ON THE EXTENSION OF INTERVAL FUNCTIONS

BY

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Introduction. The problem of extending the range of definition of a function defined on a class of elementary figures—intervals, rectangles—has been treated in various ways in the literature. In the theory of Lebesgue measure a particular function—length of interval (area of rectangle)—is extended in a completely additive way to an additive class of sets. In the general extension problem we start, say, with a function (real, single-valued, and finite) of intervals $\phi(I)$ and extend the range of definition to an additive class of sets obtaining a function $\Phi(E)$ which is completely additive and which has the property that $\Phi(E)=\phi(I)$ whenever "E is the set I." But what is the interval I? A priori $\phi(I)$ is defined on a class of intervals I, where I is considered neither open nor closed but merely as an interval. From the viewpoint of $\Phi(E)$ an interval I must be considered as a definite point set—a closed interval, an open interval, a semi-open interval, and so on. Corresponding to open intervals and to closed intervals, $\Phi(E)$ gives rise to two interval functions: $\phi_1(I)=\Phi(I')$, $\phi_2(I)=\Phi(I^0)$ where I' is understood to be closed and I^0 open. If $\phi(I)=\phi_1(I)$ identically, then $\Phi(E)$ is an extension of $\phi(I)$ considered as a function of closed intervals; if $\phi(I)=\phi_2(I)$ identically, then $\Phi(E)$ is an extension of $\phi(I)$ considered as a function of open intervals. As a starting point in the general extension problem, the function $\phi(I)$ has been considered, somewhat artificially and arbitrarily perhaps, a function either of closed intervals or of open intervals (see, for example, [10])[1]. Extensions $\Phi(E)$ which have the property that $\Phi(I')=\Phi(I^0)$ identically are of particular interest since then $\Phi(E)$ is an extension of $\phi(I)$ whether I be considered open or closed.

The main results of the paper concern the existence of B-extensions, a precise definition of which is given in §1.6. Suffice it to say here that if $\Phi(E)$ is a B-extension of $\phi(I)$ then $\Phi(I')=\Phi(I^0)=\phi(I)$. The idea of a B-extension was suggested by a result of Burkill [2] which we shall review in §1.5. Burkill's theorem on extension is stated in terms of a sufficient condition while our results on B-extensions are stated in terms of necessary and sufficient conditions.

In Part 1 we explain notation, define terms, and summarize results. In Part 2 we present a proof of a theorem (Theorem 1) which states a necessary and sufficient condition that a non-negative function of closed intervals

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(1) Numbers in brackets indicate references in the bibliography at the end of the paper.

admit a non-negative completely additive extension. This theorem was proved in [10] using the results of Radon [9]. The present proof makes use of the theory of outer measure in the sense of Carathéodory. Theorem 2 extends the result of Theorem 1 to the case of a function of arbitrary sign. Theorem 3 concerns the uniqueness and characterization of an extension of a function of intervals. Part 3 contains proofs of our results (Theorems 4, 5, and 6) on the B-extension. We present necessary and sufficient conditions that a non-negative function of intervals, a function of intervals of arbitrary sign, and an indefinite integral of a function of intervals, respectively, admit B-extensions.

1. **Preliminaries and summary.**

1.1. In modern literature intervals are considered as k-dimensional where k is a positive integer. We shall consider functions of intervals in the xy-plane, that is, our intervals are two-dimensional. The word interval is used in the sequel only in the accepted point set sense: Given two points (x_1, y_1), (x_2, y_2) an *interval* is the set of points (x, y) such that $x_1 \leqq x \leqq x_2$, $y_1 \leqq y \leqq y_2$; an *open interval* is the set of points (x, y) such that $x_1 < x < x_2$, $y_1 < y < y_2$ (Saks [12, p. 57]). Let R_0 be the interval $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$. All sets considered in this paper are subsets of R_0 unless otherwise stated. We use the letters I, J, R to denote intervals. The symbol I^0 denotes the open interval which corresponds to I, that is, I^0 is the interior of the set I. The letter C is used to denote a class of intervals. A capital script letter, as $\mathcal{E}$, is used to denote an *elementary system* of intervals, that is, a finite set of intervals $I_1, I_2, \cdots, I_k$ such that $I_i^0 \cdot I_j^0 = 0$ whenever $i \neq j$. A capital script letter is also used as an operator in the sense that $\mathcal{E}I$ denotes an elementary system of intervals $I_1, I_2, \cdots, I_k$ which constitutes a *subdivision* of I, that is, $\sum_{i=1}^k I_i = I$. The *parameter of regularity* of an interval I, denoted by $p(I)$, is the ratio of the length of the shorter side of I to the length of the longer side of I. The *norm* of an interval I, denoted by $\|I\|$, is the length of the diameter of I; the symbol $\|\mathcal{E}\|$ is defined as the largest of the numbers $\|I\|$ where $I \in \mathcal{E}$. The measure of an interval I is denoted by $|I|$; the symbol $|\mathcal{E}|$ is defined as the number $\sum |I|$ where the sum is taken over $I \in \mathcal{E}$. The *boundary* of an interval I, denoted by $b(I)$, is the set $I - I^0$.

1.2. A class of sets in R_0 is said to be *closed* (relative to R_0)—and is generically denoted by K—if the following conditions are satisfied:

(i) Every set open relative to R_0 (denoted generically by O) is in K.

(ii) If a set E is in K, then the complement of E relative to R_0 (denoted by CE) is also in K.

(iii) If $\{E_n\}$ is a sequence of sets in K, then $\sum_n E_n$ is also a set in K.

Clearly every closed class K contains all Borel sets $E \subset R_0$. In fact, the class of all Borel sets in R_0, which we denote by B, is identical with the product of all closed classes in R_0.

Let λ be a fixed number satisfying the relation $0 \leqq \lambda < 1$, and let C_λ denote the class of all intervals I such that $p(I) \geqq \lambda$. A subscript λ as in I_λ and $\mathcal{E}_\lambda$

indicates that $I_\lambda \in C_\lambda$ and that $I \in C_\lambda$ for every $I \in \mathcal{E}_\lambda$. The symbol C_0 will denote the class of all intervals I such that $I \subset R_0$. Let $\phi(I)$ denote a real, finite, single-valued function which is defined for every $I \in C_\lambda$. This function is denoted briefly by the symbol $[\phi, C_\lambda]$ in which the first letter denotes the function and the second letter denotes the range of definition of the function.

A function $\Phi(E)$ which is defined on a closed class K, that is, the function $[\Phi, K]$, is a *completely additive extension* of the function $[\phi, C_\lambda]$ if the following conditions are satisfied:

(i) $[\Phi, K]$ is a completely additive set function.

(ii) $\Phi(I) = \phi(I)$ for every $I \in C_\lambda$.

In Part 2 we shall prove the following theorems:

THEOREM 1. (See [10, Theorem 3].) *A necessary and sufficient condition that a non-negative function of intervals $[\phi, C_\lambda]$ have a non-negative completely additive extension is that it satisfy the following condition $\mathfrak{C}$: If $\{I_n\}$ is any sequence of intervals in C_λ such that $I_i \cdot I_j = 0$ when $i \neq j$ and if $\{J_m\}$ is any sequence of intervals in C_λ such that $\sum_m J_m \supset \sum_n I_n$ then $\sum_m \phi(J_m) \geqq \sum_n \phi(I_n)$.*

THEOREM 2. *A necessary and sufficient condition that a function of intervals $[\phi, C_\lambda]$, of arbitrary sign, have a completely additive extension is that it be the difference of two non-negative functions each of which satisfies condition $\mathfrak{C}$.*

THEOREM 3. *If $[\Phi, K]$ is a completely additive extension of a function of intervals $[\phi, C_\lambda]$, then the value of the number $\Phi(E)$, where E is any Borel set, is uniquely determined by the function $[\phi, C_\lambda]$. If $[\phi, C_\lambda]$ is non-negative, then $[\Phi, K]$ is also non-negative and for every Borel set E we have the characterization: $\Phi(E) = \text{g.l.b.} \sum_n \phi(I_n)$ for all sequences $\{I_n\}$ such that $\sum_n I_n \supset E$ and $I_n \in C_\lambda$, $n = 1, 2, \cdots$.*

1.3. Given a function of intervals $[\phi, C_\lambda]$, we extend the range of definition of $[\phi, C_\lambda]$ to include all elementary systems of intervals $\mathcal{E}_\lambda$ as follows: $\phi(\mathcal{E}_\lambda) = \sum \phi(I)$ where the sum is taken over $I \in \mathcal{E}_\lambda$. The function $[\phi, C_\lambda]$ is *additive* if for every I_λ and every $\mathcal{E}_\lambda I_\lambda$ it is true that $\phi(E_\lambda I_\lambda) = \phi(I_\lambda)$. If we replace $\phi(\mathcal{E}_\lambda I_\lambda) = \phi(I_\lambda)$ by $\phi(\mathcal{E}_\lambda I_\lambda) \geqq \phi(I_\lambda)$ and then by $\phi(\mathcal{E}_\lambda I_\lambda) \leqq \phi(I_\lambda)$ we obtain the definitions of a function which *increases by subdivision* and *decreases by subdivision* respectively. The function $[\phi, C_\lambda]$ is *continuous* if for every number $\epsilon > 0$ there exists a number $\delta > 0$ such that (i) $|I_\lambda| < \delta$ implies $|\phi(I_\lambda)| < \epsilon$ and (ii) $I_{\lambda 1} \subset I_{\lambda 2}$, $|I_{\lambda 2} - I_{\lambda 1}| < \delta$ imply $|\phi(I_{\lambda 1}) - \phi(I_{\lambda 2})| < \epsilon$. It is observed that condition (ii) in this definition is a consequence of condition (i) if the function is additive and $\lambda = 0$. The function $[\phi, C_\lambda]$ is *absolutely continuous* if for every number $\epsilon > 0$ there exists a number $\delta > 0$ such that $|\mathcal{E}_\lambda| < \delta$ implies $|\phi(\mathcal{E}_\lambda)| < \epsilon$.

1.4. Given a function of intervals $[\phi, C_\lambda]$, we define for every interval I the following two numbers:

(i) $L(\phi, I) = \liminf \phi(\mathcal{E}_\lambda I)$ as $\|\mathcal{E}_\lambda I\| \to 0$,

(ii) $U(\phi, I) = \limsup \phi(\mathcal{E}_\lambda I)$ as $\|\mathcal{E}_\lambda I\| \to 0$,

and call these numbers, which are finite or infinite, the lower and upper integrals of $[\phi, C_\lambda]$ over the interval I respectively. Given any interval I, any number λ such that $0 \leqq \lambda < 1$, and any number $\delta > 0$, it is easily shown that there exists an $\mathcal{E}_\lambda I$ such that $\|\mathcal{E}_\lambda I\| < \delta$. Consequently the lower and upper integrals are defined for every $I \in C_0$. In our bracket notation these functions (not necessarily finite) may be denoted by $[L(\phi), C_0]$ and $[U(\phi), C_0]$ respectively. In case $L(\phi, I) = U(\phi, I)$ is a finite number, we denote the common value by $F(\phi, I)$ and call it the integral of ϕ over I. Defining the integral in this manner, that is, for a function of intervals defined on a class C_λ, makes it sufficiently flexible to include the integral in the extended sense of Burkill, the strong integral of Saks, and the regular integral of Kempisty, by suitably choosing λ(2). If $[\phi, C_\lambda]$ is integrable over R_0 then it is integrable over every $I \in C_0$. The indefinite integral, which we may denote by $[F(\phi), C_0]$, is an additive function of intervals.

A function of intervals $[\phi, C_0]$ is said to be *absolutely continuous in the restricted sense*, briefly *RAC*, if the function $[U(|\phi|), C_0]$ is continuous.

1.5. The following theorem was stated and proved by Burkill [2, p. 289] for an integral which, under the stated assumptions, reduces to the integral as defined in 1.4 if $\lambda = 0$.

THEOREM. *If a function of intervals $[\phi, C_\lambda]$ is absolutely continuous and integrable, if E is a measurable set, and if ϵ_n, $n = 1, 2, \cdots$, is any decreasing sequence of positive numbers approaching* 0, *and corresponding to each n, E is decomposed into $\mathcal{E}_n + e_n' - e_n''$ where e_n' and e_n'' are measurable sets such that $|e_n'|$ and $|e_n''|$ are each less than ϵ_n, and $\mathcal{E}_n$ is an elementary system of intervals, then as $n \to \infty$, $F(\phi, \mathcal{E}_n)$ approaches a limit which we call $F(E)$ and which is independent of the particular decomposition of E for any n.*

Burkill showed that the function $F(E)$, which is defined for all measurable subsets of R_0, is an absolutely continuous, completely additive function of measurable sets, which for intervals reduces to the integral. This is a strong type of extension in the sense that if any interval I is given, and if E is any set satisfying the relation $I^0 \subset E \subset I$, then $F(E) = F(\phi, I)$. This property is a direct implication of the absolute continuity and additivity of the function $[F(\phi), C_0]$.

1.6. Burkill's result suggested the following type of extension. A completely additive set function $[\Phi, B]$ defined for all Borel sets in R_0 (briefly, an additive function of Borel sets) is a *B-extension* of a function $[\phi, C_\lambda]$ if $\Phi(E) = \phi(I)$ for every $I \in C_\lambda$ and for every Borel set E such that $I^0 \subset E \subset I$. In Part 3 we shall establish the following theorems:

(2) For Burkill's definition, see [2, p. 279]; for Saks's definition, see [11, p. 212]; for Kempisty's definition, see [6, p. 212].

THEOREM 4. *A necessary and sufficient condition that a non-negative function of intervals $[\phi, C_\lambda]$ admit a non-negative B-extension is that it be an additive, continuous function.*

THEOREM 5. *A necessary and sufficient condition that a function of intervals $[\phi, C_\lambda]$ admit a B-extension is that it be additive and RAC.*

THEOREM 6. *A necessary and sufficient condition that the indefinite integral of an integrable function of intervals $[\phi, C_\lambda]$ admit a B-extension is that $[\phi, C_\lambda]$ be RAC.*

2. Completely additive extensions of functions of intervals.

2.1. The necessity of condition $\mathfrak{C}$ in Theorem 1 is an immediate consequence of the following property of a completely additive set function: If $[\Phi, K]$ is any non-negative completely additive set function, if $\{e_n\}$ is any sequence of mutually exclusive sets in K, and if $\{E_m\}$ is any sequence of sets in K such that $\sum_m E_m \supset \sum_n e_n$ then $\sum_m \Phi(E_m) \geqq \sum_n \Phi(e_n)$.

2.2. Let $[\phi, C_\lambda]$ be a non-negative function of intervals which satisfies condition $\mathfrak{C}$. For every set $E \subset R_0$ we define

$$\bar{\Phi}(E) = \text{g.l.b.} \sum_n \phi(I_{\lambda n})$$

for all sequences $\{I_{\lambda n}\}$ such that $\sum_n I_{\lambda n} \supset E$. Obviously $\bar{\Phi}(E)$ is a non-negative function. We shall show that it is an outer measure in the sense of Carathéodory (see [12, p. 43]), that is, we shall show that it satisfies the following conditions:

(i) $\bar{\Phi}(E_1) \leqq \bar{\Phi}(E_2)$ whenever $E_1 \subset E_2$.

(ii) $\bar{\Phi}(\sum_n E_n) \leqq \sum_n \bar{\Phi}(E_n)$ for every sequence $\{E_n\}$ of sets.

(iii) $\bar{\Phi}(E_1+E_2) = \bar{\Phi}(E_1) + \bar{\Phi}(E_2)$ whenever the distance from E_1 to E_2, which we denote by $d(E_1, E_2)$, is greater than 0. Conditions (i) and (ii) follow directly from the definition of $\bar{\Phi}(E)$ and from condition $\mathfrak{C}$. We proceed to establish condition (iii).

A *transversal* of R_0, denoted generically by t, is a closed line segment which is parallel to either the x-axis (a horizontal transversal) or to the y-axis (a vertical transversal), and which satisfies the following two conditions: (i) the end points of t lie on the boundary of R_0, (ii) the set which consists of t less its end points—denoted by t^0—is contained in R_0^0.

Given any interval $I \in C_0$ we say that a subdivision $\mathcal{E}_\lambda I$ is a *ϕ'-subdivision* of I if the set of intervals in $\mathcal{E}_\lambda I$ can be segregated into two sets, say $I_1, I_2, \cdots, I_n$ and $J_1, J_2, \cdots, J_m$, such that $I_i \cdot I_j = 0$ for $i \neq j$ and $J_i \cdot J_j = 0$ for $i \neq j$. Given a transversal t, it may be verified that there exist intervals I such that $I \supset t$, $I^0 \supset t^0$, and such that I has a ϕ'-subdivision. A ϕ'-subdivision of an interval I, where $I \supset t$, $I^0 \supset t^0$, is called a ϕ'-covering of t. For every transversal t we define $\phi'(t) = \text{g.l.b.}\ \phi(\mathcal{E}_\lambda I)$ for all elementary systems $\mathcal{E}_\lambda I$

which are ϕ'-coverings of t. It follows from condition $\mathfrak{C}$ that there are at most a denumerable number of transversals t for which $\phi'(t)>0$.

Let I_0 be any interval in C_λ; let $t_1, t_2, \cdots, t_n$ be a finite number of horizontal transversals and $t_{n+1}, \cdots, t_m$ a finite number of vertical transversals such that $t_i \cdot I_0^0 \neq 0$ for $i=1, \cdots, m$. These transversals determine a subdivision of I_0 into $(n+1)(m+1)$ intervals, say $I_1, I_2, \cdots, I_k$. We shall say that this elementary system is a *regular subdivision* of I_0 if $\phi'(t_i)=0$ for $i=1, 2, \cdots, m$, and $I_i \in C_\lambda$ for $i=1, 2, \cdots, k$. Given a number $\delta>0$ it may be shown that there exists a regular subdivision of I_0—say $\mathcal{E}I_0$—such that $\|\mathcal{E}I_0\|<\delta$.

Let I be one of the intervals in a regular subdivision of an interval $I_0 \in C_\lambda$. Let I^* denote the set $I^0+b(I_0)\cdot I$. The set $I-I^*$ is the sum of k, $1 \leqq k \leqq 4$, line segments. Let $t_1, t_2, \cdots, t_k$ be the set of transversals such that t_i, $i=1, \cdots, k$, contains one of the line segments in $I-I^*$. Given $\epsilon>0$ let $\mathcal{E}_{\lambda i}J_i$ be a ϕ'-covering of t_i such that $\phi(\mathcal{E}_{\lambda i}J_i)<\epsilon/4$. Let $J \in C_\lambda$ be an interval such that $I^* \supset J$ and $I \subset J+J_1+ \cdots +J_k$. It follows from condition $\mathfrak{C}$ that $\phi(I) \leqq \phi(J)+\phi(\mathcal{E}_{\lambda 1}J_1)+ \cdots +\phi(\mathcal{E}_{\lambda k}J_k)<\phi(J)+\epsilon$.

Let $I_1, I_2, \cdots, I_k$ be the intervals in a regular subdivision of $I_0 \in C_\lambda$. Given $\epsilon>0$, let J_i, $i=1, \cdots, k$, be an interval in C_λ such that $I_i^* \supset J_i$ and $\phi(I_i)<\phi(J_i)+\epsilon/k$. From condition $\mathfrak{C}$ it follows that

$$\phi(I_0) \leqq \sum_{i=1}^{k} \phi(I_i) < \sum_{i=1}^{k} \phi(J_i) + \epsilon \leqq \phi(I_0) + \epsilon.$$

Since $\epsilon>0$ is arbitrary it follows that $\phi(I_0)=\phi(I_1)+ \cdots +\phi(I_k)$. Thus $[\phi, C_\lambda]$ is additive over regular subdivisions.

Let E be any set in R_0 and, given $\epsilon>0$, let $\{I_n\}$ be a sequence of intervals in C_λ such that $\sum_n I_n \supset E$ and $\bar{\phi}(E)>\sum_n \phi(I_n)-\epsilon$. Given $\delta>0$ let $\mathcal{E}_{\lambda n}I_n$, $n=1, 2, \cdots$, be a regular subdivision of I_n such that $\|\mathcal{E}_{\lambda n}I_n\|<\delta$. Arrange the set of intervals J such that $J \in \mathcal{E}_{\lambda n}I_n$ for some integer n into a sequence $\{J_m\}$. Then $\sum_m J_m \supset E$; $\bar{\phi}(E)>\sum_m \phi(J)-\epsilon$; and $\|J_m\|<\delta$ for $m=1, 2, \cdots$.

Let E_1 and E_2 be any two sets such that $d(E_1, E_2)=\delta>0$, and let there be given a number $\epsilon>0$. Let $\{I_n\}$ be a sequence of intervals in C_λ such that $\sum_n I_n \supset E_1+E_2$, $\|I_n\|<\delta$, $n=1, 2, \cdots$, and $\bar{\phi}(E_1+E_2)>\sum_n \phi(I_n)-\epsilon$. Let I_{1i}, $i=1, 2, \cdots$, be the intervals in $\{I_n\}$ such that $I_{1i}\cdot E_1 \neq 0$. Let I_{2j}, $j=1, 2, \cdots$, be the remainder of the intervals in $\{I_n\}$. Then $\sum_i I_{1i} \supset E_1$; $\sum_j I_{2j} \supset E_2$ and $\bar{\phi}(E_1)+\bar{\phi}(E_2) \leqq \sum_i \phi(I_{1i})+\sum_j \phi(I_{2j})=\sum_n \phi(I_n)<\bar{\phi}(E_1+E_2)+\epsilon$. Since $\epsilon>0$ is arbitrary it follows that $\bar{\phi}(E_1)+\bar{\phi}(E_2) \leqq \bar{\phi}(E_1+E_2)$. This result together with condition (ii) establishes condition (iii) in the definition of outer Carathéodory measure.

Applying the theory of outer Carathéodory measure (see [12, chap. 2]) we may now complete our proof of Theorem 1. A set E is ϕ-measurable if $\bar{\phi}(P+Q)=\bar{\phi}(P)+\bar{\phi}(Q)$ for every pair of sets P and Q contained, respectively, in the set E and in its complement CE. The class of all sets which are ϕ-measurable—we denote it by K_ϕ—is an additive class in the sense of Saks, that is, (i) it contains the empty set, (ii) if $E \in K_\phi$ then $CE \in K_\phi$, and (iii) if $\{E_n\}$ is a sequence of sets in K_ϕ, then $\sum_n E_n$ is a set in K_ϕ. Furthermore, the class K_ϕ contains all sets which are open relative to R_0. Thus K_ϕ is a closed class of sets (1.2). The function $[\bar{\phi}, K_\phi]$ is a completely additive set function; it is non-negative; and it follows from condition $\mathfrak{C}$ that $\bar{\phi}(I) = \phi(I)$ for every $I \in C_\lambda$. Thus $[\bar{\phi}, K_\phi]$ is a non-negative completely additive extension of the function $[\phi, C_\lambda]$. This completes a proof of Theorem 1; we proceed to outline a proof of Theorem 2.

2.3. Let $[\phi, C_\lambda]$ be a function of intervals and let $[\Phi, K]$ be a completely additive extension of $[\phi, C_\lambda]$. Then $[\Phi, K]$ is the difference of two non-negative completely additive set functions (see [13, p. 90]), call them $[\Phi_1, K]$ and $[\Phi_2, K]$. We suppose that $\Phi(E) = \Phi_1(E) - \Phi_2(E)$ for every set $E \in K$. Each of the functions $[\Phi_i, K]$, $i = 1, 2$, satisfies the condition stated in 2.1; in particular, each of the functions $[\Phi_i, C_\lambda]$ satisfies condition $\mathfrak{C}$. But $\phi(I_\lambda) = \Phi(I_\lambda) = \Phi_1(I_\lambda) - \Phi_2(I_\lambda)$; thus the condition in Theorem 2 is necessary.

Let $[\phi, C_\lambda]$, $[\phi_1, C_\lambda]$, $[\phi_2, C_\lambda]$ be finite, single-valued functions of intervals. We assume that $[\phi_1, C_\lambda]$ and $[\phi_2, C_\lambda]$ are non-negative functions each of which satisfies condition $\mathfrak{C}$ and that $\phi(I) = \phi_1(I) - \phi_2(I)$ for every $I \in C_\lambda$. Let $[\Phi_1, K_1]$ and $[\Phi_2, K_2]$ be non-negative completely additive extensions of $[\phi_1, C_\lambda]$ and $[\phi_2, C_\lambda]$ respectively. Let K denote the closed class $K_1 \cdot K_2$. For every set $E \in K$ we define $\Phi(E) = \Phi_1(E) - \Phi_2(E)$. The function $[\Phi, K]$ is completely additive, and furthermore, the relation

$$\phi(I) = \phi_1(I) - \phi_2(I) = \Phi_1(I) - \Phi_2(I) = \Phi(I)$$

holds for every $I \in C_\lambda$. Thus $[\Phi, K]$ is a completely additive extension of $[\phi, C_\lambda]$. This establishes the sufficiency of the condition in Theorem 2.

2.4(3). Let $[\Phi, B]$ be a non-negative additive function of Borel sets (1.6). For every set $E \in B$ define $\bar{\phi}(E) = \text{g.l.b.} \sum_n \Phi(I_n)$ for all sequences $\{I_n\}$ of intervals in C_λ such that $\sum_n I_n \supset E$. The function $[\Phi, B]$ is a non-negative completely additive extension of the function $[\Phi, C_\lambda]$. Therefore $[\Phi, C_\lambda]$ satisfies condition $\mathfrak{C}$ and the function $\bar{\phi}(E)$ is completely additive on the class of all Borel sets. It follows from the definition of $\bar{\phi}(E)$ and from condition $\mathfrak{C}$ that $\bar{\phi}(R_0) = \Phi(R_0)$ and that $\bar{\phi}(E) \geqq \Phi(E)$ for every $E \in B$. Thus, for $E \in B$, we have $\bar{\phi}(E) \geqq \Phi(E)$, $\bar{\phi}(CE) \geqq \Phi(CE)$, and $\bar{\phi}(E) + \bar{\phi}(CE) = \bar{\phi}(R_0)$ $= \Phi(R_0) = \Phi(E) + \Phi(CE)$. Obviously then, $\bar{\phi}(E) = \Phi(E)$.

2.5. Let $[\Phi_1, B]$ and $[\Phi_2, B]$ be any two non-negative additive functions of Borel sets and suppose that $\Phi_1(I) \geqq \Phi_2(I)$ for every $I \in C_\lambda$. For $E \in B$ and for $i = 1, 2$, define $\bar{\phi}_i(E) = \text{g.l.b.} \sum_n \Phi_i(I_n)$ for all sequences $\{I_n\}$ such that $\sum_n I_n \supset E$. From 2.4 it follows that $\bar{\phi}_i(E) = \Phi_i(E)$ for every $E \in B$. But $\bar{\phi}_1(E) \geqq \bar{\phi}_2(E)$, and hence $\Phi_1(E) \geqq \Phi_2(E)$.

(3) The proof of Theorem 3 as presented here in §§2.4–2.7 was suggested by Professor Earl Mickle. For another treatment of the uniqueness of a completely additive extension, see [9].

2.6. Let $[\Phi_1, B]$ and $[\Phi_2, B]$ be any two additive functions of Borel sets such that $\Phi_1(I_\lambda) \geqq \Phi_2(I_\lambda)$ for every $I_\lambda \in C_\lambda$. We express each of the functions Φ_1, Φ_2 as the difference of two non-negative additive functions of Borel sets:

$$\Phi_1(E) = \Phi_{11}(E) - \Phi_{12}(E); \qquad \Phi_2(E) = \Phi_{21}(E) - \Phi_{22}(E).$$

Then for $I \in C_\lambda$ we have $\Phi_{11}(I) - \Phi_{12}(I) = \Phi_1(I) \geqq \Phi_2(I) = \Phi_{21}(I) - \Phi_{22}(I)$ and $\Phi_{11}(I) + \Phi_{22}(I) \geqq \Phi_{12}(I) + \Phi_{21}(I)$. From 2.5 it follows that if $E \in B$ then $\Phi_{11}(E) + \Phi_{22}(E) \geqq \Phi_{12}(E) + \Phi_{21}(E)$. From this it follows immediately that $\Phi_1(E) \geqq \Phi_2(E)$.

2.7. Let $[\Phi_1, K]$ and $[\Phi_2, K]$ be two completely additive extensions of a function $[\phi, C_\lambda]$. Then $\Phi_1(I) = \Phi_2(I) = \phi(I)$ for every $I \in C_\lambda$. It follows from 2.6 that $\Phi_1(E) = \Phi_2(E)$ for every $E \in B$. In other words, if $[\Phi, K]$ is a completely additive extension of a function $[\phi, C_\lambda]$, and if $E \in B$, then the value of $\Phi(E)$ is uniquely determined by the function $[\phi, C_\lambda]$. If $[\phi, C_\lambda]$ is non-negative, the number $\Phi(E)$ has the characterization as defined in 2.2. Thus we have established Theorem 3.

3. B-**extensions.** This part contains our results on the B-extension as stated in §1.6. In §§3.1 and 3.2 we state and prove two lemmas which are used in the proofs of the main theorems.

3.1. Let $[\Phi, B]$ be a completely additive extension of a function of intervals $[\phi, C_\lambda]$. A necessary and sufficient condition that $[\Phi, B]$ be a B-extension of $[\phi, C_\lambda]$ is that $\Phi(I) = \Phi(I^0)$ for every $I \in C_0$.

Proof. Let $[\Phi, B]$ be a B-extension of $[\phi, C_\lambda]$. Then $\Phi(I) = \Phi(I^0)$ for every $I \in C_\lambda$. Let $I \in C_0$ and let $\mathcal{E}_\lambda$ be a subdivision of I into the intervals $I_1, \cdots, I_n$. Express the set $I - I^0$ as the sum of mutually exclusive Borel sets $E_1 + E_2 + \cdots + E_n$ where $E_i \subset I_i - I_i^0$; $i = 1, \cdots, n$. Since $\Phi(E_i) = 0$ it follows that $\Phi(I - I^0) = 0$ and that $\Phi(I) = \Phi(I^0)$.

Let $[\Phi, K]$ be a completely additive extension of $[\phi, C_\lambda]$ and assume that $\Phi(I) = \Phi(I^0)$ for every $I \in C_0$. Let i, $i \subset R_0$, be any closed linear interval which is parallel to either the x- or the y-axis. Let $I_1 \supset I_2 \supset \cdots$ be a descending sequence of intervals in C_0 such that $\prod_n I_n = i$ and $\prod_n I_n^0$ is the empty set. It follows from the additivity of $[\Phi, B]$ that

$$0 = \lim_n \Phi(\overset{0}{I_n}) = \lim_n \Phi(I_n) = \Phi(i).$$

Let t be a transversal or a boundary segment of R_0. The function $\Phi(E)$ where $E \in B$, $E \subset t$ is a completely additive extension of the function of linear intervals $\Phi(i)$ where $i \subset t$. But such an extension is unique on Borel sets. Therefore, since $\Phi(i) = 0$, it follows that $\Phi(E) = 0$ for $E \in B$, $E \subset t$. Let I be any interval in C_λ and let $E \in B$ be any set such that $I^0 \subset E \subset I$. Then $\Phi(I - E) = 0$ and it follows that

$$\phi(I) = \Phi(I) = \Phi(I - E) + \Phi(E) = \Phi(E).$$

Thus $[\Phi, B]$ is a B-extension of $[\phi, C_\lambda]$.

3.2. If a function of intervals $[\phi, C_\lambda]$ admits a B-extension, then $[\phi, C_\lambda]$ is additive.

Proof. Let $[\Phi, B]$ be a B-extension of $[\phi, C_\lambda]$. Let I_λ and $\mathcal{E}_\lambda I_\lambda$ be given. Denote the intervals in $\mathcal{E}_\lambda I_\lambda$ by $I_1, I_2, \cdots, I_n$. Then

$$\phi(I_\lambda) = \Phi(\overset{0}{I_\lambda}) \geqq \sum_{i=1}^{n} \Phi(\overset{0}{I_i}) = \sum_{i=1}^{n} \phi(I_i).$$

But $\sum_{i=1}^{n} I_i \supset I_\lambda$. It follows from condition $\mathfrak{C}$ that $\phi(I_\lambda) \leqq \sum_{i=1}^{n} \phi(I_i)$. Thus $[\phi, C_\lambda]$ is additive. If we extend the range of definition of $[\phi, C_\lambda]$ from C_λ to C_0 by defining $\phi(I) = \Phi(I)$, then the function $[\phi, C_0]$ is also additive.

3.3. We proceed to a proof of Theorem 4. Let $[\phi, C_\lambda]$ be a non-negative function of intervals and let $[\Phi, B]$ be a non-negative B-extension of $[\phi, C_\lambda]$. Extend the range of definition of $[\phi, C_\lambda]$ from C_λ to C_0 by defining $\phi(I) = \Phi(I)$ for all $I \in C_0$. Let R_* be a fixed interval such that $R_*^0 \supset R_0$. Let C_* denote the class of all intervals $I \subset R_*$. Define the function $[\phi_*, C_*]$ by the relation $\phi_*(I) = \phi(I \cdot R_0)$ if $I \cdot R_0$ is an interval in C_0; by the relation $\phi_*(I) = 0$ for all other $I \in C_*$. Let B_* denote the class of all Borel sets $E \subset R_*$ and define the function $[\Phi_*, B_*]$ by the relation $\Phi_*(E) = \Phi(E \cdot R_0)$ for every $E \in B_*$. Then $[\Phi_*, B_*]$ is a non-negative B-extension of $[\phi_*, C_*]$. Let t be any transversal of R_0 or a closed boundary segment of R_0. Let $\{I_n\}$ be a sequence of intervals in C_* such that $\prod_{n=1}^{\infty} I_n^0 = t$. Then

$$\lim_n \phi_*(I_n) = \lim_n \Phi_*(I_n) = \lim_n \Phi_*(\overset{0}{I_n}) = \Phi_*(t) = \Phi(t) = 0.$$

Given $\epsilon > 0$ let each vertical transversal of R_0 and each of the two vertical sides of R_0 be covered by an open interval I^0 such that $I \in C_*$ and $\phi_*(I) < \epsilon$. Then the closed set R_0 is covered by this class of open intervals. A finite number of these open intervals, say $I_1^0, I_2^0, \cdots, I_n^0$, suffice to cover R_0. Let t_1, t_2 denote the vertical boundary segments of R_0 and let $t_3, t_4, \cdots, t_m$ be the set of vertical transversals of R_0 which lie on the boundaries of the intervals I_i, $i = 1, \cdots, n$. Let δ_1 be the minimum of the numbers $d(t_i \cdot t_j)$ where $i \neq j$ and $i, j = 1, 2, \cdots, m$. Let $I \in C_0$ be any interval whose horizontal dimension is less than δ_1. Then $I \subset I_i$ for some integer i, $i = 1, \cdots, n$, and it follows that $\phi(I) = \phi_*(I) \leqq \phi_*(I_i) < \epsilon$. Similarly we may show that there exists a number $\delta_2 > 0$ such that if $I \in C_0$ is an interval whose vertical dimension is less than δ_2 then $\phi(I) < \epsilon$. Let $\delta_3 = \min(\delta_1, \delta_2)$, let δ be a number such that $0 < \delta < \delta_3^2$, and let $I \in C_0$ be any interval such that $|I| < \delta$. Then at least one dimension of I is less than δ_3 and it follows that $\phi(I) < \epsilon$. Condition (ii) in the definition of continuity follows from the additivity of $[\phi, C_\lambda]$ which was established in 3.2. Thus the condition in Theorem 4 is necessary.

3.4. Let $[\phi, C_\lambda]$ be a non-negative, additive, continuous function of intervals. Let I be any interval in C_0 and let $\mathcal{E}_\lambda I$ and $\mathcal{J}_\lambda I$ be any two subdivisions of I. Complete the elementary system $\mathcal{E}_\lambda I$ with an elementary system of intervals $\mathcal{G}_\lambda$ to a subdivision of R_0. The system $\mathcal{J}_\lambda+\mathcal{G}_\lambda$ also forms a subdivision of R_0 and we have that $\phi(\mathcal{E}_\lambda)+\phi(\mathcal{G}_\lambda)=\phi(R_0)=\phi(\mathcal{J}_\lambda)+\phi(\mathcal{G}_\lambda)$. Thus $\phi(\mathcal{E}_\lambda)=\phi(\mathcal{J}_\lambda)$. We extend the range of definition of $[\phi, C_\lambda]$ from C_λ to C_0 by defining $\phi(I)=\phi(\mathcal{E}_\lambda I)$, where $\mathcal{E}_\lambda I$ is any subdivision of $I\in C_0$. The function $[\phi, C_0]$ is non-negative and additive. Using the definitional properties of continuity on the class C_λ, the additivity on the class C_0, and a technique similar to that employed in 3.3, it may be proved that $[\phi, C_0]$ is a continuous function.

From the non-negative and additive properties of $[\phi, C_0]$ it follows that $[\phi, C_0]$ satisfies the following condition $\mathfrak{C}'$: *If $I_1, I_2, \cdots, I_n$ is a finite set of intervals in C_0 such that $I_i\cdot I_j=0$ when $i\neq j$, and if $J_1, J_2, \cdots, J_m$ is a finite set of intervals in C_0 such that $\sum_{j=1}^m J_j\supset\sum_{i=1}^n I_i$, then $\sum_{j=1}^m\phi(J_j)\geqq\sum_{i=1}^n\phi(I_i)$.*

We shall show that $[\phi, C_0]$ satisfies condition $\mathfrak{C}$ as stated in Theorem 1. Let R_* be a fixed interval such that $R_*^0\supset R_0$. Let the class C_* and the function $[\phi_*, C_*]$ be defined as in 3.3. Then $[\phi_*, C_*]$ is a continuous, additive function. Let $I_1, I_2, \cdots, I_n$ be a finite set of mutually exclusive intervals in C_0 and let $J_1, J_2, \cdots$ be an infinite sequence of intervals in C_0 such that $\sum_{j=1}^\infty J_j\supset\sum_{i=1}^n I_i$. Given $\epsilon>0$, let R_i, $i=1, 2, \cdots$, be an interval in C_* such that $R_i^0\supset J_i$ and $\phi_*(R_i)<\phi_*(J_i)+\epsilon/2^i$. Then $\sum_{j=1}^\infty R_j^0\supset\sum_{i=1}^n I_i$. Since the set $\sum_{i=1}^n I_i$ is closed it follows that there is an integer m such that $\sum_{j=1}^m R_j^0\supset\sum_{i=1}^n I_i$. Thus we have

$$\sum_{i=1}^n\phi(I_i)=\sum_{i=1}^n\phi_*(I_i)\leqq\sum_{i=1}^m\phi_*(R_i)<\sum_{i=1}^\infty\phi_*(J_i)+\epsilon=\sum_{i=1}^\infty\phi(J_i)+\epsilon.$$

Since $\epsilon>0$ is arbitrary, it follows that $\sum_{i=1}^n\phi(I_i)\leqq\sum_{j=1}^\infty\phi(J_j)$. Thus condition $\mathfrak{C}'$ is fulfilled when the finite set of J's is replaced by a sequence of J's. That it may be further extended by replacing the finite set of I's by a sequence of I's is obvious. Thus the function $[\phi, C_0]$ satisfies condition $\mathfrak{C}$. Let $[\Phi, B]$ be the non-negative completely additive extension of $[\phi, C_0]$ to the class B. We shall show that $[\Phi, B]$ is a B-extension of $[\phi, C_0]$. Let I be any interval in C_0. Given a number $\epsilon>0$, let $J\in C_0$ be an interval such that $J\subset I^0$ and $\phi(J)>\phi(I)-\epsilon$. Then

$$\Phi(I)=\phi(I)<\phi(J)+\epsilon=\Phi(J)+\epsilon=\Phi(I^0)+\epsilon.$$

Since $\epsilon>0$ is arbitrary, it follows that $\Phi(I)=\Phi(I^0)$ and from 3.1 it follows that $[\Phi, B]$ is a B-extension of $[\phi, C_0]$; obviously it is also a B-extension of the function $[\phi, C_\lambda]$. Thus the condition in Theorem 4 is sufficient.

3.5. Given a function of intervals $[\phi, C_\lambda]$ and a transversal t of R_0 we define(4):

$A(t)=\lim\sup\ [\phi(\mathcal{E}_\lambda)-\phi(\mathcal{J}_\lambda)]$ for all $\mathcal{E}_\lambda$ and $\mathcal{J}_\lambda$ such that $\|\mathcal{E}_\lambda\|\to 0$,

(4) This definition is an extension of the concept of ecart as employed by Saks for functions of linear intervals. See [11, p. 211].

$\|\mathcal{J}_\lambda\| \to 0$, and $I \cdot t \neq 0$ for all $I \in \mathcal{E}_\lambda$ and $I \in \mathcal{J}_\lambda$.

$a(t) = \liminf [\phi(\mathcal{E}_\lambda) - \phi(\mathcal{J}_\lambda)]$ for all $\mathcal{E}_\lambda$ and $\mathcal{J}_\lambda$ as in the definition of $A(t)$.

The numbers $\Omega(t) = 2^{-1}[A(t) + |A(t)|] \geqq 0$ and $\omega(t) = 2^{-1}[a(t) - |a(t)|] \leqq 0$ are called the non-negative and non-positive *ecarts* of $[\phi, C_\lambda]$ on t. If both $\Omega(t) = 0$ and $\omega(t) = 0$, then t is a transversal of zero ecart; otherwise t is a transversal of nonzero ecart.

Let $[\phi, C_\lambda]$ be a function of intervals which is RAC, that is, the function $[U(|\phi|), C_0]$ is continuous. Let t be a transversal of R_0. Given $\epsilon > 0$, let I be an interval in C_0 such that $I^0 \supset t^0$, $I \supset t$, and $U(|\phi|, I) < \epsilon/4$. Let $\eta > 0$ be a number such that $\|\mathcal{E}_\lambda I\| < \eta$ implies $|\phi|(\mathcal{E}_\lambda I) < U(|\phi|, I) + \epsilon/4$. Then if $\mathcal{J}_{\lambda 1}$ and $\mathcal{J}_{\lambda 2}$ are any two elementary systems such that $\|\mathcal{J}_{\lambda i}\| < \eta$, $i = 1, 2$, and $J \subset I$ for every interval $J \in \mathcal{J}_{\lambda i}$, we have

$$|\phi|(\mathcal{J}_{\lambda i}) < U(|\phi|, I) + \epsilon/4 < \epsilon/2.$$

Thus $|\phi(\mathcal{J}_{\lambda 1}) - \phi(\mathcal{J}_{\lambda 2})| \leqq |\phi|(\mathcal{J}_{\lambda 1}) + |\phi|(\mathcal{J}_{\lambda 2}) < \epsilon$. Since $\epsilon > 0$ is arbitrary, it follows that $A(t) = a(t) = 0$ and R_0 has no transversals of nonzero ecart.

3.6. If $[\phi, C_\lambda]$ increases (decreases) by subdivision and is RAC, then $U(\phi, I)$ $(L(\phi, I))$ is finite and additive.

Proof. Since $[\phi, C_\lambda]$ increases by subdivision, it follows from a theorem of Kempisty that $U(\phi, I) > -\infty$ for all $I \in C_0$ and that $U(\phi, I)$ is additive. It follows from the continuity of $U(|\phi|, I)$ that $U(\phi, I)$ is also a continuous function. Let $\delta > 0$ be a number such that $|I| < \delta$ implies $U(\phi, I) < 1$. Let $\mathcal{E}I_0$ be a subdivision of a fixed interval I_0 into intervals I such that $|I| < \delta$. It follows that $U(\phi, I_0)$ is less than the number of intervals in $\mathcal{E}I_0$.

If $[\phi, C_\lambda]$ decreases by subdivision, the result follows as a corollary if we consider the function $[-\phi, C_\lambda]$.

3.7. If $[\phi, C_\lambda]$ is RAC and increases (decreases) by subdivision, then $[\phi, C_\lambda]$ is integrable.

Proof. Let $[\phi, C_\lambda]$ be an RAC function which increases by subdivision. Then by 3.6, $U(\phi, R_0)$ is a finite number. Given $\epsilon > 0$, let $\mathcal{E}_\lambda R_0$ be a subdivision of R_0 such that $\phi(\mathcal{E}_\lambda R_0) > U(\phi, R_0) - \epsilon$. Let $\{\mathcal{E}_{\lambda m} R_0\}$ be a sequence of subdivisions of R_0 such that $\|\mathcal{E}_{\lambda m} R_0\| \to_m 0$ and $\phi(\mathcal{E}_{\lambda m} R_0) \to_m L(\phi, R_0)$. Let $t_1, t_2, \cdots, t_k$ be the set of all transversals of R_0 such that t_i, $i = 1, \cdots k$, contains at least one boundary segment of an interval $I \in \mathcal{E}_\lambda R_0$. Let $\mathcal{J}_m$, $m = 1, 2, \cdots$, be the subdivision of R_0 which is formed by the transversals t_i, $i = 1, 2, \cdots, k$, and all of the boundary segments of intervals $I \in \mathcal{E}_{\lambda m}$. Let $T = \sum_{i=1}^k t_i$ and let $\mathcal{E}_{\lambda m}^T$ be the elementary system consisting of all the intervals I such that $I \in \mathcal{E}_{\lambda m}$ and $I \cdot T \neq 0$. Each interval I in $\mathcal{J}_m$ which is in the class $C_0 - C_\lambda$ is contained in an interval $I \in \mathcal{E}_{\lambda m}^T$. We replace each such interval $I \in \mathcal{J}_m$ by a subdivision $\mathcal{K}_\lambda I$ and denote the resulting elementary system by $\mathcal{J}_{\lambda m}$. Let $\mathcal{J}_{\lambda m}^T$ denote the elementary system of intervals I such that $I \in \mathcal{J}_{\lambda m}$ and I is contained in some interval $J \in \mathcal{E}_{m\lambda}^T$. The elementary systems $\mathcal{E}_{\lambda m} - \mathcal{E}_{\lambda m}^T$ and $\mathcal{J}_{\lambda m} - \mathcal{J}_{\lambda m}^T$ are identical. Therefore $\phi(\mathcal{E}_{\lambda m}) - \phi(\mathcal{J}_{\lambda m})$

$=\phi(\mathcal{E}^T_{\lambda m})-\phi(\mathcal{J}^T_{\lambda m})$. It follows from 3.5 that $\lim_m [\phi(\mathcal{E}^T_{\lambda m})-\phi(\mathcal{J}^T_{\lambda m})]=0$. Thus $\lim_m \phi(\mathcal{J}_{\lambda m})=\lim_m \phi(\mathcal{E}_{\lambda m})=L(\phi, R_0)$. Since $[\phi, C_\lambda]$ increases by subdivision we have

$$\phi(\mathcal{J}_{\lambda m}) \geqq \phi(\mathcal{E}_\lambda) > U(R_0) - \epsilon, \qquad m = 1, 2, \cdots .$$

Therefore $L(\phi, R_0)=\lim_m \phi(\mathcal{J}_{\lambda m})>U(\phi, R_0)-\epsilon$ and it follows that $[\phi, C_\lambda]$ is integrable. The case in which $[\phi, C_\lambda]$ decreases by subdivision follows as an immediate corollary since the function $[-\phi, C_\lambda]$ increases by subdivision.

3.8. We proceed to the proof of Theorem 5. Let $[\Phi, B]$ be a B-extension of the function of intervals $[\phi, C_\lambda]$. Express the function $[\Phi, B]$ as the difference of two non-negative additive functions of Borel sets, say $[\Phi_1, B]$ and $[\Phi_2, B]$, where $\Phi(E)=\Phi_1(E)-\Phi_2(E)$, $E\in B$. For $i=1, 2$ and $I\in C_0$ define $\phi_i(I)=\Phi_i(I)$. Then $[\phi_1, C_0]$ and $[\phi_2, C_0]$ are non-negative additive functions of intervals, and $[\Phi_1, B]$, $[\Phi_2, B]$ are non-negative B-extensions of $[\phi_1, C_0]$, $[\phi_2, C_0]$ respectively. It follows from Theorem 4 that $[\phi_1, C_0]$, $[\phi_2, C_0]$ are continuous functions. Since $|\phi(I_\lambda)|\leqq\phi_1(I_\lambda)+\phi_2(I_\lambda)$ for every $I\in C_\lambda$ it follows that $U(|\phi|, I)\leqq U(\phi_1, I)+U(\phi_2, I)=\phi_1(I)+\phi_2(I)$ for every $I\in C_0$. Thus $U(|\phi|, I)$ is a continuous function, that is, the function $[\phi, C_\lambda]$ is RAC. The necessity of the condition in Theorem 5 follows from this result and 3.2.

Let $[\phi, C_\lambda]$ be an additive, RAC function of intervals. For every $I\in C_\lambda$ define $\phi_1(I)=2^{-1}[\phi(I)+|\phi(I)|]$, $\phi_2(I)=2^{-1}[|\phi(I)|-\phi(I)]$. Then $[\phi_1, C_\lambda]$, $[\phi_2, C_\lambda]$ are non-negative functions and $\phi(I_\lambda)=\phi_1(I_\lambda)-\phi_2(I_\lambda)$ for every $I\in C_\lambda$. It is readily proved that $[\phi_1, C_\lambda]$, $[\phi_2, C_\lambda]$ are RAC and increase by subdivision. It follows from 3.7 that they are integrable. Their indefinite integrals, denoted by $[F_1, C_0]$ and $[F_2, C_0]$ respectively, are continuous, additive functions. Let $[\Phi_1, B]$ and $[\Phi_2, B]$ denote the B-extensions of $[F_1, C_0]$ and $[F_2, C_0]$ respectively. For every $E\in B$ define $\Phi(E)=\Phi_1(E)-\Phi_2(E)$. For $I\in C_\lambda$ we have $\phi(I)=\phi_1(I)-\phi_2(I)=\Phi_1(I)-\Phi_2(I)=\Phi(I)$. For $I\in C_0$ we have $\Phi(I)=\Phi_1(I)-\Phi_2(I)=\Phi_1(I^0)-\Phi_2(I^0)=\Phi(I^0)$. It follows from 3.1 that $[\Phi, B]$ is a B-extension of $[\phi, C_\lambda]$.

3.9. If $[\phi, C_\lambda]$ is integrable, then $U(|\phi|, I)=U(|F(\phi)|, I)$ for every $I\in C_0$.

Proof. Given a number $\epsilon>0$ let $\delta>0$ be a number such that $\|\mathcal{E}_\lambda\|<\delta$ implies $|\phi(\mathcal{E}_\lambda)-F(\phi, \mathcal{E}_\lambda)|<\epsilon/2$ (see [6, Theorem 3]). Given an interval $I_0\in C_0$ let $\mathcal{E}_\lambda I_0$ be a subdivision of I_0 which consists of the intervals $I_1, \cdots, I_n$, and which is such that $\|\mathcal{E}_\lambda I_0\|<\delta$. Let $\mathcal{E}_1$ consist of the intervals $I\in\mathcal{E}_\lambda I_0$ which satisfy the relation $\phi(I)-F(\phi, I)\geqq 0$ and let $\mathcal{E}_2$ consist of the intervals $I\in\mathcal{E}_\lambda I_0$ which satisfy the relation $\phi(I)-F(\phi, I)<0$. Then $|\phi(\mathcal{E}_i)-F(\phi, \mathcal{E}_i)|<\epsilon/2$, $i=1, 2$, and $\sum_{i=1}^n|\phi(I_i)-F(\phi, I_i)|<\epsilon$. But $|F(\phi, I_i)|\leqq|\phi(I_i)|+|F(\phi, I_i)-\phi(I_i)|$;

$$|\phi(I_i)| \leqq |F(\phi, I_i)| + |F(\phi, I_i) - \phi(I_i)|.$$

Therefore $||F(\phi, I_i)|-|\phi(I_i)||\leqq|F(\phi, I_i)-\phi(I_i)|$, and $\sum_{i=1}^n||F(\phi, I_i)|$

$-|\phi(I_i)|| < \epsilon$. It follows that $U(|\phi|, I_0) = U(|F(\phi)|, I_0)$.

3.10. Since the integral of a function of intervals is additive, it follows from Theorem 5 that: *A necessary and sufficient condition that the integral of a function of intervals admit a B-extension is that the integral be RAC.* Theorem 6 follows immediately from 3.9.

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