ON ABSOLUTE CONVERGENCE OF MULTIPLE FOURIER SERIES

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Introduction. The results of this paper are extensions of corresponding results for simple Fourier series, given by one of the authors (cf. [5])(1). The main problem was to study the relationship between the mean modulus of a function $f(x)$ and series of the type $\sum |c_n|^\beta$, $\beta > 0$, where the c_n are the Fourier coefficients of $f(x)$. We obtain here analoguous results, employing spherical means of a function of several variables. These means were first used by Bochner [1] in the study of summation of multiple Fourier series.

A particular result is: if $a_{n_1 \cdots n_\kappa}$ are the Fourier coefficients of $f(x_1, \cdots, x_\kappa)$, and f satisfies a Lipschitz condition of degree α, then $\sum |a_{n_1 \cdots n_\kappa}|^\beta < \infty$ for $\beta > 2\kappa/(\kappa + 2\alpha)$, while the series may be divergent for $\beta = 2\kappa/(\kappa+2\alpha)$. For some previous results concerning the absolute convergence of double Fourier series cf. [3].

1. **Notations.** We denote by capital letters vectors in the κ-dimensional space, so that $X = (x_1, x_2, \cdots, x_\kappa)$, $N = (n_1, n_2, \cdots, n_\kappa)$; $|N| = (\sum_1^\kappa n_\nu^2)^{1/2}$ is the norm of N; $NX = \sum_1^\kappa n_\nu x_\nu$ is the scalar product of N and X. The $x_1, \cdots, x_\kappa$ are real variables, the $n_1, \cdots, n_\kappa$ are integers. $f(x_1, \cdots, x_\kappa) = f(X)$ is a real-valued integrable function of period 2π in each variable. The formal Fourier series of $f(X)$ is

$$(1.1) \qquad f(X) \sim \sum_{n_1} \overset{-\infty,\infty}{\cdots} \sum_{n_\kappa} c_{n_1, \cdots, n_\kappa} e^{i(n_1x_1 + \cdots + n_\kappa x_\kappa)} = \sum c_N e^{iNX},$$

where

$$(1.2) \qquad c_N = \frac{1}{(2\pi)^\kappa} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} f(X) e^{-iNX} dX.$$

$J_\mu(x)$ is the Bessel function of order $\mu \geqq 0$:

$$J_\mu(x) = \sum_{\nu=0}^{\infty} (-1)^\nu \frac{(x/2)^{\mu+2\nu}}{\nu!\Gamma(\mu+\nu+1)};$$

we put

$$\alpha_\mu(x) = \frac{2^\mu \Gamma(\mu+1) J_\mu(x)}{x^\mu} = \sum_{\nu=0}^{\infty} (-1)^\nu \frac{x^{2\nu}\Gamma(\mu+1)}{4^\nu \nu! \Gamma(\mu+\nu+1)},$$

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(1) Numbers in brackets refer to the bibliography at the end of the paper.

$$A_n(X) = \sum_{|N|^2=n} c_N \exp(iNX), \quad \text{so that} \quad f(X) \sim \sum_{n=0}^{\infty} A_n(X).$$

We shall denote by $\omega(t)$ a positive function of t, decreasing to zero as $t \downarrow 0$.

2. **Lemmas.** We give here some auxiliary theorems.

LEMMA 1. *If $R_\kappa(n)$ is the number of lattice points in the sphere $\sum_1^n x_\nu^2 \leqq n$, then*

$$R_\kappa(n) = O(n^{\kappa/2}) \equiv On^{\kappa/2}, \qquad \text{as } n \to \infty. \tag{2.1}$$

Actually the sharper estimate is known [cf. 2, p. 825]:

$$R_\kappa(n) = \frac{\pi^{\kappa/2} n^{\kappa/2}}{\Gamma(1+\kappa/2)} + On^{\kappa(\kappa-1)/2(\kappa+1)}.$$

LEMMA 2. *For $\mu \geqq 0$, x real or complex,*

$$\begin{aligned} \alpha_\mu(x) &= \frac{2\Gamma(\mu+1)}{\Gamma(\mu+1/2)\Gamma(1/2)} \int_0^{\pi/2} \cos(x\cos t)\sin^{2\mu} t\,dt \\ &= \frac{\Gamma(\mu+1)}{\Gamma(\mu+1/2)\Gamma(1/2)} \int_0^{\pi} e^{ix\cos t} \sin^{2\mu} t\,dt. \end{aligned} \tag{2.2}$$

The proof follows on using the cosine series or exponential series and integrating termwise [6, pp. 47–48].

COROLLARY. *For real x*

$$|\alpha_\mu(x)| \leqq \frac{2\Gamma(\mu+1)}{\Gamma(\mu+1/2)\Gamma(1/2)} \int_0^{\pi/2} \sin^{2\mu} t\,dt = \alpha_\mu(0) = 1. \tag{2.3}$$

For $\mu = 0$, (2.3) reduces to $|J_0(x)| \leqq 1$, an inequality given by Hansen [6, p. 31].

LEMMA 3. *For any $u > 0$ and a corresponding constant $b(u) > 0$, $b(u) < 1 - \alpha_\mu(x) < 2$ for $x > u$; moreover*

$$1 - \alpha_\mu(x) > \frac{x^2}{\pi^2(\mu+1)} \qquad \text{for } 0 < x < \pi,$$

and

$$1 - \alpha_\mu(x) < (x/2)^2 \frac{1}{\mu+1} \qquad \text{for } x > 0.$$

Proof. From (2.2) and (2.3), putting $2\Gamma(\mu+1)/\Gamma(\mu+1/2)\Gamma(1/2) = \gamma(\mu)$, we have

$$1 - \alpha_\mu(x) = \gamma(\mu) \int_0^{\pi/2} \{1 - \cos(x\cos t)\} \sin^{2\mu} t\,dt > 0 \qquad \text{for } x > 0. \tag{2.4}$$

It is known that $J_\mu(x)\to 0$ as $x\to\infty$, hence $\alpha_\mu(x)\to 0$; thus for some $b(u)>0$

$$1-\alpha_\mu(x) > b(u) \qquad \text{for } x>u.$$

Furthermore from (2.4) and (2.3)

$$1-\alpha_\mu(x) < 2\gamma(\mu)\int_0^{\pi/2}\sin^{2\mu} t dt = 2, \qquad \text{for } x>0.$$

Finally, for $0<x<\pi$,

$$1-\cos(x\cos t) = 2\sin^2\left(\frac{x}{2}\cos t\right)\begin{cases} > \dfrac{2x^2}{\pi^2}\cos^2 t \\ < \dfrac{x^2}{2}\cos^2 t, \end{cases}$$

hence

$$1-\alpha_\mu(x) > \frac{2\gamma(\mu)}{\pi^2}x^2\int_0^{\pi/2}\cos^2 t\sin^{2\mu} t dt = \frac{2\gamma(\mu)}{\pi^2}x^2\left\{\frac{1}{\gamma(\mu)}-\frac{1}{\gamma(\mu+1)}\right\}$$
$$= \frac{x^2}{\pi^2(\mu+1)},$$

and

$$1-\alpha_\mu(x) < \frac{\gamma(\mu)x^2}{2}\int_0^{\pi/2}\cos^2 t\sin^{2\mu} t dt = \frac{x^2}{4(\mu+1)};$$

this proves the lemma.

LEMMA 4. *Let h be real, $r>0$, $\delta>0$, then the following statements are equivalent*:

$$\sum_{n=1}^{\infty} n^{rh-1}\omega(\delta n^{-r}) < \infty, \tag{2.5}$$

$$\sum_{\lambda=1}^{\infty} 2^{\lambda rh}\omega(\delta\cdot 2^{-\lambda r}) < \infty, \tag{2.5$'$}$$

$$\int_1^\infty t^{h-1}\omega\left(\frac{1}{t}\right)dt < \infty. \tag{2.6}$$

Proof. We have for $rh\geqq 1$

$$2^{(\lambda-1)rh}\omega(\delta\cdot 2^{-\lambda r}) < \sum_{\nu=2^{\lambda-1}}^{2^\lambda-1}\nu^{rh-1}\omega(\delta\cdot\nu^{-r}) < 2^{\lambda rh}\omega(\delta\cdot 2^{-(\lambda-1)r}),$$

hence

$$2^{-rh}\sum_{\lambda=1}^{\infty}2^{\lambda rh}\omega(\delta\cdot 2^{-\lambda r}) < \sum_{n=1}^{\infty}n^{rh-1}\omega(\delta\cdot n^{-r}) < 2^{rh}\sum_{0}^{\infty}2^{\lambda rh}\omega(\delta\cdot 2^{-\lambda r}),$$

with similar inequalities for $rh<1$; hence (2.5) and (2.5′) are equivalent. We also have for $rh\leqq 1$

$$\int_n^{n+1} x^{rh-1}\omega(\delta x^{-r})dx < n^{rh-1}\omega(\delta n^{-r}) < \int_{n-1}^{n} x^{rh-1}\omega(\delta x^{-r})dx,$$

hence

$$\int_1^{\infty} x^{rh-1}\omega(\delta x^{-r})dx < \sum_1^{\infty} n^{rh-1}\omega(\delta n^{-r}) < \int_0^{\infty} x^{rh-1}\omega(\delta x^{-r})dx,$$

with similar inequalities for $rh>1$; the substitution $x^r=\delta t$ yields the equivalence of (2.5) and (2.6). This proves the lemma.

In view of (2.6), r and δ are not necessarily the same in the different statements.

COROLLARY. *The following statements are equivalent:*

$$\sum n^{(h/\kappa)-1}\omega(\delta n^{-1/\kappa}) < \infty$$

and

$$\sum 2^{\lambda h/2}\omega(\delta\cdot 2^{-\lambda/2}) < \infty.$$

This follows on putting $r=1/\kappa$ in (2.5), and $r=1/2$ in (2.5′).

LEMMA 5. *If $a_\nu\geqq 0$, and $r>0$, then the two statements are equivalent:*

$$\sum_1^{\infty} a_\nu \left| 1-\alpha_\mu(t\nu^{1/2})\right|^r = O\omega(t) \qquad \text{as } t\rightarrow 0, \tag{2.7}$$

and

$$n^{-r}\sum_1^{n} \nu^r a_\nu + \sum_{n+1}^{\infty} a_\nu = O\omega(\delta n^{-1/2}) \qquad \text{as } n\rightarrow\infty, \tag{2.8}$$

δ being an arbitrary positive number.

Assume first that (2.8) holds; given $t>0$ choose

$$n = \lfloor \delta^2 t^{-2}\rfloor \leqq \delta^2 t^{-2} < n+1;$$

then from Lemma 3

$$\sum_1^{n+1} a_\nu \left| 1-\alpha_\mu(t\nu^{1/2})\right|^r < \frac{t^{2r}}{4^r(\mu+1)^r}\sum_1^{n+1}\nu^r a_\nu = Ot^{2r}n^r\omega(\delta(n+1)^{-1/2}) = O\omega(t),$$

and

$$\sum_{n+2}^{\infty} a_\nu \left| 1-\alpha_\mu(t\nu^{1/2})\right|^r < 2\sum_{n+2}^{\infty} a_\nu = O\omega(\delta(n+1)^{-1/2}) = O\omega(t).$$

Conversely, if (2.7) holds, choose for a given n and $\delta>0$

$$t = \min(\pi n^{-1/2}, \delta n^{-1/2}),$$

then

$$\sum_1^n a_\nu \left| 1 - \alpha_\mu(t\nu^{1/2}) \right|^r > \frac{t^{2r}}{\pi^{2r}(\mu+1)^r} \sum_1^n \nu^r a_\nu,$$

hence

$$n^{-r} \sum_1^n \nu^r a_\nu = O\omega(t) = O\omega(\delta n^{-1/2}).$$

Furthermore, using again Lemma 3, we have

$$\sum_{n+1}^\infty a_\nu \left| 1 - \alpha_\mu(t\nu^{1/2}) \right|^r > b \sum_{n+1}^\infty a_\nu \qquad (b \text{ a constant}),$$

hence

$$\sum_{n+1}^\infty a_\nu = O\omega(t) = O\omega(\delta n^{-1/2}).$$

This proves the lemma. It follows that if (2.8) holds for some $\delta > 0$, it holds for any $\delta > 0$.

LEMMA 6. *Assume that for some* $\delta > 0$

$$\sum_{\lambda=1}^\infty \omega(\delta 2^{-\lambda} n^{-1/2}) = O\omega(\delta n^{-1/2}), \qquad \text{as } n \to \infty, \tag{2.9}$$

and let $r > 0$, $a_\nu \geqq 0$; *then the following statements are equivalent*:

$$n^{-r} \sum_1^n \nu^r a_\nu + \sum_{n+1}^\infty a_\nu = O\omega(\delta n^{-1/2}), \qquad n \to \infty, \tag{2.10}$$

$$n^{-r} \sum_1^n \nu^r a_\nu = O\omega(\delta n^{-1/2}), \tag{2.11}$$

$$\sum_1^\infty a_\nu \left| 1 - \alpha_\mu(t\nu^{1/2}) \right|^r = O\omega(t), \qquad t \to 0. \tag{2.12}$$

The equivalence of (2.10) and (2.11) follows from Lemma (2.5) in [5]; the equivalence of (2.11) and (2.12) follows from Lemma 5. This proves Lemma 6.

LEMMA 7. *Young-Hausdorff inequality. If* $1 < p \leqq 2$, *and*

$$f(X) \sim \sum c_N \exp(iNX),$$

then

$$\left\{ \sum \left| c_N \right|^{p'} \right\}^{1/p'} \leqq M_p(f) \equiv M_p f, \tag{2.13}$$

and

$$M_{p'}^{p}f \leqq \sum |c_N|^p,$$

where $1/p+1/p'=1$, *and*

$$M_p^p f = \frac{1}{(2\pi)^\kappa}\int_{-\pi}^{\pi} |f(X)|^p dX$$

(cf. [4]).

Denote by $f(X;t)$ the spherical mean of $f(X)$ over the surface of the sphere of radius t and center x; then [1, p. 177]

$$\begin{aligned} f(X;t) &= (2\pi)^{-\kappa/2}\Gamma\left(\frac{\kappa}{2}\right)\int_\sigma f(x_1+t\xi_1,\cdots,x_\kappa+t\xi_\kappa)d\sigma_\xi \\ &\backsim \sum c_N\alpha_\mu(t|N|)\exp(iNX) \qquad (2.14)\\ &\backsim \sum_{n=0}^{\infty}\alpha_\mu(tn^{1/2})A_n(x), \qquad \mu=(\kappa-2)/2; \end{aligned}$$

σ denotes the unit sphere $\xi_1^2+\cdots+\xi_\kappa^2=1$, $d\sigma_\xi$ its $(\kappa-1)$-dimensional volume element. Thus, putting $f(X;t)-f(X)=\phi(X;t)$, we have

$$\begin{aligned} \phi(X;t) &\backsim \sum c_N\{\alpha_\mu(t|N|)-1\}\exp(iNX) \\ &\backsim \sum_{n=0}^{\infty}\{\alpha_\mu(tn^{1/2})-1\}A_n(x). \end{aligned}$$

LEMMA 8. *If* $M_1\phi(X;t)=O\omega(t)$ *as* $t\to 0$, *then for any* $\delta>0$

$$c_N = O\omega\left(\frac{\delta}{|N|}\right) \qquad \textit{as } |N|\to\infty.$$

It follows from (1.1), (1.2) and (2.14) that

$$c_N\{\alpha_\mu(t|N|)-1\} = (2\pi)^{-\kappa}\Gamma\left(\frac{\kappa}{2}\right)\int_{-\pi}^{\pi}\phi(X;t)\exp(-iNX)dX,$$

hence

$$|c_N||1-\alpha_\mu(t|N|)| \leqq M_1\phi(X;t) = O\omega(t).$$

Lemma 8 now follows from Lemma 3, on putting $t|N|=\delta$.

LEMMA 9. *Let* $P_n(z)=\sum_0^n c_\nu z^\nu$, $1\leqq p\leqq\infty$; *if*

$$M_pP_n(z)\leqq 1 \qquad \textit{for } |z|\leqq 1,$$

then

$$M_pP_n'(z)\leqq n$$

(cf. [5, p. 385]).

Note. For $p=\infty$, $M_pP(z)=\max|P(z)|$ for $|z|\leqq 1$.

We shall frequently use Hölder's and Minkowski's well known inequalities for multiple series and integrals (cf. Hardy, Littlewood, and Pólya, *Inequalities*, Cambridge, 1934).

3. A theorem on absolute convergence. We now present our main criterion for absolute convergence.

THEOREM 1. *If, with the notations of* §2, $1 \leqq p \leqq 2, f(X) \in L_p$,

$$M\phi(X;t) = O\omega(t) \qquad \textit{as } t \to 0, \tag{3.1}$$

and

$$\sum_1^\infty n^{-\beta/p'}\omega^\beta(\delta n^{-1/\kappa}) < \infty \qquad \textit{for some } \beta > 0, \tag{3.2}$$

then

$$\sum |c_N|^\beta < \infty. \tag{3.3}$$

By (3.1) and Lemma 7 for $1 < p \leqq 2$, $\sum |c_N|^{p'} |1-\alpha_\mu(t|N|)|^{p'} = O\omega^{p'}(t)$, or

$$\sum_1^\infty \rho_n^{p'} \left| 1 - \alpha_\mu(tn^{1/2}) \right|^{p'} = O\omega^{p'}(t), \tag{3.4}$$

where $\rho_n = \rho_n(p)$ is defined by

$$\rho_n^{p'} = \sum_{|N|^2=n} |c_N|^{p'} = \sum |c_{n_1 \cdots n_\kappa}|^{p'} \qquad (n_1^2 + \cdots + n_\kappa^2 = n).$$

By Lemma 5, (3.4) is equivalent to

$$n^{-p'}\sum_1^n \nu^{p'}\rho_\nu^{p'} + \sum_{n+1}^\infty \rho_\nu^{p'} = O\omega^{p'}(\delta n^{-1/2}),$$

hence

$$\sum_{n+1}^{2n} \rho_\nu^{p'} = O\omega^{p'}(\delta n^{-1/2}). \tag{3.5}$$

By the Hölder inequality for $q > 1$, $1/q + 1/q' = 1$,

$$\sum_n^{2n} \rho_\nu^\beta = \sum_{n \leqq |N|^2 \leqq 2n} |c_N|^\beta \leqq \left(\sum |c_N|^{\beta q}\right)^{1/q}\left(\sum 1\right)^{1/q'};$$

let first $\beta < p'$; choose

$$\beta q = p', \quad \text{hence} \quad q' = \frac{q}{q-1} = \frac{p'}{p'-\beta}.$$

Now, from (3.5) and (2.1)

$$\sum_{n}^{2n} \rho_\nu^\beta = O\omega^\beta(\delta n^{-1/2})(R_\kappa(2n))^{1-\beta/p'} = On^{(1-\beta/p')\kappa/2}\omega^\beta(\delta n^{-1/2}).$$

Putting $n=2^\lambda$, $\lambda=0, 1, \cdots$, and summing over λ yields

$$\sum_{1}^{\infty} \rho_\nu^\beta = O\sum_{\lambda=0}^{\infty} 2^{\lambda\kappa(1-\beta/p')/2}\omega^\beta(\delta 2^{-\lambda/2});$$

the right side is convergent by the corollary to Lemma 4 (with $h=\kappa(1-\beta/p')$) and by (3.2). Hence (3.3) holds.

Next if $\beta=p'>1$, then (3.3) follows from (2.13), if we assume only, instead of (3.1), that $f(X)\in L_p$; a fortiori

$$\sum | c_N |^\beta < \infty \qquad \text{for } \beta \geqq p'.$$

Finally let $p=1$; (3.2) becomes

$$\sum \omega^\beta(\delta n^{-1/\kappa}) < \infty. \tag{3.6}$$

Denote by $r_\kappa(n)$ the number of lattice points on the circle $\sum_1^\kappa x_\nu^2=n$; thus $\sum_0^n r_\kappa(\nu)=R_\kappa(n)$.

From Lemma 8, for any $\delta>0$

$$\sum_{|N|^2=n} | c_N |^\beta = O\sum \omega^\beta(\delta | N |^{-1}) = Or_\kappa(n)\omega^\beta(\delta n^{-1/2});$$

furthermore from (3.6) and Lemma 4 (with $h=\kappa$)

$$\sum_{1}^{\infty} n^{\kappa/2-1}\omega^\beta(\delta n^{-1/2}) < \infty.$$

Now, using (2.1), we have

$$\begin{aligned}\sum_{1}^{n} r_\kappa(\nu)\omega^\beta(\delta\nu^{-1/2}) &= \sum_{1}^{n} R_\kappa(\nu)\omega^\beta(\delta\nu^{-1/2}) - \sum_{0}^{n-1} R_\kappa(\nu)\omega^\beta(\delta(\nu+1)^{-1/2})\\ &\leqq R_\kappa(n)\omega^\beta(\delta n^{-1/2}) + \sum_{1}^{n-1} R_\kappa(\nu)\left\{\omega^\beta(\delta\nu^{-1/2}) - \omega^\beta(\delta(\nu+1)^{-1/2})\right\}\\ &= On^{\kappa/2}\omega^\beta(\delta n^{-1/2}) + O\sum_{1}^{n-1} \nu^{\kappa/2}\left\{\omega^\beta(\delta\nu^{-1/2}) - \omega^\beta(\delta(\nu+1)^{-1/2})\right\}\\ &= O\sum_{1}^{n}\left\{\nu^{\kappa/2} - (\nu-1)^{\kappa/2}\right\}\omega^\beta(\delta\nu^{-1/2})\\ &= O(1), \qquad \text{as } n\to\infty.\end{aligned}$$

This completes the proof of Theorem 1.

Actually we can prove for $\beta = p'$ that

$$\sum \rho_n^{p'} \log n < \infty.$$

4. **Converse theorems.** We give here two theorems to be employed in subsequent sections.

THEOREM 2. *Let* $1 \leqq p \leqq 2$; *assume that*

$$\sum_{\lambda=1}^{\infty} \omega^p(\delta 2^{-\lambda} n^{-1/2}) = O\omega^p(\delta n^{-1/2}), \qquad \text{as } n \to \infty, \tag{4.1}$$

and that

$$\sum_{1}^{n} \nu^p \rho_\nu^p = O n^p \omega^p(\delta n^{-1/2}), \qquad \text{as } n \to \infty; \tag{4.2}$$

then

$$M_{p'}\phi(X; t) = O\omega(t), \qquad \text{as } t \to 0.$$

Note. If $p=1$, $p'=\infty$, then $M_{p'}$ means the effective upper bound of $|\phi(X; t)|$ in the region of X.

Proof. By Lemma 6, (4.2) is equivalent to

$$\sum_{1}^{\infty} \rho_n^p \left| 1 - \alpha_\mu(tn^{1/2}) \right|^p = O\omega^p(t),$$

that is,

$$\sum | c_N |^p \left| 1 - \alpha_\mu(t | N |) \right|^p = O\omega^p(t).$$

Now from (2.14) and Lemma 7 (which holds also for $p=1$)

$$M_{p'}\phi(X; t) = O\omega(t) \qquad \text{as } t \to 0;$$

this proves the theorem.

Note that (4.2) means:

$$\sum_{|N|^2 \leq n} | N |^p | c_N |^p = O n^p \omega^p(\delta n^{-1/2}).$$

THEOREM 3. *Assume that* $\omega(t) \downarrow 0$ *as* $t \downarrow 0$, *and that*

$$\sum_{\lambda=1}^{\infty} \omega^2(2^{-\lambda}\delta n^{-1/2}) = O\omega^2(\delta n^{-1/2}) \qquad \text{as } n \to \infty.$$

Then a necessary and sufficient condition that

$$M_2\phi(X; t) = O\omega(t) \qquad \text{as } t \to 0, \tag{4.3}$$

is that

$$\sum_{1}^{n} \nu^2 \rho_\nu^2 = O n^2 \omega^2(\delta n^{-1/2}) \qquad \text{as } n \to \infty. \tag{4.4}$$

First if (4.4) holds then (4.3) follows by Theorem 2 (for $p=2$). Conversely if (4.3) holds, then from (2.14) and Lemma 7

$$\sum |c_N|^2 |1 - \alpha_\mu(t|N|)|^2 = O\omega^2(t),$$

which by Lemma 6 is equivalent to (4.4).

5. **Counter examples.** For $\beta=1$, Theorem 1 becomes:

THEOREM 1′. *If $M_p\phi(t) = O\omega(t)$ as $t \to 0$, and*

$$\sum n^{-1/p'} \omega(\delta n^{-1/\kappa}) < \infty, \qquad 1 \leqq p \leqq 2, \tag{5.1}$$

then

$$\sum |c_N| < \infty.$$

To show that this result is the best possible we shall prove:

THEOREM 4. *Let $\omega(t)$, in addition to having the property $\omega(t) \downarrow 0$ as $t \downarrow 0$, be such that*

$$\int_1^u \omega(t^{-1})dt = O u\omega(u^{-1}) \qquad \text{as } u \to \infty, \tag{5.2}$$

while

$$\sum n^{-1/p'} \omega(\delta n^{-1/\kappa}) = \infty, \qquad \text{where } 1 \leqq p \leqq 2. \tag{5.3}$$

Then there exists a function $f(X) \in L_p$, such that

$$M_p\phi(X; t) = O\omega(t), \tag{5.4}$$

while

$$\sum |c_N| = \infty.$$

By Lemma 4 and its corollary (with $h=\kappa/p$) (5.1) is equivalent to

$$\sum_{\lambda=1}^{\infty} 2^{\kappa\lambda/p} \omega(\delta 2^{-\lambda}) < \infty,$$

while (5.3) is equivalent to

$$\sum 2^{\kappa\lambda/p} \omega(\delta 2^{-\lambda}) = \infty. \tag{5.3′}$$

We define $\epsilon_n = \omega(2^{-n}\delta)$, $\lambda_n = 2^{n+1} + n - 2$,

$$g_n(z) = 2^{-n(1+1/p')} \left(\sum_{0}^{2^n} z^\nu \right)^2, \qquad n = 0, 1, 2, \cdots; \tag{5.5}$$

so that

$$\lambda_{n+1} - \lambda_n = 2^{n+1} + 1.$$

Construct the power series

$$G(Z) = \sum_{n=0}^{\infty} (\epsilon_n - \epsilon_{n+1})(z_1 \cdots z_\kappa)^{\lambda_n} \prod_{\nu=1}^{\kappa} g_n(z_\nu);$$

then $G(Z)$ has the formal power series

$$G(Z) = \sum_{n_1=0}^{\infty} \cdots \sum_{n_\kappa=0}^{\infty} \gamma_{n_1 \cdots n_\kappa} z_1^{n_1} \cdots z_\kappa^{n_\kappa}.$$

It is clear from the construction that $\gamma_N \geqq 0$; putting $Z=1$ we find

$$\sum \gamma_N > \sum_{n=0}^{m} (\epsilon_n - \epsilon_{n+1}) 2^{-\kappa n(1+1/p')} (2^{n+1} - 1)^{2\kappa} > \sum_{0}^{m} (\epsilon_n - \epsilon_{n+1}) 2^{\kappa n/p}$$

$$> \sum_{1}^{m} (\epsilon_n - \epsilon_{m+1}) 2^{\kappa n/p} (1 - 2^{-\kappa/p}).$$

For a given integer l choose m so large that $\epsilon_l > 2\epsilon_{m+1}$, then

$$\sum \gamma_N > \frac{1}{2}(1 - 2^{-\kappa/p}) \sum_{1}^{l} \epsilon_n 2^{\kappa n/p} \to \infty \qquad \text{as } l \to \infty,$$

by (5.3′). Hence

$$\sum \gamma_N = \infty.$$

We next show that for $z_\nu = e^{ix_\nu}$, $\nu = 1, 2, \cdots, \kappa$, $G(Z)$ becomes the Fourier power series of a function $F(X) \in L_p$. Write

$$u_n(Z) = (\epsilon_n - \epsilon_{n+1})(z_1 \cdots z_\kappa)^{\lambda_n} \prod_{\nu=1}^{\kappa} g_n(z_\nu), \tag{5.6}$$

then for $z_\nu = e^{ix_\nu}$

$$\begin{aligned} M_p u_n &= (\epsilon_n - \epsilon_{n+1}) \frac{1}{(2\pi)^\kappa} 2^{-\kappa n(1+1/p')} \left(\int_{-\pi}^{\pi} \left| \sum_{0}^{2^n} e^{i\nu x} \right|^{2p} dx \right)^{\kappa/p} \\ &= O(\epsilon_n - \epsilon_{n+1}) 2^{-\kappa n(1+1/p')} \left(\int_0^{\pi} \left| \frac{\sin (2^n + 1)x/2}{x} \right|^{2p} dx \right)^{\kappa/p} \\ &= O(\epsilon_n - \epsilon_{n+1}) 2^{-\kappa n(1+1/p')} 2^{n(2p-1)\kappa/p} = O(\epsilon_n - \epsilon_{n+1}); \end{aligned} \tag{5.7}$$

hence, by Minkowski's inequality,

$$M_p G \leqq \sum_{n=0}^{\infty} M_p u_n = O(1).$$

We shall finally prove (5.4); we have

$$F(X;t) - F(X) = \sum_{\nu=0}^{\infty} \{u_\nu(X;t) - u_\nu(X)\},$$

hence $M_p\phi \leqq \sum_{\nu=0}^{\infty} M_p\{u_\nu(X;t) - u_\nu(X)\} = \sum_{\nu=0}^{n} + \sum_{n+1}^{\infty} = S_1 + S_2$, say. Now, by Minkowski's inequality and (2.14),

$$M_p u_\nu(X;t) = \Gamma\left(\frac{\kappa}{2}\right) 2^{-\kappa-1}\pi^{3\kappa/2}\left(\int_{-\pi}^{\pi}\left|\int_\sigma u_\nu(x_1 + t\xi_1, \cdots, x_\kappa + t\xi_\kappa)d\sigma_\xi\right|^p dX\right)^{1/p}$$

$$\leqq \Gamma\left(\frac{\kappa}{2}\right) 2^{-\kappa-1}\pi^{3\kappa/2}\int_\sigma\left(\int_{-\pi}^{\pi} |u_\nu(x_1 + t\xi_1, \cdots, x_\kappa + t\xi_\kappa)|^p dX\right)^{1/p} d\sigma_\xi$$

$$= 2^{-1}\Gamma\left(\frac{\kappa}{2}\right)\pi^{-\kappa/2}\int_\sigma M_p(u_\nu)d\sigma_\xi;$$

hence, if we use (5.7), $S_2 = O\epsilon_n$. Furthermore

$$M_p\{u_\nu(X;t) - u_\nu(X)\} \leqq 2^{-1}\Gamma\left(\frac{\kappa}{2}\right)\pi^{-\kappa/2}\int_\sigma M_p\{u_\nu(X;t) - u_\nu(X)\}d\sigma_\xi;$$

from the mean value theorem

$$u_\nu(X;t) - u_\nu(X) = t\sum_{\lambda=1}^{\kappa} \xi_\lambda \frac{\partial u_\nu}{\partial x_\lambda}(X;\theta t), \quad \text{where } 0 < \theta < 1;$$

hence from Minkowski's inequality

$$M_p\{u_\nu(X;t) - u_\nu(X)\} \leqq t\sum_{\lambda=1}^{\kappa} |\xi_\lambda| M_p \frac{\partial u_\nu}{\partial x_\lambda}(X;\theta t)$$

$$\leqq (2\pi)^{-\kappa/2} t\int_\sigma M_p \frac{\partial u_\nu}{\partial x_\lambda}(X)d\sigma_\xi.$$

We now employ Lemma 9; thus from (5.5) and (5.6)

$$M_p\{u_\nu(X;t) - u_\nu(X)\} = tO(\epsilon_\nu - \epsilon_{\nu+1})(2^{\nu+1} + \lambda_\nu) = tO2^\nu(\epsilon_\nu - \epsilon_{\nu+1}).$$

It follows that

$$S_1 = tO\sum_0^n 2^\nu(\epsilon_\nu - \epsilon_{\nu+1}) = Ot\sum_0^n (2^{\nu+1} - 2^\nu)\epsilon_\nu$$

$$= Ot\sum_0^n (2^{\nu+1} - 2^\nu)\omega(\delta 2^{-\nu}) = Ot\int_0^{2^n} \omega(\delta x^{-1})dx$$

$$= Ot2^n\omega(\delta 2^{-n}),$$

by (5.2). We now choose n so that for a given positive $t<\delta$

$$2^{n-1} < \delta t^{-1} \leqq 2^n, \qquad n \geqq 1;$$

then

$$S_1 = O\omega(t), \quad \text{and} \quad S_2 = O\omega(t),$$

and the proof of Theorem 4 is complete.

A simpler example, but of a special type, is

$$G(Z) = \sum_{n=0}^{\infty} (\epsilon_n - \epsilon_{n+1}) \sum_{\nu=1}^{\kappa} z_\nu^{\lambda_n} g_n(z_\nu).$$

6. **The case $p=2$ and arbitrary $\beta>0$.** For the case $p=2$, Theorem 1 becomes:

THEOREM 1″. *If $M_2\phi(t)=O\omega(t)$, and for some $\beta>0$*

$$\sum n^{-\beta/2}\omega^\beta(\delta n^{-1/\kappa}) < \infty,$$

then

$$\sum |c_N|^\beta < \infty.$$

We now prove:

THEOREM 5. *Let $\omega(t)$, in addition to having the property $\omega(t)\downarrow 0$ as $t\downarrow 0$, be such that*

(6.1)
$$\int_1^u x\omega^2(x^{-1})dx = Ou^2\omega^2(u^{-1}) \qquad \text{as } u\to\infty,$$

while for a given positive $\beta<2$

(6.2)
$$\sum n^{-\beta/2}\omega^\beta(\delta n^{-1/\kappa}) = \infty.$$

Then there exists a function $f(X)\in L_2$, such that

(6.3)
$$M_2\phi(X;t) = O\omega(t), \qquad t\to 0,$$

but

$$\sum |c_N|^\beta = \infty.$$

We employ again the polynomial (5.5), where now $p'=2$, and the polynomial (5.6), replacing the factor $\epsilon_n-\epsilon_{n+1}$ by

$$(\epsilon_n^\beta - \epsilon_{n+1}^\beta)^{1/\beta} = \alpha_n,$$

say.

As before $\epsilon_n=\omega(\delta 2^{-n})$. On writing

(6.4)
$$G(Z) = \sum_{n=0}^{\infty} u_n(Z) = \sum \gamma_N z_1^{n_1}\cdots z_\kappa^{n_\kappa},$$

we have again $\gamma_N \geqq 0$. Now

$$\sum \gamma_N^\beta > \sum \alpha_n^\beta 2^{-3n\beta\kappa/2}\left(\sum_1^{2^n} \nu^\beta\right)$$

$$> \frac{1}{(\beta+1)^\kappa}\sum_{n=0}^m \alpha_n^\beta 2^{n\kappa(1-\beta/2)} = \frac{1}{(\beta+1)^\kappa}\sum_0^m (\epsilon_n^\beta - \epsilon_{n+1}^\beta)2^{n\kappa(1-\beta/2)}$$

$$> (2^{\kappa(1-\beta/2)} - 1)\sum_1^m (\epsilon_n^\beta - \epsilon_{m+1}^\beta)2^{(n-1)(1-\beta/2)\kappa}.$$

Hence for $\epsilon_l^\beta > 2\epsilon_{m+1}^\beta$

$$\sum \gamma_N^\beta > \frac{1}{2}(2^{\kappa(1-\beta/2)} - 1)\sum_1^l \epsilon_n^\beta 2^{\kappa(n-1)(1-\beta/2)}.$$

By the corollary to Lemma 4, (6.2) is equivalent to

$$(6.5)\qquad \sum_{\lambda=1}^\infty 2^{(1-\beta/2)\lambda\kappa}\omega^\beta(\delta 2^{-\lambda}) = \infty, \quad \text{or} \quad \sum_\lambda 2^{(1-\beta/2)\lambda\kappa}\epsilon_\lambda^\beta = \infty,$$

hence $\sum \gamma_N^\beta = \infty$.

Next, in the same manner as in §5, one can prove that

$$(6.6)\qquad M_2^2(u_n) = O\alpha_n^2;$$

it is easily seen that [5, formula (6.14)]

$$(6.7)\qquad \alpha_n^2 = (\epsilon_n^\beta - \epsilon_{n+1}^\beta)^{2/\beta} = O(\epsilon_n^2 - \epsilon_{n+1}^2),$$

hence

$$M_2^2(G) = \sum_{n=0}^\infty M_2^2(u_n) = O\sum_n (\epsilon_n^2 - \epsilon_{n+1}^2) = O(1).$$

Finally, to prove (6.3), write

$$M_2^2\phi = \sum_0^\infty M_2^2\{u_\nu(X;t) - u_\nu(X)\} = \sum_0^n + \sum_{n+1}^\infty = T_1 + T_2,$$

say. From (6.6) and (6.7), $T_2 = O\epsilon_n^2$, while, if we employ Lemma 9 (as in §5)

$$T_1 = t^2 O\sum_0^n 2^{2\nu}\alpha_\nu^2 = t^2 O\sum_0^n 2^{2\nu}(\epsilon_\nu^2 - \epsilon_{\nu+1}^2)$$

$$= t^2 O\sum_1^n \epsilon_\nu^2(2^{2\nu} - 2^{2\nu-2}) = t^2 O\sum_1^n (2^\nu - 2^{\nu-1})2^\nu\epsilon_\nu^2$$

$$= t^2 O\int_1^{2^n} x\omega^2(\delta x^{-1})dx = tO\int_1^{\delta^{-1}2^n} y\omega^2(y^{-1})dy.$$

Employing (6.1), we now get

$$T_1 = t^2 O 2^{2n}\omega^2(\delta 2^{-n}).$$

Given a positive t, choose n so that

$$2^n < \delta/t \leqq 2^{n+1};$$

then

$$T_1 = t^2 O t^{-2}\omega^2(t) = O\omega^2(t), \quad \text{and} \quad T_2 = O\omega^2(t),$$

hence

$$M_2\phi(t) = O\omega(t) \qquad \text{as } t \to 0.$$

This proves Theorem 5.

Remark. The conditions (5.2) and (6.1) are equivalent (cf. [5, Remark 6.1]).

7. **A continuous function as counter example.** In [5, §6] we have employed polynomials

$$g(z) = \sum_{\nu=0}^{2(q-1)} a_\nu^{(q)} z^\nu, \qquad q \text{ a prime} \equiv 1 \pmod 4, \tag{7.1}$$

with the following properties

$$|g(z)| \leqq 1 \qquad \text{for } |z| \leqq 1,$$
$$|a_\nu^{(q)}| = q^{-3/2}(\nu + 1), \qquad \nu = 0, 1, \cdots, q - 2.$$

On putting $g(z_1) \cdots g(z_\kappa) = \sum b_N z_1^{n_1} \cdots z_\kappa^{n_\kappa}$, it follows that

$$\begin{aligned} \sum |b_N|^\beta &> (\sum |a_\nu^{(q)}|^\beta)^\kappa > q^{-3\beta\kappa/2}(1^\beta + 2^\beta + \cdots + (q-1)^\beta)^\kappa \\ &> \frac{1}{\kappa+1} q^{-3\beta\kappa/2}(q-1)^{\kappa(\beta+1)}. \end{aligned} \tag{7.2}$$

Let $1 < q_1 < q_2 < \cdots$ be a sequence of primes congruent to 1 (mod 4), and such that for all large n

$$2^{n-1} < q_n < 2^n; \tag{7.3}$$

denote by $g_n(z)$ the polynomial (7.1) with $q = q_n$, and let

$$\lambda_1 = 0, \quad \lambda_{n+1} = 2(q_1 + \cdots + q_n) - n, \qquad n \geqq 1; \tag{7.4}$$

ϵ_n, α_n, and u_n are defined as in §6. We assume that $\omega(t)$ satisfies the conditions of Theorem 5 and, in case $1 < \beta < 2$, the additional conditions

$$\int_1^\infty x^{-1}\omega(x^{-1})dx = \int_0^1 \tau^{-1}\omega(\tau)d\tau < \infty, \tag{7.5}$$

$$\int_{t^{-1}}^{\infty} x^{-1}\omega(x^{-1})dx = \int_0^t \tau^{-1}\omega(\tau)d\tau = O\omega(t) \qquad \text{as } t \to 0. \tag{7.6}$$

Now, as shown in [5, §6],

$$\sum_1^n 2^\nu \alpha_\nu < 2\int_1^{2^n} \omega(x^{-1})dx, \tag{7.7}$$

and

$$\sum_{n+1}^{\infty} \alpha_\nu < \begin{cases} \epsilon_{n+1} & \text{for } 0 < \beta \leqq 1, \\ 2\displaystyle\int_{2^n}^{\infty} x^{-1}\omega(x^{-1})dx & \text{for } 1 < \beta < 2. \end{cases} \tag{7.8}$$

We define as before

$$G(Z) = \sum_1^{\infty} u_n(Z) = \sum \gamma_N z_1^{n_1} \cdots z_\kappa^{n_\kappa}. \tag{7.9}$$

By (7.1)

$$| u_n(Z) | \leqq \alpha_n \qquad \text{for } | z_1 | \leqq 1, \cdots, | z_\kappa | \leqq 1, \tag{7.10}$$

hence the simple series in (7.9) converges uniformly and defines a continuous function in $|z_1| \leqq 1, \cdots, |z_\kappa| \leqq 1$. Putting $z_\nu = \exp(ix_\nu)$, $\nu = 1, \cdots, \kappa$, (7.9) becomes the Fourier power series of a continuous function $F(x_1, \cdots, x_\kappa)$. Furthermore, using (7.2) and (7.3), we have

$$\begin{aligned} \sum | \gamma_N |^\beta &> \frac{1}{\kappa + 1} \sum_n \alpha_n^\beta q_n^{-3\kappa\beta/2}(q_n - 1)^{\kappa(\beta+1)} \\ &> b \sum_n (\epsilon_n^\beta - \epsilon_{n+1}^\beta) 2^{n\kappa(1-\beta/2)}, \qquad b \text{ a constant,} \end{aligned}$$

and the divergence of this series follows from (6.2) as in §6.

We shall finally show that the modulus of continuity of $F(X)$ is majorized by $\omega(t)$. We define the modulus of continuity of $F(X)$ by

$$\max_{|H| \leqq t} \max_{(X)} | F(X + H) - F(X) | = \zeta(t),$$

where $|H| = (h_1^2 + \cdots + h_\kappa^2)^{1/2}$, and each x_ν varies in $(-\pi, \pi)$. Now, in view of (7.9),

$$\begin{aligned} | F(X + H) - F(X) | &\leqq \sum_1^{\infty} | u_\nu(e^{i(x_1+h_1)}, \cdots, e^{i(x_\kappa+h_\kappa)}) - u_\nu(e^{ix_1}, \cdots) | \\ &= \sum_1^n + \sum_{n+1}^{\infty} = V_1 + V_2, \end{aligned}$$

say. From (7.10) and (7.8)

$$V_2 < 2\sum_{n+1}^{\infty} \alpha_\nu < \begin{cases} 2\epsilon_n & \text{for } 0 < \beta \leq 1, \\ 2\int_{2^n}^{\infty} x^{-1}\omega(x^{-1})dx & \text{for } 1 < \beta < 2; \end{cases}$$

in view of (7.6) we have in either case

$$V_2 = O\omega(\delta 2^{-n}).$$

To estimate V_1, we employ as in §5 the mean value theorem, and Lemma 9 for $p = \infty$. We then get

$$V_1 < \left(\sum_1^{\kappa} |h_\nu|\right)\left(\sum_1^{n} \alpha_\nu \lambda_{\nu+1}\right)$$

and, using (7.7) and (5.2) (which is equivalent to (6.1)),

$$V_1 = O\,|\,H\,|\,2^n\omega(\delta 2^{-n}).$$

For $|H| \leq t$ choose n so that $2^{n-1} < t^{-1} \leq 2^n$, then

$$V_1 = O\omega(t) \quad \text{and} \quad V_2 = O\omega(t)$$

hence

$$\zeta(t) = O\omega(t).$$

We have thus proved the theorem:

THEOREM 6. *If the assumptions of Theorem* 5 *are satisfied and if* $0 < \beta \leq 1$, *then there exists a continuous function* $F(X)$ *with modulus of continuity* $\zeta(t) < \omega(t)$, *while* $\sum |c_N|^\beta = \infty$. *The same result holds for* $1 < \beta < 2$ *under the additional assumptions* (7.5) *and* (7.6).

As an example choose $\omega(t) = t^\alpha$, $0 < \alpha < 1$; it is seen easily that now (6.1), (7.5), and (7.6) hold. Theorem 1'' yields the convergence of $\sum |c_N|^\beta$ whenever $M_2\phi = Ot^\alpha$, and if $\beta > 2\kappa/(\kappa + 2\alpha)$. For $\beta = 2\kappa/(\kappa + 2\alpha)$, however, there exists a continuous function whose modulus of continuity is less than t^α, while $\sum |c_N|^\beta = \infty$.

Closing remark. In a similar manner the convergence of the series $\sum |N|^\alpha |c_N|^\beta$ can be discussed. The mode of procedure applies as well to Fourier integrals. We may also consider instead of the spherical mean (2.14) the more general average

$$f_p(X; t) = \frac{c_p}{t^\kappa}\int_0^t \left(1 - \frac{r^2}{t^2}\right)^{p-1} f(X; r) r^{\kappa-1} dr.$$

Finally, if we denote the linear operator which transforms $f(X)$ into $f(X; t) - f(X)$ by $\Delta f(X; t)$, iteration yields

$$\Delta_m f(X; t) \sim \sum c_N(\alpha_\mu(t|N|) - 1)^m \exp(iNX), \qquad m = 1, 2, 3, \cdots,$$

and in Theorem 1 the assumption $M_p\phi(X; t) \equiv M_p\Delta_1 f(X; t) = O\omega(t)$ can be replaced by $M_p\Delta_m f(X; t) = O\omega(t)$.

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