

AN INDIRECT SUFFICIENCY PROOF FOR THE PROBLEM OF BOLZA IN NONPARAMETRIC FORM

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1. Introduction. Until recently sufficiency theorems in the calculus of variations have been established by direct methods, either by the construction of a Mayer field or by an expansion theorem. Indirect proofs have now been devised. The first indirect proof was given in 1942 by McShane [2]⁽¹⁾ who established a very general sufficiency theorem for a weak minimum in the problem of Bolza. The methods used by McShane were extended by Myers [3] so as to obtain a sufficiency theorem for a semistrong minimum in the nonparametric case. In certain special cases Myers obtained a strong sufficiency theorem. In a recent paper the author [6] showed the methods introduced by McShane can be modified so as to yield a strong sufficiency theorem for the parametric problem of Bolza. Although these results are phrased in terms of the parametric problem they are equally applicable to the nonparametric case. As yet the general sufficiency theorems given by McShane, Myers and the author have not been established by direct methods.

Although the sufficiency theorems given by the author are applicable to the nonparametric problem of Bolza by transforming this problem to a parametric one, the details of the method used do not appear to be sufficiently general to enable one to obtain an indirect proof of the sufficiency theorem for the nonparametric case without recourse to this transformation. It is the purpose of the present paper to remedy this situation and to obtain certain additional results that are of interest. The difficulties encountered earlier were overcome mainly by a suitable choice of a function $K(C, C_0)$ which measures the deviation of a comparison arc C from the arc C_0 under consideration.

The problem to be studied in this paper is that of minimizing a function

$$I(C) = g(a) + \int_C f(a, x, y, \dot{y}) dx$$

in a class of arcs

$$C: \quad a^h, y^i(x) \quad (x^1 \leq x \leq x^2; h = 1, \dots, r; i = 1, \dots, n)$$

satisfying conditions of the form

$$(1.1) \quad \phi^\beta(a, x, y, \dot{y}) = 0 \quad (\beta = 1, \dots, m < n),$$

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⁽¹⁾ Numbers in brackets refer to the references cited at the end of the paper.

$$(1.2) \quad x^s = X^s(a), \quad y^i(x^s) = Y^{is}(a) \quad (s = 1, 2).$$

The components a^h of C are independent of x . It will be shown in §8 below that if C_0 is an arc satisfying conditions (1.1) and (1.2) and the conditions described in §2 there is a neighborhood \mathfrak{F} of C_0 and a constant $\rho > 0$ such that the inequality

$$(1.3) \quad I(C) - I(C_0) \geq \rho K(C, C_0)$$

holds for every admissible arc in \mathfrak{F} satisfying the conditions (1.1) and (1.2). Here

$$(1.4) \quad K(C, C_0) = (a^h - a_0^h)(a^h - a_0^h) + \int_c \{ [1 + (\dot{y}^i - \dot{y}_0^i)(\dot{y}^i - \dot{y}_0^i)]^{1/2} - 1 \} dx$$

and $a_0, y_0^i(x)$ are the functions defining C_0 , extended if necessary so that the integral (1.4) is well defined. As was remarked above, the integral (1.4) can be considered a measure of the deviation of the arc C from C_0 , provided C satisfies the end conditions (1.2). Further inequalities of the form (1.3) are also given in §8.

One of the interesting features of the sufficiency proof given below is that the problem is reduced in §§3, 4 to the study of one in which there are no differential side conditions (cf. [8]). The problem so obtained is of a type which, as far as we know, has not been treated heretofore and to which the usual field theory does not appear to be applicable. An analogue of the theorem of Lindeberg is given in §5.

The methods here used are applicable with simple modifications to the case when the arcs C are also required to satisfy isoperametric conditions of the form

$$I^\sigma(C) = g^\sigma(a) + \int_c f^\sigma(a, x, y, \dot{y}) dx = 0 \quad (\sigma = 1, \dots, p).$$

The modifications necessary are like those made by the author in the parametric case (see [6]) and will be left to the reader.

In carrying out our sufficiency proof it will be convenient at times to assume that the functions $X^1(a)$ and $X^2(a)$ given in (1.1) and (1.2) are constants. This hypothesis greatly simplifies our computations. Moreover no generality is lost thereby since this situation can be realized by replacing x by a new variable t defined by the equation

$$x = X^1(a) + t[X^2(a) - X^1(a)],$$

provided $X^2(a) > X^1(a)$, as we shall suppose.

2. A sufficiency theorem. The terminology and notations to be used are

essentially those used by the author for the parametric case [6]. For example, we shall use the vector notation

$$C: \quad a, \quad y(x) \quad (x^1 \leq x \leq x^2)$$

for an arc C . The lengths of vectors a , y , p , \dots will be denoted by $|a|$, $|y|$, $|p|$, \dots . A repeated index in a term will denote summation, except for the indices q and s .

It will be assumed that the functions $f(a, x, y, p)$, $\phi^\beta(a, x, y, p)$ and their derivatives with respect to a^h , y^i , p^i are continuous and have continuous derivatives on an open set \mathfrak{R} in (a, x, y, p) -space. The functions $g(a)$, $X^s(a)$, $Y^u(a)$ are assumed to be of class C'' on \mathfrak{R} . We suppose that $X^1(a) < X^2(a)$ on \mathfrak{R} . The subset of \mathfrak{R} on which $\phi^\beta = 0$ will be denoted by \mathfrak{D} .

By an *admissible arc* C will be meant one for which the functions $y^i(x)$ defining C are absolutely continuous and have integrable square derivatives and whose elements $[a, x, y(x), \dot{y}(x)]$ are in \mathfrak{R} for almost all x on x^1x^2 . We shall say that C satisfies the differential equations $\phi^\beta = 0$ if $\phi^\beta[a, x, y(x), \dot{y}(x)] = 0$ for almost all x on x^1x^2 .

We shall be concerned with a particular admissible arc

$$C_0: \quad a_0, \quad y_0(x) \quad (x^1 \leq x \leq x^2)$$

of class C'' that satisfies the conditions (1.1) and (1.2). It will be assumed that the matrix $\|\phi_{p^i}^\beta\|$ has rank m along C_0 and that there exists a set of multipliers, not identically zero,

$$(2.1) \quad \lambda^0 \geq 0, \quad \lambda^\beta(x)$$

of class C' such that if we write

$$F(a, x, y, p, \lambda) = \lambda^0 f + \lambda^\beta \phi^\beta, \quad G(a) = \lambda^0 g(a)$$

then the equations

$$(2.2) \quad (d/dx)F_{p^i} = F_{y^i},$$

$$(2.3) \quad G_h + [(F - \dot{y}^i F_{p^i})X_h^s + F_{p^i} Y_h^{is}]_{s=1}^{s=2} + \int_{C_0} F_{a^h} dx = 0$$

hold along C_0 . Here and elsewhere the subscript h on $G(a)$, $X^s(a)$, $Y^u(a)$ denotes the derivative of these functions with respect to a^h at $a=a_0$. The quantity in the brackets is to be evaluated at the initial end point of C_0 when $s=1$ and at the final end point when $s=2$.

We shall make a further restriction on our choice of multipliers (2.1). To this end let $E_F(a, x, y, p, q)$ be the E -function

$$E_F = F(a, x, y, q) - F(a, x, y, p) - (q^i - p^i)F_{p^i}(a, x, y, p)$$

and denote by $E_L(p, q)$ the corresponding E -function for the integrand

$$L(p) = (1 + p^i p^i)^{1/2}$$

of the length integral. Recall that \mathfrak{D} is the set of all elements (a, x, y, p) in \mathfrak{R} for which $\phi^\beta = 0$. It will be assumed that there is a neighborhood \mathfrak{D}_0 of C_0 relative to \mathfrak{D} and a constant $\tau > 0$ such that the inequality

$$E_F(a, x, y, p, q) \geq \tau E_L(p, q)$$

holds whenever (a, x, y, p) is in \mathfrak{D}_0 and (a, x, y, q) is in \mathfrak{D} . This is the condition of E -dominance recently introduced by the author. It is equivalent to the strengthened condition of Weierstrass together with the condition of nonsingularity⁽²⁾. The latter condition states that the determinant

$$\begin{vmatrix} F_{p^i p^j} & \phi_{p^i}^\gamma \\ \phi_{p^j}^\beta & 0 \end{vmatrix}$$

is different from zero along C_0 .

If it is possible to choose the multipliers (2.1) so that $\lambda^0 = 1$ we shall suppose that they have been so chosen. Then apart from an arbitrary positive constant the totality of multipliers that satisfy the conditions described above are expressible in the form

$$(2.4) \quad \lambda^0 \geq 0, \quad \lambda^\beta(x, b) = \lambda^\beta(x) + \lambda^{\beta\sigma}(x)b^\sigma \quad (\sigma = 1, \dots, l).$$

It is to be understood that we may have $l=0$. Here the functions $\lambda^{\beta\sigma}(x)$ form with $\lambda^{0\sigma}=0$ a maximal set of linearly independent multipliers of the form $\lambda^0=0, \lambda^\beta(x)$ with which C_0 satisfies equations (2.1) and (2.3). If $\lambda^0=0$ in (2.4), then $\lambda^\beta(x)$ is a linear combination of the set $\lambda^{\beta\sigma}(x)$ ($\sigma=1, \dots, l$). The functions $\lambda^{\beta\sigma}(x)$ are of class C' and the matrix $\|\lambda^{\beta\sigma}(x)\|$ has rank l on $x^1 x^2$. In what follows we frequently shall make statements about the parameters $b=(b^1, \dots, b^l)$ appearing in (2.4), thereby tacitly implying that $l \geq 1$. These statements are to be considered to be vacuously true when $l=0$.

In the following pages we shall be interested only in the multipliers belonging to this family. In view of this fact it is convenient to introduce the notation

$$(2.5) \quad H(a, x, y, p, b) = \lambda^0 f + \lambda^\beta(x, b)\phi^\beta + \theta(a, x, y, p)\phi^\beta\phi^\beta.$$

Here $\theta(a, x, y, p)$ is for the moment an arbitrary function of class C'' on \mathfrak{R} . It is readily verified that the conditions described above as well as the further condition imposed on C_0 in Theorem 2.1 below is independent of our choice of θ . The reason for its introduction will be made clear in Theorem 2.2 and in the next section.

⁽²⁾ The proof of these remarks can be found in the author's paper [4] on the Weierstrass E -function. This paper deals with the parametric problem. However the hypotheses are such that the results are equally applicable to the nonparametric case.

The E -function for H will be denoted by $E_H(a, x, y, p, b, q)$. Let B be the set of all elements $b = (b^1, \dots, b^i)$ for which there is a constant $\tau > 0$ and a neighborhood \mathfrak{D}_0 of C_0 relative to \mathfrak{D} such that the inequality

$$(2.6) \quad E_H(a, x, y, p, b, q) \geq \tau E_L(p, q)$$

holds whenever (a, x, y, p) is in \mathfrak{D}_0 and (a, x, y, q) is in \mathfrak{D} . The set B is an open set. To prove this observe that whenever (a, x, y, p) and (a, x, y, q) are in \mathfrak{D} we have

$$E_H(a, x, y, p, b, q) = E_F(a, x, y, p, q) + \lambda^{\sigma\beta} b^\sigma E_{\phi\beta}(a, x, y, p, q),$$

where $E_{\phi\beta}$ is the E -function for ϕ^β . Moreover, as has been shown by the author [4, p. 59], there is a constant $\tau_1 > 0$ and a neighborhood \mathfrak{D}_1 of C_0 relative to \mathfrak{D} such that the inequality

$$E_L(p, q) \geq \tau_1 |E_{\phi\beta}(a, x, y, p, q)|$$

holds whenever (a, x, y, p) is in \mathfrak{D}_1 and (a, x, y, q) is in \mathfrak{D} . From these remarks it follows readily that if (2.6) is effective for an element b_0 with a given constant τ and neighborhood \mathfrak{D}_0 in \mathfrak{D}_1 , then it will remain effective on a neighborhood of b_0 provided τ is replaced by $\tau/2$. Consequently B is open, as was to be proved. It is clear that the conditions (2.2) and (2.3) with F replaced by H hold on C_0 . It should also be observed that (2.6) implies (see [4, p. 57]) that the inequality

$$H_{p^i p^j} \pi^i \pi^j > 0$$

holds along C_0 whenever b is in B and $\pi \neq 0$ satisfies the conditions $\phi_p^{\beta i} \pi^i = 0$.

We next introduce the function

$$(2.7) \quad J(C, b) = G(a) + \int_C H(a, x, y, \dot{y}, b) dx$$

and observe that $J(C, b) = \lambda^0 I(C)$ whenever the equation $\phi^\beta = 0$ holds along C . The second variation $J_2(\gamma, b)$ of $J(C, b)$ along C_0 takes the form

$$J_2(\gamma, b) = 2Q(\alpha) + \int_{x^1}^{x^2} 2\Omega(\alpha, x, \eta, \dot{\eta}, b) dx.$$

Here $2\Omega(\alpha, x, \eta, \pi, b)$ is the second differential of $H(a, x, y, p, b)$ on C_0 with respect to the variables a^h, y^i, p^i . The function $2Q(\alpha)$ is a quadratic form in $\alpha = (\alpha^1, \dots, \alpha^r)$ such that the coefficient of $\alpha^h \alpha^k$ is

$$\begin{aligned} G_{hk} + [(H - \dot{y}^i H_{p^i}) X_{hk}^s + H_{p^i} y^{is}]_{s=1}^{s=2} \\ + [(H_x - \dot{y}^i H_{y^i}) X_h^s X_k^s + H_{y^i} (X_h^s Y_k^{is} + X_k^s Y_h^{is}) + H_{a^h} X_k^s + H_{a^k} X_h^s]_{s=1}^{s=2}. \end{aligned}$$

As before the subscripts h, k denote derivatives with respect to a^h and a^k

at $a=a_0$. When $s=1$ the quantity in the brackets is to be evaluated at the initial point of C_0 and when $s=2$ at the final end point of C_0 . If $X^1(a)$ and $X^2(a)$ are constant this coefficient takes the simpler form

$$(2.8) \quad G_{hk} + [H_p Y_{hk}^{is}]_{s=1}^{s=2}.$$

The symbol γ denotes a set of constants and absolutely continuous functions

$$\gamma: \quad \alpha^h, \eta^i(x) \quad (x^1 \leq x \leq x^2; h = 1, \dots, r; i = 1, \dots, n)$$

whose derivatives $\dot{\eta}^i(x)$ are integrable square on $x^1 x^2$. Such a system will be called an *admissible variation*. We shall be interested in admissible variations γ that satisfy along C_0 the differential equations

$$(2.9) \quad \Phi^\beta(\alpha, x, \eta, \dot{\eta}) = \phi_{\alpha^h}^\beta \alpha^h + \phi_{\eta^i}^\beta \dot{\eta}^i + \phi_{\dot{\eta}^i}^\beta \ddot{\eta}^i = 0$$

for almost all x on $x^1 x^2$ and the end conditions

$$(2.10) \quad \dot{\eta}^i(x^s) = (Y_h^{is} - \dot{y}^i(x) X_h^s) \alpha^h \quad (s = 1, 2).$$

Again if $X^1(a)$ and $X^2(a)$ are constants, these equations take the simpler form

$$\dot{\eta}^i(x^s) = Y_h^{is} \alpha^h.$$

As a first sufficiency theorem we have:

THEOREM 2.1. *Let C_0 have the properties described above. Suppose further that for every non-null admissible variation γ satisfying the conditions (2.9) and (2.10) there is an element b in B such that the inequality $J_2(\gamma, b) > 0$ holds. Then there is a neighborhood \mathfrak{F} of C_0 in axy -space such that the inequality $I(C) > I(C_0)$ holds for every admissible arc $C \neq C_0$ in \mathfrak{F} satisfying the differential equations (1.1) and end conditions (1.2).*

This result, stated in somewhat different form, was conjectured by McShane [2, p. 346]. As was stated in the introduction it is a corollary of the corresponding result for the parametric case. The hypothesis used by the author appears to be weaker. However they are equivalent. In this theorem the function $\theta(a, x, y, p)$ appearing in the definition of $J(C, b)$ is of no consequence. The function θ can be used to establish a new proof of Theorem 2.1 by showing that the following more general result is true.

THEOREM 2.2. *Let C_0 have the properties described in Theorem 2.1. The function $\theta(a, x, y, p)$ used in the definition of the function $J(C, b)$ can be chosen so that there is a neighborhood \mathfrak{F} of C_0 in axy -space such that given an admissible arc $C \neq C_0$ in \mathfrak{F} satisfying the end conditions (1.2) there exists an element b in B such that $J(C, b) > J(C_0, b)$.*

In this theorem we do not require the arc C to satisfy the differential equations (1.1). It is clear that Theorem 2.2 implies Theorem 2.1 in case $\lambda^0 = 1$. If $\lambda^0 = 0$ then $\dot{J}(C, b) = J(C_0, b) = 0$ for every arc C satisfying the conditions $\phi^\beta = 0$. Hence C_0 is the only arc in \mathfrak{F} satisfying the conditions (1.1) and (1.2). The conclusion in Theorem 2.1 is therefore vacuously true and Theorem 2.1 holds in either case.

3. Preliminary lemmas. The selection of the function $\theta(a, x, y, p)$ appearing in the definition of H will be made in the next section. In doing so we shall make use of certain results which we shall now establish. We begin with the following lemma.

LEMMA 3.1. *Under the hypothesis of Theorem 2.1, there is for each element b in B a constant θ_b such that if $\theta \geq \theta_b$ the inequality*

$$(3.1) \quad H_{p^i p^i} \pi^i \pi^i > 0$$

holds along C_0 whenever $\pi \neq 0$. Moreover if

$$\gamma_q: \quad \alpha_q, \eta_q(x) \quad (x^1 \leq x \leq x^2; q = 1, 2, \dots)$$

is a set of admissible variations converging uniformly on $x^1 x^2$ to a variation

$$\gamma_0: \quad \alpha_0, \quad \eta_0(x) \quad (x^1 \leq x \leq x^2)$$

then

$$(3.2) \quad \liminf_{q \rightarrow \infty} J_2(\gamma_q, b) \geq J_2(\gamma_0, b)$$

provided b is in B and $\theta \geq \theta_b$.

The first statement has been established by Reid [1, p. 679] and the author [4, p. 57]. The second follows from the fact that when (3.1) holds the function $J_2(\gamma, b)$ is lower semi-continuous (see [2, p. 358]).

LEMMA 3.2. *Under the hypotheses of Theorem 2.1 there is a compact subset B_0 of B and a constant θ_0 such that if $\theta(a, x, y, p) \geq \theta_0$ and γ is a non-null admissible variation satisfying the end conditions (2.10), there is an element b in B_0 such that $J_2(\gamma, b) > 0$. Moreover the constant θ_0 can be chosen so that the inequality (3.1) holds along C_0 whenever b is in B_0 and $\pi \neq 0$, provided $\theta \geq \theta_0$.*

In this lemma we do not require the variation γ to satisfy the differential equations (2.9). In order to prove this result let $P(\gamma, b)$ be the function obtained from $J_2(\gamma, b)$ by setting $\theta = 0$. Since $\phi^\beta = 0$ along C_0 we have

$$J_2(\gamma, b) = P(\gamma, b) + 2 \int_{x^1}^{x^2} \theta \Phi^\beta \Phi^\beta dx$$

where Φ^β is given by (2.9). Consequently if we set

$$Q(\gamma) = 2 \int_{x^1}^{x^2} \Phi^\theta \Phi^\theta dx$$

we have

$$J_2(\gamma, b) \geq P(\gamma, b) + \theta_0 Q(\gamma)$$

whenever $\theta \geq \theta_0$, the equality holding when $\theta = \theta_0$.

Since B is an open set it is the union of a denumerable set of compact sets B_1, B_2, \dots such that $B_q \subset B_{q+1}$. If the first statement of the lemma were false there would exist for every integer q a non-null admissible variation

$$\gamma_q: \quad \overset{h}{\alpha}_q, \quad \overset{i}{\eta}_q(x) \quad (x^1 \leq x \leq x^2)$$

satisfying the end conditions (2.10) such that the inequality

$$(3.3) \quad P(\gamma_q, b) + qQ(\gamma_q) \leq J_2(\gamma_q, b) \leq 0 \quad (\theta \geq q)$$

holds for every element b in B_q . Since the functions at hand are homogeneous in γ we can suppose that γ_q has been chosen so that

$$(3.4) \quad |\alpha_q|^2 + \int_{x^1}^{x^2} \{ |\eta_q(x)|^2 + |\dot{\eta}_q(x)|^2 \} dx = 1,$$

where a pair of vertical bars denotes the length of the vector at hand. As has been shown by McShane [2, pp. 353–355, 375] we can replace the sequence $\{\gamma_q\}$ by a subsequence, which we take to be the original sequence, which converges to a variation

$$\gamma_0: \quad \overset{h}{\alpha}_0, \quad \overset{i}{\eta}_0(x) \quad (x^1 \leq x \leq x^2)$$

in the sense that

$$(3.5) \quad \lim_{q \rightarrow \infty} \overset{h}{\alpha}_q = \overset{h}{\alpha}_0, \quad \lim_{q \rightarrow \infty} \overset{i}{\eta}_q(x) = \overset{i}{\eta}_0(x) \quad \text{uniformly on } x^1, x^2.$$

This variation obviously satisfies the end conditions (2.10).

As a next step we shall show that γ_0 satisfies the differential equations (2.9). To this end consider an element b in B and let q_b be chosen so that b is in B_{q_b} whenever $q \geq q_b$. Using Lemma 3.1, we see that there is a constant θ_b such that

$$(3.6) \quad \liminf_{q \rightarrow \infty} [P(\gamma_q, b) + \theta_b Q(\gamma_q)] \geq P(\gamma_0, b) + \theta_b Q(\gamma_0).$$

Using the inequality (3.3), which holds when $q \geq q_b$, together with the relations (3.6) and $Q(\gamma) \geq 0$, we see that

$$\liminf_{q \rightarrow \infty} Q(\gamma_q) \leq 0.$$

But since the Legendre condition holds for $Q(\gamma)$ we have (see [2, p. 358])

$$\liminf_{q \rightarrow \infty} Q(\gamma_q) \geq Q(\gamma_0) \geq 0.$$

Consequently $Q(\gamma_0) = 0$. From the definition of Q it is clear that this can be the case only if γ_0 satisfies the differential equations $\Phi^\theta = 0$ for almost all values of x on $x^1 x^2$, as was to be proved.

We shall show next that γ_0 must be the null variation. Suppose this were not the case. Then by our hypotheses there is an element b in B such that $J_2(\gamma_0, b) > 0$. Using this value of b in the last paragraph it is seen by (3.6), with $Q(\gamma_0) = 0$ and $P(\gamma_0, b) > 0$, that for large values of q one has

$$P(\gamma_q, b) + \theta_b Q(\gamma_q) > 0,$$

contradicting the inequality (3.3) which also holds for large values of q . It follows that γ_0 is the null variation.

We shall complete the proof of the lemma by showing that γ_0 cannot be the null variation. For suppose that this is the case. Let b be an element in B and take $\theta = \theta_b$, where θ_b is chosen as described in Lemma 3.1. Then by virtue of (3.6) we have

$$\liminf_{q \rightarrow \infty} J_2(\gamma_q, b) = \liminf_{q \rightarrow \infty} [P(\gamma_q, b) + \theta_b Q(\gamma_q)] \geq 0$$

since $P(\gamma_0, b) = Q(\gamma_0) = 0$. Using (3.3), we see that the equality must hold. Consequently, by virtue of (3.5) with $\alpha_0^h = \eta_0^i = 0$ and the definition of $J_2(\gamma, b)$ we have

$$(3.7) \quad 0 = \liminf_{q \rightarrow \infty} J_2(\gamma_q, b) = \liminf_{q \rightarrow \infty} \int_{x^1}^{x^2} H_{p^i p^j} \dot{\eta}_q^i \dot{\eta}_q^j dx.$$

Since by Lemma 3.1 the last integrand is a positive definite form there is a constant $c > 0$ such that inequality

$$H_{p^i p^j} \pi^i \pi^j \geq c \pi^i \pi^i \geq c |\pi|^2$$

holds. Consequently equation (3.7) implies that

$$\liminf_{q \rightarrow \infty} \int_{x^1}^{x^2} |\dot{\eta}_q|^2 dx = 0.$$

Using (3.4) and (3.5) we see that

$$\lim_{q \rightarrow \infty} \int_{x^1}^{x^2} |\dot{\eta}_q|^2 dx = 1.$$

This contradiction completes the proof of the first statement of Lemma 3.2.

In order to complete the proof of Lemma 3.2 observe that by continuity considerations it is seen that if $\theta = \theta_b$ is effective as described in Lemma 3.1

for an element $b = b_0$ it is effective for a neighboring element b . Hence by the Heine-Borel theorem there is a value θ_1 such that the inequality (3.1) holds for every element b in B_0 provided $\theta \geq \theta_1$ and $\pi \neq 0$. Increase θ_0 if necessary so that $\theta_0 \geq \theta_1$. The last statement in Lemma 3.2 is then valid, as was to be proved.

In the proof of Theorem 8.1 below we shall use the result given in the following lemma.

LEMMA 3.3. *Let θ_0 and B_0 be chosen as described in Lemma 3.2 and let*

$$(3.8) \quad J_2(\gamma, b, \sigma) = J_2(\gamma, b) + \sigma I_2(\gamma, b),$$

where $I_2(\gamma, b)$ is a function of γ, b of the same form as $J_2(\gamma, b)$. There exists a number $\sigma_0 > 0$ such that given a non-null admissible variation γ satisfying the end conditions (2.10), there is an element b in B_0 such that the inequality

$$(3.9) \quad J_2(\gamma, b, \sigma) > 0$$

holds whenever $|\sigma| < \sigma_0$.

Observe first that since $I_2(\gamma, b)$ is of the same form as $J_2(\gamma, b)$ there is a constant τ such that

$$(3.10) \quad |I_2(\gamma, b)| \leq \tau \int_{x^1}^{x^2} |\alpha|^2 + \int_{x^1}^{x^2} \{ |\eta|^2 + |\dot{\eta}|^2 \} dx$$

whenever b is in B_0 . Suppose now the lemma is false. Then there exists for every integer q a value σ_q , not exceeding $1/q$ in absolute value, and a non-null admissible variation

$$\gamma_q: \quad \alpha_q, \quad \eta_q(x) \quad (x^1 \leq x \leq x^2)$$

such that

$$J_2(\gamma_q, b, \sigma_q) \leq 0$$

whenever b is in B_0 . We can suppose that γ_q satisfies condition (3.4). Then $|I_2(\gamma_q, b)| \leq \tau$ by (3.10). Since $\lim_{q \rightarrow \infty} \sigma_q = 0$ we have

$$(3.11) \quad \liminf_{q \rightarrow \infty} J_2(\gamma_q, b, \sigma_q) = \liminf_{q \rightarrow \infty} J_2(\gamma_q, b) \leq 0$$

for every b in B_0 . As in the proof of Lemma 3.2 we can replace the sequence $\{\gamma_q\}$ by a subsequence, again denoted by $\{\gamma_q\}$, which converges uniformly on $x^1 x^2$ to an admissible variation γ_0 satisfying the end conditions (2.10). By (3.11) and (3.2), which holds because of our choice of θ_0 , we have $J_2(\gamma_0, b) \leq 0$ for every element b in B_0 . It follows from the last lemma that γ_0 is null. Hence equation (3.7) holds. As was seen in the proof of Lemma 3.2 this is impossible. This contradiction establishes Lemma 3.3.

4. **Choice of the function $\theta(a, x, y, p)$.** We are now in position to select

a function θ that will be effective as described in Theorem 2.2. This choice will be made in the following:

LEMMA 4.1. *Let C_0 satisfy the conditions described in Theorem 2.1 and select a compact set B_0 of B and a constant θ_0 having the properties described in Lemma 3.2. There exists a function $\theta(a, x, y, p) \geq \theta_0$ of class C'' such that when this function is used in the definition (2.5) of $H(a, x, y, p, b)$ there is a neighborhood \mathfrak{R}_0 of C_0 relative to \mathfrak{R} and a constant $\tau > 0$ such that the inequality*

$$(4.1) \quad E_H(a, x, y, p, b, q) \geq \tau E_L(p, q)$$

holds whenever b is in B_0 , (a, x, y, p) is in \mathfrak{R}_0 and (a, x, y, q) is in \mathfrak{R} .

The proof of this result is like that used by the author [4, §8] and Miss Lewis [8, §4] in a similar situation. Select $\theta = \theta_0$ in H . We first observe that there is a constant $\tau_1 > 0$ and a neighborhood \mathfrak{R}_1 of C_0 such that the inequality

$$(4.2) \quad E_H(a, x, y, p, b, q) \geq \tau_1 E_L(p, q)$$

holds whenever b is in B_0 , (a, x, y, q) is in \mathfrak{D} and (a, x, y, p) is in the intersection of \mathfrak{D} and \mathfrak{R}_1 . For by the argument like that following (2.6) it is seen for each element b_0 in B_0 the constant τ_1 and the neighborhood \mathfrak{R}_1 can be chosen so as to be effective for neighboring elements b . It follows from the Heine-Borel theorem and the compactness of B_0 that \mathfrak{R}_1 and τ_1 can be chosen so as to be effective for every b in B_0 , as described above. By virtue of Taylor's theorem and the continuity of our functions it follows from (3.1) that we can diminish τ_1 and \mathfrak{R}_1 so that the inequality (4.2) also holds whenever b is in B_0 and (a, x, y, p) , (a, x, y, q) are in \mathfrak{R}_1 but not necessarily in \mathfrak{D} .

We shall show next that a constant τ can be chosen so that there exists a second neighborhood \mathfrak{R}_0 of \mathfrak{R}_1 whose closure is interior to \mathfrak{R}_1 such that the inequality (4.1) holds whenever (a, x, y, p) is in \mathfrak{R}_0 , (a, x, y, q) is in \mathfrak{D} and b is in B_0 . To this end we select first a neighborhood \mathfrak{R}^* of C_0 whose closure is in \mathfrak{R}_1 . Then there is a positive constant $\epsilon < 1$ such that if (a, x, y, p) is in \mathfrak{R}^* and (a, x, y, q) is exterior to \mathfrak{R}_1 we have

$$(4.3) \quad 3\epsilon L(q) \leq E_L(p, q) \leq 2L(q).$$

We next select \mathfrak{R}_0 so small that there exist n continuous functions $r^i(a, x, y, p)$ defined on \mathfrak{R}_0 such that if (a, x, y, p) is in \mathfrak{R}_0 , then (a, x, y, r) is in \mathfrak{R}^* and $\phi^{\theta}(a, x, y, r) = 0$. Here we are using the notation r^i for $r^i(a, x, y, p)$. We have

$$(4.4) \quad E_H(a, x, y, p, b, q) = E_H(a, x, y, r, b, q) + R$$

where

$$\begin{aligned} R = & q^i [H_{p^i}(a, x, y, r, b) - H_{p^i}(a, x, y, p, b)] \\ & + [p^i H_{p^i}(a, x, y, p, b) - r^i H_{p^i}(a, x, y, r, b)] \\ & + H(a, x, y, r, b) - H(a, x, y, p, b). \end{aligned}$$

If \mathfrak{R}_0 is taken sufficiently small we have

$$|R| < \epsilon \tau_1 L(q)$$

for every element (a, x, y, p) in \mathfrak{R}_0 and b in B_0 . It follows from (4.2) and (4.3) that if b is in B_0 , (a, x, y, p) is in \mathfrak{R}_0 and (a, x, y, q) is in \mathfrak{D} but not in \mathfrak{R}_1 we have

$$E_H(a, x, y, p, b, q) \geq \tau_1 E_L(r, q) - \epsilon \tau_1 L(q) \geq 2\epsilon \tau_1 L(q) \geq \epsilon \tau_1 E_L(p, q).$$

Setting $\tau = \epsilon \tau_1$ in (4.1) we obtain the result described at the beginning of this paragraph.

We now select open sets $\mathfrak{R}_2, \mathfrak{R}_3, \dots$ whose union is \mathfrak{R} and which have the property that the closure of \mathfrak{R}_j is in \mathfrak{R}_{j+1} for $j=1, 2, \dots$. Let $\theta_j(a, x, y, p)$ be functions of class C'' such that

$$(4.5) \quad \theta_j = 0 \text{ on } \mathfrak{R}_{j-1}, \quad \theta_j \geq 0 \text{ on } \mathfrak{R}_j, \quad \theta_j = 1 \text{ on } \mathfrak{R} - \mathfrak{R}_j.$$

In what follows it will be understood that b is in B_0 and (a, x, y, p) is in \mathfrak{R}_0 . If (a, x, y, q) is in $\mathfrak{R}_{j+1} - \mathfrak{R}_j$ ($j \geq 1$), then (4.1) holds if $\phi^p(a, x, y, q) = 0$ and hence, by continuity, if

$$(4.6) \quad \phi^p(a, x, y, q) \phi^p(a, x, y, q) < \epsilon_j E_L(p, q),$$

where ϵ_j is a small positive constant. Select a constant δ_j such that

$$(4.7) \quad E_H(a, x, y, p, b, q) > \delta_j$$

whenever (a, x, y, q) is in $\mathfrak{R}_{j+1} - \mathfrak{R}_j$. Let d_j be a positive constant such that the inequality

$$(4.8) \quad d_j \epsilon_j E_L(p, q) + \delta_j > \tau_1 E_L(p, q) \quad (j \text{ not summed})$$

holds on this set. Set

$$(4.9) \quad \theta(a, x, y, p) = \theta_0 + d_j \theta_j \quad (j = 1, 2, \dots; j \text{ summed}).$$

We then have by (4.5)

$$(4.10) \quad \theta - \theta_0 = 0 \text{ on } \mathfrak{R}_0, \quad \theta - \theta_0 \geq 0 \text{ on } \mathfrak{R}, \quad \theta - \theta_0 \geq d_j \text{ on } \mathfrak{R} - \mathfrak{R}_j.$$

Setting $h = (\theta - \theta_0) \phi^p \phi^p$ we have $h \equiv 0$ on \mathfrak{R}_0 and

$$E_h(a, x, y, p, q) = h(a, x, y, q) \geq 0$$

provided (a, x, y, p) is in \mathfrak{R}_0 , as we have supposed. Let $H^* = H + h$. We then have

$$(4.11) \quad E_{H^*}(a, x, y, p, b, q) = E_H(a, x, y, p, b, q) + h(a, x, y, q)$$

whenever (a, x, y, p) is in \mathfrak{R}_0 . If (a, x, y, q) is in \mathfrak{R}_1 then (4.1) holds so that

$$(4.12) \quad E_{H^*}(a, x, y, p, b, q) \geq \tau E_L(p, q).$$

The same is true if (a, x, y, q) is in $\mathfrak{R}_{j+1} - \mathfrak{R}_j$ ($j \geq 1$) and (4.6) holds. If (4.6) fails to hold, then by (4.10)

$$h(a, x, y, q) \geq (\theta - \theta_0)\epsilon_j E_L(p, q) \geq d_j \epsilon_j E_L(p, q) \quad (j \text{ not summed}).$$

It follows from (4.8) and (4.11) that (4.12) holds in this case also. The function θ defined by (4.9) accordingly has the properties described in Lemma 4.1.

It is not difficult to show that there is a constant $\sigma > 0$ such that

$$L(p) \geq \sigma L(p - \dot{y}_0(x)) \quad (x^1 \leq x \leq x^2).$$

As has been shown by the author [4, p. 59] this implies the existence of a constant τ_1 and a neighborhood \mathfrak{R}_0 of C_0 such that the inequality

$$(4.13) \quad E_L(p, q) \geq \tau_1 E_L(p - \dot{y}_0(x), q - \dot{y}_0(x))$$

holds. Hence when (2.6) holds we have

$$E_H(a, x, y, p, b, q) \geq \tau \tau_1 E_L(p - \dot{y}_0(x), q - \dot{y}_0(x)).$$

We shall be interested in the particular case when $p = \dot{y}_0(x)$. This gives:

LEMMA 4.2. *Under the hypotheses of Lemma 4.1 there exists a constant $\tau > 0$ and a neighborhood \mathfrak{F} of C_0 in axy-space such that the inequality*

$$(4.14) \quad E_H(a, x, y, \dot{y}_0(x), b, q) \geq \tau E_L(0, q - \dot{y}_0(x)) = \frac{\tau |q - \dot{y}_0(x)|^2}{1 + L(q - \dot{y}_0(x))}$$

holds, whenever b is in B_0 , (a, x, y) is in \mathfrak{F} and (a, x, y, q) is in \mathfrak{R} .

5. Theorem of Lindeberg. Before proceeding to the proof of Theorem 2.2 it will be convenient to establish an analogue of the theorem of Lindeberg. This will be done in Theorem 5.2 below. The results here obtained are of interest apart from the application to be made in the next section. In the proof we do not utilize all the properties of $J(C, b)$. In fact we shall use only the property described in the conclusion of Lemma 4.1.

The results obtained below are based on the following lemma.

LEMMA 5.1. *Let $p(a)$, $M(a, x, y, b)$, $N_i(a, x, y, b)$ be continuous functions and let*

$$J^*(C, b) = p(a) + \int_C \{M(a, x, y, b)dx + N_i(a, x, y, b)dy^i\}.$$

Let B_0 be the subset of B described in Lemma 4.1. Given a constant $\epsilon > 0$ there exists a neighborhood \mathfrak{F} of C_0 in axy-space such that for every admissible arc C in \mathfrak{F} satisfying the end conditions (1.2) and every element b in B_0 one has

$$(5.1) \quad |J^*(C, b) - J^*(C_0, b)| \leq \epsilon[1 + E_H^*(C, b)],$$

where

$$(5.2) \quad E_H^*(C, b) = \int_C E_H[a, x, y, \dot{y}_0(x), b, \dot{y}] dx.$$

Using the fact that B_0 is compact the proof can be made by the argument used by the author [5, pp. 75–78] to establish a similar theorem for the parametric case. In the proof one can assume without loss of generality that $X^1(a)$ and $X^2(a)$ are constants.

The lower semicontinuity of $J(C, b)$ is given in the following theorem.

THEOREM 5.1. *Let B_0 be the compact subset of B described in Lemma 4.1. Given a constant $\epsilon > 0$ there is a neighborhood \mathfrak{F} of C_0 in axy -space such that the inequality*

$$(5.3) \quad J(C, b) \geq J(C_0, b) - \epsilon$$

holds whenever b is in B_0 and C is an admissible arc in \mathfrak{F} satisfying the end conditions (1.2).

In order to prove this result let

$$(5.4) \quad J^*(C, b) = G(a) + \int_C \{ H(a, x, y, \dot{y}_0(x), b) + (\dot{y} - \dot{y}_0(x)) H_{p^i}(a, x, y, \dot{y}_0(x), b) \} dx.$$

We then have

$$(5.5) \quad J(C, b) = J^*(C, b) + E_H^*(C, b),$$

where $E_H^*(C, b)$ is given by (5.2). Using the relation $J^*(C_0, b) = J(C_0, b)$ it is seen that

$$(5.6) \quad J(C, b) - J(C_0, b) = J^*(C, b) - J^*(C_0, b) + E_H^*(C, b).$$

Given a positive constant $\epsilon < 1$ let \mathfrak{F} be chosen so that (5.1) holds as described in Lemma 5.1. It follows that

$$J(C, b) - J(C_0, b) \geq -\epsilon + (1 - \epsilon)E_H^*(C, b) \geq -\epsilon$$

whenever b is in B_0 and C is an admissible arc in \mathfrak{F} satisfying the end conditions (1.2). This proves Theorem 5.1.

We are now in position to prove an analogue of the theorem of Lindeberg. To this end consider a second function

$$(5.7) \quad I(C, b) = P(a) + \int_C R(a, x, y, \dot{y}, b) dx$$

where $P(a)$ and $R(a, x, y, \dot{y}, b)$ have the continuity and differentiability prop-

erties on \mathfrak{R} prescribed for $G(a)$ and $H(a, x, y, p, b)$. It is assumed that the E -function E_R for R is such that there is a constant $\tau_1 > 0$ and a neighborhood \mathfrak{R}_0 of C_0 in axy -space such that the inequality

$$(5.8) \quad E_H(a, x, y, p, b, q) \geq \tau_1 |E_R(a, x, y, p, b, q)|$$

holds whenever (a, x, y, p) is in \mathfrak{R}_0 , (a, x, y, q) is in \mathfrak{R} and b is in the set B_0 described in Lemma 4.1. The length integral

$$\int_C L(\dot{y}) dx$$

has this property.

THEOREM 5.2. *Let B_0 be the compact subset of B described in Lemma 4.1. If $I(C, b)$ has the properties described above there is a constant $\rho > 0$ such that given a constant $\delta > 0$ there is a neighborhood \mathfrak{F} of C_0 in axy -space such that the inequality*

$$(5.9) \quad J(C, b) - J(C_0, b) \geq 2\rho \{ |I(C, b) - I(C_0, b)| - \delta \}$$

holds for every admissible arc C in \mathfrak{F} satisfying the end conditions (1.2) and for every b in B_0 . If C also satisfies the condition

$$(5.10) \quad |I(C, b) - I(C_0, b)| \geq 2\delta$$

then

$$(5.11) \quad J(C, b) - J(C_0, b) \geq \rho |I(C, b) - I(C_0, b)|.$$

In order to prove this result choose constants τ, τ_1 and a neighborhood \mathfrak{R}_0 of C_0 so that the inequalities (5.8) and (4.1) hold as stated. Choose $\rho = \tau_1/2$. Set

$$\begin{aligned} H_\sigma(a, x, y, p, b) &= H(a, x, y, p, b) + \sigma R(a, x, y, p, b), \\ G_\sigma(a) &= G(a) + \sigma P(a) \end{aligned}$$

where $\sigma = \pm \rho$ and consider the function

$$(5.12) \quad J(C, b, \sigma) = G_\sigma + \int_C H_\sigma dx = J(C, b) + \sigma I(C, b).$$

By (5.7) and (4.1) we have

$$(5.13) \quad E_{H_\sigma}(a, x, y, p, b, q) \geq (\tau/2) E_L(p, q)$$

whenever (a, x, y, p) is in \mathfrak{R}_0 , (a, x, y, q) is in \mathfrak{R} and b is in B_0 . It follows that $J(C, b, \sigma)$ satisfies the conditions upon which the proof of Theorem 5.1 is based. Given a constant $\epsilon > 0$ there is accordingly a neighborhood \mathfrak{F} (effective for $\sigma = \rho$ and for $\sigma = -\rho$) such that the inequality

$$J(C, b, \sigma) - J(C_0, b, \sigma) = J(C, b) - J(C_0, b) + \sigma[I(C, b) - I(C_0, b)] \geq -\epsilon$$

holds whenever b is in B_0 and C is an admissible arc in \mathfrak{F} satisfying the end conditions (1.2). Taking $\sigma = \pm \rho$ one obtains the inequality (5.9) with $\delta = \epsilon/2\rho$.

The last statement in Theorem 5.2 follows at once from (5.9). This proves the theorem.

By virtue of the inequality (4.14) the function

$$(5.14) \quad K(C, C_0) = |a - a_0|^2 + \int_{x^1}^{x^2} \{L(\dot{y}(x) - \dot{y}_0(x)) - 1\} dx,$$

considered as a function of C with C_0 held fast, has the properties prescribed for $I(C, b)$. This integral is a measure of the difference between C and C_0 and will be used in the next section. As a consequence of the last theorem we have the following corollary.

COROLLARY. *There exists a constant $\rho > 0$ such that given a constant $\sigma > 0$ there is a neighborhood \mathfrak{F} of C_0 in axy -space such that the inequality*

$$(5.15) \quad J(C, b) - J(C_0, b) \geq \rho K(C, C_0)$$

holds for every admissible arc C in \mathfrak{F} satisfying the end conditions (1.2) and the relation

$$(5.16) \quad K(C, C_0) \geq \sigma.$$

Before closing this section it will be convenient to establish the following result which will be useful in the next section.

THEOREM 5.3. *Let*

$$C_q: \quad a_q, \quad y_q(x) \quad (x_1 \leq x \leq x_2; q = 1, 2, \dots)$$

be a sequence of admissible arcs satisfying the end conditions (1.2) and having

$$(5.17) \quad \lim_{q \rightarrow \infty} a_q = a_0, \quad \lim_{q \rightarrow \infty} y_q(x) = y_0(x) \quad \text{uniformly on } x^1 x^2.$$

If for an element b in B_0

$$(5.18) \quad \limsup_{q \rightarrow \infty} J(C_q, b) \leq J(C_0, b)$$

then

$$(5.19) \quad \lim_{q \rightarrow \infty} K(C_q, C_0) = 0$$

and

$$(5.20) \quad \lim_{q \rightarrow \infty} \dot{y}_q(x) = \dot{y}_0(x) \quad \text{in measure on } x^1 x^2.$$

Given a constant $\rho > 0$ there is a constant $\delta > 0$ and an integer q_0 such that if M is a subset of x^1x^2 of measure at most δ and $q \geq q_0$ then

$$(5.21) \quad 0 \leq \int_M l_q(x) dx < \rho$$

where

$$(5.22) \quad l_q(x) = 1 + L(\dot{y}_q(x) - \dot{y}_0(x)).$$

The relation (5.19) follows from Theorem 5.2 with $I(C, b) = K(C, C_0)$ and (5.20) follows from (5.19). To prove the last statement in the theorem let ρ be a positive constant and choose q_0 such that when $q \geq q_0$ one has $K(C_q, C_0) < \rho/2$ and hence

$$(5.23) \quad \int_{x^1}^{x^2} \{l_q(x) - 2\} dx < \rho/2.$$

Let $\delta = \rho/4$ and consider a subset M of x^1x^2 of measure at most δ . Since the integrand of (5.23) is nonnegative we have

$$\int_M \{l_q(x) - 2\} dx < \rho/2$$

and hence

$$\int_M l_q(x) dx < 2\delta + \rho/2 = \rho,$$

as was to be proved.

6. The variation γ_0 . Theorem 2.2 will be established by showing that if it were false there would exist a non-null variation γ_0 satisfying end conditions (2.10) such that $J_2(\gamma_0, b) \leq 0$ for every element b in B_0 , contrary to the result described in Lemma 3.2. It is understood that the function $\theta(a, x, y, p)$ has been chosen so that Lemmas 3.2 and 4.1 are valid. We assume also that the functions $X^1(a)$ and $X^2(a)$ are constants.

Suppose now the conclusion in Theorem 2.2 is false. Then given an integer q there exists an admissible arc

$$C_q: \quad a_q, \quad y_q(x) \quad (x^1 \leq x \leq x^2)$$

in the $(1/q)$ -neighborhood of C_0 satisfying the end conditions (1.2) and having

$$(6.1) \quad J(C_q, b) \leq J(C_0, b)$$

whenever b is in B_0 . The arc C_q is taken distinct from C_0 . In view of Theorem

5.3 we have $\lim_{q \rightarrow \infty} \dot{y}_q(x) = \dot{y}_0(x)$ in measure. By replacing our sequence by a suitably chosen subsequence we can obtain the relations

$$(6.2) \quad \begin{aligned} \lim_{q \rightarrow \infty} a_q &= a_0, & \lim_{q \rightarrow \infty} y_q(x) &= y_0(x) & \text{uniformly on } x^1 x^2, \\ \lim_{q \rightarrow \infty} \dot{y}_q(x) &= \dot{y}_0(x) & & \text{almost uniformly on } x^1 x^2. \end{aligned}$$

Let k_q be the positive number defined by the equation

$$(6.3) \quad k_q^2 = K(C_q, C_0) = |a_q - a_0|^2 + \int_{x^1}^{x^2} \{L(\dot{y}_q - \dot{y}_0) - 1\} dx$$

where a pair of vertical bars denotes the length of the vector at hand. Following McShane and Myers we consider the variations

$$\gamma_q: \quad \alpha_q = (a_q - a_0)/k_q, \quad \eta_q(x) = (y_q(x) - y_0(x))/k_q.$$

Using the fact that we can replace the original sequence $\{C_q\}$ by a subsequence, it can be brought about that the sequence $\{\gamma_q\}$ will have the properties described in the following lemma.

LEMMA 6.1. *The variation γ_q satisfies the relation*

$$(6.4) \quad |\alpha_q|^2 + \int_{x^1}^{x^2} |\dot{\eta}_q|^2 / l_q dx = 1,$$

where

$$(6.5) \quad l_q(x) = 1 + L(\dot{y}_q(x) - \dot{y}_0(x)).$$

Moreover, given a constant $\epsilon > 0$ there is a constant $\delta > 0$ and an integer q_0 such that if $q \geq q_0$ and $x'_\sigma x''_\sigma$ ($\sigma = 1, \dots, N$) are N nonoverlapping subintervals of $x^1 x^2$ the sum of whose lengths does not exceed δ , then

$$(6.6) \quad \sum_{\sigma=1}^N |\eta_q(x'_\sigma) - \eta_q(x''_\sigma)| < \epsilon.$$

Equation (6.4) follows from (6.3) and the relation

$$L(\dot{y}_q - \dot{y}_0) - 1 = |\dot{y}_q - \dot{y}_0|^2 / l_q.$$

In order to prove the second statement in the lemma let ϵ be a positive constant and set $\rho = \epsilon^2$. Choose $\delta < \rho$ and q_0 related to ρ as described in Theorem 5.3. Let M be a set of nonoverlapping subintervals $x'_\sigma x''_\sigma$ ($\sigma = 1, \dots, N$) of $x^1 x^2$ whose length sum does not exceed δ . Then by Theorem 5.3

$$(6.7) \quad \int_M l_q dx < \rho = \epsilon^2.$$

But

$$\sum_{\sigma=1}^N |\eta_q(x_{\sigma}'') - \eta_q(x_{\sigma}')| = \sum_{\sigma=1}^N \left| \int_{x_{\sigma}'}^{x_{\sigma}''} \dot{\eta}_q dx \right| \leq \int_M |\dot{\eta}_q| dx.$$

By the inequality of Schwarz

$$\int_M |\dot{\eta}_q| dx \leq \left[\int_M |\dot{\eta}_q|^2 / l_q dx \right]^{1/2} \left[\int_M l_q dx \right]^{1/2}.$$

Hence

$$(6.8) \quad \sum_{\sigma=1}^N |\eta_q(x_{\sigma}'') - \eta_q(x_{\sigma}')| \leq \left[\int_M |\dot{\eta}_q|^2 / l_q dx \right]^{1/2} \left[\int_M l_q dx \right]^{1/2}.$$

It follows from (6.4) and (6.7) that (6.6) holds, as was to be proved.

LEMMA 6.2. *The sequence of arcs $\{C_q\}$ can be chosen so that there exists an admissible variation*

$$\gamma_0: \quad \alpha_0, \quad \eta_0(x) \quad (x^1 \leq x \leq x^2)$$

satisfying the end conditions (2.10) and having

$$(6.9) \quad \lim_{q \rightarrow \infty} \alpha_q = \alpha_0, \quad \lim_{q \rightarrow \infty} \eta_q(x) = \eta_0(x) \quad \text{uniformly on } x^1 x^2.$$

Moreover for every measurable subset M of $x^1 x^2$ on which $\{\dot{\eta}_q(x)\}$ converges uniformly to $\dot{\eta}_0(x)$ one has

$$(6.10) \quad \int_M |\dot{\eta}_0|^2 dx \leq \liminf_{q \rightarrow \infty} \int_M |\dot{\eta}_q(x)|^2 dx \leq 2.$$

Replace the original sequence $\{C_q\}$ by a subsequence, again denoted by $\{C_q\}$, such that $\{\alpha_q^h\}$ converges to a value α_0^h and $\{\eta_q^t(x)\}$ has a finite or an infinite limit on a denumerable dense subset D of $x^1 x^2$. By virtue of (6.4), the quantity α_0^h is finite and since the arcs C_q satisfy the end conditions (1.2) we have ($s=1, 2$)

$$(6.11) \quad \lim_{q \rightarrow \infty} \dot{\eta}_q^t(x^s) = \lim_{q \rightarrow \infty} [Y^{ts}(a_q) - Y^{ts}(a_0)]/k_q = Y_h^{ts} \alpha_0^h,$$

where Y_h^{ts} is the derivative of $Y^{ts}(a)$ with respect to a^h at $a=a_0$. Consequently $\{\eta_q^t(x)\}$ converges to a finite value at $x=x^1$ and $x=x^2$. It follows from the last lemma and the Heine-Borel theorem that the sequence $\{\eta_q^t(x)\}$ is bounded and hence converges to a finite value at each point of D .

Consider now a constant $\rho > 0$ and let δ and q_0 be related to $\epsilon = \rho/3$ as described in the last lemma. By the Heine-Borel theorem the interval $x^1 x^2$ can be covered by a finite number of subintervals of lengths δ with centers at

points x_1, \dots, x_w in D . Increase q_0 if necessary such that

$$|\eta_q(x_j) - \eta_r(x_j)| < \rho/3 \quad (q \geq r \geq q_0, j = 1, \dots, w).$$

Consider a point x on x^1x^2 and choose j so that $|x_j - x| < \delta$, then by Lemma 6.1 with $N=1$ we have

$$|\eta_q(x) - \eta_q(x_j)| < \rho/3 \quad (q \geq q_0).$$

From these inequalities it follows that

$$|\eta_q(x) - \eta_r(x)| \leq |\eta_q(x) - \eta_q(x_j)| + |\eta_q(x_j) - \eta_r(x_j)| + |\eta_r(x_j) - \eta_r(x)| < \rho,$$

provided $q \geq r \geq q_0$. The sequence $\{\eta_q(x)\}$ therefore converges uniformly to a function $\eta_0(x)$. Using this fact it follows from the inequality (6.6) that $\eta_0(x)$ is absolutely continuous on x^1x^2 . Since $X^*(a)$ are assumed to be constants it is seen from (6.11) that γ_0 satisfies the end conditions (2.10).

We shall show next that the functions $\eta_0^k(x)$ have integrable square derivatives. Following McShane we subdivide the interval x^1x^2 into 2^k equal intervals by points

$$x_1 = x^1 < x_2 < \dots < x_{2^k+1} = x^2.$$

Let $p_k^i(x)$ be continuous functions which coincide with $\eta_0^i(x)$ at these points and are linear between them. Then

$$(6.12) \quad \int_{x^1}^{x^2} |p_k|^2 dx = \sum_{\sigma=1}^{2^k} |\eta_0(x_{\sigma+1}) - \eta_0(x_\sigma)|^2 / (x_{\sigma+1} - x_\sigma).$$

From the relation (6.8) with $N=1$ we see that

$$\frac{|\eta_q(x_{\sigma+1}) - \eta_q(x_\sigma)|^2}{x_{\sigma+1} - x_\sigma} \leq \int_{x_\sigma}^{x_{\sigma+1}} \frac{|\dot{\eta}_q|^2}{l_q} dx \left[\frac{\int_{x_\sigma}^{x_{\sigma+1}} l_q dx}{x_{\sigma+1} - x_\sigma} \right].$$

But

$$\lim_{q=\infty} \int_{x_\sigma}^{x_{\sigma+1}} l_q dx = 2(x_{\sigma+1} - x_\sigma).$$

Hence

$$\sum_{\sigma=1}^{2^k} \frac{|\eta_0(x_{\sigma+1}) - \eta_0(x_\sigma)|^2}{x_{\sigma+1} - x_\sigma} \leq 2 \liminf_{q=\infty} \int_{x^1}^{x^2} \frac{|\dot{\eta}_q|^2}{l_q(x)} dx \leq 2.$$

It follows from (6.12) that

$$\liminf_{q=\infty} \int_{x^1}^{x^2} 2 |\dot{\eta}_q|^2 / l_q dx \geq \int_{x^1}^{x^2} |p_k|^2 dx.$$

Hence by Fatou's lemma $|\dot{\eta}_0(x)|^2$ is integrable and

$$2 \geq \liminf_{q=-\infty} \int_{x^1}^{x^2} 2 |\dot{\eta}_q|^2 / l_q dx \geq \liminf_{q=-\infty} \int_{x^1}^{x^2} |\dot{p}_k|^2 dx \geq \int_{x^1}^{x^2} |\dot{\eta}_0(x)|^2 dx.$$

The relation (6.10) follows from this relation and the fact that $l_q(x)$ converges uniformly to 2 on every subset M of x^1x^2 on which $\{\dot{\eta}_q(x)\}$ converges uniformly to $\dot{\eta}_0(x)$.

In the sequel it will be assumed that the sequence $\{C_q\}$ has been chosen as described in the last lemma. As a further result we have:

LEMMA 6.3. *Let M be a measurable subset of x^1x^2 on which $\{\dot{\eta}_q(x)\}$ converges uniformly to $\dot{\eta}_0(x)$. If $g(x)$ is integrable together with its square, then*

$$(6.13) \quad \lim_{q=-\infty} \int_M g(x)(\eta_q^i - \eta_0^i) dx = \lim_{q=-\infty} \int_M g(x)(\dot{\eta}_q^i - \dot{\eta}_0^i) dx = 0.$$

If in addition $g(x)$ is bounded, this equation holds for every measurable subset M of x^1x^2 . If $N_{ir}(x)$ ($r=0, 1, 2, \dots$) are continuous functions of x on x^1x^2 such that

$$(6.14) \quad \lim_{q=-\infty} N_{iq}(x) = N_{i0}(x) \quad \text{uniformly on } x^1x^2$$

then

$$(6.15) \quad \lim_{q=-\infty} \int_M N_{iq} \dot{\eta}_q^i dx = \int_M N_{i0} \dot{\eta}_0^i dx$$

for every measurable subset M of x^1x^2 .

The proof is a simple modification of a proof of a similar result given by McShane [2, p. 356]. By virtue of the relation

$$\left| \int_M g(x)(\eta_q^i - \eta_0^i) dx \right| \leq (\max |\eta_q - \eta_0|) \int_M |g(x)| dx$$

it follows that the first limit in (6.13) has the value zero.

We consider next the second limit. Let $g(x)$ and M have the properties described in the lemma. We can suppose $g(x)=0$ on the complement of M . Let $p(x)$ be a function of class C' such that

$$\int_{x^1}^{x^2} |g(x) - p(x)|^2 dx < \epsilon^2/128.$$

We shall show first that there exists an integer q_0 such that

$$(6.16) \quad \int_{x^1}^{x^2} |g(x) - p(x)|^2 l_q(x) dx < \epsilon^2/16.$$

To prove this let P be a bound for $|g(x) - p(x)|^2$ on x^1x^2 if $g(x)$ is bounded

and on the complement of M if g is unbounded. By Theorem 5.3 there is a constant $\delta > 0$ and an integer q_0 such that if $q \geq q_0$ the inequality $\int_{M_1} l_q(x) dx < \epsilon^2/32P$ holds for every set M_1 of measure at most δ . Let M_1 be chosen of measure at most δ such that $\{\dot{y}_q(x)\}$ converges uniformly to $\dot{y}_0(x)$ on its complement M_2 . We can suppose $M_2 \supset M$ if $g(x)$ is unbounded. Increase q_0 if necessary so that $l_q \leq 4$ on M_2 when $q \geq q_0$. We then have

$$\begin{aligned} \int_{M_1} |g(x) - p(x)|^2 l_q dx &\leq P \int_{M_1} l_q(x) dx < \epsilon^2/32, \\ \int_{M_2} |g(x) - p(x)|^2 l_q(x) dx &\leq 4 \int_{x^1}^{x^2} |g(x) - p(x)|^2 dx < \epsilon^2/32. \end{aligned}$$

Hence (6.16) holds when $q \geq q_0$, as stated.

Observe next that by virtue of (6.4), (6.10) and the relation $l_q \geq 2$ we have

$$\int_{x^1}^{x^2} |\dot{\eta}_q - \dot{\eta}_0|^2 / l_q dx \leq 2 \int_{x^1}^{x^2} |\dot{\eta}_q|^2 / l_q dx + 2 \int_{x^1}^{x^2} |\dot{\eta}_0|^2 / l_q dx \leq 4.$$

Using this result together with the inequality (6.16) it is seen by the inequality of Schwarz that

$$\begin{aligned} (6.17) \quad & \left| \int_{x^1}^{x^2} (g - p)(\dot{\eta}_q^i - \dot{\eta}_0^i) dx \right| \\ & \leq \left[\int_{x^1}^{x^2} (g - p)^2 l_q dx \right]^{1/2} \left[\int_{x^1}^{x^2} |\dot{\eta}_q - \dot{\eta}_0|^2 / l_q dx \right]^{1/2} < \epsilon/2, \end{aligned}$$

provided $q \geq q_0$. Moreover since by an integration by parts

$$\int_{x^1}^{x^2} p(\dot{\eta}_q^i - \dot{\eta}_0^i) dx = [p(\eta_q^i - \eta_0^i)]_{x^1}^{x^2} - \int_{x^1}^{x^2} p'(\eta_q^i - \eta_0^i) dx$$

we can increase q_0 if necessary so that

$$(6.18) \quad \left| \int_{x^1}^{x^2} p(\dot{\eta}_q^i - \dot{\eta}_0^i) dx \right| < \epsilon/2 \quad (q \geq q_0).$$

Combining this result with (6.17) we have for $q \geq q_0$

$$\left| \int_{x^1}^{x^2} g(\dot{\eta}_q^i - \dot{\eta}_0^i) dx \right| \leq \left| \int_{x^1}^{x^2} p(\dot{\eta}_q^i - \dot{\eta}_0^i) dx \right| + \left| \int_{x^1}^{x^2} (g - p)(\dot{\eta}_q^i - \dot{\eta}_0^i) dx \right| < \epsilon.$$

Since g was taken to be identically zero on the complement of M we have accordingly

$$\lim_{q \rightarrow \infty} \int_{x^1}^{x^2} g(\dot{\eta}_q^i - \dot{\eta}_0^i) dx = \lim_{q \rightarrow \infty} \int_M g(\dot{\eta}_q^i - \dot{\eta}_0^i) dx = 0,$$

as was to be proved.

In order to prove the last statement in the theorem we write

$$(6.19) \quad \int_M N_{iq} \dot{\eta}_q^i dx = \int_{x^1}^{x^2} N_{i0} \dot{\eta}_q^i dx + \int_{x^1}^{x^2} (N_{iq} - N_{i0}) \dot{\eta}_q^i dx.$$

By the second statement in the lemma we have

$$(6.20) \quad \lim_{q \rightarrow \infty} \int_{x^1}^{x^2} N_{i0} \dot{\eta}_q^i dx = \int_{x^1}^{x^2} N_{i0} \dot{\eta}^i dx.$$

By the inequality of Schwarz and (6.4) we have

$$\begin{aligned} \left| \int_{x^1}^{x^2} (N_{iq} - N_{i0}) \dot{\eta}_q^i dx \right| &\leq \left[\int_{x^1}^{x^2} |N_q - N_0|^2 l_q dx \right]^{1/2} \left[\int_{x^1}^{x^2} |\dot{\eta}_q|^2 / l_q dx \right]^{1/2} \\ &\leq \max |N_q - N_0| \left[\int_{x^1}^{x^2} l_q dx \right]^{1/2} \end{aligned}$$

It follows that the last term in (6.19) has the limit zero. Using (6.20) it is seen that (6.15) holds as stated.

7. Proof of Theorem 2.2. As the next step in the proof of Theorem 2.2 we shall show that

$$(7.1) \quad \lim_{q \rightarrow \infty} \frac{J^*(C_q, b) - J^*(C_0, b)}{k_q^2} = \frac{1}{2} J_2(\gamma_0, b) - \frac{1}{2} \int_{x^1}^{x^2} H_{p^i p^i} \dot{\eta}_0^i \dot{\eta}_0^i dx,$$

where $J^*(C, b)$ is given by (5.4), $J_2(\gamma, b)$ is the second variation of $J(C, b)$ along C_0 and the last integral is to be evaluated along C_0 . In order to establish this result we observe that by Taylor's theorem the integrand H_q^* of $J^*(C_q, b)$ can be put in the form

$$(7.2) \quad H_q^* = H + k_q \{ H_{a^i a^i} \dot{\eta}_q^i + H_{y^i y^i} \dot{\eta}_q^i + H_{p^i p^i} \dot{\eta}_q^i \} + (k_q^2/2) \{ M_q + \dot{\eta}_q^i N_{iq} \},$$

where H and its derivatives are to be evaluated along C_0 and

$$\begin{aligned} \lim_{q \rightarrow \infty} M_q &= 2\Omega(\alpha_0, x, \eta_0(x), 0, b) && \text{uniformly on } x^1 x^2 \\ \lim_{q \rightarrow \infty} N_{iq} &= 2\Omega_{\tau^i}(\alpha_0, x, \eta_0(x), 0, b) && \text{uniformly on } x^1 x^2. \end{aligned}$$

Using Lemma 6.3 it is seen that

$$\begin{aligned} (7.3) \quad \lim_{q \rightarrow \infty} \int_{x^1}^{x^2} \{ M_q + \dot{\eta}_q^i N_{iq} \} dx \\ = \int_{x^1}^{x^2} \{ 2\Omega(\alpha_0, x, \eta_0, \dot{\eta}_0, b) - H_{p^i p^i} \dot{\eta}_0^i \dot{\eta}_0^i \} dx. \end{aligned}$$

Similarly

$$(7.4) \quad \begin{aligned} G(a_q) &= G(a_0) + k_q G_h \alpha_q^h + (k_q^2/2) G_{2q}, \\ y^{is}(a_q) &= y^{is}(a_0) + k_q Y_h^{is} \alpha_q^h + (k_q^2/2) Y_q^{is}, \end{aligned}$$

where

$$(7.5) \quad \lim_{q \rightarrow \infty} G_{2q} = G_{hk} \alpha_0^h \alpha_0^k, \quad \lim_{q \rightarrow \infty} Y_q^{is} = Y_{hk}^{is} \alpha_0^h \alpha_0^k.$$

By the use of the Euler equations (2.2) and the transversality condition (2.3) with $F=H$ we find by (7.2) and (7.4) that

$$\frac{J^*(C_q, b) - J^*(C_0, b)}{k_q^2} = \frac{1}{2} \left\{ G_{2q} + [H_{p^i} Y_q^{is}]_{s=1}^{s=2} + \int_{x^1}^{x^2} \{ M_q + \dot{\eta}_q^i N_{iq} \} dx \right\}.$$

In view of (7.3) and (7.5) it follows that the relation (7.1) holds, as was to be proved.

We shall show next that

$$(7.6) \quad \liminf_{q \rightarrow \infty} k_q^{-2} \int_{x^1}^{x^2} E_H(a_q, x, y_q, \dot{y}_0, b, \dot{y}_q) dx \geq \frac{1}{2} \int_{x^1}^{x^2} H_{p^i p^j} \dot{\eta}_0^i \dot{\eta}_0^j dx,$$

where the second integral is to be evaluated along C_0 . To this end let M be a subset of $x^1 x^2$ on which the sequence $\{\dot{y}_q(x)\}$ converges uniformly to $\dot{y}_0(x)$. Then for large values of q we have, by Taylor's theorem,

$$(7.7) \quad E_H(a_q, x, y_q, \dot{y}_0, b, \dot{y}_q) = k_q^2 A_{ijq}(x) \dot{\eta}_q^i \dot{\eta}_q^j \quad (x \text{ on } M),$$

where

$$(7.8) \quad \lim_{q \rightarrow \infty} A_{ijq}(x) = A_{ij}(x) \quad \text{uniformly on } M$$

and

$$(7.9) \quad 2A_{ij} = H_{p^i p^j}(a_0, x, y_0(x), \dot{y}_0(x), b).$$

Consider now the equation

$$\int_M A_{ijq} \dot{\eta}_q^i \dot{\eta}_q^j dx = \int_M A_{ij} \dot{\eta}_q^i \dot{\eta}_q^j dx + \int_M (A_{ijq} - A_{ij}) \dot{\eta}_q^i \dot{\eta}_q^j dx.$$

By virtue of the relations (7.8) and (6.10) the last integral has the limit zero as q becomes infinite. It follows that

$$\liminf_{q \rightarrow \infty} \int_M A_{ijq} \dot{\eta}_q^i \dot{\eta}_q^j dx = \liminf_{q \rightarrow \infty} \int_M A_{ij} \dot{\eta}_q^i \dot{\eta}_q^j dx.$$

By the use of Lemma 6.3 we have

$$\lim_{q \rightarrow \infty} \int_M A_{ij} \dot{\eta}_0^i \dot{\eta}_q^j = \int_M A_{ij} \dot{\eta}_0^i \dot{\eta}_0^j dx.$$

Hence

$$\liminf_{q \rightarrow \infty} \int_M A_{ij} \dot{\eta}_q^i \dot{\eta}_q^j dx = \int_M A_{ij} \dot{\eta}_0^i \dot{\eta}_0^j + \liminf_{q \rightarrow \infty} \int_M A_{ij} (\dot{\eta}_q^i - \dot{\eta}_0^i) (\dot{\eta}_q^j - \dot{\eta}_0^j) dx.$$

The last term is non-negative since, by (7.9) and (3.1), the quadratic form $A_{ij} \pi^i \pi^j$ is positive definite. It follows from (7.7) and (7.9) that

$$\liminf_{q \rightarrow \infty} k_q^{-2} \int_M E_H(a_q, x, y_q, y_0, b, y_q) dx \geq 2^{-1} \int_M H_{p^i p^i} \dot{\eta}_0^i \dot{\eta}_0^j dx.$$

For large values of q we have

$$E_H(a_q, x, y_q(x), y_0(x), b, y_q(x)) \geq 0,$$

except on a set of measure zero. Hence

$$\liminf_{q \rightarrow \infty} k_q^{-2} E_H^*(C_q, b) \geq 2^{-1} \int_M H_{p^i p^i} \dot{\eta}_0^i \dot{\eta}_0^j dx.$$

Since $\{\dot{y}_q^i(x)\}$ converges to $\dot{y}_0^i(x)$ almost uniformly on $x^1 x^2$ it follows from our choice of M and the integrability of $|\dot{\eta}_0|^2$ that (7.6) holds, as stated.

Combining (7.6), (7.1), (6.1) and (5.6) it is found that

$$(7.10) \quad 0 \geq \liminf_{q \rightarrow \infty} \frac{J(C_q, b) - J(C_0, b)}{k_q^2} \geq \frac{J_2(\gamma_0, b)}{2}$$

for every element b in B_0 . If γ_0 were not null, there would exist, by Lemma 3.2, an element b in B_0 such that $J_2(\gamma_0, b) > 0$. It follows that γ_0 is null. Using (7.1) and (5.6) again, we see from (7.10) that

$$\liminf_{q \rightarrow \infty} k_q^{-2} E_H^*(C_q, b) \leq 0.$$

But by (4.13) there is a constant $\tau > 0$ such that

$$0 \geq \liminf_{q \rightarrow \infty} k_q^{-2} E_H^*(C_q, b) \geq \tau \liminf_{q \rightarrow \infty} \int_{x^1}^{x^2} |\dot{\eta}_q|^2 / l_q dx.$$

This is impossible in view of equation (6.4). This proves Theorem 2.2.

8. Further results. Consider now a second integral $I(C, b)$ having the properties described in the paragraph preceding Theorem 5.2. Suppose in addition that C_0 satisfies the Euler equations and transversality conditions for $I(C, b)$, whenever b is in B_0 . We can now prove the following theorem.

THEOREM 8.1. *If $I(C, b)$ has the properties described above there exists a*

constant $\rho > 0$ and a neighborhood \mathfrak{F} of C_0 such that given an admissible arc C in \mathfrak{F} satisfying the end conditions (1.2), there is an element b in B_0 such that

$$(8.1) \quad J(C, b) - J(C_0, b) \geq \rho |I(C, b) - I(C_0, b)|.$$

As in the proof of Theorem 5.2 we consider the function

$$J(C, b, \sigma) = J(C, b) + \sigma I(C, b).$$

It is clear that C_0 satisfies the Euler equations and the transversality conditions for $J(C, b, \sigma)$ since it satisfies the corresponding conditions for each of the functions $J(C, b)$ and $I(C, b)$. Moreover by the proof of Theorem 5.2 there is a constant $\rho > 0$ and a neighborhood \mathfrak{R}_0 of C_0 such that if $|\sigma| \leq \rho$ the inequality (5.13) holds whenever b is in B_0 , (a, x, y, p) is in \mathfrak{R}_0 and (a, x, y, q) is in \mathfrak{R} .

Consider next the second variation

$$J_2(\gamma, b, \sigma) = J_2(\gamma, b) + \sigma I_2(\gamma, b).$$

By virtue of Lemma 3.3 there is a constant $\sigma_0 > 0$ such that given a non-null variation γ satisfying the end conditions (2.10) there is a value b in B_0 such that $J_2(\gamma, b, \sigma) > 0$ if $|\sigma| \leq \sigma_0$. Decrease the number ρ chosen above so that $\rho \leq \sigma_0$. For $\sigma = \pm \rho$ the function $J(C, b, \sigma)$ has the properties of $J(C, b)$ upon which the proof of Theorem 2.2 is based. It follows that the conclusion in Theorem 2.2 is valid for $J(C, b, \sigma)$ as well as for $J(C, b)$. Choose a neighborhood \mathfrak{F} effective for $J(C, b, \sigma)$ with $\sigma = \pm \rho$ as described in Theorem 2.2. Then given an admissible arc C in \mathfrak{F} satisfying the end conditions (1.2) there is an element b in B_0 such that

$$0 \leq J(C, b, \pm \rho) - J(C_0, b, \pm \rho) = J(C_0, b) - J(C, b) \pm \rho [I(C, b) - I(C_0, b)].$$

The inequality (8.1) therefore holds as was to be proved.

Observing that the function $K(C, C_0)$ defined by (5.14) has the properties of $I(C, b)$ we obtain the following corollary.

COROLLARY 1. *Under the hypotheses of Theorem 2.1 there exists a constant $\rho > 0$ and a neighborhood \mathfrak{F} of C_0 in axy -space such that given an admissible arc C in \mathfrak{F} satisfying the end conditions (1.2) there is an element b in B_0 such that*

$$J(C, b) - J(C_0, b) \geq \rho K(C, C_0).$$

Moreover if C satisfies the differential equations (1.1) then

$$I(C) - I(C_0) \geq \rho K(C, C_0).$$

This result can be considered to be a generalization of the theorem of Osgood (cf. [6, p. 99]).

REFERENCES

1. W. T. Reid, *Isoperimetric problems of Bolza in nonparametric form*, Duke Math. J. vol. 5 (1939) pp. 675-691.
2. E. J. McShane, *Sufficient conditions for a weak relative minimum in the problem of Bolza*, Trans. Amer. Math. Soc. vol. 52 (1942) pp. 344-379.
3. F. G. Myers, *Sufficient conditions for the problem of Lagrange*, Duke Math. J. vol. 10 (1943) pp. 73-97.
4. M. R. Hestenes, *The Weirstrass E-function in the calculus of variations*, Trans. Amer. Math. Soc. vol. 60 (1946) pp. 51-71.
5. ———, *The theorem of Lindeberg in the calculus of variations*, Trans. Amer. Math. Soc. vol. 60 (1946) pp. 72-92.
6. ———, *Sufficient conditions for the isoperimetric problem of Bolza in the calculus of variations*, Trans. Amer. Math. Soc. vol. 60 (1946) pp. 93-118.
7. ———, *An alternate sufficiency proof for the normal problem of Bolza*, Trans. Amer. Math. Soc. vol. 61 (1947) pp. 256-264.
8. A. L. Lewis, *Sufficiency proofs for the problem of Bolza in the calculus of variations*, Dissertation, The University of Chicago, 1943.

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