_{(1)} \sin\theta + \overset{.a}{i}_{(2)} \cos\theta \quad (\epsilon = \pm 1), \tag{1.8}$$

then

$$\bar{\delta}_1 = \cos\theta\,\delta_1 + \sin\theta\,\delta_2, \qquad \epsilon\bar{\delta}_2 = - \sin\theta\,\delta_1 + \cos\theta\,\delta_2; \tag{1.9}$$

$$\begin{aligned} \bar{\alpha} &= -(\delta_1\theta - \beta)\sin\theta + (\delta_2\theta + \alpha)\cos\theta, \\ -\epsilon\bar{\beta} &= (\delta_1\theta - \beta)\cos\theta + (\delta_2\theta + \alpha)\sin\theta. \end{aligned} \tag{1.10}$$

From (1.9) and (1.10) we have

$$\epsilon(\bar{\delta}_2\bar{\alpha} - \bar{\delta}_1\bar{\beta}) = \delta_2\alpha - \delta_1\beta + \Delta_2\theta. \tag{1.11}$$

Therefore, if $i^a_{(1)}$, $i^a_{(2)}$ are an isometric net, then $\bar{i}^a_{(1)}$, $\bar{i}^a_{(2)}$ are also one if and only if $\Delta_2\theta = 0$.

For the linear element (that is, fundamental form)

$$ds^2 = Edu^2 + Gdv^2, \tag{1.12}$$

we have

$$\delta_1 = \frac{\partial_u}{E^{1/2}}, \qquad \delta_2 = \frac{\partial_v}{G^{1/2}} \qquad \left(\partial_u \equiv \frac{\partial}{\partial u}, \ \partial_v \equiv \frac{\partial}{\partial v}\right),$$

$$\alpha = \frac{1}{E^{1/2}} \partial_u \log G^{1/2}, \qquad \beta = \frac{1}{G^{1/2}} \partial_v \log E^{1/2}, \tag{1.13}$$

$$K = -\frac{1}{(EG)^{1/2}}\left[\partial_u\left(\frac{\partial_u G^{1/2}}{E^{1/2}}\right) + \partial_v\left(\frac{\partial_v E^{1/2}}{G^{1/2}}\right)\right].$$

For the linear element

$$ds^2 = 2Fdudv, \tag{1.14}$$

we have

$$\Delta_2 = \frac{2}{F}\partial_v\partial_u, \qquad K = -\frac{1}{F}\partial_v\partial_u \log F. \tag{1.15}$$

Closely connected with (1.14) is the Liouville equation (cf. Knoblauch [10, p. 543])

$$\partial_v\partial_u \log \lambda = A\lambda^B \qquad (A, B \text{ constants}), \tag{1.16}$$

the solution of which is

$$\lambda^B = -\frac{2U'V'}{AB(1 + UV)^2}, \tag{1.17}$$

where U, V are respectively arbitrary non-constant functions of u, v alone, and the prime denotes differentiation.

2.1. **The curvature tensor $H_{cb}^{\cdot\cdot\kappa}$.** We shall use the general notation of Schouten-Struik [15, chaps. 2, 3]. Let y^κ ($\kappa, \lambda, \cdots = 1, \cdots, 4$) be the coordinates in S_4(3), with positive definite fundamental tensor $a_{\lambda\kappa}$, and let $\Gamma_{\lambda\kappa}^{\mu}$, ∇_μ be the Christoffel symbol of the second kind and the symbol of covariant differentiation with respect to $a_{\lambda\kappa}$, respectively.

In S_4 we introduce a surface V_2 by the equations

$$y^\kappa = y^\kappa(u^a) \qquad (a, b = 1, 2). \tag{2.1}$$

Then the first fundamental tensor induced in V_2 by S_4 is

$$g_{cb} = a_{\lambda\kappa} B_c^\lambda B_b^\kappa, \tag{2.2}$$

(3) Although the definitions and results in §§2.1–2.3 are stated for a V_2 in an S_4, most of them also hold for a V_2 in any Riemannian 4-space.

where

$$B_b^{\kappa} = \partial_b y^{\kappa}. \tag{2.3}$$

At each point P of V_2, the connecting tensor B_b^{κ} spans the tangent plane to V_2 at P. Let $i_{(p)}^{\kappa}$ ($p, q = 3, 4$) be two mutually orthogonal unit vectors normal to V_2, defined at each point of V_2. Then

$$\overset{(p)}{h}_{cb} = - B_c^{\lambda} B_b^{\kappa} \nabla_{\lambda} \overset{(p)}{i}_{\kappa}, \tag{2.4}$$

where $i_{\kappa}^{(p)} = a_{\kappa\lambda} i_{(p)}^{\lambda}$, is the second fundamental tensor of V_2 in S_4 for the normal $i_{(p)}^{\kappa}$. The *curvature tensor*

$$H_{cb}^{\cdot\cdot\kappa} = \overset{(3)}{h}_{cb} \, i_{(3)}^{\kappa} + \overset{(4)}{h}_{cb} \, i_{(4)}^{\kappa} \tag{2.5}$$

is independent of the choice of $i_{(p)}^{\kappa}$.

Two directions i^a, j^a in V_2 are called *conjugate* if

$$H_{cb}^{\cdot\cdot\kappa} i^c j^b = 0. \tag{2.6}$$

In particular, if a direction j^a satisfies

$$H_{cb}^{\cdot\cdot\kappa} j^c j^b = 0, \tag{2.7}$$

it is called an *asymptotic* direction. In general, there are two and only two conjugate directions but no asymptotic direction.

Let (C) be a curve on V_2 passing through P and with unit tangent vector i^a at P. Then the component

$$\overline{PQ} = u^{\kappa}$$

in the normal plane of V_2 of the curvature vector of (C) with respect to S_4 is

$$u^{\kappa} = H_{cb}^{\cdot\cdot\kappa} i^c i^b, \tag{2.8}$$

which depends only on i^a and is called the *normal curvature vector* of V_2 in S_4 for the direction i^a at P. The locus (G) of the end point Q as i^a takes all the possible positions in the tangent plane is an ellipse, proper or degenerate, in the normal plane. We shall call (G) the *curvature conic* of V_2 in S_4 at P.

From (2.5) it follows that the equations of (G) in the rectangular Cartesian coordinates z, t with axes $i_{(3)}^{\kappa}$, $i_{(4)}^{\kappa}$ are

$$z = \overset{(3)}{h}_{cb} i^c i^b, \qquad t = \overset{(4)}{h}_{cb} i^c i^b. \tag{2.9}$$

2.2. **The tensor $H_{cb}^{\cdot\cdot\kappa}$ and the conic (G).** Owing to the arbitrary choice of $i_{(3)}^{\kappa}$ in (2.5), $H_{cb}^{\cdot\cdot\kappa}$ may be considered as a pencil of tensors $\lambda h_{cb}^{(3)} + \mu h_{cb}^{(4)}$. We shall now proceed to prove the following relationship between the characteristics of this pencil (in the sense of the theory of elementary divisors[4]) and

[4] See, for example, Bôcher [1, chaps. 20–21].

the geometric properties of (G) and P.

THEOREM 2.1. *If the pencil $H_{cb}^{\cdot\cdot\kappa}$ is not singular, we denote as usual the characteristics of the pencil by squared brackets. Then for:*

[11], *(G) is an ellipse not passing through P, which degenerates into a line segment if and only if the conjugate directions of V_2 are orthogonal to each other.*

[(11)], *(G) is a line segment passing through P but not ending at P.*

[2], *(G) is a non-degenerate ellipse passing through P.*

If the pencil of tensors is singular, (G) is a line segment ending at P.

Proof. We first suppose that there is a nonsingular tensor in the pencil. Let it be $h_{cb}^{(3)}$. Then we have the following cases.

Case 1. [11]. A coordinate system(5) u^a in V_2 exists so that at the point P under consideration the equations (2.9) of (G) take the form

$$z = \epsilon_1(i^1)^2 + \epsilon_2(i^2)^2, \qquad t = \sigma_1(i^1)^2 + \sigma_2(i^2)^2, \tag{2.10}$$

where $\epsilon_1, \epsilon_2 = \pm 1$; σ_1, σ_2 are scalars, and i^a is a variable unit vector, that is, its components i^1, i^2 are subject only to the condition

$$g_{cb}i^c i^b = 1. \tag{2.11}$$

The conjugate directions of V_2 at P are evidently

$$\overset{a}{j}_{(1)} = (1, 0), \qquad \overset{a}{j}_{(2)} = (0; 1). \tag{2.12}$$

From (2.10) and (2.11) it follows that (G) is an ellipse. If (G) passes through P, the equations $z=t=0$ are satisfied by some i^a. This requires that $\sigma_1/\epsilon_1=\sigma_2/\epsilon_2\equiv\sigma$. Then the subcase [(11)] arises, and (2.10) become

$$z = \epsilon_1(i^1)^2 + \epsilon_2(i^2)^2, \qquad t = \sigma z, \tag{2.13}$$

showing that (G) is a line segment not ending at P.

Let us now return to the general case (2.10). The condition for (G) to be a line segment not passing through P is that λ, μ exist such that $\lambda z+\mu t=1$, that is, by (2.10). that

$$(\lambda\epsilon_1 + \mu\sigma_1)(i^1)^2 + (\lambda\epsilon_2 + \mu\sigma_2)(i^2)^2 = 1$$

should be true for some i^1, i^2 satisfying (2.11). Subtraction of the above equation and (2.11) gives

$$(\lambda\epsilon_1 + \mu\sigma_1 - g_{11})(i^1)^2 - 2g_{12}i^1i^2 + (\lambda\epsilon_2 + \mu\sigma_2 - g_{22})(i^2)^2 = 0.$$

Since this should be true for all values of i^1, i^2, we have

$$g_{12} = 0, \tag{2.14}$$

$$\lambda\epsilon_1 + \mu\sigma_1 = g_{11}, \qquad \lambda\epsilon_2 + \mu\sigma_2 = g_{22}. \tag{2.15}$$

(5) We note that u^a are either real or complex conjugate. If they are complex conjugate, $\epsilon_1=\epsilon_2$; and σ_1, σ_2; i^1, i^2 are complex conjugate, z, t being always real.

Equation (2.14) shows that the conjugate directions (2.12) are orthogonal to each other. If $\sigma_1/\epsilon_1 \neq \sigma_2/\epsilon_2$, the two equations (2.15) have a unique solution for λ, μ; for, this is not true only when $g_{11}=g_{22}=0$, but then we have, because of (2.14), $g_{cb}=0$. If $\sigma_1/\epsilon_1=\sigma_2/\epsilon_2\equiv\sigma$, then $z=\sigma t$, and we have a line segment passing through P.

We have now proved our assertions in Theorem 2.1 for the subcases [11] and [(11)]. Incidentally we remark that *for the subcase* [11] *the two tangents from* P *to* (G) *are the normal curvature vectors for the conjugate directions* (2.12). In fact, the line $\lambda z+\mu t=0$ cuts (G) in two points for which

$$(\lambda\epsilon_1+\mu\sigma_1)(\dot{i}^1)^2+(\lambda\epsilon_2+\mu\sigma_2)(\dot{i}^2)^2=0,$$

and therefore if the two points coincide, they do so either at $\dot{i}^1=0$, or at $\dot{i}^2=0$.

Case 2. [2]. A (real) coordinate system u^a in V_2 exists such that at P the equations (2.9) of (G) take the form

$$z=2\dot{i}^1\dot{i}^2, \qquad t=2\sigma_3\dot{i}^1\dot{i}^2+\sigma_2(\dot{i}^2)^2 \qquad (\sigma_2\neq 0). \tag{2.16}$$

The $\dot{i}^1$, $\dot{i}^2$ satisfy (2.11), where $g_{11}\neq 0$ since g_{cb} is positive definite and the coordinates u^a are real. The (G) as given by (2.16) is an ellipse passing through P; for, $\dot{i}^2=0$ is part of a solution of the equations $z=t=0$ and (2.11). The point P on (G) corresponds to the asymptotic direction $\dot{j}^a_{(1)}=(1,0)$. If (G) were a line segment, we should have $t/z=\sigma_3+\sigma_2\dot{i}^2/\dot{i}^1=\text{const}$. But this is impossible since $\sigma_2\neq 0$. Hence (G) is a proper ellipse.

If all the tensors in the pencil $H_{cb}^{\cdot\cdot\kappa}$ are singular, there exists a (real) coordinate system u^a in V_2 such that the equations (2.9) of (G) take the form

$$z=0, \qquad t=\sigma_1(\dot{i}^1)^2. \tag{2.17}$$

The conic (G) is therefore a line segment ending at P.

2.3. **Some formulas concerning** (G). Let all the tensors in V_2 be expressed in terms of two mutually orthogonal unit vectors $\dot{i}^a_{(1)}$, $\dot{i}^a_{(2)}$ and some scalars. Thus,

$$\begin{aligned}\overset{(3)}{h}_{cb}&=\rho_1\overset{\cdot(1)}{i}_c\overset{\cdot(1)}{i}_b+\rho_2\overset{\cdot(2)}{i}_c\overset{\cdot(2)}{i}_b+\rho_3(\overset{\cdot(1)}{i}_c\overset{\cdot(2)}{i}_b+\overset{\cdot(2)}{i}_c\overset{\cdot(1)}{i}_b),\\ \overset{(4)}{h}_{cb}&=\sigma_1\overset{\cdot(1)}{i}_c\overset{\cdot(1)}{i}_b+\sigma_2\overset{\cdot(2)}{i}_c\overset{\cdot(2)}{i}_b+\sigma_3(\overset{\cdot(1)}{i}_c\overset{\cdot(2)}{i}_b+\overset{\cdot(2)}{i}_c\overset{\cdot(1)}{i}_b),\end{aligned} \tag{2.18}$$

$$\dot{i}^a=\dot{i}^a_{(1)}\cos\phi+\dot{i}^a_{(2)}\sin\phi. \tag{2.19}$$

The equations of the curvature conic (G) are therefore, by (2.19),

$$\begin{aligned}z&=2^{-1}(\rho_1+\rho_2)+2^{-1}(\rho_1-\rho_2)\cos 2\phi+\rho_3\sin 2\phi,\\ t&=2^{-1}(\sigma_1+\sigma_2)+2^{-1}(\sigma_1-\sigma_2)\cos 2\phi+\sigma_3\sin 2\phi.\end{aligned} \tag{2.20}$$

Elementary calculations show that

$$d^2=4^{-1}[(\rho_1+\rho_2)^2+(\sigma_1+\sigma_2)^2], \tag{2.21}$$

where d is the distance from P to the center of (G).

(2.22) $$\text{Area of } (G) = \pm 2^{-1}\pi[(\rho_1 - \rho_2)\sigma_3 - (\sigma_1 - \sigma_2)\rho_3].$$

(2.23) $$r^2 = 4^{-1}[(\rho_1 - \rho_2)^2 + (\sigma_1 - \sigma_2)^2] + \rho_3^2 + \sigma_3^2,$$

where r is the radius of the director circle of (G).

From these, it follows that the power of P with respect to the director circle of (G) is

$$d^2 - r^2 = \rho_1\rho_2 + \sigma_1\sigma_2 - \rho_3^2 - \sigma_3^2.$$

When V_2 is in an S_4 with constant scalar curvature C, we see from the Gauss equation $(3.6)_1$, which is the first equation in equations (3.6) of §3, that the right-hand member of the above equation is the Gaussian curvature K of V_2 minus C. Hence we have the following theorem.

THEOREM 2.2. *At any point P of a V_2 in an S_4 (or R_4), the power of P with respect to the director circle of (G) is equal to the Gaussian curvature of V_2 minus the scalar curvature of S_4 (or to the Gaussian curvature of V_2). In particular, at any P of a V_2 in an R_4, the Gaussian curvature of V_2 is greater than* 0, *equal to* 0, *or less than* 0 *according as P lies outside, on, or inside the director circle of* (G).

From the equations (2.20) of (G) and the Gauss equation $(3.6)_1$ of V_2 in S_4, it can be shown by elementary calculation that the angle Ω subtended by (G) at P is given by

(2.24) $$(\tan \Omega)^2 = \Delta/(K - C)^2,$$

where

(2.25) $$\Delta = (\sigma_1\rho_2 - \sigma_2\rho_1)^2 + 4(\sigma_3\rho_1 - \sigma_1\rho_3)(\sigma_3\rho_2 - \sigma_2\rho_3).$$

Putting $K = C$ in (2.24) we get a partial verification of Theorem 2.2. A further consequence of (2.24) is that (G) *passes through P if and only if* $\Delta = 0$.

Another interesting formula concerning (G) is the one that gives the angle ω between the two conjugate directions of V_2 in S_4:

(2.26) $$(\tan \omega)^2 = \Delta\pi^2/4(\text{Area of } (G))^2.$$

This can be proved as follows. By definition (2.6), the conjugate directions i^a, j^a are given by

(2.27) $$\overset{(3)}{h}_{cb}\, i^c j^b = 0, \qquad \overset{(4)}{h}_{cb}\, i^c j^b = 0,$$

and hence by

(2.28) $$\begin{vmatrix} \overset{(3)}{h}_{cb}\, i^c \\ \overset{(4)}{h}_{cb}\, i^c \end{vmatrix} = 0.$$

But from (2.18) and (2.19), we have

$$\overset{(3)}{h}_{cb}\,\dot{i}^{c} = (\rho_1 \cos\phi + \rho_3 \sin\phi)\overset{(1)}{\dot{i}_b} + (\rho_2 \sin\phi + \rho_3 \cos\phi)\overset{(2)}{\dot{i}_b},$$

$$\overset{(4)}{h}_{cb}\,\dot{i}^{c} = (\sigma_1 \cos\phi + \sigma_3 \sin\phi)\overset{(1)}{\dot{i}_b} + (\sigma_2 \sin\phi + \sigma_3 \cos\phi)\overset{(2)}{\dot{i}_b}.$$

Therefore (2.28) is equivalent to

$$\begin{vmatrix} \rho_1 \cos\phi + \rho_3 \sin\phi, & \rho_2 \cos\phi + \rho_3 \sin\phi \\ \sigma_1 \cos\phi + \sigma_3 \sin\phi, & \sigma_2 \cos\phi + \sigma_3 \sin\phi \end{vmatrix} = 0,$$

that is

$$\begin{vmatrix} \rho_1 & \rho_3 \\ \sigma_1 & \sigma_3 \end{vmatrix} \cos^2\phi + \begin{vmatrix} \rho_1 & \rho_2 \\ \sigma_1 & \sigma_2 \end{vmatrix} \cos\phi \sin\phi + \begin{vmatrix} \rho_3 & \rho_2 \\ \sigma_3 & \sigma_2 \end{vmatrix} \sin^2\phi = 0. \tag{2.29}$$

The roots of this equation for ϕ give the conjugate directions, and formula (2.26) can be proved easily from (2.29) and (2.22).

A consequence of (2.26) is that *the two conjugate directions are orthogonal to each other if and only if* (G) *degenerates into a line segment*; in particular, *the conjugate directions are indeterminate if and only if this line segment passes through* P. This is in accordance with Theorem 2.1.

The Kommerell conic (K) is the locus of the point, apart from P, at which the normal plane of V_2 at P is intersected by the neighboring normal planes. It can be shown that the polar line (with respect to the unit circle in the normal plane) of the point ϕ on (G) touches (K) at the point that is the intersection of the normal plane at P by the normal plane at a point near P in the direction ϕ: $i^a = i^a_{(1)} \cos\phi + i^a_{(2)} \sin\phi$. (K) *is therefore the polar reciprocal of* (G) *with respect to the unit circle at* P, and its equation is

$$(\rho_1 z + \sigma_1 t - 1)(\rho_2 z + \sigma_2 t - 1) = (\rho_3 z + \sigma_3 t)^2.$$

We point out, for future reference, that when (G) has a focus at P, (K) is a circle, and conversely.

Although in many literatures the Kommerell conic has been the central figure in the curvature theory of V_2 in R_4, it will not appear again in this paper, the curvature conic (G) having taken its place.

3. **Fundamental equations for V_2 in S_4.** We now continue the theory of V_2 in S_4 which was commenced in §2.1. The fundamental equations for a V_2 in S_4 consist of the following two groups of equations (Schouten-Struik [15, chaps. 2–3]; Eisenhart [8, chaps. 4–5]):

$$\begin{aligned} \partial_c y^\kappa &= B_c^\kappa, \\ \partial_c B_b^\kappa &= \Gamma^a_{cb} B_a^\kappa + \Gamma^\kappa_{\mu\lambda} B_c^\mu B_b^\lambda + \overset{(3)}{h}_{cb}\, \dot{i}^\kappa_{(3)} + \overset{(4)}{h}_{cb}\, \dot{i}^\kappa_{(4)}, \\ \partial_c \dot{i}^\kappa_{(3)} &= -\Gamma^\kappa_{\mu\lambda} B_c^\mu \dot{i}^\lambda_{(3)} - v_c \dot{i}^\kappa_{(4)} - \overset{(3)}{h}_c{}^b B_b^\kappa, \\ \partial_c \dot{i}^\kappa_{(4)} &= -\Gamma^\kappa_{\mu\lambda} B_c^\mu \dot{i}^\lambda_{(4)} + v_c \dot{i}^\kappa_{(3)} - \overset{(4)}{h}_c{}^b B_b^\kappa; \end{aligned} \tag{3.1}$$

(3.2)
$$2(C-K)=\sum_{p=3}^{4}\left[h_a^{(p)b}h_b^{(p)a}-(h_a^{(p)a})^2\right],$$
$$h_{a[b,c]}^{(3)}=-h_{a[b}^{(4)}v_{c]},\qquad h_{a[b,c]}^{(4)}=h_{a[b}^{(3)}v_{c]},\qquad v_{[b,c]}=h_{a[b}^{(4)}h_{c]}^{(3)a}\text{(6)};$$

where

(3.3)
$$h_{cb}^{(p)}=-B_c^{\lambda}B_b^{\kappa}\nabla_\lambda \dot{i}_\kappa^{(p)},\qquad v_c=B_c^{\lambda}\dot{i}_{(3)}^{\kappa}\nabla_\lambda \dot{i}_\kappa^{(4)},$$

$C=-R/12$ is the (constant) scalar curvature of S_4 (in the sense of Schouten[7]), and the comma denotes covariant differentiation with respect to the fundamental tensor (2.2) of V_2. We observe that these fundamental equations, taken as a whole, are independent of the choice of the mutually orthogonal unit normal vectors $i_{(3)}^{\kappa}$, $i_{(4)}^{\kappa}$ to V_2.

Eisenhart [8, p. 212] proved that *given tensors* g_{cb}, $h_{cb}^{(3)}$, $h_{cb}^{(4)}$ *and a vector* v_c *satisfying* (3.2), *a* V_2 *in* S_4 *is determined* (*by* (3.1)) *to within a "motion" in* S_4 *such that its first and second fundamental tensors are the given tensors.*

By a *motion* in S_4 we mean a transformation of the points in S_4 which preserves the linear element of S_4. In particular, a motion in a Euclidean space R_4 reduces to rotations and reflections. Hereafter, we shall call two V_2 in S_4 *congruent* if one is obtainable from the other by a motion in S_4.

As in §2.3, we express $h_{cb}^{(3)}$, $h_{cb}^{(4)}$, v_c, i^a in terms of two mutually orthogonal unit vectors $i_{(1)}^a$, $i_{(2)}^a$ in V_2:

(3.4)
$$h_{cb}^{(3)}=\rho_1\dot{i}_c^{(1)}\dot{i}_b^{(1)}+\rho_2\dot{i}_c^{(2)}\dot{i}_b^{(2)}+\rho_3(\dot{i}_c^{(1)}\dot{i}_b^{(2)}+\dot{i}_c^{(2)}\dot{i}_b^{(1)}),$$
$$h_{cb}^{(4)}=\sigma_1\dot{i}_c^{(1)}\dot{i}_b^{(1)}+\sigma_2\dot{i}_c^{(2)}\dot{i}_b^{(2)}+\sigma_3(\dot{i}_c^{(1)}\dot{i}_b^{(2)}+\dot{i}_c^{(2)}\dot{i}_b^{(1)}),$$
$$v_c=\nu_1\dot{i}_c^{(1)}+\nu_2\dot{i}_c^{(2)},\qquad \dot{i}^a=\dot{i}_{(1)}^a\cos\phi+\dot{i}_{(2)}^a\sin\phi.$$

Then the equations of (G), referred to the rectangular coordinate axes $i_{(3)}^{\kappa}$, $i_{(4)}^{\kappa}$, are

(3.5)
$$z=2^{-1}(\rho_1+\rho_2)+2^{-1}(\rho_1-\rho_2)\cos 2\phi+\rho_3\sin 2\phi,$$
$$t=2^{-1}(\sigma_1+\sigma_2)+2^{-1}(\sigma_1-\sigma_2)\cos 2\phi+\sigma_3\sin 2\phi,$$

and the fundamental equations (3.2) take the invariant form

(3.6)
$$C-K=\rho_3^2-\rho_1\rho_2+\sigma_3^2-\sigma_1\sigma_2,$$
$$\delta_2\rho_1-\delta_1\rho_3+\beta(\rho_1-\rho_2)-2\alpha\rho_3-\nu_1\sigma_3+\nu_2\sigma_1=0,$$
$$\delta_1\rho_2-\delta_2\rho_3-\alpha(\rho_1-\rho_2)-2\beta\rho_3-\nu_2\sigma_3+\nu_1\sigma_2=0,$$
$$\delta_2\sigma_1-\delta_1\sigma_3+\beta(\sigma_1-\sigma_2)-2\alpha\sigma_3+\nu_1\rho_3-\nu_2\rho_1=0,$$
$$\delta_1\sigma_2-\delta_2\sigma_3-\alpha(\sigma_1-\sigma_2)-2\beta\sigma_3+\nu_2\rho_3-\nu_1\rho_2=0,$$
$$\delta_2\nu_1-\delta_1\nu_2+\beta\nu_1-\alpha\nu_2+(\rho_1-\rho_2)\sigma_3-(\sigma_1-\sigma_2)\rho_3=0,$$

(6) Here we write, for example, $v_{b,c}-v_{c,b}=v_{[b,c]}$.

(7) Eisenhart [8] called R the scalar curvature.

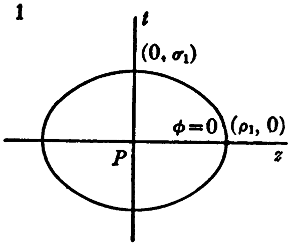
where

$$\alpha = -\, i^{b}_{(1)} i^{c}_{(2)} i^{(2)}_{b,c}, \qquad \beta = -\, i^{b}_{(2)} i^{c}_{(1)} i^{(1)}_{b,c}, \qquad \delta_1 f = i^{c}_{(1)} f_{,c}, \qquad \delta_2 f = i^{c}_{(2)} f_{,c}$$

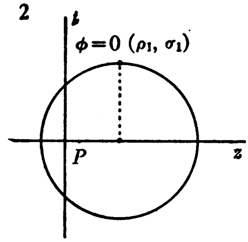
are as defined in §1.2. Formulas (3.6) are obtained after a straightforward but rather lengthy calculation.

If the conic (G) and its relation to the point P are of certain nature all over V_2, we can make use of this particular nature to simplify the equations of (G), and consequently the fundamental equations (3.6), by suitable choice of $i^{\kappa}_{(3)}$ and $i^{a}_{(1)}$. This choice is evidently equivalent to the choice of the z-axis in the plane of (G) and the position of the point $(\phi=0)$ on (G). Consider, for example, the special V_2 for which (1) P is the center of (G), (2) (G) is a circle, (3) P is a focus of (G), (4) P lies on (G), and (5) (G) is a line segment subtending a right angle at P, respectively. By choosing the z-axis and the point $(\phi=0)$ on (G) as in the following figures,

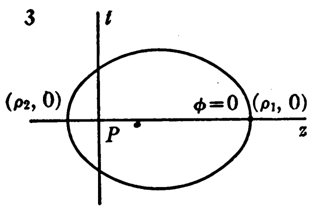
1



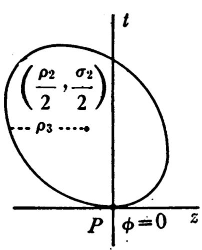
2



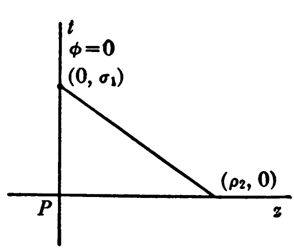
3



4



5



the simplified equations of (G) and the equations satisfied by the ρ's and σ's for the respective cases are:

$(3.7)_1 \quad z = \rho_1 \cos 2\phi, \quad t = \sigma_3 \sin 2\phi; \quad \rho_2 = -\rho_1, \quad \rho_3 = \sigma_1 = \sigma_2 = 0.$

$(3.7)_2 \quad z = \rho_1 + \sigma_1 \sin 2\phi, \ t = \sigma_1 \cos 2\phi; \ \sigma_2 = -\sigma_1, \ \rho_3 = \sigma_1, \ \rho_2 = \rho_1, \ \sigma_3 = 0.$

$$(3.7)_3 \quad \begin{aligned} &z = 2^{-1}(\rho_1 + \rho_2) + 2^{-1}(\rho_1 - \rho_2)\cos 2\phi, \quad t = (-\rho_1\rho_2)^{1/2}\sin 2\phi; \\ &\rho_3 = \sigma_1 = \sigma_2 = 0, \quad \sigma_3^2 = -\rho_1\rho_2. \end{aligned}$$

$$(3.7)_4 \quad \begin{aligned} &z = 2^{-1}\rho_2(1 - \cos 2\phi) + \rho_3 \sin 2\phi, \quad t = 2^{-1}\sigma_2(1 - \cos 2\phi); \\ &\rho_1 = \sigma_1 = \sigma_3 = 0. \end{aligned}$$

$(3.7)_5 \quad z = 2^{-1}\rho_2(1 - \cos 2\phi), \ t = 2^{-1}\sigma_1(1 + \cos 2\phi); \ \rho_1 = \rho_3 = \sigma_2 = \sigma_3 = 0.$

It is seen that for any special V_2 there is a system of equations (3.7) in the ρ's and σ's analogous to the second set of equations in $(3.7)_1$ for case 1. Then from the above-mentioned Eisenhart's theorem it follows that the existence of this particular V_2 depends on the consistency of equations (3.6) and (3.7) in $i^a_{(c)}$ and the ρ's, σ's, ν's. If these equations are consistent, the V_2 in question exists, and all its intrinsic properties can be derived from (3.6) and (3.7). In particular, when the solution of the latter equations can be found explicitly, the tensors g_{cb}, $h^{(3)}_{cb}$, $h^{(4)}_{cb}$, v_c are determined. And then equations (3.1) give the manner of imbedding of this V_2 in S_4. In the case where (3.1) can actually be solved, we obtain the finite equations of V_2.

Since for any V_2 in S_4, the second fundamental tensors $h^{(p)}_{cb}$ admit an orthogonal transformation, which arises from the permissible change of the mutually orthogonal unit normals $i^\kappa_{(p)}$ to V_2, the arbitrariness of the imbedding of a V_2 in S_4, to within motion in S_4, is equal to the arbitrariness of the solution of (3.6) and (3.7) for the $i^a_{(c)}$, ρ's, σ's and ν's minus the arbitrariness of $h^{(p)}_{cb}$ (that is, of the $i^\kappa_{(p)}$) in compliance with (3.7). It is obvious from the very origin of (3.7) that the latter equations usually limit the arbitrariness of $i^\kappa_{(p)}$ to mere changes of signs. In fact, the only case to the contrary is when (G) is a circle with center at P, that is, when $\sigma_3 = \rho_1$ in $(3.7)_1$; this is the case of an R-surface (cf. §4).

The rest of this paper will be devoted to the study of some special types of surfaces, particularly the above-mentioned ones.

4.1. **Minimal surfaces in S_4.** A *minimal* V_2 in S_4 is a V_2 whose mean normal curvature vector is everywhere zero, that is, whose curvature conic (G) at each point P always has its center at P. A minimal V_2 is called an *R-surface* (a *plane surface*[8]) if its curvature conic is always a circle of nonzero (zero) radius. A minimal V_2 that is not an R-surface nor a plane surface is called a *general* minimal surface.

The R-surface in an R_4 was first studied by Kwietniewski [12] and Kommerell [11], and the minimal surface in R_4 by Eisenhart [6]. The R-surface in an S_4 was later studied by Borůvka [2]. The first three authors based

(8) In general, a subspace in a Riemannian n-space V_n is said to be *plane* if its normal curvature vector in V_n is zero for every direction at every point.

their studies on the finite equations of the surface, which are available for a minimal V_2 in R_4. The finite equations of a minimal V_2 in an S_4, however, are still to be found, and Borůvka used a method which agrees in spirit with the one we outlined in the preceding section. In what follows we shall obtain a necessary and sufficient condition in invariant form for a linear element ds^2 to be one of a minimal V_2 (or R-surface) in S_4. We shall give also generalizations to some of Kommerell and Eisenhart's results.

4.2. **Fundamental equations. R-surface.** For a minimal V_2, P is the center of (G) and, therefore, with suitable choice of $i^{\kappa}_{(3)}$, $i^{a}_{(1)}$, the equations of the conic (G) are (cf. $(3.7)_1$)

$$z = \rho_1 \cos 2\phi, \qquad t = \sigma_3 \sin 2\phi,$$

so that

$$(4.1) \qquad \rho_2 = -\rho_1, \qquad \rho_3 = \sigma_1 = \sigma_2 = 0;$$

$$(4.2) \qquad \overset{(3)}{h}_{cb} = \rho_1(\overset{(1)}{i_c}\overset{(1)}{i_b} - \overset{(2)}{i_c}\overset{(2)}{i_b}), \qquad \overset{(4)}{h}_{cb} = \sigma_3(\overset{(1)}{i_c}\overset{(2)}{i_b} + \overset{(2)}{i_c}\overset{(1)}{i_b}).$$

Then the fundamental equations (3.6) become

$$(4.3) \qquad C - K = \overset{2}{\rho_1} + \overset{2}{\sigma_3},$$

$$(4.4) \qquad \begin{aligned} &\delta_2\rho_1 + 2\beta\rho_1 - \nu_1\sigma_3 = 0, \qquad -\delta_1\rho_1 - 2\alpha\rho_1 - \nu_2\sigma_3 = 0, \\ &-\delta_1\sigma_3 - 2\alpha\sigma_3 - \nu_2\rho_1 = 0, \qquad -\delta_2\sigma_3 - 2\beta\sigma_3 + \nu_1\rho_1 = 0, \end{aligned}$$

$$(4.5) \qquad \delta_2\nu_1 - \delta_1\nu_2 + \beta\nu_1 - \alpha\nu_2 + 2\rho_1\sigma_3 = 0.$$

Eliminations of ν_2 from $(4.4)_{2,3}$ (that is, the second and third equations in (4.4)) and of ν_1 from $(4.4)_{1,4}$ give, respectively,

$$(4.6) \qquad \delta_1(\overset{2}{\rho_1} - \overset{2}{\sigma_3}) + 4\alpha(\overset{2}{\rho_1} - \overset{2}{\sigma_3}) = 0, \qquad \delta_2(\overset{2}{\rho_1} - \overset{2}{\sigma_3}) + 4\beta(\overset{2}{\rho_1} - \overset{2}{\sigma_3}) = 0.$$

Consider first the R-surface, for which (G) is a circle of nonzero radius. In this case $\sigma_3 = \pm\rho_1 \neq 0$. But we may suppose that

$$(4.1)_a \qquad \sigma_3 = \rho_1 \equiv \rho \neq 0,$$

after reversing the sense of $i^{a}_{(2)}$ if necessary. Then (4.6) are identically satisfied, and (4.3) and (4.4) become

$$(4.7) \qquad C - K = 2\rho^2,$$

$$(4.8) \qquad \nu_1 = \delta_2 \log \rho + 2\beta, \qquad -\nu_2 = \delta_1 \log \rho + 2\alpha.$$

If we substitute these values for ν_1, ν_2 in (4.5), we have

$$(\delta_1\delta_1 + \delta_2\delta_2 + \alpha\delta_1 + \beta\delta_2) \log \rho + 2\rho^2 + 2(\delta_1\alpha + \delta_2\beta + \alpha^2 + \beta^2) = 0.$$

Using (1.5), (1.6) and the value of ρ from (4.7), this becomes

$$(4.9) \qquad \Delta_2 \log (C - K) + 2C - 6K = 0,$$

where Δ_2 indicates the second differential parameter with respect to the linear element ds^2 of V_2. For the R-surface in an R_4, $C=0$ and (4.9) reduces to

$$\Delta_2 \log(-K) - 6K = 0. \tag{4.10}$$

Equation (4.9), which is in an invariant form, is a necessary condition for a ds^2 to be the linear element of an R-surface in an S_4. To prove the sufficiency of this condition, let there be given a ds^2 satisfying it with some constant C. We may choose any mutually orthogonal unit vectors $i^{a}_{(1)}$, $i^{a}_{(2)}$ in V_2, and, after having obtained a value for ρ from (4.7), we find from (4.8) the values for ν_1, ν_2. Then we have a solution of the fundamental equations (4.3)–(4.5), and therefore an R-surface in S_4 with this ds^2 as linear element. Hence we have the following theorem.

THEOREM 4.1. *A ds^2 is the linear element of an R-surface in an S_4 if and only if* (4.9) *is satisfied for some constant C. Then C is the* (*constant*) *scalar curvature of the S_4. A ds^2 is the linear element of an R-surface in an R_4 if and only if* (4.10) *is satisfied.*

If $K=\text{const.}$, (4.9) and (4.7) reduce to $K=C/3=\rho^2>0$. Hence, we have proved in a different way the following result due to Borůvka [2]:

THEOREM 4.2. *In an S_4 of scalar curvature $C>0$, there exists only one type of R-surface of constant Gaussian curvature K. For this, $K=C/3$, and the curvature circle* (*G*) *is of constant radius $(K)^{1/2}$ all over the surface. In an R_4 an R-surface has negative Gaussian curvature, which cannot be constant.*

The last part of the theorem follows at once from (4.7) and (4.10).

Kommerell [11, §11] proved the rigidity of an R-surface in an R_4. We shall now prove this result for the more general case of an R-surface in an S_4, using a different method.

THEOREM 4.3. *If two R-surfaces V_2, $\overline{V}_2$ in an S_4 are applicable, then they are congruent.*

Proof. We use a dash to indicate quantities and equations belonging to $\overline{V}_2$ and suppose that the unit vectors $i^{\kappa}_{(p)}$, $i^{a}_{(c)}$, $\bar{i}^{\kappa}_{(p)}$, $\bar{i}^{a}_{(c)}$ have been so chosen that equations (4.1), $(4.1)_a$, (4.1^-), $(4.1^-)_a$ are satisfied. Then since $d\bar{s}^2=ds^2$, we have from earlier results in this section:

$$C - K = 2\rho^2, \tag{4.11}$$

$$C - K = 2\bar{\rho}^2, \tag{4.11$^-$}$$

$$\begin{aligned} \overset{(3)}{h}_{cb} &= \rho(\overset{(1)}{i}_c\overset{(1)}{i}_b - \overset{(2)}{i}_c\overset{(2)}{i}_b), \\ \overset{(4)}{h}_{cb} &= \rho(\overset{(1)}{i}_c\overset{(2)}{i}_b + \overset{(2)}{i}_c\overset{(1)}{i}_b), \end{aligned} \tag{4.12}$$

$$\begin{aligned} \overset{(3)}{h}_{cb} &= \bar{\rho}(\overset{(1)}{\bar{i}}_c\overset{(1)}{\bar{i}}_b - \overset{(2)}{\bar{i}}_c\overset{(2)}{\bar{i}}_b), \\ \overset{(4)}{\bar{h}}_{cb} &= \bar{\rho}(\overset{(1)}{\bar{i}}_c\overset{(2)}{\bar{i}}_b + \overset{(2)}{\bar{i}}_c\overset{(1)}{\bar{i}}_b), \end{aligned} \tag{4.12$^-$}$$

$$\begin{aligned} v_c = {} & (\delta_2 \log\rho + 2\beta)\overset{(1)}{i}_c \\ & - (\delta_1 \log\rho + 2\alpha)\overset{(2)}{i}_c, \end{aligned} \tag{4.13}$$

$$\begin{aligned} \bar{v}_c = {} & (\bar{\delta}_2 \log\bar{\rho} + 2\bar{\beta})\overset{(1)}{\bar{i}}_c \\ & - (\bar{\delta}_1 \log\bar{\rho} + 2\bar{\alpha})\overset{(2)}{\bar{i}}_c. \end{aligned} \tag{4.13$^-$}$$

Let us express $\bar{i}_c^{(1)}, \bar{i}_c^{(2)}$ in terms of $i_c^{(1)}, i_c^{(2)}$; thus

$$(4.14)\quad \bar{i}_c^{(1)} = i_c^{(1)} \cos\theta + i_c^{(2)} \sin\theta,\quad \epsilon\bar{i}_c^{(2)} = -i_c^{(1)} \sin\theta + i_c^{(2)} \cos\theta \quad (\epsilon = \pm 1).$$

From (4.11) and (4.11⁻) we have

$$(4.15)\qquad \bar{\rho} = \epsilon'\rho \qquad (\epsilon' = \pm 1).$$

If we use (4.14) and (4.15) in (4.12⁻) and (4.13⁻), and then take account of (4.12) and (4.13), we have by straightforward calculation:

$$(4.16)\qquad \begin{aligned} \bar{h}_{cb}^{(3)} &= \epsilon'(h_{cb}^{(3)} \cos 2\theta + h_{cb}^{(4)} \sin 2\theta), \\ \bar{h}_{cb}^{(4)} &= \epsilon\epsilon'(-h_{cb}^{(3)} \sin 2\theta + h_{cb}^{(4)} \cos 2\theta), \end{aligned}$$

$$(4.17)\qquad \bar{v}_c = \epsilon[v_c - 2(i_c^{(1)}\delta_1\theta + i_c^{(2)}\delta_2\theta)] = \epsilon[v_c - B_c^\lambda\nabla_\lambda(2\theta)].$$

Let us now replace the normal vectors $i_{(3)}^\kappa, i_{(4)}^\kappa$ to V_2 by the following mutually orthogonal unit vectors

$$'i_{(3)}^\kappa = \epsilon'(i_{(3)}^\kappa \cos 2\theta + i_{(4)}^\kappa \sin 2\theta),\qquad 'i_{(4)}^\kappa = \epsilon\epsilon'(-i_{(3)}^\kappa \sin 2\theta + i_{(4)}^\kappa \cos 2\theta),$$

which are normal to V_2. Then the corresponding tensors $'h_{cb}^{(3)}, 'h_{cb}^{(4)}, 'v_c$ for V_2 are

$$\begin{aligned} 'h_{cb}^{(3)} &= -B_c^\lambda B_b^\kappa \nabla_\lambda' i_\kappa^{(3)} = \epsilon'(h_{cb}^{(3)} \cos 2\theta + h_{cb}^{(4)} \sin 2\theta), \\ 'h_{cb}^{(4)} &= -B_c^\lambda B_b^\kappa \nabla_\lambda' i_\kappa^{(4)} = \epsilon\epsilon'(-h_{cb}^{(3)} \sin 2\theta + h_{cb}^{(4)} \cos 2\theta), \\ 'v_c &= \epsilon[v_c - B_c^\lambda\nabla_\lambda(2\theta)]. \end{aligned}$$

Since these tensors are identical with $\bar{h}_{cb}^{(3)}, \bar{h}_{cb}^{(4)}, \bar{v}_c$, respectively, we may conclude by Eisenhart's theorem that $\overline{V}_2$ differs from V_2 only by a motion in S_4.

4.3. **General minimal surface.** For a general minimal V_2 (cf. §4.1), we have (4.1)–(4.5) and $\rho_1^2 \neq \sigma_3^2$. In conformity with (4.3) we write

$$(4.18)\qquad \rho_1 = (C-K)^{1/2}\cos w,\qquad \sigma_3 = (C-K)^{1/2}\sin w,$$

where w is a scalar, so that

$$(4.19)\qquad \rho_1^2 - \sigma_3^2 = (C-K)\cos 2w \neq 0.$$

We observe that since ρ_1, σ_3 are numerically equal to the semi-axes of the conic (G), and $\sigma_3/\rho_1 = \tan w$, w determines the shape of (G).

On account of (4.18), equations (4.6) become

$$(4.20)\quad \delta_1 \log[(C-K)\cos 2w] + 4\alpha = 0,\ \delta_2 \log[(C-K)\cos 2w] + 4\beta = 0.$$

Or, if we write for convenience

$$(4.21)\qquad \xi = 4^{-1}\log[(C-K)\cos 2w],$$

$$(4.20')\qquad \delta_1\xi + \alpha = 0,\qquad \delta_2\xi + \beta = 0.$$

The integrability condition of (4.20′) is

$$\delta_2\alpha - \delta_1\beta = 0. \tag{4.22}$$

This shows that the congruences $i^a_{(1)}$, $i^a_{(2)}$ form an isometric net in V_2. The curves of the congruences $i^a_{(1)}$, $i^a_{(2)}$ are called the *lines of curvature* of the general minimal V_2 in S_4 for the reason that the normal curvature of V_2 in S_4 has stationary values in the directions $i^a_{(1)}$, $i^a_{(2)}$, being equal to the semi-axes of (G). Hence we have the following theorem.

THEOREM 4.4. *The lines of curvature of a general minimal V_2 in S_4 form an isometric net.*

To eliminate $i^a_{(1)}$, $i^a_{(2)}$ from (4.20′), we need only calculate $\Delta_2\xi$, thus

$$\Delta_2\xi = (\delta_1\delta_1 + \delta_2\delta_2 + \alpha\delta_1 + \beta\delta_2)\xi = -(\delta_1\alpha + \delta_2\beta + \alpha^2 + \beta^2) = K.$$

Therefore,

$$\Delta_2\xi = K; \tag{4.23′}$$

or, substituting back the value (4.21) for ξ,

$$\Delta_2 \log \left[(C - K)\cos 2w\right] = 4K. \tag{4.23}$$

Let us now solve $(4.4)_{1,2}$ for ν_1, ν_2, then use the values of ρ_1 and σ_3 from (4.18), and finally eliminate $(C-K)$ by means of (4.20). Then the result is, after simplification,

$$\begin{aligned} \nu_1 &= \frac{\delta_2 w}{\cos 2w} = 2^{-1}\delta_2 \log \frac{1+\sin 2w}{\cos 2w}, \\ \nu_2 &= \frac{-\delta_1 w}{\cos 2w} = -2^{-1}\delta_1 \log \frac{1+\sin 2w}{\cos 2w}. \end{aligned} \tag{4.24}$$

Using these in (4.5), we get

$$\Delta_2 \log \frac{1+\sin 2w}{\cos 2w} + 2(C-K)\sin 2w = 0. \tag{4.25}$$

Equations (4.23) and (4.25) can be rewritten as

$$\begin{aligned} \Delta_2 \log \cos 2w &= \Delta_2 \log (1 + \sin 2w) + 2(C-K)\sin 2w \\ &= -\left[\Delta_2 \log (C-K) - 4K\right]. \end{aligned} \tag{4.26}$$

We now prove the following theorem.

THEOREM 4.5. *A necessary and sufficient condition for a ds^2 to be the linear element of a general minimal V_2 in an S_4 of scalar curvature C is that there exist a solution w of the equations* (4.23) *and* (4.25). *If the condition is satisfied, then w determines the shape of the curvature conic of V_2 in S_4.*

Before we proceed to prove this theorem, we remark that it is highly desirable to be able to eliminate w from (4.26), thus replacing the condition (4.26) by one that involves only invariant quantities for ds^2. But so far the author has not succeeded in doing this.

Proof. We have seen that the condition stated in the theorem is necessary. To prove that it is also sufficient, we need only show that given a ds^2 satisfying the condition and given a solution w of (4.23) and (4.25), there exist two mutually orthogonal unit vectors $i^a_{(1)}$, $i^a_{(2)}$ satisfying (4.20). For, if this is true, we may obtain ρ_1, σ_3 from (4.18), and ν_1, ν_2 from (4.24). Then, on account of (4.25), these scalars, together with $i^a_{(1)}$, $i^a_{(2)}$, satisfy the equations (4.3)–(4.5).

Let $i^{*a}_{(1)}$, $i^{*a}_{(2)}$ be an isometric net of ds^2, and indicate quantities belonging to them by a star. Then

$$\delta_2^*\alpha^* - \delta_1^*\beta^* = 0, \tag{4.22*}$$

and consequently, there exists a scalar ξ^* satisfying

$$\delta_1^*\xi^* + \alpha^* = 0, \qquad \delta_2^*\xi^* + \beta^* = 0. \tag{4.20*}$$

This ξ^* is a solution of (4.23′) for ξ, as is evident from the way this equation was derived. Now by hypothesis (4.23), the ξ defined by (4.21) is also a solution of (4.23′). Therefore

$$\xi = \xi^* + \psi, \tag{4.27}$$

where ψ is a scalar such that

$$\Delta_2\psi = (\delta_1^*\delta_1^* + \delta_2^*\delta_2^* + \alpha^*\delta_1^* + \beta^*\delta_2^*)\psi = 0. \tag{4.28}$$

We wish to find the most general pair of mutually orthogonal unit vectors

$$\begin{aligned} i^a_{(1)} &= i^{*a}_{(1)} \cos\theta + i^{*a}_{(2)} \sin\theta, \\ i^a_{(2)} &= \epsilon(- i^{*a}_{(1)} \sin\theta + i^{*a}_{(2)} \cos\theta) \qquad (\epsilon = \pm 1) \end{aligned} \tag{4.29}$$

satisfying (4.20′). Written in terms of $i^{*a}_{(1)}$, $i^{*a}_{(2)}$, equations (4.20′) are (cf. (1.8)–(1.10))

$$\epsilon(-\sin\theta\,\delta_1^* + \cos\theta\,\delta_2^*)\xi - \epsilon[(-\beta^* + \delta_1^*\theta)\cos\theta + (\alpha^* + \delta_2^*\theta)\sin\theta] = 0,$$
$$(\cos\theta\,\delta_1^* + \sin\theta\,\delta_2^*)\xi + [-(-\beta + \delta_1^*\theta)\sin\theta + (\alpha^* + \delta_2^*\theta)\cos\theta] = 0,$$

which are equivalent to

$$\delta_1^*\xi + \alpha^* + \delta_2^*\theta = 0, \qquad \delta_2^*\xi + \beta^* - \delta_1^*\theta = 0.$$

These become, on account of (4.27) and (4.20*),

$$\delta_1^*\psi + \delta_2^*\theta = 0, \qquad \delta_2^*\psi - \delta_1^*\theta = 0. \tag{4.30}$$

The integrability condition of (4.30) for θ is (4.28), and is therefore satisfied. Hence (4.30) admits a unique solution, to within an additive constant, for θ.

In other words, for the solution (4.21) of (4.23′), equations (4.20′) have solutions for $i^a_{(1)}$, $i^a_{(2)}$, and, if $i^{*a}_{(1)}$, $i^{*a}_{(2)}$ is a solution, the most general solution is (4.29) with $\theta=$const. Theorem 4.5 is thus completely proved.

Incidentally we observe that all the results so far obtained for the general minimal V_2 remain true when $\sigma_3=0$, that is, when $w=0$. But $w=0$ implies that $h^{(4)}_{cb}=0$ and $v_c=0$, as follows from (4.1) and (4.24). Therefore, this is the case of a minimal surface in an S_3 of scalar curvature C (cf. (7.2) and Schouten-Struik [15, p. 150]). Since $\rho_1\neq 0$ by (4.19), this minimal surface in S_3 is not a plane surface. Hence we have from (4.26) the following corollary to Theorem 4.5:

COROLLARY 4.5. *A necessary and sufficient condition for a ds^2 to be the linear element of a non-plane minimal surface in an S_3 of scalar curvature C is that $\Delta_2 \log (C-K)=4K$ be satisfied.*

This result can of course be proved much more easily by using the fundamental equations (7.2) for a V_2 in S_3 and should be compared with one of Ricci's [14, p. 340] which states that a necessary and sufficient condition for a ds^2 to be the linear element of a V_2 with *constant* mean curvature c in an R_3 is that $\Delta_2 \log (c^2-K)=4K$ be satisfied.

By using the last part of the proof of Theorem 4.5 we can prove the following theorem.

THEOREM 4.6. *In an S_4 a general minimal V_2^* which is not a minimal surface in an S_3 can be deformed continuously into ∞^1 non-congruent minimal V_2's with equal curvature conics at corresponding points. Any minimal surface $\overline{V}_2$ in S_4 applicable but not congruent to V_2^* and with curvature conics equal to those of V_2^* at corresponding points is congruent to one of these ∞^1 minimal V_2's. The image on V_2^* of the lines of curvature of any one of these V_2's makes a constant angle with the lines of curvature of V_2^*.*

Proof. In an S_4 let V_2^* be a general minimal surface which is not a minimal surface in an S_3, and let $\overline{V}_2$ be another which is applicable but not congruent to V_2^* such that the curvature conics of V_2^* and $\overline{V}_2$ are equal at corresponding points. We use a star and a dash to indicate the quantities and equations belonging to V_2^*, $\overline{V}_2$, respectively, and suppose that equations (4.1*), (4.1⁻) are satisfied. Then since ρ_1^*, σ_3^* ($\bar{\rho}_1$, $\bar{\sigma}_3$) are numerically equal to the semi-axes of (G^*) $((\overline{G}))$, the vectors $\bar{i}^\kappa_{(p)}$, which have been essentially determined by (4.1⁻), can be readjusted so that we have

$$(4.31) \qquad \bar{w} = w^*, \qquad \bar{\rho}_1 = \rho_1^* \neq 0, \qquad \bar{\sigma}_3 = \sigma_3^* \neq 0.$$

From this and (4.21) we have

$$(4.32) \qquad \bar{\xi} = \xi^*.$$

Since $i^{*a}_{(1)}$, $i^{*a}_{(2)}$ satisfy (4.20′) with $\xi=\xi^*$, the most general solution for

$\dot{i}^a_{(1)}, \dot{i}^a_{(2)}$ of (4.20′) with $\xi=\xi^*$ is, as we have seen,

$$\text{(4.33)} \qquad \dot{i}^a_{(1)} = \dot{i}^{*a}_{(1)} \cos\gamma + \dot{i}^{*a}_{(2)} \sin\gamma, \qquad \epsilon\dot{i}^a_{(2)} = -\dot{i}^{*a}_{(1)} \sin\gamma + \dot{i}^{*a}_{(2)} \cos\gamma,$$

where $\gamma=$const. and $\epsilon=\pm 1$. The $ds^2=ds^{*2}$, $w=w^*$, and the above $\dot{i}^a_{(1)}, \dot{i}^a_{(2)}$ give a solution of the fundamental equations for the imbedding of a V_2 with linear element ds^2 in S_4 as a minimal V_2 whose curvature conics are, by (4.18), equal to those of V_2^* at corresponding points. The second fundamental tensors for V_2 are, by (4.31) and (4.33),

$$\text{(4.34)} \qquad \begin{aligned} h^{(3)}_{cb} &= \rho_1(\dot{i}^{(1)}_c\dot{i}^{(1)}_b - \dot{i}^{(2)}_c\dot{i}^{(2)}_b) = h^{*(3)}_{cb} \cos 2\gamma + h^{*(4)}_{cb} \frac{\rho_1^*}{\sigma_3^*} \sin 2\gamma, \\ h^{(4)}_{cb} &= \sigma_3(\dot{i}^{(1)}_c\dot{i}^{(2)}_b + \dot{i}^{(2)}_c\dot{i}^{(1)}_b) = -h^{*(3)}_{cb} \frac{\sigma_3^*}{\rho_1^*} \sin 2\gamma + h^{*(4)}_{cb} \cos 2\gamma. \end{aligned}$$

To find v_c, we have from (4.24) and (4.33)

$$\nu_1 = \frac{\delta_2 w}{\cos 2w} = \epsilon(\nu_2^* \sin\gamma + \nu_1^* \cos\gamma), \qquad \nu_2 = \frac{-\delta_1 w}{\cos 2w} = +\nu_2^* \cos\gamma - \nu_1^* \sin\gamma,$$

and therefore

$$\text{(4.35)} \qquad v_c = \nu_1\dot{i}^{(1)}_c + \nu_2\dot{i}^{(2)}_c = \epsilon(\nu_1^*\dot{i}^{*(1)}_c + \nu_2^*\dot{i}^{*(2)}_c) = \epsilon v^*_c.$$

Since $(\sigma_3^*/\rho_1^*)^2 \neq 1$, it is seen from (4.34) that unless $\sin 2\gamma=0$, $h^{(3)}_{cb}$ and $h^{(4)}_{cb}$ cannot be obtained from $h^{*(3)}_{cb}$ and $h^{*(4)}_{cb}$ by a change of $\dot{i}^{*\kappa}_{(3)}$ and $\dot{i}^{*\kappa}_{(4)}$. Therefore for any value of γ such that $\sin 2\gamma\neq 0$, the corresponding V_2 is not congruent to V_2^*.

As a consequence of (4.32), the minimal V_2 will become the given $\overline{V}_2$ for some (constant) value $\bar{\gamma}$ of γ. And the last part of Theorem 4.6 follows at once from (4.33) with $\gamma=\bar{\gamma}$.

Although direct verification may not be easy, it is conceivable that for the case of an R_4, the ∞^1 minimal V_2's in Theorem 4.6 are the *associate* minimal surfaces to V_2^* defined by Eisenhart [6, §8].

It follows from the preceding results that the problem of the deformation of a general minimal V_2 in S_4 now reduces to the following problem: Given a ds^2 for which equations (4.26) have a known particular solution for w, to find the most general solution. The author has not yet been able to solve the latter problem.

4.4. **Further consequences of** (4.23) **and** (4.25). Equation (4.25) shows that w cannot be constant. Hence we have the following theorem.

THEOREM 4.7. *The curvature conics of an R-surface in S_4 are all circles, and those of a minimal V_2 in an S_3, considered as a V_2 in S_4, are all line segments. No other minimal V_2 in an S_4 has all its curvature conics at different points similar to one another.*

We now prove the following theorem.

THEOREM 4.8. *In an S_4 a plane surface and an R-surface with constant Gaussian curvature are the only minimal V_2 for which the sum of the axes of the curvature conic is constant all over the surface.*

Proof. That a plane surface in S_4 has the property mentioned in the theorem is obvious, and that an R-surface with constant Gaussian curvature has the same property follows at once from Theorem 4.3. Excluding these two cases we now suppose that the ρ_1, σ_3 for a general minimal V_2 are both positive (with suitable choice of the senses of $i^{\kappa}_{(p)}$). Then we have from (4.18) that the sum of the semi-axes of G) is

$$\rho_1 + \sigma_3 = (C - K)^{1/2}(\cos w + \sin w).$$

If this is constant, we have, by squaring, that

$$(C - K)(1 + \sin 2w) = \text{const.} = A. \tag{4.36}$$

Now equation (4.25) can be written as

$$\Delta_2 \log \frac{(C - K)(1 + \sin 2w)}{(C - K) \cos 2w} + 2(C - K)(1 + \sin 2w) - 2(C - K) = 0,$$

which, in consequence of the preceding equation and (4.23), reduces to

$$- 4K + 2A - 2(C - K) = 0, \quad \text{that is,} \quad K = A - C = \text{const.}$$

This together with (4.36) would require w to be a constant, and that contradicts (4.25). Therefore Theorem 4.8 is proved.

Finally we shall prove the following theorem.

THEOREM 4.9. *There exists no minimal V_2 in S_4, other than an R-surface with constant Gaussian curvature or a minimal V_2 in an S_3 considered as a V_2 in S_4, for which the curvature conics all have equal area.*

Proof. For the first exceptional case stated in the theorem, the area of all the curvature conics is equal to $2c\pi$ (cf. Theorem 4.3); for the second exceptional case, the curvature conic is always a line segment, and therefore its area is zero. Excluding these two cases, if the curvature conic (G) of a general minimal V_2 in S_4 has constant area, we have, from (4.18),

$$\rho_1\sigma_3 = (C - K) \sin w \cos w = \text{const.} \equiv 2^{-1}A,$$

that is,

$$(C - K) \sin 2w = A. \tag{4.37}$$

We first note from this that K *cannot be constant,* otherwise $w = \text{const.}$ Equations (4.23), (4.25) can be written, respectively, as

$$\Delta_2 \log [(C-K)\cos 2w]^2 = 8K,$$

$$\Delta_2 \log \left(\frac{1+\sin 2w}{\cos 2w}\right)^2 + 4(C-K)\sin 2w = 0.$$

In virtue of (4.37), these become

$$\Delta_2 \log [A^2 - (C-K)^2] = 8K, \tag{4.38}$$

$$\Delta_2 \log \frac{A+(C-K)}{A-(C-K)} = -4A, \tag{4.39}$$

which are equivalent to

$$\Delta_2 \log (A+C-K) = 2(2K-A), \quad \Delta_2 \log (A-C+K) = 2(2K+A).$$

Since $\Delta_2 f(K) = f'\Delta_2 K + f''\Delta_1 K$, the preceding equations can be written

$$(K-C-A)\Delta_2 K - \Delta_1 K = 2(2K-A)(K-C-A)^2,$$
$$(K-C+A)\Delta_2 K - \Delta_1 K = 2(2K+A)(K-C+A)^2.$$

Solving these for $\Delta_2 K$, $\Delta_1 K$, we have after simplification

$$\begin{aligned} \Delta_2 K &= \phi(K) \equiv 2(5K^2 - 6CK + C^2 + A^2) = 10K^2 + \cdots, \\ \Delta_1 K &= f(K) \equiv 2(3K-C)[(K-C)^2 - A^2] = 6K^3 + \cdots, \end{aligned} \tag{4.40}$$

where the dots indicate terms of lower degrees in K.

It is known (cf. Eisenhart [7, p. 97]) that a ds^2 for which some function K satisfies equations of the form (4.40) can be reduced to

$$ds^2 = EdK^2 + Gdt^2 \equiv \frac{1}{f}\left(dK^2 + \exp\left(2\int \phi f^{-1} dK\right)dt^2\right).$$

For this, the Gaussian curvature K is given by

$$K = -\frac{1}{(EG)^{1/2}}\left[\frac{1}{E^{1/2}}(G^{1/2})'\right]',$$

where the prime denotes differentiation with respect to K. Substituting here the values of E, G, we have after straightforward calculation

$$Kf + (\phi - f')(\phi - 2^{-1}f') + (\phi' - 2^{-1}f'')f = 0. \tag{4.41}$$

In consequence of (4.40), this is a polynomial equation in K, whose term of the highest degree is $10K^4$. Therefore K must be constant. But this has been seen to be impossible. Hence Theorem 4.9 is proved.

4.5. **Linear element of a minimal V_2 in an R_4.** Let us now find the linear element

$$ds^2 = 2Fdudv \tag{4.42}$$

of a minimal V_2 in an R_4. We have for (4.42)

$$(4.43)\qquad \Delta_2 = \frac{2}{F}\partial_v\partial_u, \qquad K = -\frac{1}{F}\partial_v\partial_u \log F.$$

Consider first *the case of an R-surface.* On account of (4.43), equation (4.10) can be written

$$2F^{-1}\partial_v\partial_u \log(-K) = -6F^{-1}\partial_v\partial_u \log F,$$

that is,

$$\partial_v\partial_u \log(-KF^3) = 0.$$

Therefore

$$-KF^3 = U_1V_1,$$

where(9) U_1, V_1 are nonzero functions of u, v alone. Substitution for K from (4.43) gives

$$\partial_v\partial_u \log F = U_1V_1F^{-2}.$$

Reducing this to Liouville's form by putting $\lambda = F(U_1V_1)^{-1/2}$, we see that the solution for F is (cf. (1.16), (1.17))

$$F^{-2} = \frac{U_2'V_2'}{U_1V_1(1+U_2V_2)},$$

where U_2, V_2 are non-constant functions of u, v alone. Therefore

$$ds^2 = 2(1+U_2V_2)\left(\frac{U_1V_1}{U_2'V_2'}\right)^{1/2} dudv,$$

or, after an obvious transformation on u, v:

$$(4.44)\qquad ds^2 = 2(1+UV)dudv,$$

U, V being arbitrary non-constant functions of u, v alone. Hence we have the following theorem.

THEOREM 4.10. *A necessary and sufficient condition for a ds^2 to be the linear element of an R-surface in R_4 is that it be reducible to the form* (4.44).

It can readily be shown that by a suitable change of parameters (4.44) can be reduced to the form

$$(4.45)\qquad ds^2 = (1+z_x^2+t_y^2)(dx^2+dy^2) \qquad \left(z_x = \frac{\partial z}{\partial x},\ t_y = \frac{\partial t}{\partial y}\right),$$

where z, t are the real and imaginary parts of an arbitrary analytic function

(9) The V_2 here which denotes a function of v should not be confused with the usual V_2 which denotes a surface.

$F(x+(-1)^{1/2}y)$. In an R_4 with rectangular Cartesian coordinates (X, Y, Z, T), (4.45) is the linear element of the surface

$$X = x, \qquad Y = y, \qquad Z = z(x, y), \qquad T = t(x, y).$$

It was in this way that the R-surface in R_4 was first studied by Kwietniewski [12] and Kommerell [11].

It is not difficult to prove from (4.44) or directly from (4.10) the following theorem.

THEOREM 4.11. *If ds^2 is the linear element of an R-surface in R_4, then $e^{2\sigma}ds^2$ will also be one if $\Delta_2\sigma = 0$.*

We now consider *the ds^2 for a general minimal V_2 in R_4.* We rewrite equations (4.23) and (4.25) with $C=0$ as

$$\Delta_2 \log \cos 2w + \Delta_2 \log (-K) - 4K = 0, \tag{4.46}$$

$$\Delta_2 \log (1 + \sin 2w) + \Delta_2 \log (-K) - 2K - 2K(1 + \sin 2w) = 0. \tag{4.47}$$

On account of (4.43), equation (4.46) can be reduced to

$$\partial_v\partial_u \log (-KF^2 \cos 2w) = 0.$$

Therefore

$$-KF^2 \cos 2w = U(u)V(v). \tag{4.48}$$

Similarly, equation (4.47) can be written

$$2F^{-1}\partial_v\partial_u \log [-FK(1 + \sin 2w)] + 2F^{-1}[-FK(1 + \sin 2w)] = 0.$$

Omitting the factor F^{-1}, this equation takes Liouville's form and therefore has the solution

$$-FK(1 + \sin 2w) = \frac{2\overline{U}'\overline{V}'}{(1 + \overline{U}\overline{V})^2} \neq 0, \tag{4.49}$$

where $\overline{U}$, $\overline{V}$ are arbitrary non-constant functions of u, v alone. Eliminating w from (4.48) and (4.49) we get

$$-FK = \frac{(UV)^2}{4\overline{U}'\overline{V}'}(1 + \overline{U}\overline{V})^2F^{-2} + \frac{\overline{U}'\overline{V}'}{(1 + \overline{U}\overline{V})^2},$$

which can be written, by using (4.43),

$$\partial_v\partial_u \log [(1 + \overline{U}\overline{V})F^{-1}]^2 = -\frac{(UV)^2}{4\overline{U}'\overline{V}'}[(1 + \overline{U}\overline{V})F^{-1}]^2.$$

This again has the form of a Liouville's equation, and therefore its solution is

$$\frac{(UV)^2}{4\overline{U}'\overline{V}'}[(1 + \overline{U}\overline{V})F^{-1}]^2 = \frac{2U^{*\prime}V^{*\prime}}{2(1 + U^*V^*)^2} \neq 0.$$

Solving this for F, we see that a suitable transformation on u, v will reduce ds^2 to

$$(4.50)\qquad ds^2 = 2(1 + U_1V_1)(1 + U_2V_2)dudv,$$

where U_1, U_2; V_1, V_2 are some functions of u, v respectively, none of them being a constant.

When $U_2 = U_1$, $V_2 = V_1$, formula (4.50) becomes

$$(4.51)\qquad ds^2 = 2(1 + U_1V_1)^2 dudv,$$

which happens to be the well known linear element of a non-plane minimal V_2 in an R_3 (cf. Eisenhart [7, formula (100), p. 257]). We observe further that when $w=0$, elimination of K from (4.48) and (4.49) also leads to (4.51). Hence we have proved the following theorem.

THEOREM 4.12. *A necessary and sufficient condition for a ds^2 to be the linear element of a general minimal surface in an R_4 (or a non-plane minimal surface in an R_3) is that it be reducible to the form* (4.50)(or (4.51)).

4.6. **More properties of the R-surface.** From the preceding results we have noticed the special position which the R-surface occupies among the minimal V_2 in S_4. We shall see further that the R-surface also occupies a very special position among some other types of surfaces in S_4. We first prove the following theorem.

THEOREM 4.13. *Among the surfaces in S_4 whose curvature conic at each point P is a circle (G), the one for which the center of (G) is at a constant distance d from P is either an R-surface (for which $d=0$) or a sphere*([10]) *in a plane S_3 in S_4 (for which the circle degenerates into a point).*

Proof. Let (G) be a circle, then by choosing $i^{\kappa}_{(3)}$, $i^{a}_{(1)}$ suitably (cf. $(4.7)_2$) we have

$$(4.52)\qquad \sigma_2 = -\sigma_1, \qquad \rho_3 = \sigma_1, \qquad \rho_2 = \rho_1, \qquad \sigma_3 = 0,$$

and the equations of (G) are

$$z = \rho_1 + \sigma_1 \sin 2\phi, \qquad t = \sigma_1 \cos 2\phi.$$

The fundamental equations (3.6) become

$$(4.53)\qquad C - K = 2\sigma_1^2 - \rho_1^2,$$

$$(4.54)\qquad \begin{aligned} &\delta_2\rho_1 - \delta_1\sigma_1 - 2\alpha\sigma_1 + \nu_2\sigma_1 = 0, \qquad &&\delta_1\rho_1 - \delta_2\sigma_1 - 2\beta\sigma_1 - \nu_1\sigma_1 = 0,\\ &\delta_2\sigma_1 + 2\beta\sigma_1 + \nu_1\sigma_1 - \nu_2\rho_1 = 0, \qquad &&-\delta_1\sigma_1 - 2\alpha\sigma_1 + \nu_2\sigma_1 - \nu_1\rho_1 = 0,\end{aligned}$$

$$(4.55)\qquad \delta_2\nu_1 - \delta_1\nu_2 + \beta\nu_1 - \alpha\nu_2 - 2\sigma_1^2 = 0.$$

([10]) A surface in a Riemannian 3-space V_3 is called a *sphere* if its normal curvature in V_3 is constant for every direction at every point.

If $\rho_1=0$, we have the case of an R-surface. If $\rho_1\neq 0$, we have on adding $(4.54)_{2,3}$ and substracting $(4.54)_{1,4}$,

$$(4.56)\qquad \nu_1 = -\delta_2 \log \rho_1, \qquad \nu_2 = \delta_1 \log \rho_1,$$

respectively. Substitution of these values for ν_1, ν_2 in (4.55) gives

$$-(\delta_1\delta_1 + \delta_2\delta_2 + \beta\delta_2 + \alpha\delta_1)\log \rho_1 - 2\overset{2}{\sigma_1} = 0,$$

that is

$$(4.57)\qquad \Delta_2 \log \rho_1 + 2\overset{2}{\sigma_1} = 0.$$

From this it follows that if $\rho_1=\text{const.}\neq 0$, σ_1 must be zero. This, together with (4.52), shows that the V_2 is a sphere in a plane S_3 in S_4 (cf. Schouten-Struik [15, pp. 99–101, 148–150]). Our theorem is thus proved.

Next we prove the following theorem.

THEOREM 4.14. *Among the surfaces V_2 in S_4 whose curvature conic (G) at each point P has a focus at P, the one for which (G) is of constant shape is an R-surface. A V_2 can be imbedded in an S_4 such that the curvature conic at each point P has a focus at P without making V_2 an R-surface, if and only if the linear element of V_2 is reducible to the form*

$$(4.58)\qquad ds^2 = du^2/\text{ch}^2\,\psi + dv^2/\text{sh}^2\,\psi,$$

where ψ satisfies a certain partial differential equation of the fourth order in the independent variables u, v.

Proof. From $(3.7)_3$ we can always choose $i^{\kappa}_{(3)}$, $i^{a}_{(1)}$ so that

$$(4.59)\qquad \rho_3 = \sigma_1 = \sigma_2 = 0, \qquad \overset{2}{\sigma_3} = -\rho_1\rho_2 \neq 0.$$

We have an R-surface if and only if $\rho_2=-\rho_1$. The fundamental equations (3.6) become

$$(4.60)\qquad C - K = 2\overset{2}{\sigma_3},$$

$$(4.61)\qquad \begin{aligned} &\delta_2\rho_1 + \beta(\rho_1-\rho_2) - \nu_1\sigma_3 = 0, \qquad \delta_1\rho_2 - \alpha(\rho_1-\rho_2) - \nu_2\sigma_3 = 0,\\ &\delta_1\sigma_3 + 2\alpha\sigma_3 + \nu_2\rho_1 = 0, \qquad \delta_2\sigma_3 + 2\beta\sigma_3 + \nu_1\rho_2 = 0,\end{aligned}$$

$$(4.62)\qquad \delta_2\nu_1 - \delta_1\nu_2 + \beta\nu_1 - \alpha\nu_2 + (\rho_1-\rho_2)\sigma_3 = 0.$$

Multiply $(4.61)_{1,4}$ by ρ_2, σ_3 respectively, and add. The result is

$$\rho_2\delta_2\rho_1 + \beta\rho_2(\rho_1-\rho_2) + 2^{-1}\delta_2\overset{2}{\sigma_3} + 2\beta\overset{2}{\sigma_3} = 0.$$

Using here the value for σ_3 from (4.59), we have after simplification

$$(4.63)_1\qquad 2^{-1}\delta_2 \log\frac{\rho_1}{\rho_2} - \beta\left(\frac{\rho_2}{\rho_1}+1\right) = 0.$$

Similarly from $(4.61)_{2,3}$ we can prove that

$$(4.63)_2 \qquad 2^{-1}\delta_1 \log \frac{\rho_2}{\rho_1} - \alpha\left(\frac{\rho_1}{\rho_2} + 1\right) = 0.$$

If (G) is of constant shape,

$$(4.64) \qquad \rho_2/\rho_1 = \text{const.} = -A \neq 0.$$

Then (4.63) become

$$\beta(\rho_2/\rho_1 + 1) = 0, \qquad \alpha(\rho_1/\rho_2 + 1) = 0.$$

Therefore either $\rho_1+\rho_2=0$ so that V_2 is an R-surface, or $\alpha=\beta=0$. For the latter case, Liouville's formula (1.6) tells us that $K=0$. Therefore it follows from (4.59)–(4.61) that

$$A\rho_1^2 = 2\sigma_3^2 = C = \text{const.}, \qquad \nu_1 = \nu_2 = 0.$$

These, however, contradict (4.62). Hence the R-surface is the only V_2 of this type for which (G) is of constant shape.

Now excluding the case of an R-surface, we can put

$$(4.65) \qquad \rho_1/\rho_2 = -\operatorname{th}^2\psi \qquad (\psi \neq \text{const.}).$$

Then equations (4.63) can be reduced to

$$(4.66) \qquad \delta_1\psi = -\alpha \operatorname{th}\psi, \qquad \delta_2\psi = -\beta \coth\psi.$$

Taking $i^a_{(1)}$, $i^a_{(2)}$ as parametric curves, we have

$$ds^2 = Edu^2 + Gdv^2,$$

and equations (4.66) are equivalent to (cf. (1.13))

$$\partial_u \log (G^{1/2} \operatorname{sh}\psi) = 0, \qquad \partial_v \log (E^{1/2} \operatorname{ch}\psi) = 0,$$

which give

$$G \operatorname{sh}^2\psi = 1, \qquad E \operatorname{ch}^2\psi = 1,$$

after an obvious change of u, v. Hence (4.58) is proved.

From (4.59) and (4.65) it follows that

$$(4.67) \qquad \rho_1 = \sigma_3 \operatorname{th}\psi, \qquad \rho_2 = -\sigma_3 \coth\psi.$$

Putting these in $(4.61)_{3,4}$ and solving for ν_1, ν_2 we have

$$(4.68) \qquad \nu_1 = (\delta_2 \log \sigma_3 + 2\beta) \operatorname{th}\psi, \qquad \nu_2 = -(\delta_1 \log \sigma_3 + 2\alpha) \coth\psi.$$

Now substitute in (4.62) these values for ν_1 and ν_2, the values for α, β from (4.66), and finally the value for σ_3 from (4.59), where K is to be calculated from (4.58) and is therefore a differential expression of the second order in ψ. Then the result is a differential equation of the fourth order in ψ with independent variables u, v. That this differential equation be satisfied is evidently a necessary and sufficient condition for the V_2 with linear element (4.58) to be imbeddable in S_4 in the manner required in Theorem 4.14. Hence the theorem is proved.

Let us remark that the Kommerell conic of this type of V_2 is always a circle (see end of §2.3), and that a detailed discussion of this type of V_2 in an R_4 can be found in Calapso's papers [3, 4].

5.1. **Ruled surface in S_4 (cf. Coburn [5]).** By definition a *ruled* surface in S_4 is a V_2 which contains a congruence of geodesics of S_4. In particular, it is a *developable* if the geodesics of this congruence envelope a curve in V_2. A ruled V_2 which is not a developable will be called a *skew* ruled V_2. It is well known that a geodesic of a Riemannian V_n is also a geodesic of any subspace in which it lies, and that the normal curvature vector of the subspace in V_n for the direction of the geodesic is zero. Therefore the curvature conic (G) at each point P of a ruled V_2 in S_4 passes through P, the point P on (G) being the end point of the zero normal curvature vector of V_2 for the geodesic of the congruence at P.

As we saw in $(3.7)_4$, we can choose $i^{\kappa}_{(p)}$, $i^{a}_{(c)}$ for a ruled V_2 in S_4 so that

(5.1) $$\rho_1 = \sigma_1 = \sigma_3 = 0,$$

(5.2) $$\beta = 0.$$

We know (Schouten-Struik [15, pp. 99–101, 148–150]) that a surface in an S_4 is a developable if and only if the curvature $H_{cb}^{\cdot\cdot\kappa}$ is of rank 1. Therefore it follows from (5.1) that a ruled V_2 in S_4 is a developable if and only if $\rho_3 = 0$.

Let us first study *the skew ruled V_2 in S_4*. Equations $(3.6)_1$ is

(5.3) $$C - K = \rho_3^2.$$

Equation $(3.6)_4$ is $\nu_1\rho_3 = 0$, which gives, since $\rho_3 \neq 0$,

(5.4) $$\nu_1 = 0.$$

Then equation $(3.6)_2$ becomes

(5.5) $$\delta_1\rho_3 + 2\alpha\rho_3 = 0.$$

Substitution of the value for ρ_3 from (5.3) gives

(5.5′) $$\delta_1 \log (C - K) + 4\alpha = 0.$$

Equation $(3.6)_3$ is

(5.6) $$\delta_1\rho_2 + \alpha\rho_2 - \delta_2\rho_3 = 0.$$

Solving equation $(3.6)_5$ for ν_2, we have

$$\nu_2 = -\frac{1}{\rho_3}(\delta_1\sigma_2 + \alpha\sigma_2),$$

which, because of (5.5), can be written

$$\nu_2 = -\delta_1\lambda + \alpha\lambda, \quad \text{where} \quad \lambda = \sigma_2/\rho_3. \tag{5.7}$$

Substituting this in the last equation of (3.6), we have

$$\delta_1\delta_1\lambda - \lambda(\delta_1\alpha + \alpha^2 - \rho_3^2) = 0.$$

But since $\beta = 0$, equation (1.6) reduces to $K = -(\delta_1\alpha + \alpha^2)$. On account of this and (5.3), the preceding equation simplifies into

$$\delta_1\delta_1\lambda = C\lambda. \tag{5.8}$$

Equations (5.1)–(5.8) are the fundamental equations for a skew ruled V_2 in S_4. Equation (5.5′) involves only K and the congruences $i^a_{(1)}$, $i^a_{(2)}$. Let this be satisfied by a V_2 and a geodesic congruence $i^a_{(1)}$ of it. Then when V_2 is referred to $i^a_{(1)}$, $i^a_{(2)}$ as parametric curves v=const. and u=const., equations (5.7) and (5.8) determine ρ_2 and λ to within one and two arbitrary functions of v, respectively. And for any λ thus determined, equation (5.7) gives a unique value for ν_2. Therefore (5.5′) is a necessary and sufficient condition for a given V_2 with a given geodesic congruence $i^a_{(1)}$ to be imbeddable in S_4 as a skew ruled V_2 with $i^a_{(1)}$ as rulings. Moreover, with $\rho_3 \neq 0$ and given $i^a_{(1)}$, equations (5.1) will determine $i^\kappa_{(p)}$ to within signs. Hence the arbitrariness of the imbedding is equal to the arbitrariness of the solution for the scalars ρ's, σ's and ν's. It is seen further that if we put $\lambda = 0$, the equations (5.1)–(5.8) reduce to a consistent system of equations; in fact, they reduce to the fundamental equations for a ruled V_2 in an S_3 of constant scalar curvature $C/3$ with second fundamental tensors $h_{cb} = \rho_1 i^{(1)}_c i^{(1)}_b + \rho_2 i^{(2)}_c i^{(2)}_b + \rho_3(i^{(1)}_c i^{(2)}_b + i^{(2)}_c i^{(1)}_b)$ (cf. (7.2) below). Hence we have the following theorem.

THEOREM 5.1. *Given on a V_2 a geodesic congruence $i^a_{(1)}$, V_2 can be imbedded in an S_4 of scalar curvature C as a skew ruled surface with $i^a_{(1)}$ as rulings if and only if* (5.5′) *is satisfied. If this condition is satisfied, the imbedding, to within a motion in S_4, depends on three arbitrary functions of a single variable. A skew ruled V_2 in S_4 can be deformed into a ruled V_2 in an S_3 of scalar curvature equal to that of S_4 such that rulings become rulings and that one of the second fundamental tensors of V_2 in S_4 is identical with the second fundamental tensor of V_2 in S_3 (cf. Theorem* 7.1).

5.2. **Certain pair of applicable skew ruled V_2 in S_4.** Let V_2, $\bar{V}_2$ be two applicable skew ruled surfaces in S_4 such that rulings correspond to rulings and the curvature conics of V_2, $\bar{V}_2$ at corresponding points are equal. We suppose that $i^\kappa_{(p)}$, $i^a_{(c)}$, $\bar{i}^\kappa_{(p)}$, $\bar{i}^a_{(c)}$ have been so chosen that equations (5.1)–(5.8), (5.1⁻)–(5.8⁻) hold. Here and in what follows we use a dash to indicate quantities and equations belonging to $\bar{V}_2$. Since $d\bar{s}^2 = ds^2$ and rulings correspond, the vectors $\bar{i}^a_{(c)}$, $i^a_{(c)}$ differ at most by signs, and therefore we may readjust the senses of $\bar{i}^a_{(c)}$ so that $\bar{i}^a_{(c)} = i^a_{(c)}$. Then

$$\bar{K} = K, \qquad \bar{\delta}_c = \delta_c, \qquad \bar{\alpha} = \alpha, \qquad \bar{\beta} = \beta = 0. \tag{5.9}$$

Now the curvature conics (G), $(\bar{G})$ of V_2, $\bar{V}_2$ at corresponding points are equal if and only if

$$\bar{\sigma}_2\bar{\rho}_3 = \pm\, \sigma_2\rho_3, \qquad \bar{\rho}_2^2 + \bar{\sigma}_2^2 + 4\bar{\rho}_3^2 = \rho_2^2 + \sigma_2^2 + 4\rho_3^2 \tag{5.10}$$

are satisfied. These two equations express the fact that (G), $(\bar{G})$ have equal areas and director circles (cf. (2.22), (2.23)).

Equations (5.1⁻) determine $\bar{i}^{\kappa}_{(p)}$ to within signs. Equations (5.3), (5.3⁻) and (5.9) give $\bar{\rho}_3 = \pm\rho_3$, from which and (5.10) we see that the senses of $\bar{i}^{\kappa}_{(p)}$ can be so adjusted that

$$\bar{\rho}_3 = \rho_3, \qquad \bar{\sigma}_2 = \sigma_2, \qquad \bar{\rho}_2 = \pm\, \rho_2. \tag{5.11}$$

If $\bar{\rho}_2 = \rho_2$, it follows from (5.11), (5.1)–(5.8) that the scalars ρ's, σ's, ν's are equal for V_2 and $\bar{V}_2$, and consequently V_2 and $\bar{V}_2$ are congruent.

If $\bar{\rho}_2 = -\rho_2$, (5.11) and $(3.7)_4$ show that the normal curvature of V_2 at P for the direction ϕ is equal to that of $\bar{V}_2$ at $\bar{P}$ for the direction $\bar{\phi} = -\phi$. In this case it follows from (5.9), (5.11), (5.6) and (5.6⁻) that equation (5.6) breaks up into the two equations

$$\delta_2\rho_3 = 0, \qquad \delta_1\rho_2 + \alpha\rho_2 = 0. \tag{5.12}$$

On account of (5.3), equation $(5.12)_1$ is the condition that the congruence of curves $K = \text{const.}$ is identical with $i^{a}_{(2)}$, that is, it is the orthogonal congruence to the rulings. Moreover, the integrability condition of (5.12) and (5.5) is $\delta_2\alpha = 0$, showing that each curve $K = \text{const.}$ has constant geodesic curvature.

It is readily seen from (5.6) that equations (5.12) hold for a skew ruled V_2 in S_4 if and only if $\rho_2 = 0$ is permissible as part of a solution for (5.1)–(5.8). Now if we put $\rho_2 = \lambda = 0$ in (5.1)–(5.8), the latter equations reduce to the fundamental equations of a *minimal* ruled V_2 in an S_3 (in S_4) whose scalar curvature is equal to twice that of S_4; this can be seen from (7.2). Hence for a skew ruled V_2 in S_4, equations (5.12) hold if and only if V_2 is deformable into a minimal ruled V_2 in an S_3 such that rulings correspond to rulings.

Summing up the preceding results we have the following theorems.

THEOREM 5.2. *On a skew ruled V_2 in S_4, the curves along which the Gaussian curvature K is constant are orthogonal to the rulings if and only if V_2 is deformable into a minimal ruled V_2 in an S_3 such that rulings become rulings. In such a ruled V_2 each of the curves $K = \text{const.}$ has constant geodesic curvature.*

THEOREM 5.3. *Let V_2, $\bar{V}_2$ be two applicable skew ruled surfaces in an S_4 with rulings corresponding to rulings and equal curvature conics at corresponding points. If V_2 is not deformable into a minimal ruled surface in an S_3 such that rulings become rulings, then V_2 and $\bar{V}_2$ are congruent. If V_2 is deformable in the manner just described, V_2 and $\bar{V}_2$ may or may not be congruent. If they are not then the normal curvature of V_2 at any point P for any direction i^a is equal to that* *of $\overline{V}_2$ at the corresponding point $\overline{P}$ for the direction whose reflection with respect to the ruling of $\overline{V}_2$ at $\overline{P}$ corresponds to i^α.*

When P is a vertex of (G), $\rho_2=0$ and λ determines the shape of (G), as is seen from figure 4. It follows from (5.8) that $\lambda=\text{const.}\neq 0$ is permissible as part of a solution to (5.1)–(5.8) when and only when $C=0$. Hence we have the following theorem.

THEOREM 5.4. *In an S_4 there exists a skew ruled surface whose curvature conic at each point P is a nondegenerate ellipse of constant shape having a vertex at P if and only if S_4 is an R_4.*

5.3. **The linear element of a skew ruled V_2 in S_4.** This linear element can be found from (5.5′). Referring to $i^\alpha_{(c)}$ as parametric curves, we have

$$ds^2 = du^2 + G dv^2, \tag{5.13}$$

$v=\text{const.}$ being the geodesic congruence $i^\alpha_{(1)}$. For (5.11),

$$\delta_1 = \partial_u, \ \delta_2 = \frac{\partial_v}{G^{1/2}}, \quad \alpha = \partial_u \log G^{1/2}, \ \beta = 0. \ K = -\frac{1}{G^{1/2}}\partial_u\partial_u G^{1/2}. \tag{5.14}$$

Equation (5.5′) is

$$\partial_u \log (C - K) + 2\partial_u \log G = 0,$$

that is,

$$(C - K)G^2 = - V_1, \tag{5.15}$$

where and throughout this section V_1, V_2[11], and so on, denote arbitrary functions of v alone. Using the value for K from (5.12), this becomes

$$CG^2 + (G^{1/2})^3\partial_u\partial_u G^{1/2} = - V_1,$$

integration of which gives

$$CG + (\partial_u G^{1/2})^2 = V_2 + V_1/G,$$

that is,

$$(\partial_u G)^2 = 4(- CG^2 + V_2 G + V_1). \tag{5.16}$$

In particular, when $C=0$, we have from this

$$G = V_2 u^2 + 2V_3 u + V_4. \tag{5.17}$$

From (5.3) and $(5.12)_1$, if the congruence $K=\text{const.}$ is orthogonal to the rulings, K is a function of u alone. This is the case if and only if $G=U(u)V(v)$, as is seen from (5.15) and (5.14). Hence we have the following theorem.

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(11) See footnote 9.

THEOREM 5.5. *A V_2 can be imbedded in an S_4 as a skew ruled surface with* v=const. *as rulings if and only if its linear element is reducible to the form* (5.13) *in which G satisfies* (5.16). *The congruence* K=const. *is orthogonal to the rulings if and only if G, as given by* (5.16), *is a product of a function of u alone by a function of v alone.*

5.4. **Developable in S_4.** The curvature conic at each point P of a developable V_2 in S_4 is a line segment (G) ending at P, and the congruence along which V_2 has zero normal curvature in S_4 is geodesic in V_2. Therefore by choosing $i^{\kappa}_{(3)}$ along (G) and $i^{a}_{(1)}$ as the geodesic congruence, we have

$$\rho_1 = \rho_3 = \sigma_1 = \sigma_2 = \sigma_3 = 0, \qquad \beta = 0. \tag{5.18}$$

Here ρ_2 is or is not equal to zero according as V_2 is or is not plane in S_4.

If $\rho_2 \neq 0$ and $i^{a}_{(1)}$ is given, (5.16) will determine $i^{\kappa}_{(p)}$ to within signs. For this case the fundamental equations (3.6) become

$$C - K = 0, \tag{5.19}$$

$$\delta_1\rho_2 + \alpha\rho_2 = 0, \qquad \nu_1 = 0, \qquad \delta_1\nu_2 + \alpha\nu_2 = 0. \tag{5.20}$$

Equation (5.19) shows that V_2 has constant Gaussian curvature. Referring V_2 to $i^{a}_{(1)}$, $i^{a}_{(2)}$ as parametric curves v=const., u=const., we see that ρ_2, ν_2 are determined by $(5.20)_{1,3}$, each to within an arbitrary function of v. Furthermore, we note that $\rho_2 = \nu_1 = \nu_2 = 0$ is a particular solution of (5.20).

For a plane V_2 in S_4, all the ρ's and σ's are zero, and the fundamental equations (3.6) reduce to (5.19) and

$$\nu_1 = \text{arbitrary}, \qquad \delta_1\nu_2 + \alpha\nu_2 = 0. \tag{5.21}$$

Hence summing up the preceding results we have the following theorem.

THEOREM 5.6. *A V_2 can be imbedded as a developable in an S_4 of scalar curvature C if and only if it has constant Gaussian curvature C. Given arbitrarily a curve (C) in such V_2, it is possible to imbed V_2 in S_4 as a developable with the geodesics tangent to (C) as rulings. The imbedding depends, to within a motion in S_4, on two arbitrary functions, one of one variable and the other of two variables, or both of two variables, according as the developable is or is not plane. A developable in S_4 can be deformed into a plane V_2 in an S_3 whose scalar curvature is equal to twice that of S_4.*

It is well known that an R_2 (that is, a V_2 with $K=0$) in an R_3 is a developable. This is no longer the case for an R_2 in R_4; for, the curvature tensor $H_{cb}^{\cdot\cdot\kappa}$ of an R_2 in R_4 is not necessarily of rank 1 (cf. Theorem 2.2). But for a *ruled* R_2 in an R_4 we have $K=C=0$, so that $\rho_3=0$ is a consequence of (5.1) and $(3.6)_1$. Hence we have the following theorem.

THEOREM 5.7. *A ruled R_2 in an R_4 is a developable.*

Let V_2 and $\overline{V}_2$ be two non-plane developables in S_4 which are applicable to each other such that rulings correspond to rulings and the curvature conics (which are line segments) at corresponding points are equal. Then as in §5.2, we have (5.9) and $\bar{\rho}_2=\rho_2$. Since ν_2 has only to satisfy equation $(5.20)_3$ and therefore depends on an arbitrary function of one variable, we have proved the following counterpart to Theorem 5.3:

THEOREM 5.8. *A non-plane developable in an S_4 can be deformed into infinitely many developables in S_4 such that rulings become rulings and the curvature conics at corresponding points are equal. The deformation depends on one arbitrary function of one variable.*

6. **Surface in R_4 with an orthogonal net of Voss.** A V_2 in an R_4 is said to have a *net of Voss* if its two conjugate congruences are uniquely determined and the curves of the two congruences are geodesics of V_2. A few properties of a V_2 in R_4 with an *orthogonal* net of Voss were given by Zitto [17] (cf. also Schouten-Struik [15, pp. 150–153]). Here we obtain the finite equations of such a V_2 by actually integrating its fundamental equations.

Since the conjugate net is unique and orthogonal, the curvature conic (G) at each point P is a line segment not passing through P (see paragraph below (2.29)). And since V_2 has mutually orthogonal congruences of geodesics, it follows from Liouville's formula (1.6) that $K=0$, that is, the V_2 is an R_2. Consequently, the line segment (G) at each point P of V_2 subtends a right angle at P (cf. Theorem 2.2).

Now if we choose $i^{\kappa}_{(p)}$ suitably and take $i^{a}_{(c)}$ as the unit tangent vectors to the net, we have, as in $(3.7)_5$,

$$(6.1)\qquad \rho_1=\rho_3=\sigma_2=\sigma_3=0.$$

Since $i^{a}_{(c)}$ are geodesic congruences,

$$(6.2)\qquad \alpha=\beta=0.$$

Furthermore, since the line segment (G) does not pass through P, neither ρ_2 nor σ_1 can be zero. Then the fundamental equations (3.6) reduce to

$$(6.3)\qquad \delta_1\rho_2=\delta_2\sigma_1=0,\qquad \nu_1=\nu_2=0.$$

For an R_2, equations (6.1)–(6.3) have solutions for $i^{a}_{(c)}$ and the ρ's, σ's, ν's. Therefore, there exist in an R_4 surfaces with an orthogonal net of Voss.

We now proceed to find the finite equations of such V_2's. With $i^{a}_{(c)}$ as parametric curves, the linear element of R_2 is

$$(6.4)\qquad ds^2=du^2+dv^2,$$

so that $\Gamma^{a}_{cb}=0$, and

$$(6.5)\qquad i^{a}_{(1)}=i^{(1)}_{a}=(1,0),\qquad i^{a}_{(2)}=i^{(2)}_{a}=(0,1),\qquad \delta_1=\partial_u,\qquad \delta_2=\partial_v.$$

Then it follows from (6.1)–(6.3) that

$$h^{(3)}_{cb} = V^0 \overset{.(2)}{i_c} \overset{.(2)}{i_b}, \qquad h^{(4)}_{cb} = U^0 \overset{.(1)}{i_c} \overset{.(1)}{i_b}, \qquad v_c = 0, \tag{6.6}$$

where U^0 $(=\sigma_1)$, V^0 $(=\rho_2)$ are respectively nonzero functions of u, v alone. Therefore the fundamental equations (3.1) become

$$\begin{aligned} \delta_c y^\kappa &= B^\kappa_c, & \delta_c B^\kappa_b &= V^0 \overset{.(2)}{i_c} \overset{.(2)}{i_{(2)}} \overset{.\kappa}{i_{(3)}} + U^0 \overset{.(1)}{i_c} \overset{.(1)}{i_b} \overset{.\kappa}{i_{(4)}}, \\ \delta_c \overset{.\kappa}{i_{(3)}} &= - V^0 \overset{.(2)}{i_c} \overset{.b}{i_{(2)}} B^\kappa_b, & \delta_c \overset{.\kappa}{i_{(4)}} &= - U^0 \overset{.(1)}{i_c} \overset{.b}{i_{(1)}} B^\kappa_b, \end{aligned} \tag{6.7}$$

where y^κ are rectangular Cartesian coordinates in R_4.

We shall now solve this system of equations. Equations $(6.7)_{3,4}$ for $c=1, 2$, respectively, are $\partial_u i^\kappa_{(3)}=0$, $\partial_v i^\kappa_{(4)}=0$. Therefore

$$\overset{.\kappa}{i_{(3)}} = V^\kappa, \qquad \overset{.\kappa}{i_{(4)}} = U^\kappa, \tag{6.8}$$

where V^κ, U^κ are vectors in R_4 depending on v, u alone, respectively. Then $(6.7)_{3,4}$ for $c=2, 1$, respectively, are

$$(V^\kappa)' = - V^0 B^\kappa_2, \qquad (U^\kappa)' = - U^0 B^\kappa_1. \tag{6.9}$$

These together with $(6.7)_1$ give

$$- y^\kappa = \int \frac{(U^\kappa)'}{U^0} du + \int \frac{(V^\kappa)'}{V^0} dv. \tag{6.10}$$

Substituting in $(6.7)_2$ the values for $i^\kappa_{(3)}$, $i^\kappa_{(4)}$, and B^κ_c from (6.8) and (6.9) we have

$$((U^\kappa)'/U^0)' = - U^0 U^\kappa, \qquad ((V^\kappa)'/V^0)' = - V^0 V^\kappa. \tag{6.11}$$

If we multiply the first equation by $2(U^\kappa)'/U^0$ and integrate, the result is

$$((U^\kappa)'/U^0)^2 + (U^\kappa)^2 = \text{const.} \equiv (A^\kappa)^2.$$

Therefore we may put

$$U^\kappa = A^\kappa \cos \theta^\kappa, \qquad (U^\kappa)'/U^0 = A^\kappa \sin \theta^\kappa. \tag*{$(6.12)_1$}$$

To determine θ we need only substitute the first equation in the second one and find that $(\theta^\kappa)' = - U^0$. Hence

$$\theta^\kappa = - \int U^0 du + C^\kappa \qquad (C^\kappa = \text{const.}) \tag*{$(6.12)_2$}$$

Similarly, we derive from $(6.11)_2$

$$V^\kappa = B^\kappa \cos \phi^\kappa, \quad (V^\kappa)'/V^0 = B^\kappa \sin \phi^\kappa, \quad \phi^\kappa = - \int V^0 dv + D^\kappa \quad (B^\kappa, D^\kappa \text{ const.}). \tag{6.13}$$

Using (6.12) and (6.13) in (6.10), we have

$$(6.14)\qquad \begin{aligned} -y^{\kappa} &= A^{\kappa}\int \sin\left(-\int U^0 du + C^{\kappa}\right) du \\ &\quad + B^{\kappa}\int \sin\left(-\int V^0 dv + D^{\kappa}\right) dv. \end{aligned}$$

Suppose that suitable constants of integration have been added to

$$(6.15)\qquad \int U^0 du \equiv -U, \qquad \int V^0 dv \equiv -V$$

so that $U=0$, $V=0$ at some point $P_0(u_0, v_0)$. Then at P_0 we have

$$(6.16)\qquad \begin{aligned} &(\overset{\kappa}{\theta})_0 = \overset{\kappa}{C}, && (\overset{\kappa}{\phi})_0 = \overset{\kappa}{D}, \\ &(\overset{\kappa}{i}_{(3)})_0 = (\overset{\kappa}{V})_0 = \overset{\kappa}{B}\cos \overset{\kappa}{D}, && (\overset{\kappa}{i}_{(4)})_0 = (\overset{\kappa}{U})_0 = \overset{\kappa}{A}\cos \overset{\kappa}{C}, \\ &(\overset{\kappa}{B}_1)_0 = -\overset{\kappa}{A}\sin \overset{\kappa}{C}, && (\overset{\kappa}{B}_2)_0 = -\overset{\kappa}{B}\sin \overset{\kappa}{D}. \end{aligned}$$

Now according to Eisenhart's theorem quoted in §3, the vectors $(B_1^{\kappa})_0$, $(B_2^{\kappa})_0$, $(i_{(3)}^{\kappa})_0$, $(i_{(4)}^{\kappa})_0$ can be chosen arbitrarily as long as they satisfy the following relations

$$(6.17)\qquad \sum_{\kappa} \overset{\kappa}{B}_c \overset{\kappa}{B}_b = \delta_{cb}, \quad \overset{\kappa}{B}_c \overset{(p)}{i}_{\kappa} = 0, \quad \overset{\kappa}{i}_{(p)} \overset{(q)}{i}_{\kappa} = \delta_p^q \quad (b, c = 1, 2;\ p, q = 3, 4).$$

Therefore we may choose $-A^{\kappa}\sin C^{\kappa} = (B_1^{\kappa})_0 = (0, 1, 0, 0)$, $-B^{\kappa}\sin D^{\kappa} = (B_2^{\kappa})_0 = (1, 0, 0, 0)$, $B^{\kappa}\cos D^{\kappa} = (i_{(3)}^{\kappa})_0 = (0, 0, -1, 0)$, $A^{\kappa}\cos C^{\kappa} = (i_{(4)}^{\kappa})_0 = (0, 0, 0, -1)$; that is,

$$(6.18)\qquad \begin{aligned} A^1 &= A^3 = 0, & B^2 &= B^4 = 0, \\ A^2 \sin C^2 &= -1, & A^4 \sin C^4 &= 0, \\ A^2 \cos C^2 &= 0, & A^4 \cos C^4 &= -1, \\ B^1 \sin D^1 &= -1, & B^3 \sin D^3 &= 0, \\ B^1 \cos D^1 &= 0, & B^3 \cos D^3 &= -1. \end{aligned}$$

A solution of these equations is

$$(6.19)\qquad \begin{aligned} &A^{\kappa} = (0, -1, 0, -1), \qquad B^{\kappa} = (-1, 0, -1, 0), \\ &C^2 = 2^{-1}\pi, \qquad C^4 = 0, \qquad D^1 = 2^{-1}\pi, \qquad D^3 = 0. \end{aligned}$$

If we use these in (6.15) and simplify, the result is

$$(6.20)\qquad y^{\kappa} = \left(\int \cos V dv,\ \int \cos U du,\ \int \sin V dv,\ \int \sin U du\right),$$

where U, V are as defined by (6.15) and are therefore arbitrary nonconstant functions of u, v, respectively. Other solutions of (6.18) lead to the same result (6.20). Hence we have the following theorem.

THEOREM 5.1. *Any surface in R_4 with an orthogonal net of Voss is congruent to one of the surfaces* (6.20), *where y^κ are rectangular Cartesian coordinates in R_4 and U, V, are respectively nonconstant functions of u, v alone.*

The simplest of such surfaces is

$$y^\kappa = (\sin v, \sin u, -\cos v, -\cos u), \tag{6.21}$$

which is obtained by putting $V=v$, $U=u$ in (6.20). This surface is well known for the fact that it is a closed R_2 in R_4 (cf. Tompkins [16]).

7. A correspondence between a V_2 in an S_3 and an isometric V_2 in an S_4. We propose to solve the following problem: *Given a V_2 in an S_3, to find the imbedding of an isometric V_2 in an S_4 such that one of the second fundamental tensors of V_2 in S_4 is identical with the second fundamental tensor of V_2 in S_3.*

N. Coburn [5] gave two examples of such a correspondence, one for an arbitrary V_2 in S_3 and one for a ruled V_2 in S_3. Here we give a complete solution of the problem by proving the following theorem.

THEOREM 7.1. *Let the constant scalar curvatures of an S_3 and an S_4 be C' and C, respectively. Then our problem has the following, and only the following, solutions (where the imbedding is described to within a motion in S_4):*

(1a) $C<C'$. *For an arbitrarily given V_2 in S_3, there is in general one and only one solution. The curvature conic of the isometric V_2 in S_4 at each point P of this V_2 is a line segment at the constant distance $(C'-C)^{1/2}$ from P, and the imbedding of this V_2 in S_4 is unique.*

(1b) $C\neq C'$. *If the linear element of V_2 in S_3 is reducible to*

$$ds^2 = \frac{\epsilon_1 du^2 + \epsilon_2 e^{2\theta} dv^2}{1 - \epsilon e^{2\theta}} \quad (\theta \neq \text{const.};\ \epsilon,\ \epsilon_1,\ \epsilon_2 = \pm 1)$$

when referred to its lines of curvature, there is another solution than (1a). *The imbedding of this isometric V_2 in S_4 is unique or depends on one arbitrary constant according as the lines of curvature of V_2 in S_3 do not form or form an isometric net, that is, according as $\theta\neq U+V$ or $\theta=U+V$, where U, V are functions of u, v alone. The curvature conic* (G) *of this V_2 in S_4 is always a line segment.*

(2) $C=C'$. (2a) *For any V_2 in S_3, there is a trivial solution; the V_2 in S_3 may be considered as a surface in S_4.*

(2b) *For an arbitrarily given V_2 in S_3, there is in general one and only one solution other than the trivial one* (1a). *The imbedding of this isometric V_2 in S_4 depends on an arbitrary function of one variable and on an arbitrary solution of a linear partial differential equation of second order in two independent variables. The curvature conic of this V_2 in S_4 is in general not degenerate.*

(2c) *If the V_2 in S_3 has a geodesic congruence of lines of curvature, there is another solution than* (2a) *and* (2b). *The imbedding of this isometric V_2 in S_4 depends on an arbitrary function of a single variable, and the curvature conic of this V_2 in S_4 is always a line segment.*

(2d) *If the V_2 in S_3 is a ruled surface, there is a solution other than* (2a) *and* (2b). *The imbedding of this isometric V_2 in S_4 depends on two arbitrary functions of one variable. This V_2 in S_4 is also a ruled surface, and its curvature conic is in general not degenerate.*

Proof. For a V_2 in an S_3, let K be its Gaussian curvature; $i^a_{(1)}$, $i^a_{(2)}$ two mutually orthogonal congruences; β, α their geodesic curvatures (cf. §1.2); and

$$h_{cb} = \rho_1 \overset{(1)}{i_c} \overset{(1)}{i_b} + \rho_2 \overset{(2)}{i_c} \overset{(2)}{i_b} + \rho_3 (\overset{(1)}{i_c} \overset{(2)}{i_b} + \overset{(2)}{i_c} \overset{(1)}{i_b}) \tag{7.1}$$

the second fundamental tensor. Then the fundamental equations of V_2 in S_3 are

$$\begin{aligned} C' - K &= \overset{2}{\rho_3} - \rho_1\rho_2, \\ \delta_2\rho_1 - \delta_1\rho_3 + \beta(\rho_1 - \rho_2) - 2\alpha\rho_3 &= 0, \\ \delta_1\rho_2 - \delta_2\rho_3 - \alpha(\rho_1 - \rho_2) - 2\beta\rho_3 &= 0, \end{aligned} \tag{7.2}$$

where $C' = -R'/6$ is the constant scalar curvature of S_3. We notice that these equations are obtainable from (3.6) by putting $\sigma_1 = \sigma_2 = \sigma_3 = 0$, $\nu_1 = \nu_2 = 0$, and $C = C'$.

Let us suppose that a V_2, isometric to the V_2 in S_3, is imbedded in an S_4 of scalar curvature $C/6$ such that its second fundamental tensor $h^{(3)}_{cb}$ in S_4 is identical with the second fundamental tensor (7.1) of V_2 in S_3. Using the same $i^a_{(1)}$, $i^a_{(2)}$ for both cases, we have, by comparing (7.2) with (3.6),

$$C - C' = \overset{2}{\sigma_3} - \sigma_1\sigma_2, \tag{7.3}$$

$$\nu_1\sigma_3 = \nu_2\sigma_1, \qquad \nu_1\sigma_2 = \nu_2\sigma_3, \tag{7.4}$$

$$\begin{aligned} \delta_2\sigma_1 - \delta_1\sigma_3 + \beta(\sigma_1 - \sigma_2) - 2\alpha\sigma_3 + \nu_1\rho_3 - \nu_2\rho_1 &= 0, \\ \delta_1\sigma_2 - \delta_2\sigma_3 - \alpha(\sigma_1 - \sigma_2) - 2\beta\sigma_3 + \nu_2\rho_3 - \nu_1\rho_2 &= 0, \end{aligned} \tag{7.5}$$

$$\delta_2\nu_1 - \delta_1\nu_2 + \beta\nu_1 - \alpha\nu_2 + (\rho_1 - \rho_2)\sigma_3 - (\sigma_1 - \sigma_2)\rho_3 = 0. \tag{7.6}$$

Our problem is to solve equations (7.3)–(7.6) for the σ's and ν's, the congruences $i^a_{(1)}$, $i^a_{(2)}$ and the ρ's being given and satisfying (7.2). Consider the two cases $C \neq C'$ and $C = C'$ separately.

Case 1. $C \neq C'$. In this case, equations (7.3) and (7.4) are equivalent to

$$C - C' = \overset{2}{\sigma_3} - \sigma_1\sigma_2 \neq 0, \qquad \nu_1 = \nu_2 = 0. \tag{7.7}$$

Let us choose $i^a_{(1)}$, $i^a_{(2)}$ to be the lines of curvature of V_2 in S_3, so that

$$\rho_3 = 0. \tag{7.8}$$

Then equation (7.6) becomes $(\rho_1-\rho_2)\sigma_3=0$. If $\rho_2=\rho_1$, our choice of $i^a_{(1)}$ is still arbitrary. We may choose it so that $\sigma_3=0$. Hence from the preceding equation we always have that

$$(7.9) \qquad \sigma_3 = 0.$$

Equations (7.8) and (7.9) show that the curvature conic (G) of the V_2 in S_4 is a line segment.

On account of (7.7)–(7.9), equations (7.3) and (7.5) become

$$(7.10) \qquad \sigma_1\sigma_2 = C' - C \equiv \epsilon m^2 \neq 0 \qquad (\epsilon = \pm 1, m = \text{const.}),$$

$$(7.11) \qquad \delta_2\sigma_1 + \beta(\sigma_1 - \sigma_2) = 0, \qquad \delta_1\sigma_2 - \alpha(\sigma_1 - \sigma_2) = 0.$$

In accordance with (7.10), let us put

$$(7.12) \qquad \sigma_1 = me^{\theta}, \qquad \sigma_2 = \epsilon m e^{-\theta}.$$

Then equations (7.11) become

$$(7.13) \qquad \delta_1\theta = \alpha(1 - \epsilon e^{2\theta}), \qquad \delta_2\theta = \beta(\epsilon e^{-2\theta} - 1).$$

Thus we have a desirable imbedding of V_2 in S_4 if and only if the lines of curvature $i^a_{(1)}$, $i^a_{(2)}$ of V_2 in S_3 are such that equations (7.13) admit a solution for θ.

The integrability condition of (7.13) is

$$(7.14) \qquad (1 - \epsilon e^{-2\theta})\left[(\delta_1\beta - \alpha\beta) - e^{2\theta}(\delta_2\alpha - \alpha\beta)\right] = 0.$$

This condition can be satisfied in three ways.

Firstly, we may have $1-\epsilon e^{-2\theta}=0$, that is,

$$(7.15) \qquad \epsilon = 1, \qquad \theta = 0.$$

Then equations (7.13) are identically satisfied, and equations (7.12) and (7.10) give

$$(7.16) \qquad \sigma_1 = \sigma_2 = m = \text{const.}, \qquad C' - C = m^2 > 0.$$

In (7.7), (7.9) and (7.16) we have a solution of equations (7.3)–(7.6), and therefore a desirable imbedding of V_2 in S_4. The given V_2 in S_3 is entirely unrestricted, but the scalar curvature of S_4 has to be smaller than twice that of S_3, as is seen from (7.16). The curvature conic at each point P of such V_2 in S_4 is a line segment at a constant distance $\sigma_1=m=(C'-C)^{1/2}$ from P. This proves (1a) of the theorem.

To satisfy (7.14), we may also have

$$(7.17) \qquad \delta_2\alpha - \alpha\beta = 0, \qquad \delta_1\beta - \alpha\beta = 0.$$

Condition (7.14) being now identically satisfied, equations (7.13) are completely integrable. Consequently, σ_1 and σ_2 are uniquely determined when the value of θ at a certain point of V is given. This is a desirable imbedding of V_2

in S_4 depending on one arbitrary constant. We note from (7.17) that the lines of curvature $i^a_{(1)}$, $i^a_{(2)}$ of V_2 in S_3 form an isometric net (cf. (1.7)). Conversely, if there exists a non-constant scalar θ satisfying (7.13) in which $i^a_{(1)}$, $i^a_{(2)}$ form an isometric net, then (7.17) are true. For, in this case, $\delta_2\alpha=\delta_1\beta$, so that (7.17) is a consequence of (7.14). The curvature conic (G) of the V_2 in S_4 is the line segment:

$$(7.18)\qquad \begin{aligned} z &= \rho_1 \cos^2 \phi + \rho_2 \sin^2 \phi, \\ t &= \sigma_1 \cos^2 \phi + \sigma_2 \sin^2 \phi \qquad (\sigma_1\sigma_2 = C' - C = \text{const.}). \end{aligned}$$

An example of such V_2 in S_3 is obtained by putting $\alpha=\beta=0$. Then $K=0$ and the fundamental equations (7.2) reduce to $C'=-\rho_1\rho_2$, $\delta_2\rho_1=0$, $\delta_1\rho_2=0$, which are equivalent to

$$C' \neq 0, \qquad \rho_1 = \text{const.} \neq 0, \qquad \rho_2 = -C'/\rho_1 = \text{const.},$$

or to

$$C' = 0, \qquad \rho_1 = 0, \qquad \delta_1\rho_2 = 0.$$

The latter case gives a developable surface in an R_3.

Finally, if none of the equations in (7.17) is satisfied, the last possible solution of (7.14) is

$$(7.19)\qquad \epsilon e^{2\theta} = \frac{\delta_1\beta - \alpha\beta}{\delta_2\alpha - \alpha\beta} \neq 1, \quad \text{that is,} \quad 2\theta = \log \frac{\epsilon(\delta_1\beta - \alpha\beta)}{(\delta_2\alpha - \alpha\beta)}\cdot$$

Using this in (7.13) we have

$$(7.20)\qquad \begin{aligned} \delta_1 \log \frac{\epsilon(\delta_1\beta - \alpha\beta)}{(\delta_2\alpha - \alpha\beta)} &= 2\alpha \frac{\delta_2\alpha - \delta_1\beta}{\delta_2\alpha - \alpha\beta} \neq 0, \\ \delta_2 \log \frac{\epsilon(\delta_1\beta - \alpha\beta)}{(\delta_2\alpha - \alpha\beta)} &= 2\beta \frac{\delta_2\alpha - \delta_1\beta}{\delta_1\beta - \alpha\beta} \neq 0. \end{aligned}$$

If the lines of curvature of V_2 in S_3 satisfy these equations, the function θ and consequently σ_1, σ_2, are uniquely determined. Therefore, we have here a uniquely determined desirable imbedding of the isometric V_2 in S_4. The curvature conic for this V_2 in S_4 is the line segment (7.18).

From (7.13) we can find the form of the ds^2, referred to the lines of curvature, for the V_2 of the cases (7.17) and (7.20). Take $i^a_{(1)}$, $i^a_{(2)}$ as parametric curves, then we have (1.12) and (1.13), and equations (7.13) become

$$\frac{\partial_u\theta}{1 - \epsilon e^{2\theta}} = \partial_u \log G^{1/2}, \qquad \frac{\partial_v\theta}{\epsilon e^{-2\theta} - 1} = \partial_v \log E^{1/2}.$$

Integration of these gives

$$E = \frac{U^*}{1 - \epsilon e^{2\theta}}, \qquad G = \frac{V^* e^{2\theta}}{1 - \epsilon e^{2\theta}},$$

where U^*, V^* are respectively some functions of u, v alone. Therefore after an obvious transformation on u and v, we have

$$(7.21)\qquad ds^2 = \frac{\epsilon_1 du^2 + \epsilon_2 e^{2\theta} dv^2}{1 - \epsilon e^{2\theta}}, \qquad \epsilon, \epsilon_1, \epsilon_2 = \pm 1.$$

We have seen that the case (7.17) arises when $\theta \neq$const. and $i^a_{(1)}$, $i^a_{(2)}$ form an isometric net, that is, when

$$G/E = \epsilon_1\epsilon_2 e^{2\theta} \neq \text{const.}$$

is the product of a function of u with a function of v. Hence the linear element of this V_2 is

$$(7.22)\qquad ds^2 = \frac{du^2 \pm dv^2}{U + V},$$

where $U=U(u)$, $V=V(v)$ are not both constant. We have thus completed the proof of the first part of the theorem.

Case 2. $C'=C$. Equations (7.3) and (7.4) are equivalent to

$$(7.23)\qquad \sigma_3^2 - \sigma_1\sigma_2 = 0, \qquad \nu_1\sigma_3 = \nu_2\sigma_1.$$

If all the σ's are zero, it follows from (7.6) that

$$(7.24)\qquad \delta_2\nu_1 - \delta_1\nu_2 + \beta\nu_1 - \alpha\nu_2 = 0.$$

In consequence of this, we can choose the normals $i^\kappa_{(3)}$, $i^\kappa_{(4)}$ to V_2 in S_4 so that $\nu_1=\nu_2=0$. For, from (7.24) and (1.4) it follows that a scalar f exists so that

$$\nu_1 = \delta_1 f, \qquad \nu_2 = \delta_2 f, \quad \text{that is,} \quad \nu_c = \partial_c f.$$

Now if we use

$$\overset{*\kappa}{i}_{(3)} = \overset{\kappa}{i}_{(3)} \cos f + \overset{\kappa}{i}_{(4)} \sin f, \qquad \overset{*\kappa}{i}_{(4)} = - \overset{\kappa}{i}_{(3)} \sin f + \overset{\kappa}{i}_{(4)} \cos f$$

as the normals to V_2 in S_4, the corresponding vector $v_c{}^*$ is (cf. (2.3))

$$\overset{*}{v}_c = B^\lambda_c \overset{*\kappa}{i}_{(3)} \nabla_\lambda \overset{*(4)}{i}_\kappa = v_c - B^\lambda_c \nabla_\lambda f = \partial_c f - \partial_c f = 0,$$

as was to be proved. This being the case, equations (7.3)–(7.6) are now all satisfied. But we have here only a trivial imbedding of an isometric V_2 in S_4, because the V_2 in S_3 may be considered as a V_2 in S_4. This proves (2a) of the theorem.

Let us suppose that not all the σ's are zero. Then we may choose $i^a_{(1)}$ so that

$$(7.25)\qquad \sigma_3 = 0.$$

Equation (7.23) now gives

$$(7.26)\qquad \sigma_2 = 0, \qquad \sigma_1 \neq 0, \qquad \nu_2 = 0,$$

and thence equations (7.5) and (7.6) become

$$(7.27)\quad \delta_2\sigma_1 + \beta\sigma_1 + \nu_1\rho_3 = 0,\ \delta_2\nu_1 + \beta\nu_1 - \sigma_1\rho_3 = 0,\ \alpha\sigma_1 + \nu_1\rho_2 = 0.$$

To study the consistency of these equations in σ_1 and ν_1, we have to distinguish whether $\nu_1=0$ or $\nu_1\neq 0$.

If $\nu_1=0$, these equations reduce to

$$(7.28)\qquad \nu_1 = 0, \qquad \alpha = 0, \qquad \rho_3 = 0, \qquad \delta_2\sigma_1 + \beta\sigma_1 = 0.$$

Equations $(7.28)_{2,3}$ show that the lines of curvature $i^a_{(2)}$ of V_2 in S_3 are geodesics in V_2. And it follows from (7.25), (7.26) and $(7.28)_{1,4}$ that we have a desirable imbedding of an isometric V_2 in S_4 which depends on an arbitrary solution σ_1 of the linear partial differential equation $(7.28)_4$. This solution σ_1, when the linear element of V_2 is referred to $i^a_{(1)}$, $i^a_{(2)}$ as parametric curves $v=$const., $u=$const., is seen to depend on one arbitrary function of u. Hence the imbedding depends on an arbitrary function of a single variable. On account of (7.25) and $(7.28)_3$, the curvature conic of V_2 in S_4 is a line segment. This proves (2c) of Theorem 7.1.

If $\nu_1\neq 0$, equations $(7.27)_{1,2}$ can be written

$$(7.29)\qquad \delta_2 \log \sigma_1 + \beta + \rho_3\frac{\nu_1}{\sigma_1} = 0, \qquad \delta_2 \log \nu_1 + \beta - \rho_3\frac{\sigma_1}{\nu_1} = 0.$$

Subtraction gives $\delta_2 \log (\sigma_1/\nu_1) + (\nu_1/\sigma_1 + \sigma_1/\nu_1)\rho_3 = 0$, that is,

$$\rho_3 = -\frac{\delta_2 \log (\sigma_1/\nu_1)}{\nu_1/\sigma_1 + \sigma_1/\nu_1} = -\frac{\delta_2\sigma_1/\nu_1}{1 + (\sigma_1/\nu_1)^2}.$$

Therefore

$$(7.30)\qquad \rho_3 = \delta_2 \arctan (-\sigma_1/\nu_1).$$

At this stage we have to consider the cases $\alpha\neq 0$ or $\alpha=0$ separately.

If $\alpha\neq 0$, then it follows from $(7.26)_2$ and $(7.27)_3$ that $\rho_2\neq 0$. We eliminate σ_1/ν_1 from (7.30) and $(7.27)_3$ and get

$$(7.31)\qquad \rho_3 = \delta_2 \arctan (\rho_2/\alpha).$$

This is a necessary condition for (7.27) to have a solution for σ_1, ν_1. Evidently it is also sufficient. If (7.31) is satisfied, σ_1 is given by

$$(7.32)\qquad \delta_2 \log \sigma_1 + \beta - \rho_3\alpha/\rho_2 = 0,$$

which is obtained from $(7.27)_3$ and $(7.29)_1$ by elimination of ν_1/σ_1. Therefore if $\overset{0}{\sigma}_1$ is a solution of (7.32), the most general solution will be

$$(7.33)\qquad \sigma_1 = \mu\overset{0}{\sigma}_1,$$

where μ is any solution of $\delta_2\mu=0$. Hence referring V_2 to $i^a_{(1)}$, $i^a_{(2)}$ as parametric curves we see that for given $\dot{i}^a_{(1)}$, $\dot{i}^a_{(2)}$ satisfying (7.31), the solution for σ_1, ν_1 depends on an arbitrary function of a single variable.

We now show that for any V_2 in S_3 there exist orthogonal congruences $\dot{i}^a_{(1)}$, $\dot{i}^a_{(2)}$ satisfying (7.31). Let $\dot{i}^a_{(1)}$, $\dot{i}^a_{(2)}$ be expressed in terms of two given orthogonal congruences $\dot{i}^{*a}_{(1)}$, $\dot{i}^{*a}_{(2)}$:

$$\dot{i}^a_{(1)} = \dot{i}^{*a}_{(1)} \cos\theta + \dot{i}^{*a}_{(2)} \sin\theta, \qquad \dot{i}^a_{(2)} = -\dot{i}^{*a}_{(1)} \sin\theta + \dot{i}^{*a}_{(2)} \cos\theta.$$

Then $\alpha = -(\delta_1^*\theta - \beta^*) \sin\theta + (\delta_2^*\theta + \alpha^*) \cos\theta$ and α is not identically zero. Consequently equation (7.31) will become a linear differential equation of the second order in θ.

Hence we have here a solution of the problem in which the imbedding of the isometric V_2 in S_4 depends on an arbitrary function of a single variable and on an arbitrary solution of a linear partial differential equation of the second order in two independent variables. The curvature conic (G) of such V_2 in S_4 is

$$z = \rho_1 \cos^2\phi + \rho_2 \sin^2\phi + 2\rho_3 \cos\phi \sin\phi, \qquad t = \sigma_1 \cos^2\phi,$$

which is in general nondegenerate. This proves (2b) of Theorem 7.1.

Let us now return to the stage where (7.30) appeared. If $\alpha = 0$, then since $\nu_1 \neq 0$ in the case under consideration, it follows from $(7.27)_3$ that $\rho_2 = 0$. Then the second fundamental tensor of V_2 in S_3 is

$$h_{cb} = \rho_1 \dot{i}_c^{(1)} \dot{i}_b^{(1)} + \rho_3(\dot{i}_c^{(1)} \dot{i}_b^{(2)} + \dot{i}_c^{(2)} \dot{i}_b^{(1)}),$$

showing that $\dot{i}^a_{(2)}$ is an asymptotic congruence of V_2 in S_3. Since $\alpha = 0$ is the geodesic curvature of $\dot{i}^a_{(2)}$, V_2 is generated by a one-parameter family of geodesics in S_3, that is, V_2 is a ruled surface in S_3.

Consider equation (7.30). Let θ^0 be a solution of $\delta_2\theta = \rho_3$, then the most general solution of (7.30) is

$$\arctan(-\sigma_1/\nu_1) = \theta^0 + \mu,$$

where μ is an arbitrary solution of $\delta_2\mu = 0$. Therefore

$$\sigma_1/\mu_1 = -\tan(\theta^0 + \mu). \tag{7.34}$$

Using this in $(7.29)_1$ we get

$$\delta_2 \log \sigma_1 + \beta - \rho_3 \cot(\theta^0 + \mu) = 0.$$

If σ_1^0 is a solution of this equation, the most general solution is

$$\sigma_1 = \lambda \overset{c}{\sigma_1}, \tag{7.35}$$

where λ is an arbitrary solution of $\delta_2\lambda = 0$. Hence in this case we have a solution for σ_1, ν_1 depending on two arbitrary solutions of the linear differential equation $\delta_2\lambda = 0$, that is, we have a desirable imbedding of an isometric V_2

in S_4 which depends on two arbitrary functions of a single variable. The curvature conic (G) of the isometric V_2 in S_4 is

$$z = \rho_1 \cos^2 \phi + 2\rho_3 \sin \phi \cos \phi, \qquad t = \sigma_1 \cos^2 \phi,$$

which is in general nondegenerate. Since $\sigma_2 = \rho_2 = 0$, the isometric V_2 in S_4 is also a ruled surface in S_4. This proves the last part of Theorem 7.1.

References

1. M. Bôcher, *Introduction to higher algebra*, New York, 1908.

2. O. Borůvka, *Sur une classe de surfaces minima plongées dans un espace à quatre dimensions à courboure constante*, Bulletin international Česká akademie véd a umění v Praze vol. 29 (1928) pp. 256–277.

3. R. Calapso, *Sulle reti di Voss di uno spazio lineare quadri dimensionale*, Rendiconti Seminario matematico Roma (4) vol. 2 (1938) pp. 276–311.

4. ———, *Sulle deformazione delle reti di Voss di un S_4 euclideo*, Atti Accademia nazionale dei Lincei (4) vol. 28 (1939) pp. 231–236.

5. N. Coburn, *Surfaces in four-space of constant curvature*, Duke Math. J. vol. 5 (1939) pp. 30–38.

6. L. P. Eisenhart, *Minimal surfaces in Euclidean four space*, Amer. J. Math. (4) vol. 34 (1912) pp. 215–236.

7. ———, *Differential geometry*, New York, 1909.

8. ———, *Riemannian geometry*, Princeton, 1926.

9. W. Graustein, *Invariant methods in classical differential geometry*, Bull. Amer. Math. Soc. vol. 36 (1930) pp. 489–521.

10. J. Knoblauch, *Grundlagen der Differentialgeometrie*, Leipzig, 1913.

11. K. Kommerell, *Riemannsche Flächen im ebenen Raum von vier Dimensionen*, Math. Ann. vol. 60 (1905) pp. 546–596.

12. S. Kwietniewski, *Über Flächen des vierdimensionalen Raumes, deren sämtliche Tangentialebenen untereinander gleichwinklig sind, und ihre Beziehung zu den ebenen Kurven*, Dissertation, Zürich.

13. C. L. E. Moore, and E. B. Wilson, *Differential geometry of two-dimensional surfaces in hyperspace*, Proceedings of the American Academy of Arts and Sciences vol. 52 (1916) pp. 267–368.

14. G. Ricci, *Lezioni sulla teoria della superficie*, Verona and Padova, Druker, 1898.

15. J. A. Schouten, and D. J. Struik, *Einführung in die neueren Methoden der Differentialgeometrie* II, Batavia, 1938.

16. C. Tompkins, *Isometric imbedding of flat manifolds in Euclidean space*, Duke Math. J. vol. 5 (1939) pp. 58–61.

17. C. Zitto, *Reti di Voss a curvatura nulla di un S_4 euclideo*, Atti Acad. Pelororitana vol. 41 (1939) pp. 44–47.

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