MEROMORPHIC FUNCTIONS WITH SIMULTANEOUS MULTIPLICATION AND ADDITION THEOREMS

BY

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Introduction. Let $f(z)$ be a nonconstant meromorphic function. Further, let there exist two numbers $m \neq 0, 1$ and $h \neq 0$, and two rational functions R and S, such that

$$f(mz) = R[f(z)], \tag{1}$$

and

$$f(z + h) = S[f(z)]. \tag{2}$$

Then the function $f(z)$ has both a rational multiplication and a rational addition theorem. We propose to determine all such functions.

Our problem is a generalization of one treated in a paper by Ritt[1], in which all periodic meromorphic functions having a multiplication theorem were determined. This is equivalent to taking $S(z) \equiv z$ in (2). For this case, Ritt proved that $|m| \geqq 1$; and if $|m| > 1$, then $f(z)$ is a linear function of one of the functions $e^{\alpha z}$, $\cos(\alpha z+\beta)$, $\wp(z+\beta)$; when $g_3 = 0$, $\wp^2(z+\beta)$; when $g_2 = 0$, $\wp'(z+\beta)$ and $\wp^3(z+\beta)$. Here α is arbitrary while β is restricted to certain values. If $|m| = 1$, then it was shown that m is either -1 or a third, fourth or sixth root of unity. The forms for the function $f(z)$ were also explicitly given in this case. Use will be made of the results quoted and also of the methods of Ritt's paper.

We distinguish the following three cases:

(A) $|m| > 1$;

(B) $|m| \leqq 1$, with m not a rational root of unity;

(C) m a primitive nth root of unity.

For Case (A), which is the one usually considered in multiplication theorems, we can restate the problem in a way which shows its connection with the theory of a class of functions first systematically studied by Poincaré[2]. He provided an existence theorem for meromorphic functions satisfying (1) assuming that $R(z)$ has a fixed point[3] a for which $R'(a)$ has a modulus greater than unity. Thus our problem is equivalent to that of finding all Poincaré functions having a rational addition theorem.

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(1) Ritt, *Periodic functions with a multiplication theorem*, Trans. Amer. Math. Soc. vol. 23 (1922) pp. 16–25.

(2) H. Poincaré, *Sur une classe nouvelle de transcendantes uniformes*, Journal de Mathématiques (3) vol. 55 (1890).

(3) That is, a point for which $R(a) = a$.

We remark that, if $f(z)$ is a linear function of z, integral or fractional, then $f(z)$ will evidently have both a multiplication and addition theorem for every value of m and h. This solution is therefore common to all three cases and, as it turns out, is the only rational solution of our problem.

On the other hand, if $f(z)$ is transcendental, we may state the chief result of this paper for Case (A) as follows:

If a Poincaré function has a rational addition theorem, it must be a periodic function.

Although $f(z)$ is periodic in this case, the value of h in the addition theorem is not necessarily a period, and therefore the corresponding $S(z)$ need not equal z. The possibilities for h and $S(z)$ are given in detail in §4. It is found that except for the case in which $f(z)$ is a linear function of e^{az}, h, when not a period, must be a suitable half or third of a period. It is noteworthy that for each of these admissible values of h, $S(z)$ is a linear function of z.

In Case (B), $f(z)$ must be a linear function of z.

In Case (C) there appear nonperiodic solutions in addition to the types already mentioned.

It will be noted that, essentially, only the conditions that $f(mz)$ and $f(z+h)$ are uniform functions of $f(z)$ will enter into consideration.

1. **Preliminary transformations.** The case of $f(z)$ rational will be handled in §5. From this point on through §4, we assume that $f(z)$ is transcendental.

Supposing that $|m|>1$, we wish to replace $f(z)$ by another meromorphic function $g(z)$, related to it in a simple way, but having the following special properties at the origin:

(α) $g(0)=0$;

(β) $g'(0)\neq 0$;

(γ) 0 is not an exceptional point of $g(z)$; that is, $g(z)$ has an infinite number of zeros.

We further require that there exist relations of the form

$$g(Mz) = U[g(z)], \tag{3}$$

and

$$g(z+h) = V[g(z)]. \tag{4}$$

Here U and V are rational and M is a number for which $|M|>1$.

To find such a function $g(z)$, we consider $g(z)\equiv f(z+\xi)-f(\xi)$, where $f(z)$ is analytic at ξ. We shall show that ξ can be chosen so that the resulting function $g(z)$ fulfills all the conditions placed upon it.

Evidently for any value of ξ at which $f(z)$ is analytic, property (α) holds, and also a relation of type (4) exists with $V(z)\equiv S[z+f(\xi)]-f(\xi)$.

Write $\sigma(z)\equiv mz$, $\tau(z)\equiv z+h$. Let a positive integral subscript appended to a function denote the corresponding iterate of that function; for example $R_r(z)$ denotes the rth iterate of $R(z)$. Then if ξ is a finite fixed point of one of

the transformations σ or $\tau\sigma_r$ ($r=1, 2, \cdots$) and if $f(z)$ is analytic at ξ, a relation of type (3) will exist.

For if $\xi=0$, the finite fixed point of σ, we may take $M=m$ and $U(z)=R[z+f(0)]-f(0)$.

And if for a positive integral r, $\tau\sigma_r(\xi)=\xi$, we may take $M=m^r$ and $U(z)=SR_r[z+f(\xi)]-f(\xi)$. This follows since

$$g(Mz) = g[\sigma_r(z)] = f[\sigma_r(z) + \xi] - f(\xi) = f[\tau\sigma_r(z + \xi)] - f(\xi);$$

and using (1) and (2) we have

$$f[\tau\sigma_r(z + \xi)] = S\{f[\sigma_r(z + \xi)]\} = SR_r[f(z + \xi)].$$

In each of these cases $|M|>1$.

It remains to show the existence of a fixed point ξ at which $f(z)$ is analytic and for which the corresponding function $g(z)$ has properties (β) and (γ). To this end consider the sequence

$$0, \frac{h}{1-m}, \frac{h}{1-m^2}, \cdots, \frac{h}{1-m^r}, \cdots,$$

consisting of the finite fixed points of σ and $\tau\sigma_r$ ($r=1, 2, \cdots$).

These points converge to the origin since $|m|>1$. Hence there exists a point ξ in the above sequence at which $f(z)$ is analytic, at which $f'(z)$ is not zero, and which is not an exceptional point of $f(z)$ in the sense of Picard. With this value of ξ, the corresponding function $g(z)$ will have the required properties (β) and (γ).

2. **The automorphism of** $g(z)$. Adapting a method of Ritt([4]), we now establish a relation for $g(z)$ which will enable us to show that $g(z)$ is periodic.

As $g(z)$ has an infinite number of zeros, suppose that $z_1\neq 0$ is one of them. Then, since $g(0)=0$ and $g'(0)\neq 0$, there exists a neighborhood of the origin, Γ_0, in which $g(z)$ assumes no value more than once.

Since $g(z_1)=0$, there exists a neighborhood Γ_1 of z_1, in which $g(z)$ takes no value not assumed in Γ_0.

Define $\phi(z)$, for z in Γ_1, as that unique point z_2 of Γ_0 for which $g(z_2)=g(z)$. Then

$$g[\phi(z)] \equiv g(z) \tag{5}$$

for z in Γ_1.

The function $\phi(z)$ is analytic in Γ_1, not identically 0, and has a Taylor development at z_1. Since $\phi(z_1)=0$, we may write that expansion as

$$\phi(z) = \alpha_1(z - z_1) + \alpha_2(z - z_1)^2 + \cdots + \alpha_n(z - z_1)^n + \cdots. \tag{6}$$

([4]) Ritt, loc. cit. pp. 17–19.

It will now be shown that $\phi(z)$ is linear. Then evidently $\alpha_1 \neq 0$, since $\phi(z) \not\equiv 0$ and our argument will show that $\alpha_n = 0$ for $n > 1$. Take s, an integer, so great that $z_1 + h/M^s$ is in Γ_1. Then, from (5),

$$g\left[\phi\left(z_1 + \frac{h}{M^s}\right)\right] = g\left(z_1 + \frac{h}{M^s}\right). \tag{7}$$

Now by (3) and (7) with U_s denoting the sth iterate of U, we have

$$g\left[M^s\phi\left(z_1 + \frac{h}{M^s}\right)\right] = U_s\left\{g\left[\phi\left(z_1 + \frac{h}{M^s}\right)\right]\right\} = U_s\left[g\left(z_1 + \frac{h}{M^s}\right)\right],$$

or

$$g\left[M^s\phi\left(z_1 + \frac{h}{M^s}\right)\right] = g\left[M^s\left(z_1 + \frac{h}{M^s}\right)\right] = g[M^s z_1 + h]. \tag{8}$$

Using (4) and then (3), we find

$$g[M^s z_1 + h] = V[g(M^s z_1)] = VU_s[g(z_1)].$$

But, since $g(z_1) = 0$, $VU_s[g(z_1)] = VU_s(0) = V(0)$. Thus (8) becomes

$$g\left[M^s\phi\left(z_1 + \frac{h}{M^s}\right)\right] = V(0). \tag{9}$$

We have by (6)

$$M^s\phi\left(z_1 + \frac{h}{M^s}\right) = \alpha_1 h + \alpha_2 \frac{h^2}{M^s} + \cdots.$$

Then by (9)

$$g\left(\alpha_1 h + \alpha_2 \frac{h^2}{M^s} + \cdots\right) = V(0) \tag{10}$$

for sufficiently large s.

It follows from (10) that $\alpha_n = 0$ for $n > 1$, otherwise allowing s to pass through all sufficiently large values we would have an infinite number of distinct points accumulating at the point $\alpha_1 h$ for each of which $g(z)$ assumes the value $V(0)$, and $\alpha_1 h$ would be an essential singularity of $g(z)$.

Thus (5) becomes

$$g[\alpha_1(z - z_1)] \equiv g(z). \tag{11}$$

This equation, proved for z in Γ_1, must hold for the whole plane since the functions in it are analytic.

This is the desired relation.

3. **Proof of periodicity.** It now follows that $g(z)$ is periodic. For in (11),

if $\alpha_1=1$, $-z_1$ is a period of $g(z)$. If $\alpha_1\neq 1$, then by using (4) and (11) we find

$$g(z+h) = V\{g[\alpha_1(z-z_1)]\} = g[\alpha_1(z-z_1)+h].$$

But from (11) with z replaced by $z+h$, we get

$$g(z+h) = g[\alpha_1(z-z_1)+\alpha_1 h].$$

Then

$$g[\alpha_1(z-z_1)+h] = g[\alpha_1(z-z_1)+\alpha_1 h],$$

and $h(\alpha_1-1)\neq 0$ is a period of $g(z)$.

The function $f(z)$ which is $g(z)+f(0)$ or $g[z-h/(1-m^r)]+f[h/(1-m^r)]$ is, of course, also periodic. This is the result stated in the introduction.

4. **The values of h.** Having established the periodicity of our functions, our next step is to determine the values of h for which addition theorems are possible if a multiplication theorem also holds. Evidently h may be a period and then $S(z)\equiv z$. But there are other values besides periods and it is these in which we are particularly interested.

Use will be made of the fact that together with $f(z)$, $F(z) = [Af(z)+B]/[Cf(z)+D]$ will also have a multiplication and addition theorem. In what follows, we will use $F(z)$ to denote such a linearly transformed $f(z)$.

As stated in the introduction, the periodic functions with multiplication theorems are divided into six categories and each of these will be considered in turn.

Case (a): $f(z)$ *a linear function of* $e^{\alpha z}$. Here α is arbitrary and the multiplier m must be an integer.

Replacing $f(z)$ by a suitable linear function of itself, we may suppose that

$$F(z) = e^{\alpha z}.$$

Then since

$$F(z+h) = e^{\alpha h}F(z),$$

$F(z)$ and therefore $f(z)$ has an addition theorem for every value of h.

Case (b): $f(z)$ *a linear function of* $\cos(\alpha z+\beta)$. Here $\beta=k\pi/(m-1)$, k an integer, while α is arbitrary. The multiplier m must be an integer.

We may suppose

$$F(z) = \cos(\alpha z+\beta),$$

and have to determine for which values of h, $\cos(\alpha z+\alpha h+\beta)$ is a rational function of $\cos(\alpha z+\beta)$. Replacing $\alpha z+\beta$ by z and αh by γ, we must have

$$\cos(z+\gamma) = S[\cos z],$$

with S rational.

Let z_1 be any value of z and determine another value of z, z_2 so as to satisfy the congruence

$$z_1 + z_2 \equiv 0 \pmod{2\pi}. \tag{12}$$

Then, since $\cos z_1 = \cos z_2$,

$$\cos(z_1 + \gamma) = \cos(z_2 + \gamma).$$

We have the following possibilities; either

$$z_1 + \gamma \equiv z_2 + \gamma \pmod{2\pi}, \tag{13}$$

or

$$z_1 + \gamma + z_2 + \gamma \equiv 0 \pmod{2\pi}, \tag{13'}$$

or perhaps both of these congruences hold.

In any event, if (13) holds, then adding (12) to it we get

$$2z_1 \equiv 0 \pmod{2\pi};$$

that is, $z_1 = k\pi$, k an integer. If z_1, which was arbitrary, is given any value not of the form $k\pi$, only (13') can hold. We may suppose this done. Then the subtraction of (12) from (13') gives

$$2\gamma \equiv 0 \pmod{2\pi};$$

that is, $\gamma = k\pi$, k an integer. Then $h = k\pi/\alpha$.

Since $2\pi/\alpha$ is a period of $\cos(\alpha z+\beta)$, the values of h, other than the periods, must be the half-periods.

That this necessary condition for h is also sufficient is obvious. Thus, for h the half-period π/α, to which all other half-periods are congruent modulo $2\pi/\alpha$, we have

$$\cos[\alpha(z + \pi/\alpha) + \beta] = -\cos(\alpha z + \beta).$$

Case (c): $f(z)$ *a linear function of* $\wp(z+\beta)$. The multiplier m must satisfy the congruences

$$2m\omega_1 \equiv 0, \qquad 2m\omega_3 \equiv 0 \pmod{2\omega_1, 2\omega_3},$$

and the constant β is given by the equation

$$\beta = \frac{l\omega_1 + k\omega_3}{m - 1},$$

where l and k are integers, and $2\omega_1$, $2\omega_3$, a pair of primitive periods of $\wp(z)$.

By repeating the procedure of the preceding case, except that all congruences are taken modulis $2\omega_1$, $2\omega_2$ we find that

$$h = l\omega_1 + k\omega_3,$$

where l and k are any integers.

Again we get periods and half-periods for h.

To set up the addition theorem for the half-periods, it suffices to consider ω_1, ω_3 and $\omega_2 = -\omega_1 - \omega_3$, for any other half-period will be congruent to one of these modulis $2\omega_1$, $2\omega_3$.

The following formula is well known in the theory of elliptic functions ([5])

$$\wp(z + \omega_a) = e_a + \frac{(e_b - e_a)(e_c - e_a)}{\wp(z) - e_a}, \tag{14}$$

where $\wp(\omega_a) = e_a$ and a, b, c is any permutation of 1, 2, 3.

This is the required addition theorem if z is replaced by $z+\beta$.

Case (d): $f(z)$ *a linear function of* $\wp^2(z+\beta)$. Here $\wp(z)$ is lemniscatic, $m = p+qi$, with p and q integral, and

$$\beta = \frac{2l\omega_1 + 2k\omega_3}{(m-1)(1-i)},$$

where l and k are any integers.

We may replace $z+\beta$ by z and we shall suppose this done here and in the following cases. The values of h do not depend on those of β.

Taking $F(z) = \wp^2(z)$, we must have $\wp^2(z + h) = S[\wp^2(z)]$.

Since $\wp^2(iu) = \wp^2(u)$ in this case, the fourth order function $\wp^2(u)$ will take on the same value at points congruent to $i^s u$ ($s = 0, 1, 2, 3$) modulis $2\omega_1$, $2\omega_3$, and only at these points if the $i^s u$ are all incongruent.

If any two of $i^s u$ are congruent, it follows that they are all congruent, and $\wp^2(u)$ will have a point of the fourth order at any one of them; so that again $\wp^2(u)$ will take the same value only at the points $i^s u$.

Choose an arbitrary z_1, and determine z_2 so as to satisfy the congruence

$$z_2 \equiv iz_1 \pmod{2\omega_1, 2\omega_3}. \tag{15}$$

Then since $\wp^2(z_2) = \wp^2(z_1)$, it follows that

$$\wp^2(z_2 + h) = \wp^2(z_1 + h),$$

and

$$z_2 + h \equiv i^s(z_1 + h) \pmod{2\omega_1, 2\omega_3},$$

where s may be one of 0, 1, 2, 3, or perhaps any of them.

Subtracting (15) from this congruence gives

$$h(1 - i^s) \equiv (i^s - i)z_1 \pmod{2\omega_1, 2\omega_3};$$

and unless $s = 1$, this will determine z_1 as belonging to a particular residue class modulis $2\omega_1/(i^s - i)$, $2\omega_3/(i^s - i)$, whereas z_1 was chosen without restriction.

If $s = 1$, we find that

([5]) Tannery and Molk, *Théorie des functions élliptiques*, vol. 1, p. 193.

$$h = \frac{2l\omega_1 + 2k\omega_3}{1 - i},$$

where l and k are any integers.

Simplifying this expression, we get $h = (l - k)\omega_1 + (l + k)\omega_3$. Thus the values of h must be periods or suitable half-periods; namely, since $l-k$ and $l+k$ are both of the same parity, those half-periods which are congruent to ω_2 modulis $2\omega_1$, $2\omega_3$.

If we note that

$$i\omega_2 \equiv i(-\omega_1 - \omega_3) \equiv -\omega_3 + \omega_1 \equiv \omega_2 \pmod{2\omega_1, 2\omega_3},$$

and that $\wp(iu) = -\wp(u)$ for this case, then

$$e_2 = \wp(\omega_2) = \wp(i\omega_2) = -\wp(\omega_2) = -e_2,$$

and $e_2 = 0$.

Substituting $a = 2$ in (14) and squaring we get the addition formula

$$\wp^2(z + \omega_2) = \frac{e_1^2 e_3^2}{\wp^2(z)}.$$

Case (e): $f(z)$ *a linear function of* $\wp'(z+\beta)$. Here $\wp'(z)$ is equianharmonic, $m = p + qe^{2\pi i/3}$, with p and q integral and

$$(m - 1)(1 - e^{2\pi i/3})\beta \equiv 0 \pmod{2\omega_1, 2\omega_3}.$$

This case and the next one may be treated in the same way as case (d) and we simply state the results.

Since $\wp'(e^{2\pi i/3}u) = \wp'(u)$ in this case, $e^{2\pi i/3}$ takes the place of i in the preceding case and thus

$$h = \frac{2l\omega_1 + 2k\omega_3,}{1 - e^{2\pi i/3}},$$

where l and k are any integers.

Rationalizing the denominator of this fraction, we get

$$h = \frac{2\omega_1(2l - k) + 2\omega_3(l + k)}{3}.$$

Thus, apart from a period, h must be a suitable third of a period.

To get our addition theorem in the latter instance, we note that taken modulis $2\omega_1$, $2\omega_3$ all these thirds of periods are congruent to $\gamma = (2\omega_1 - 2\omega_3)/3$ or $-\gamma$. It suffices to consider γ. Since $\wp(e^{2\pi i/3}u) = e^{2\pi i/3}\wp(u)$ and

$$e^{2\pi i/3}\gamma \equiv \gamma \pmod{2\omega_1, 2\omega_3},$$

we have

$$\wp(\gamma) = \wp(e^{2\pi i/3}\gamma) = e^{2\pi i/3}\wp(\gamma)$$

and $\wp(\gamma) = 0$.

If we let $u = \gamma$ in the addition formula for $\wp(z+u)$[6] and differentiate the resulting expression with respect to z, we find

$$\wp'(z+\gamma) = \frac{\wp'(z)\wp'(\gamma) - 3[\wp'(\gamma)]^2}{\wp'(z) + \wp'(\gamma)}.$$

Case (f): $f(z)$ *a linear function of* $\wp^3(z+\beta)$. Here $\wp(z)$ is equianharmonic, the multiplier $m = p + qe^{2\pi i/3}$, with p and q integral, and

$$(m-1)(1-e^{\pi i/3})\beta \equiv 0 \pmod{2\omega_1, 2\omega_3}.$$

Since $\wp^3(e^{\pi i/3}u) = \wp^3(u)$, we may use $e^{\pi i/3}$ in the same manner as $e^{2\pi i/3}$ and i in the preceding two cases. The result is that

$$h = \frac{2l\omega_1 + 2k\omega_3}{1 - e^{\pi i/3}},$$

with l and k any integers.

This simplifies to

$$h = (1 + e^{2\pi i/3})(2l\omega_1 + 2k\omega_3).$$

Then the only values of h in this case are periods of $f(z)$.

5. **The case of $|m| \leqq 1$, m not a rational root of unity.** We first show that, in this case, $f(z)$ must be rational. For otherwise, suppose that $f(z)$ is transcendental.

Then again considering the sequence

$$0, \frac{h}{1-m}, \frac{h}{1-m^2}, \cdots, \frac{h}{1-m^r}, \cdots,$$

whose points are all distinct, we can find a point in it which is not an exceptional point of $f(z)$. If $f(z)$ is analytic at this point, the transformations of §1 will then give us a $g(z)$ having properties (α) and (γ) and satisfying relations of type (3) and (4).

If $f(z)$ has a pole at the non-exceptional point, we take $g(z) = 1/f(z)$ or $g(z) = 1/f[z + h/(1-m^r)]$ according as our point is 0 or $h/(1-m^r)$, and again get a $g(z)$ fulfilling these same four conditions. In either case, $|M| \leqq 1$, M not a rational root of unity.

If $z_1 \neq 0$ is one of the zeros of $g(z)$, then the relation

$$g(Mz) = U[g(z)]$$

shows that $M^s z_1$ is a zero of $g(z)$ for every positive integral s. If $|M| < 1$, these

[6] See, for example, Tannery and Molk, loc. cit. p. 172.

zeros will converge to the origin; if $|M|=1$, M not a rational root of unity, they will be everywhere dense on the circle $|z|=|z_1|$. Both of these results are impossible for a nonconstant meromorphic function.

Therefore, let $f(z)$ be rational(7). Then R and S are both linear. Replacing $f(z)$ by a suitable linear function of itself, which will also be rational, we may obtain the addition theorem in one of the forms:

$$F(z+h) = aF(z), \qquad a \neq 1$$

or

$$F(z+h) = F(z) + b.$$

The first of these is impossible for a nonconstant rational $F(z)$, since the existence of a zero or a pole at z_1 would imply the existence of an infinite number of zeros or poles at the points z_1+nh, n any integer.

The second gives

$$F(z) = \frac{bz}{h} + \Pi(z),$$

where $\Pi(z)$ is a meromorphic periodic function of period h.

Here $\Pi(z)$ must be rational and therefore constant. Then $F(z)$ and also $f(z)$ is a linear function of z.

6. **The case m a primitive nth root of unity.** If m is a primitive nth root of unity,

$$f(m^n z) = f(z) = R_n[f(z)],$$

and the rational function $R(z)$ must be a periodic linear function of z with period n or a divisor of n.

For every positive integral p and r, we have

$$f(m^r z + ph) = S_p[f(m^r z)] = S_p R_r[f(z)].$$

Then, if r is not a multiple of n, the function $g(z) = f(z+ph/(1-m^r))$ obeys the relation

$$g(m^r z) = S_p R_r[g(z)].$$

This shows that $S_p R_r(z)$ is a linear periodic function of z with period n or a divisor of n for every positive integral p and r, r not a multiple of n.

The function $S(z)$ itself then must be linear.

Those cases with $R(z)$ of period $d<n$ are easily handled. For with $r=d$ and $p=1$, $S_p R_r(z)$ becomes $S(z)$ which must be periodic. Then $f(z)$ is periodic(8).

(7) The following argument holds for any value of m. We thus complete our discussion of Case (A) for $f(z)$ rational.

(8) We refer to the paper of Ritt, loc. cit., for the enumeration of the functions in this case.

In a similar manner, it may be shown that $f(z)$ is periodic if the period of $S_pR(z)$ is less than n.

These cases disposed of, we may assume both $R(z)$ and $S_pR(z)$ to be of period n ($p=1, 2, \cdots$).

Then, replacing $f(z)$ by a suitable linear function of itself, we have two sub-cases according as

$$F(z+h) = aF(z), \qquad a \neq 1$$

or

$$F(z+h) = F(z) + b.$$

Subcase 1: $S(z) \equiv az$, $a \neq 1$. Suppose first that $m=-1$, that is, $n=2$. Then $R(z)$, being of period 2, is of the form $(Az+B)/(Cz-A)$. The function $SR(z)=a(Az+B)/(Cz-A)$ must also be of period 2.

If $A \neq 0$, a must be -1 and $F(z)$ is periodic.

If $A=0$, a is arbitrary. Solving the addition relation for $F(z)$ by elementary means, we get

$$F(z) = e^{\alpha z}\Pi(z),$$

where $\alpha = \log a/h$ and $\Pi(z)$ is a meromorphic function with period h.

In order for this function, $F(z)$, to satisfy the multiplication theorem, we find that $\Pi(z)$, a periodic function, must also have a multiplication theorem, namely

$$\Pi(-z) = \frac{B}{C\Pi(z)}.$$

Thus $\Pi(z)$ and also $F(z)$ are characterized.

Let us now consider the values of m for which $n>2$. Then with $R(z) = (Az+B)/(Cz+D)$, we have

$$S_pR(z) = a^p \cdot \frac{Az+B}{Cz+D};$$

and this function must be of period n for every positive integral p.

The multiplier, K, of the periodic linear transformation $S_pR(z)$ is a primitive nth root of unity which we denote by ϵ_p, the subscript indicating its dependence on p. Using the known relation between the multiplier and the coefficients of the transformation(9), we find that K is algebraic in a^p, and we may write

$$\epsilon_p = W(a^p) \qquad (p = 1, 2, \cdots) \tag{16}$$

where $W(z)$ is an algebraic function of z.

(9) See, for example, Ford, *Automorphic functions*, p. 16.

There being only a finite number of primitive nth roots of unity, at least one of them will occur in (16) as ϵ_p an infinite number of times as p grows large. This shows that a must be a rational root of unity since otherwise the algebraic function $W(z)$ would take the same value, ϵ_p, at an infinite number of distinct points $z=a^p$. Thus $S(z)$ is periodic. We conclude that if $n>2$, $f(z)$ must be periodic.

Subcase 2: $S(z)\equiv z+b$. If $b=0$, $f(z)$ is periodic. Suppose $b\neq 0$. Taking $R(z)=(Az+B)/(Cz+D)$ with $AD-BC=1$, the linear transformation $S_pR(z)=[(A+pbC)z+(B+pbD)]/(Cz+D)$, $p=1, 2, \cdots$, is periodic and hence elliptic. Using the classic condition that a linear transformation of determinant unity be elliptic([10]), we would have $|A+pbC+D|<2$ for every positive integral p and this implies that $C=0$. Then $A=\epsilon D$ where ϵ is a primitive nth root of unity. By taking a suitable linear function of $F(z)$ we may suppose our addition and multiplication theorem in the form

$$F(mz) = \epsilon F(z), \tag{17}$$

$$F(z+h) = F(z) + b. \tag{18}$$

These show that $F'(z)$ is a periodic function with a multiplication theorem. If $F'(z)$ is constant, $F(z)$ is a linear function of z. If $F'(z)$ is not constant, then, as stated in the introduction, m is either -1 or a primitive third, fourth, or sixth root of unity.

Taking the solution of (18) as

$$F(z) = bz/h + \Pi(z),$$

where $\Pi(z)$ is a meromorphic function of period h, we must determine $\Pi(z)$ so that (17) is also satisfied.

If $m=-1$, $\Pi(z)$ must be an odd function. Then, if $\Pi(z)$ is simply periodic, it is the product of $\sin(2\pi nz/h)$ by a meromorphic function of $\cos(2\pi nz/h)$, n any integer. If $\Pi(z)$ is elliptic, it is the product of $\wp'(z)$ by a rational function of $\wp(z)$.

If m is a primitive fourth root of unity, we may suppose that $m=i$. Then if $\epsilon=i$,

$$\Pi(iz) = i\Pi(z),$$

and $\Pi(z)$ is the product of $\wp'(z)$ by a rational function of $\wp^2(z)$ with $\wp(z)$ lemniscatic.

If $\epsilon=-i$, we proceed in a different manner. Let $\zeta(z)$ be the Weierstrass zeta-function corresponding to a lemniscatic $\wp(z)$ having h as a period.

The following relations hold for $\zeta(z)$.

$$\zeta(iz) = -i\zeta(z), \qquad \zeta(z+h) = \zeta(z) + \kappa_1, \qquad \zeta(z+ih) = \zeta(z) + \kappa_2.$$

Here κ_1 and κ_2 are constants each different from zero. To show this, we have

([10]) Ford, loc. cit. p. 23.

$$\zeta(iz) + \kappa_2 = \zeta(iz + ih) = -i\zeta(z + h) = -i\zeta(z) - i\kappa_1 = \zeta(iz) - i\kappa_1.$$

Thus $\kappa_2 = -i\kappa_1$, and the vanishing of either would imply the same for the other. But then $\zeta(z)$ would be elliptic, a contradiction.

If we let

$$F(z) = b\zeta(z)/\kappa_1 + \Pi(z),$$

then (17) and (18) will be satisfied only if $\Pi(z)$ is of period h and obeys the relation

$$\Pi(iz) = -i\Pi(z).$$

Then $\Pi(z)$ is a product of $\wp'''(z)$ by a rational function of $\wp^2(z)$ with $\wp(z)$ lemniscatic.

For the remaining cases the results are similar.

If m is a primitive cube root of unity,

$$F(z) = bz/h + \varphi(z)T[\wp'(z)],$$

or

$$F(z) = b\zeta(z)/\kappa_1 + \wp''(z)T[\wp'(z)],$$

where $\varphi(z)$ is equianharmonic and $T(z)$ is a rational function. If m is a primitive sixth root of unity, we replace $\varphi(z)$, $\wp''(z)$ and $\wp'(z)$ respectively by $\wp'''(z)$, $\wp^{V}(z)$, and $\wp^{3}(z)$. In each case it is understood that any linear function of $F(z)$ also has both a multiplication and addition theorem.

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